Martin's "differential" approach: some classifying remarks

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Abstract

Key-Words: Gasdynamic prospect, genuinely nonlinear Burnat type "algebraic" approach, Martin type "differential" approach, Martin linearization, hyperbolic /elliptic-hyperbolic Monge−Ampère equation, wave-wave regular interaction, nondegeneracy.

A genuinely nonlinear Burnat type "algebraic" approach [centered on a duality connection between the hodograph character and the physical character] and a Martin type "differential" approach [centered on a Monge−Ampère type representation] are quickly overviewed to begin with.

The two mentioned constructions show some distinct, complementary, valences. • The genuinely nonlinear "algebraic" approach appears to be essential for some isentropic multidimensional extensions (simple waves solutions, regular interactions of simple waves solutions) with a classifying potential. • The "differential" approach appears, in its turn, to be essential for some anisentropic descriptions.

A "differential" approach, corresponding to a particular anisentropic context, is considered via a comparison between two significant versions of it − a [hyperbolic] unsteady one-dimensional version and an [elliptic-hyperbolic] steady two-dimensional one [which appears again to show a hyperbolic character in presence of a supersonic description]. This mentioned comparison suggests an important role for a "differential" linearization.

The "differential" linearized approach is finally parallelled to the genuinely nonlinear "algebraic" approach − making evidence of some nontrivial contrasts.

Incidentally, in the mentioned particular strictly anisentropic context the "algebraic" construction must be essentially replaced by a related construction − corresponding to a "differential" linearized approach. The two mentioned constructions are related as they coincide in an isentropic context.

1. Burnat's "algebraic" approach

For the multidimensional first order ***hyperbolic*** system of a gasdynamic type [whose coefficients only depend on u]

$$\sum_{j=1}^{n}\sum_{k=0}^{m} a_{ijk}(u)\frac{\partial u_j}{\partial x_k} = 0, \quad 1 \le i \le n \tag{1}$$

the "algebraic" approach (Burnat [1]) starts with identifying *dual* pairs of directions $\vec{\beta}, \vec{\kappa}$ [we write $\vec{\kappa} \rightharpoondown \vec{\beta}$] connecting [via their duality relation] the hodograph [= in the hodograph space H of the entities u] and physical [= in the physical space E of the independent variables] characteristic details. The duality relation at $u^* \in H$ has the form:

$$\sum_{j=1}^{n}\sum_{k=0}^{m} a_{ijk}(u^*)\beta_k\kappa_j = 0, \quad 1 \le i \le n. \tag{2}$$

Here $\vec{\beta}$ is an *exceptional* direction [= orthogonal in the physical space E to a characteristic character]. A direction $\vec{\kappa}$ dual to an exceptional direction $\vec{\beta}$ is said to be a *hodograph characteristic* direction.

Example 1. For the *one-dimensional* strictly hyperbolic version of system (1) a *finite* number n of dual pairs $\vec{\kappa}_i \rightharpoondown \vec{\beta}_i$ consisting in $\vec{\kappa}_i = \vec{R}_i$ and $\vec{\beta}_i = \Theta_i(u)[-\lambda_i(u), 1]$, where $\vec{R}_i$ is a right eigenvector of the $n \times n$ matrix a and λ_i is an eigenvalue of a, are available $(i = 1, ..., n)$. Each dual pair associates in this case, at each $u^* \in \mathcal{R}$ [for a suitable $\mathcal{R} \subset H$], to a vector $\vec{\kappa}$ a *single* dual vector $\vec{\beta}$. □

Example 2 (Peradzyński [10]). For the *two-dimensional* version (3) of (1), corresponding to an isentropic description (in usual notations: c is the sound velocity, v_x, v_y are fluid velocities)

$$\begin{cases} \dfrac{\partial c}{\partial t} + v_x\dfrac{\partial c}{\partial x} + v_y\dfrac{\partial c}{\partial y} + \dfrac{\gamma-1}{2}c\left(\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y}\right) = 0 \\ \dfrac{\partial v_x}{\partial t} + v_x\dfrac{\partial v_x}{\partial x} + v_y\dfrac{\partial v_x}{\partial y} + \dfrac{2}{\gamma-1}c\,\dfrac{\partial c}{\partial x} = 0 \\ \dfrac{\partial v_y}{\partial t} + v_x\dfrac{\partial v_y}{\partial x} + v_y\dfrac{\partial v_y}{\partial y} + \dfrac{2}{\gamma-1}c\,\dfrac{\partial c}{\partial y} = 0\ , \end{cases} \tag{3}$$

an *infinite* number of dual pairs are available at each $u^* \in H$. Each dual pair associates, at the mentioned u^*, to a vector $\vec{\kappa}$ a *single* dual vector $\vec{\beta}$. □

Example 3 (Peradzyński [11]). For the isentropic description corresponding to the *three-dimensional* version of (3) an *infinite* number of dual pairs are available at each $u^* \in H$. Each dual pair associates, at the mentioned u^*, to a vector $\vec{\kappa}$ a *finite* [constant, $\neq 1$] number of k independent exceptional dual vectors $\vec{\beta}_j$, $1 \le j \le k$; and therefore has the structure $\vec{\kappa} \rightharpoondown (\vec{\beta}_1, ..., \vec{\beta}_k)$. □

Definition 4 (Burnat [1]). A curve $\mathcal{C} \subset H$ is said to be *characteristic* if it is tangent at each point of it to a characteristic direction $\vec{\kappa}$. A hypersurface $\mathcal{S} \subset H$ is said to be *characteristic* if it possesses at least a characteristic system of coordinates. □

2. Genuine nonlinearity. Simple waves solution

Remark 5. As it is well-known [Lax], in case of an one-dimensional strictly hyperbolic version of (1) any hodograph characteristic curve $\mathcal{C} \subset \mathcal{R} \subset H$, of index i, is said to be *genuinely nonlinear (gnl)* if the dual constructive pair $\vec{\kappa}_i \rightharpoondown \vec{\beta}_i$ is restricted by $\vec{\kappa}_i(u) \diamond \vec{\beta}_i(u) \equiv \vec{R}_i(u) \cdot \mathrm{grad}_u \lambda_i(u) \neq 0$ in $\mathcal{R}$; see Example 1. This condition transcribes the requirement $\frac{\mathrm{d}\vec{\beta}}{\mathrm{d}\alpha} \neq 0$ along $\mathcal{C}$. □

Definition 6. We naturally extend the *gnl* character of a hodograph characteristic curve $\mathcal{C}$ to the cases corresponding to Examples 2 and 3, by requiring along $\mathcal{C}$

$$\left|\frac{\mathrm{d}\vec{\beta}}{\mathrm{d}\alpha}\right|\neq 0$$

and, respectively,

$$\sum_{\mu=1}^{k}\left|\frac{\mathrm{d}\vec{\beta}_\mu}{\mathrm{d}\alpha}\right|\neq 0. \qquad \square$$

Definition 7*a*. A solution of (1) whose hodograph is laid along a *gnl* characteristic curve is said to be a *simple waves solution* (here below also called *wave*). □

3. A class of solutions of the system (1). Wave-wave "algebraic" regular interactions. Riemann–Burnat invariants

Let $R_1,\ldots,R_p$ be *gnl* characteristic coordinates on a given p-dimensional characteristic region $\mathcal{R}$ of a hodograph hypersurface $\mathcal{S}$ with the normal $\vec{n}$. Solutions of the system

$$\frac{\partial u_l}{\partial x_s}=\sum_{k=1}^{p}\eta_k\kappa_{kl}(u)\beta_{ks}(u),\ u\in\mathcal{R};\ 1\leq l\leq n,\ 0\leq s\leq m;\ \ \vec{\kappa}_k\perp\vec{n},\ \ 1\leq k\leq p \tag{4}$$

appear to concurrently satisfy the system (1). This indicates an "algebraic" importance of the concept of dual pair [see (2)].

Definition 7*b*. A solution of (1) whose hodograph is laid on a characteristic hypersurface is said to correspond to a *wave-wave regular interaction* if its hodograph possesses a *gnl* system of coordinates and a set of ***Riemann−Burnat invariants*** $R(x)$ exists, structuring the dependence on x of the solution u by a *regular* interaction representation

$$u_l=u_l[R_1(x_0,...,x_m),...,R_p(x_0,...,x_m)],\ 1\leq l\leq n. \tag{5}$$

Remark 8. It is easy to see that for a wave-wave regular interaction solution of (4) $R_i(x)$ must fulfil an (overdetermined and Pfaff) system

$$\frac{\partial R_k}{\partial x_s}=\eta_k\beta_{ks}[u(R)],\ \ 1\leq k\leq p,\ 0\leq s\leq m. \tag{6}$$

• Sufficient restrictions for solving (6) are proposed in [5], [10], [11]. Also see [2], [4].

• A wave-wave regular interaction reflects the *nondegenerate* nature of the *gnl* hodographs of the interacting simple waves solutions. The "algebraic" characterization of a wave-wave regular interaction will be regarded to correspond to a case of ["algebraic"] nondegeneracy.

4. Martin's "differential" approach

• Burnat's approach, associated to the construction of a simple waves solution or of a regular interaction of simple waves solutions, appears to be essentially isentropic.
• To a particular anisentropic context we associate in §§4,5 a Martin approach: in two significant versions [unsteady one-dimensional, steady two-dimensional].
• Incidentally, and partially, in the mentioned particular anisentropic context the Burnat construction could be essentially replaced by a related construction [Remark 15] − corresponding to a Martin *linearization* approach (§§6,7).
• The two mentioned [Burnat's, Martin linearization] constructions are related as they are coincident in an isentropic context (§§8,9).

4.1. An unsteady one-dimensional version

We consider the gasdynamic system

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \dfrac{\partial(\rho v_x)}{\partial x} = 0 \\ \dfrac{\partial(\rho v_x)}{\partial t} + \dfrac{\partial}{\partial x}\left(\rho v_x^2 + p\right) = 0 \\ \dfrac{\partial(\rho S)}{\partial t} + \dfrac{\partial(\rho v_x S)}{\partial x} = 0\ , \quad S = S(p,\rho) \end{cases} \tag{7}$$

(in usual notations: ρ, v_x, p, S are respectively the mass density, fluid velocity, pressure and entropy density).

We use, to begin with, the first two equations $(7)_{1,2}$ to introduce (Martin [7], [8]) the functions $\psi, \tilde{\xi}$ and ξ cf.

$$\rho=\frac{\partial\psi}{\partial x},\ \rho v_x=-\frac{\partial\psi}{\partial t};\ \rho v_x=\frac{\partial\tilde{\xi}}{\partial x},\ \rho v_x^2+p=-\frac{\partial\tilde{\xi}}{\partial t};\ \xi=\tilde{\xi}+pt.$$

We get

$$\mathrm{d}x = \frac{1}{\rho}\mathrm{d}\psi + v_x\mathrm{d}t, \quad \mathrm{d}\tilde{\xi} = v_x\mathrm{d}\psi - \rho\mathrm{d}t, \quad \mathrm{d}\xi = v_x\mathrm{d}\psi + t\mathrm{d}p. \tag{8}$$

4.2. A steady two-dimensional version

For the gasdynamic system

$$\begin{cases} \dfrac{\partial(\rho v_x)}{\partial x} + \dfrac{\partial(\rho v_y)}{\partial y} = 0 \\ \dfrac{\partial}{\partial x}\left(\rho v_x^2 + p\right) + \dfrac{\partial}{\partial y}\left(\rho v_x v_y\right) = 0 \\ \dfrac{\partial}{\partial x}\left(\rho v_x v_y\right) + \dfrac{\partial}{\partial y}\left(\rho v_y^2 + p\right) = 0 \\ \dfrac{\partial(\rho v_x S)}{\partial x} + \dfrac{\partial(\rho v_y S)}{\partial y} = 0\ , \quad S = S(p,\rho) \end{cases} \tag{9}$$

we use, to begin with, the first three equations $(9)_{1,2,3}$ to introduce (Martin [8]) the functions $\psi, \tilde{\xi}, \tilde{\eta}$ cf.

$$\rho v_x=\frac{\partial\psi}{\partial y},\ \rho v_y=-\frac{\partial\psi}{\partial x};\ \rho v_x v_y=-\frac{\partial\tilde{\xi}}{\partial x},\ \rho v_x^2+p=\frac{\partial\tilde{\xi}}{\partial y};\ \rho v_x v_y=\frac{\partial\tilde{\eta}}{\partial y},\ \rho v_y^2+p=-\frac{\partial\tilde{\eta}}{\partial x}$$

and the functions

$$\xi = \tilde{\xi} - py,\ \eta = \tilde{\eta} + px$$

to get

$$\mathrm{d}\psi = -(\rho v_y)\mathrm{d}x + (\rho v_x)\mathrm{d}y, \quad \mathrm{d}\xi = v_x\mathrm{d}\psi - y\mathrm{d}p, \quad \mathrm{d}\eta = v_y\mathrm{d}\psi + x\mathrm{d}p. \tag{10}$$

5. Anisentropic details of Martin's approach.

5.1. An unsteady one-dimensional version

A continuous [smooth] strictly adiabatic [anisentropic] flow results behind a shock discontinuity of non-constant continuous [smooth] velocity which penetrates into a region of uniform flow ([7]). For such a particular anisentropic flow, entropy $S(p,\rho)$ in $(7)_3$ is a function of ψ alone, $F(\psi)$, determined by the shock conditions. Prescription of F as a function of ψ provides an algebraic relation between p, ρ, ψ throughout the anisentropic flow region.

We seek for solutions of (7) which fulfil the (natural; see [4]) requirement

$$\frac{\partial p}{\partial t}\frac{\partial \psi}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \psi}{\partial t} \neq 0, \tag{11}$$

use (11) to select [Martin] p and ψ as new independent variables in place of x and t, and compute from (8)

$$\frac{\partial x}{\partial \psi} = v_x \frac{\partial t}{\partial \psi} + \frac{1}{\rho},\ \frac{\partial x}{\partial p} = v_x \frac{\partial t}{\partial p},\ v_x = \frac{\partial \xi}{\partial \psi},\ t = \frac{\partial \xi}{\partial p}. \tag{12}$$

On eliminating x from $(12)_{1,2}$ and taking $(12)_{3,4}$ into account it results that ξ must fulfil the hyperbolic Monge−Ampère equation

$$\frac{\partial^2 \xi}{\partial p^2}\frac{\partial^2 \xi}{\partial \psi^2} - \left(\frac{\partial^2 \xi}{\partial p \partial \psi}\right)^2 = -\zeta^2(p,\psi) \equiv \frac{\partial}{\partial p}\left(\frac{1}{\rho}\right) \equiv -\frac{1}{\rho^2 c^2} \tag{13}$$

where $\rho = \rho(p,\psi)$ and $c(p,\psi) = \sqrt{\left(\frac{\partial \rho}{\partial p}\right)_S^{-1}}$ is an ad hoc sound speed. Finally, we compute from (12)

$$x = \int \left(\frac{\partial \xi}{\partial \psi}\frac{\partial^2 \xi}{\partial p \partial \psi} + \frac{1}{\rho}\right) \mathrm{d}\psi + \left(\frac{\partial \xi}{\partial \psi}\frac{\partial^2 \xi}{\partial p^2}\right) \mathrm{d}p. \tag{14}$$

Remark 9. For *any* smooth solution $\xi(p,\psi)$ of (13) we get from (12), (14)

$$v_x = v_x(p,\psi),\ x = x(p,\psi),\ t = t(p,\psi). \tag{15}$$

On reversing $(15)_{2,3}$ into $p = p(x,t)$, $\psi = \psi(x,t)$, via (11), and carrying this into $(15)_1$ we get a form $p(x,t)$, $v_x(x,t)$, $\psi(x,t)$ of the corresponding anisentropic solution of (7). □

Remark 10 [Appendix 1]. (*a*) The hyperbolicity of (7) corresponds to the hyperbolicity of (13). (*b*) On prescribing F we will not find the streamlines $\mathcal{C}_0$ among the physical characteristic fields of (7). (*c*) The two families of characteristics $\overline{\mathcal{C}}_\mp$ of (13) in the plane p,ψ appear to correspond to the two families of sound characteristics $\mathcal{C}_\pm$ in the plane x,t. □

5.2. A steady two-dimensional version

A continuous [smooth] strictly adiabatic [anisentropic] ***rotational*** flow results behind a curved shock discontinuity from a region of uniform flow ahead ([8]). For such a particular anisentropic flow, entropy $S(p,\rho)$ in $(9)_4$ and the Bernoulli type function $e + \frac{p}{\rho} + \frac{1}{2}V^2$ [e is the density of the internal energy, $V^2 = v_x^2 + v_y^2$; according to Crocco's form of (9) (see Appendix 5)] are functions of ψ alone, $F(\psi)$, respectively $H(\psi)$, determined by the shock conditions. Prescription of F and H as functions of ψ provides two algebraic relations among p,ρ,V^2,ψ throughout the anisentropic flow region.

We seek for solutions of (9) which fulfil the [natural] requirement

$$\frac{\partial p}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial \psi}{\partial x} \neq 0, \tag{16}$$

use (16) to select [Martin] p and ψ as new independent variables in place of x and y, and compute from $(10)_{2,3}$

$$x = \frac{\partial \eta}{\partial p},\ y = -\frac{\partial \xi}{\partial p},\ v_x = \frac{\partial \xi}{\partial \psi},\ v_y = \frac{\partial \eta}{\partial \psi}, \tag{17}$$

and from $(10)_1$

$$v_x \frac{\partial y}{\partial \psi} - v_y \frac{\partial x}{\partial \psi} = \frac{1}{\rho},\quad v_x \frac{\partial y}{\partial p} - v_y \frac{\partial x}{\partial p} = 0. \tag{18}$$

We then transcribe (18) via (17) cf.

$$\frac{\partial \xi}{\partial \psi}\frac{\partial^2 \xi}{\partial p \partial \psi}+\frac{\partial \eta}{\partial \psi}\frac{\partial^2 \eta}{\partial p \partial \psi}+\frac{1}{\rho(p,\psi)}=0, \quad \frac{\partial \xi}{\partial \psi}\frac{\partial^2 \xi}{\partial p^2}+\frac{\partial \eta}{\partial \psi}\frac{\partial^2 \eta}{\partial p^2}=0 \ . \tag{19}$$

Finally we integrate $(19)_1$ with respect to p and obtain

$$\left(\frac{\partial \xi}{\partial \psi}\right)^2+\left(\frac{\partial \eta}{\partial \psi}\right)^2=\mathcal{F}(p,\psi), \quad \frac{\partial \xi}{\partial \psi}\frac{\partial^2 \xi}{\partial p^2}+\frac{\partial \eta}{\partial \psi}\frac{\partial^2 \eta}{\partial p^2}=0 \ ; \quad \mathcal{F}(p,\psi)=2\mathcal{G}(\psi)-2\int^p \frac{\mathrm{d}p}{\rho} \tag{20}$$

where $\mathcal{F}$ is a known function and $\mathcal{G}$ is arbitrary. We solve simultaneously for $\frac{\partial \eta}{\partial \psi}$ and $\frac{\partial^2 \eta}{\partial p^2}$ in (20) and carry the result into $\frac{\partial^2}{\partial p^2}\left(\frac{\partial \eta}{\partial \psi}\right)=\frac{\partial}{\partial \psi}\left(\frac{\partial^2 \eta}{\partial p^2}\right)$ in order to eliminate η in favor of ξ. We are led to a Monge−Ampère type equation for ξ [Appendix 2]:

$$4\mathcal{F}\left[\left(\frac{\partial^2 \xi}{\partial p \partial \psi}\right)^2-\frac{\partial^2 \xi}{\partial p^2}\frac{\partial^2 \xi}{\partial \psi^2}\right]-4\left(\frac{\partial \xi}{\partial \psi}\frac{\partial \mathcal{F}}{\partial p}\right)\frac{\partial^2 \xi}{\partial p \partial \psi}+2\left(\frac{\partial \xi}{\partial \psi}\frac{\partial \mathcal{F}}{\partial \psi}\right)\frac{\partial^2 \xi}{\partial p^2}+\left\{\left(\frac{\partial \mathcal{F}}{\partial p}\right)^2-2\left[\mathcal{F}-\left(\frac{\partial \xi}{\partial \psi}\right)^2\right]\frac{\partial^2 \mathcal{F}}{\partial p^2}\right\}=0 \tag{21}$$

where $\rho=\rho(p,\psi)$ and $c(p,\psi)=\sqrt{\left(\frac{\partial \rho}{\partial p}\right)_S^{-1}}$ is an ad hoc sound speed.

As the system (20) is symmetric in ξ and η it results that η must fulfil the same Monge−Ampère type equation (21). A given solution ξ of (21) is paired by a computed [cf. (20)] solution η of (21) [Appendix 2].

The characteristic directions for (21) in the plane p,ψ are given by [Appendix 3]

$$\left(\frac{\mathrm{d}p}{\mathrm{d}\psi}\right)_\pm=\frac{2\mathcal{F}\frac{\partial^2 \xi}{\partial p \partial \psi}-\frac{\partial \mathcal{F}}{\partial p}\frac{\partial \xi}{\partial \psi}\pm\sqrt{\Delta}}{-2\mathcal{F}\frac{\partial^2 \xi}{\partial p^2}} \ , \quad \Delta=\frac{4}{\rho^2 c^2}v_y^2\left(V^2-c^2\right) \ , \quad V^2=v_x^2+v_y^2 \ . \tag{22}$$

We have [Appendix 4]

$$\left.\begin{array}{l} -V^2\mathrm{d}y+\dfrac{1}{\rho c}\left[cv_x\pm v_y\sqrt{V^2-c^2}\right]\mathrm{d}\psi=0 \\ v_y\left[v_y\mathrm{d}v_x-v_x\mathrm{d}v_y\mp\dfrac{1}{\rho c}\sqrt{V^2-c^2}\mathrm{d}p\right]=0 \end{array}\right. \quad \text{along the characteristics } \overline{\mathcal{C}}_\pm \text{ of (21).} \tag{23}$$

Remark 11. In contrast with the unsteady one-dimensional case, the system (9) and the Monge−Ampère equation (21) show an ***elliptic-hyperbolic*** character generally. Still, cf. (22), they show both a *hyperbolic* character for a *supersonic* flow. This aspect pairs the one-dimensional Remark 10*a*. □

Remark 12. The Mach lines $\mathcal{C}_\pm$ of (9) in the physical plane and the characteristics $\overline{\mathcal{C}}_\pm$ [(22)] of the Monge−Ampère equation (21) are in correspondence. In fact, we get from $(23)_1$ and $(10)_1$

$$-V^2\mathrm{d}y+\frac{1}{\rho c}\left[cv_x\pm v_y\sqrt{V^2-c^2}\right](v_x\mathrm{d}y+v_y\mathrm{d}x)=0 \quad \text{along the characteristics } \overline{\mathcal{C}}_\pm \text{ of (21)}$$

which results in

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{cv_x\pm v_y\sqrt{V^2-c^2}}{cv_y\mp v_x\sqrt{V^2-c^2}}=\frac{v_xv_y\pm c\sqrt{V^2-c^2}}{v_x^2-c^2}=\lambda_\pm \quad \text{along the characteristics } \overline{\mathcal{C}}_\pm \text{ of (21)}$$

where λ_+ and λ_- are the Mach eigenvalues of the system (9). This aspect pairs the one-dimensional Remark 10*c*. □

Remark 13. On prescribing F and H we will not find the streamlines among the physical characteristic fields of (9) [Appendix 5]. This aspect pairs the one-dimensional Remark 10b. □

6. Martin linearization. Wave-wave "differential" regular interaction. Riemann–Martin invariants

6.1. An unsteady one-dimensional version

Finding a solution to the systems (7) / (9) or, alternatively, to the Monge–Ampère equations (13) / (21) is a hard task generally. This suggests considering suitable classes of solutions to the systems (7) / (9) or, alternatively, to the Monge–Ampère equations (13) / (21).

In case of the systems (7) / (9) such classes are, for example, the simple waves solutions or the wave-wave regular interactions associated to an "algebraic" construction (§§1–3; [1], [10], [11]; [2], [4]). In case of the equations (13) / (21) such classes are constructed by a *linearization* approach ([9]).

We shall distinguish for the systems (7) / (9) or, alternatively, for the Monge–Ampère equations (13) / (21) between an *isentropic* context and an *anisentropic* context.

A Riemann–Lax invariance analysis indicates for (7) that an "algebraic" construction of simple waves solutions or wave-wave regular interactions must be associated to an *isentropic* context ([2], 4.6, Example 5.1; [4]).

Some "differential" *analogues* of the simple waves solutions or wave-wave regular interaction solutions are available for $\zeta = P(p)\Psi(\psi)$ in (13) *in six cases only* ([9]):

$$\zeta_1 = 1;\quad \zeta_2 = P(p) \not\equiv 0;\quad \zeta_3 = \frac{1}{p^2};\quad \zeta_4 = \frac{\psi^{\nu-1}}{p^{\nu+1}}\ \left(\nu = -\tfrac{\gamma-1}{2\gamma};\ \nu \neq 0,1\right);\quad \zeta_5 = \frac{\psi^{-1}}{p};\quad \zeta_6 = \mathrm{e}^p\,\mathrm{e}^\psi \tag{24}$$

where γ is the specific heat ratio. The cases of ζ_4, ζ_5 and ζ_6 correspond to a *strictly anisentropic* context. For each of these six cases to the equation (13) two intermediate integrals $\mathcal{F}_\pm\left(p, \psi, \xi, \frac{\partial\xi}{\partial p}, \frac{\partial\xi}{\partial\psi}\right)$, *linear* in ξ, can be associated (Martin [9]) for which $\mathcal{F}_\pm$=constant$_\pm$=$R_\pm$ along a characteristic $\overline{\mathcal{C}}_\pm$. An extension of this result to a more general form of ζ in (13) is considered in Ludford [6].

We have to distiguish, for each of the mentioned six cases, between the circumstances (a) when $R_\pm$ depend on the characteristic $\overline{\mathcal{C}}_\pm$, and ($b$) when R_+ or R_- are overall constants.

In the case (a) we may use (Martin [9]) $R_\pm$ as new independent variables. It can be shown in this case ([9]) that the entities p^{-1}, v_x, ψ^{-1}, t fulfil various Euler–Poisson–Darboux *linear* equations

$$\frac{\partial^2 w}{\partial R_+ \partial R_-} - \frac{\nu}{R_+ - R_-}\left(\frac{\partial w}{\partial R_+} - \frac{\partial w}{\partial R_-}\right) = 0, \quad \text{constant } \nu$$

to which well-known representations are associated; we present these representations by

$$\begin{gathered} p = p(R_+, R_-),\ \psi = \psi(R_+, R_-),\ v_x = v_x(R_+, R_-), \\ t = t(R_+, R_-),\ x = x(R_+, R_-) \end{gathered} \tag{25}$$

where $x(R_+,R_-)$ results by quadratures. Reversing $(25)_2$ into $R_\pm = R_\pm(x,t)$ will induce a form of solution (25), parallel to (5) [as $R_\pm$ have a characteristic nature]. We call $R_\pm(x,t)$ ***Riemann–Martin invariants***.

In the case (b) we notice that a solution $\xi(p,\psi)$ of the *linear* equation $\mathcal{F}_+ \equiv R_+$ or $\mathcal{F}_- \equiv R_-$ will automatically fulfil (13). We have to follow, in this case, Remark 9 to describe a solution of (7); we call such solution a *pseudo simple waves solution*. See [7] for some numerical remarks.

Solution (25) might be regarded as *pseudo nondegenerate* [a formal regular interaction of *pseudo* simple waves solutions]. The image of a characteristic $\mathcal{C} \subset E$ on the hodograph of a solution of (7) will be said to be a M- characteristic. The hodograph of a formal regular interaction of pseudo simple waves solutions will be then made by glueing, along suitable M- characteristics, a hodograph (25) with some suitable hodographs of pseudo simple waves solutions; see Figure 1.

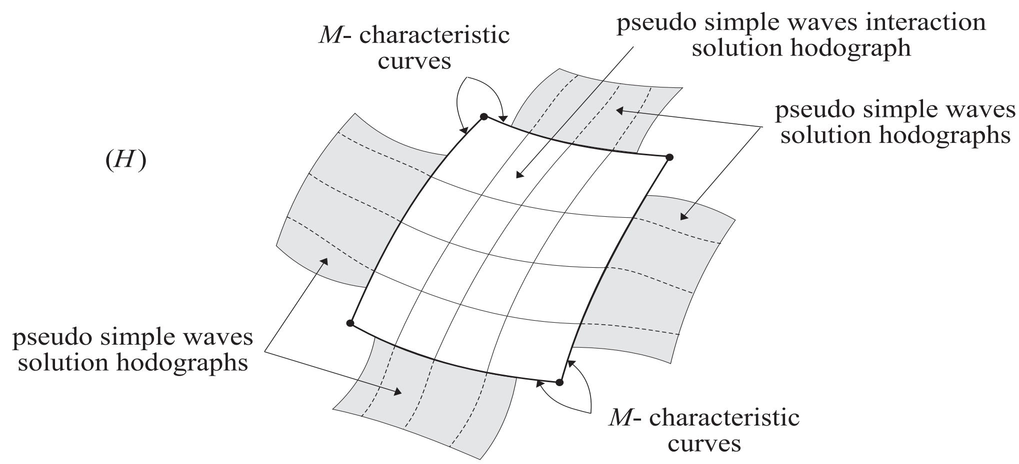

FIGURE 1

The cases (a) and (b) appear to correspond to a *Martin linearization.* In Martin [9] it is proven that for each circumstance (24) a linearization is associated with Euler−Poisson−Darboux representations, and quadratures. See Ludford [6] for an extension of the Martin linearization to a more general form of ζ in (13).

The anisentropic solutions of (7) which do not belong to the linearization list (24) will not show a regularity structure (25).

6.2. The steady two-dimensional version

For the Monge−Ampère type equation (21) the problem of a Martin linearization is still open.

7. Pseudo simple waves solution: an one-dimensional example

To ζ_4 in (24) two intermediate integrals $\mathcal{F}_{\pm} \equiv p\frac{\partial \xi}{\partial p} + \psi\frac{\partial \xi}{\partial \psi} - \xi \pm \frac{1}{\nu}\left(\frac{\psi}{p}\right)^{\nu}$ correspond. We satisfy $\mathcal{F}_{+} \equiv R_{+} = 0$ by

$$\xi = \frac{1}{\nu}\left(\frac{\psi}{p}\right)^{\nu}, \quad \nu = -\frac{\gamma-1}{2\gamma}, \quad \text{integral } \nu, \quad \nu \neq 0{,}1$$

and calculate from (12), (14)

$$x = -\frac{\nu+1}{2\nu+1}\frac{\psi^{2\nu-1}}{p^{2\nu+1}}, \quad t = -\frac{\psi^{\nu}}{p^{\nu+1}},$$
$$v_x = \frac{\psi^{\nu-1}}{p^{\nu}}, \quad \rho = (2\nu+1)\frac{p^{2\nu+1}}{\psi^{2\nu-2}},$$

which leads to

$$p = -\left(\frac{\nu+1}{2\nu+1}\right)^{\nu} \frac{t^{2\nu-1}}{(-x)^{\nu}} \;\;,\;\; v_x = \frac{2\nu+1}{\nu+1}\frac{x}{t},$$
$$\psi = -\left(\frac{\nu+1}{2\nu+1}\right)^{\nu+1} \frac{t^{2\nu+1}}{(-x)^{\nu+1}}. \tag{27}$$

This is a [local] pseudo simple waves solution of (7) corresponding to a certain region $\mathcal{D}\subset E$ (for example, a region of $t>0$, $x<0$). For this solution the assumption (11) holds, cf.

$$\frac{\partial p}{\partial t}\frac{\partial \psi}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \psi}{\partial t} \equiv -\left(\frac{\nu+1}{2\nu+1}\right)^{2\nu+1} \frac{t^{4\nu-1}}{(-x)^{2(\nu+1)}} \neq 0 \text{ in } \mathcal{D}.$$

Now, we proceed with the details. We calculate, cf. ζ_4 in (24), and (26)

$$c = \frac{1}{\zeta_4\rho} = \frac{1}{2\nu+1}\frac{\psi^{\nu-1}}{p^{\nu}} = \frac{1}{2\nu+1}v_x \tag{28}$$

and notice that the *explicit* equations of the [physical] field lines $\mathcal{C}_-, \mathcal{C}_+, \mathcal{C}_0$ [of these, only $\mathcal{C}_\pm$ have a characteristic character (see Remark 10)] through a point $(x^*, t^*) \in \mathcal{D}$ result, cf. (27), (28), by respectively integrating the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_x(x,t) - c(x,t) = k_-\frac{x}{t} \quad \text{along} \;\; \mathcal{C}_- \tag*{(29)$_-$}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_x(x,t) + c(x,t) = k_+\frac{x}{t} \quad \text{along} \;\; \mathcal{C}_+ \tag*{(29)$_+$}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_x(x,t) \qquad\qquad = k_0\frac{x}{t} \quad \text{along} \;\; \mathcal{C}_0 \tag*{(29)$_0$}$$

with

$$k_- = \frac{2\nu}{\nu+1} = -2\frac{\gamma-1}{\gamma+1}; \;\; -\frac{1}{2} < k_- < 0 \quad \text{for} \;\; 1<\gamma<\frac{5}{3}$$
$$k_+ = 2;$$
$$k_0 = \frac{2\nu+1}{\nu+1} = \frac{2}{\gamma+1}; \;\; 0<k_0<1 \quad \text{for} \;\; 1<\gamma<\frac{5}{3}\; .$$

We get from (29)

$$|x| = K_-|t|^{k_-}, \; K_- = \log\frac{|x^*|}{|t^*|^{k_-}} \;\text{ along } \mathcal{C}_- \ni (x^*, t^*), \tag*{(30)$_-$}$$

$$|x| = K_+|t|^{k_+}, \; K_+ = \log\frac{|x^*|}{|t^*|^{k_+}} \;\text{ along } \mathcal{C}_+ \ni (x^*, t^*), \tag*{(30)$_+$}$$

$$|x| = K_0|t|^{k_0}, \; K_0 = \log\frac{|x^*|}{|t^*|^{k_0}} \;\text{ along } \mathcal{C}_0 \ni (x^*, t^*). \tag*{(30)$_0$}$$

Remark 14. We notice that [in contrast with a simple waves solution] a pseudo simple waves solution has a two-dimensional hodograph [see (11)] and for it none of the characteristic fields $\mathcal{C}_\pm$ in the physical plane x, t is made of straightlines generally [see (30)]. □

The images of characteristics $\mathcal{C}_+$ are labelled by a *common* R_+ ($\equiv 0$) [as associated to correspondent characteristics $\overline{\mathcal{C}}_+$]. Still this will not affect the linearization which is already active [cf. the case (b) above]. • In case (a) mentioned above, the linearization results from the interaction of the *two* available coordinates $R_\pm$.

8. "Algebraic" approach and "differential" linearized approach: a parallel

In each of the six mentioned cases of Martin linearization a parallel is possible, independent of the already mentioned Riemann−Lax invariance analysis, between the "algebraic" approach and the "differential" approach. In [3] it is computed, at each point of the hodograph (25), the following relation between the Burnat hodograph characteristic directions $\vec{\kappa}$ and the Martin hodograph characteristic directions $\vec{\mu}$

$$\vec{\mu}_{\pm} = \left(\frac{\partial p}{\partial R_{\pm}}, \frac{\partial v_x}{\partial R_{\pm}}, \frac{\partial S}{\partial R_{\pm}}\right)^t = \eta_{\mp}\vec{\kappa}_{\mp} + \eta_0\vec{\kappa}_0 \tag{31}$$

where

$$S(R_+,R_-) \equiv F[\psi(R_+,R_-)], \;\; \eta_{\mp} = \frac{1}{\Lambda_{\mp}}\frac{\partial v_x}{\partial R_{\pm}}, \;\; \eta_0 = \frac{\partial S}{\partial R_{\pm}} \; .$$

We notice from (31) that at the points of a solution hodograph of (7) the M- characteristic fields and, respectively, the Burnat characteristic fields appear to be *distinct* generally in the mentioned strictly anisentropic context and can be shown to be *coincident* in the isentropic context (cf. $\eta_0 \equiv 0$).

Remark 15. Representation (25) corresponds, for a strictly anisentropic description, to an example of hodograph surface of (7) which *is not* a Burnat characteristic surface [Definition 4]. Still, incidentally and essentially for the linearized approach, this representation appears to be associated with an example of hodograph surface of (7) for which a characteristic character persists in a Martin sense. □

9. Isentropic details of Martin's one-dimensional approach

In an *isentropic* flow we have $S(p,\rho) \equiv$ constant in $(7)_3$. We notice in this case [Appendix 1] that restriction (11) and the Martin's differential approach are still active.
• Two intermediate integrals are available in this case − corresponding to ζ_2 in (24):

$$\mathcal{F}_{\pm} \equiv v_x \pm \int \zeta(p)\mathrm{d}p = \upsilon_{\pm}; \quad v_x = \frac{\partial \xi}{\partial \psi} \; .$$

• It is easy to see in this case that $\upsilon_{\pm}$ structure a Riemann−Burnat invariance and that the "algebraic" and "differential" approaches appear to be coincident.
• The case (b) above will correspond here to two families of simple waves solutions. This issue could suggest the terminology of pseudo simple waves solution used above in the anisentropic approach.

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Appendix

1. Nature of the particle lines of (7)

• ***At first, we verify that, for the particular anisentropic description in sections 4.1, 5.1, the particle lines are not characteristic.*** We consider a local system of coordinates (τ, ν) where $\frac{\partial}{\partial\tau}$ denotes differentiation in the particle line direction and $\frac{\partial}{\partial\nu}$ denotes differentiation in the direction which makes an angle of $+90^0$ with the first. We use

$$\frac{\partial}{\partial t} = \frac{1}{\sqrt{1+v_x^2}}\left(\frac{\partial}{\partial\tau} + v_x\frac{\partial}{\partial\nu}\right), \qquad \frac{\partial}{\partial x} = \frac{1}{\sqrt{1+v_x^2}}\left(v_x\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\nu}\right) \tag{A1.1}$$

to find from (7) the (τ, ν) form

$$\begin{cases} (1+v_x^2)\,\dfrac{\partial\rho}{\partial\tau} + \rho v_x\dfrac{\partial v_x}{\partial\tau} = \rho\dfrac{\partial v_x}{\partial\nu} \\ \rho(1+v_x^2)\,\dfrac{\partial v_x}{\partial\tau} + v_x\dfrac{\partial p}{\partial\tau} = \dfrac{\partial p}{\partial\nu} \\ \dfrac{\partial S}{\partial\tau} = 0, \quad S = S(p,\rho)\ . \end{cases} \tag{A1.2}$$

We may transcribe $(A1.2)_3$ by

$$\frac{\partial S}{\partial p}\frac{\partial p}{\partial\tau} + \frac{\partial S}{\partial\rho}\frac{\partial\rho}{\partial\tau} = 0 \quad \text{for each given } \psi \text{ [or } S] \tag{A1.3}$$

and we identify

$$c^2 = -\frac{\left(\frac{\partial S}{\partial\rho}\right)_p}{\left(\frac{\partial S}{\partial p}\right)_\rho} = \left(\frac{\partial\rho}{\partial p}\right)_S^{-1} \quad \text{for each given } \psi \text{ [or } S] \tag{A1.4}$$

as an ad hoc anisentropic sound speed. We notice that c^2 considered in $(A1.4)$ corresponds to c^2 previously taken into account in (13).

A particularity of the anisentropic description considered in section 4.1 is that a relation $S(p,\rho) = F(\psi)$ is available among p, ρ, ψ with a smooth F as a prescribed function of ψ. We use this particularity to determine $\frac{\partial \rho}{\partial \nu}$ at the points of a particle line $\mathcal{C}_0$. Precisely, we compute

$$\frac{\partial S}{\partial t} = \frac{\partial S}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial S}{\partial \rho}\frac{\partial \rho}{\partial t} = F'(\psi)\frac{\partial \psi}{\partial t} = -\rho v_x F'(\psi)$$

which could be transcribed cf. $(A1.1)$, $(A1.3)$ and $(A1.4)$, by:

$$v_x\left[\frac{\partial p}{\partial \nu} - c^2(p,\psi)\frac{\partial \rho}{\partial \nu}\right] + \frac{\rho v_x\sqrt{1+v_x^2}\,F'(\psi)}{\left(\frac{\partial S}{\partial p}\right)_\rho} = 0$$

or, cf. $(A1.2)_2$, $(A1.3)$, $(A1.4)$, by:

$$c^2(p,\psi)v_x\frac{\partial \rho}{\partial \nu} = v_x\left[\rho(1+v_x^2)\,\frac{\partial v_x}{\partial \tau} + v_x\frac{\partial p}{\partial \tau}\right] + \frac{\rho v_x\sqrt{1+v_x^2}\,F'(\psi)}{\left(\frac{\partial S}{\partial p}\right)_\rho}\ .$$

We add this relation to $(A1.2)_{1,2}$ in order to determine $\frac{\partial p}{\partial \nu}, \frac{\partial v_x}{\partial \nu}, \frac{\partial \rho}{\partial \nu}$ at the points of a particle line. ***The verification is complete.***

• The two families of characteristics $\mathcal{C}_\pm$ of (7) [could be put in correspondence with the two families of characteristics $\overline{\mathcal{C}}_\pm$ of (13). In fact we have

$$\begin{cases} \mathrm{d}\left(\dfrac{\partial \xi}{\partial \psi}\right) \pm \zeta \mathrm{d}p = 0 \\ \mathrm{d}\left(\dfrac{\partial \xi}{\partial p}\right) \mp \zeta \mathrm{d}\psi = 0 \end{cases} \qquad \text{along} \quad \overline{\mathcal{C}}_\pm \tag{$A1.5$}$$

and therefore

$$\mp \mathrm{d}t \overset{(12)_4,(A1.5)_2}{=} -\zeta \mathrm{d}\psi \overset{(8)_1}{=} -\frac{1}{c}(\mathrm{d}x - v_x \mathrm{d}t) \quad \text{along} \quad \overline{\mathcal{C}}_\pm$$

which corresponds to

$$dx = (v_x \pm c)\mathrm{d}t \quad \text{along} \quad \mathcal{C}_\pm.$$

2. The Monge–Ampère type equation (21)

In order to eliminate η we solve (20) for $\frac{\partial \eta}{\partial \psi}$ and $\frac{\partial^2 \eta}{\partial p^2}$

$$\frac{\partial \eta}{\partial \psi} = \varepsilon\left[\mathcal{F}(p,\psi) - \left(\frac{\partial \xi}{\partial \psi}\right)^2\right]^{\frac{1}{2}}, \quad \frac{\partial^2 \eta}{\partial p^2} = -\varepsilon\frac{\partial \xi}{\partial \psi}\frac{\partial^2 \xi}{\partial p^2}\left[\mathcal{F}(p,\psi) - \left(\frac{\partial \xi}{\partial \psi}\right)^2\right]^{-\frac{1}{2}}, \quad \varepsilon = \pm 1 \tag{$A2.1$}$$

and put $\frac{\partial^2}{\partial p^2}\left(\frac{\partial \eta}{\partial \psi}\right) = \frac{\partial}{\partial \psi}\left(\frac{\partial^2 \eta}{\partial p^2}\right)$ into the form

$$\Phi\left(\psi, p, \xi, \frac{\partial \xi}{\partial p}, \frac{\partial \xi}{\partial \psi}, \frac{\partial^2 \xi}{\partial p^2}, \frac{\partial^2 \xi}{\partial p \partial \psi}, \frac{\partial^2 \xi}{\partial \psi^2}\right) \equiv$$

$$\equiv 4\mathcal{F}\left[\left(\frac{\partial^2 \xi}{\partial p \partial \psi}\right)^2 - \frac{\partial^2 \xi}{\partial p^2}\frac{\partial^2 \xi}{\partial \psi^2}\right] - 4\left(\frac{\partial \xi}{\partial \psi}\frac{\partial \mathcal{F}}{\partial p}\right)\frac{\partial^2 \xi}{\partial p \partial \psi} + 2\left(\frac{\partial \xi}{\partial \psi}\frac{\partial \mathcal{F}}{\partial \psi}\right)\frac{\partial^2 \xi}{\partial p^2}$$

$$+\left\{\left(\frac{\partial \mathcal{F}}{\partial p}\right)^2 - 2\left[\mathcal{F} - \left(\frac{\partial \xi}{\partial \psi}\right)^2\right]\frac{\partial^2 \mathcal{F}}{\partial p^2}\right\} = 0. \tag{21}$$

• For a given solution ξ of (21) we compute by quadratures:

$$\eta \overset{(A2.1)}{=} \int \frac{\partial \eta}{\partial p}\mathrm{d}p + \frac{\partial \eta}{\partial \psi}\mathrm{d}\psi$$

where

$$\frac{\partial \eta}{\partial p} \overset{(A2.1)}{=} \int \frac{\partial^2 \eta}{\partial p^2}\mathrm{d}p + \frac{\partial^2 \eta}{\partial \psi \partial p}\mathrm{d}\psi \ .$$

3. The characteristic directions associated to the Monge–Ampère type equation (21)

For the characteristic directions associated to the Monge–Ampère type equation (21) we get, at a suitable point P in the space of the arguments of Φ in (21),

$$\left[\frac{\partial \Phi}{\partial\left(\frac{\partial^2 \xi}{\partial \psi^2}\right)}\beta_1^2 + \frac{\partial \Phi}{\partial\left(\frac{\partial^2 \xi}{\partial p \partial \psi}\right)}\beta_1\beta_2 + \frac{\partial \Phi}{\partial\left(\frac{\partial^2 \xi}{\partial p^2}\right)}\beta_2^2\right]_P = 0 \qquad (A3.1)$$

where

$$\vec{\beta} = (\beta_1, \beta_2) = (\mathrm{d}p, -\mathrm{d}\psi) \qquad (A3.2)$$

is an exceptional direction [orthogonal to a characteristic direction $(\mathrm{d}\psi, \mathrm{d}p)$]. We transcribe $(A3.1)$ by

$$\left[-2\mathcal{F}\frac{\partial^2 \xi}{\partial p^2}\right]_P \left(\frac{\mathrm{d}p}{\mathrm{d}\psi}\right)^2 - 2\left[2\mathcal{F}\frac{\partial^2 \xi}{\partial p \partial \psi} - \frac{\partial \mathcal{F}}{\partial p}\frac{\partial \xi}{\partial \psi}\right]_P \frac{\mathrm{d}p}{\mathrm{d}\psi} + \left[-2\mathcal{F}\frac{\partial^2 \xi}{\partial \psi^2} + \frac{\partial \mathcal{F}}{\partial \psi}\frac{\partial \xi}{\partial \psi}\right]_P = 0$$

to get finally

$$\left(\frac{\mathrm{d}p}{\mathrm{d}\psi}\right)_\pm = \frac{\left[2\mathcal{F}\frac{\partial^2 \xi}{\partial p \partial \psi} - \frac{\partial \mathcal{F}}{\partial p}\frac{\partial \xi}{\partial \psi}\right]_P \pm \sqrt{\Delta}}{\left[-2\mathcal{F}\frac{\partial^2 \xi}{\partial p^2}\right]_P} \qquad (A3.3)$$

where

$$\begin{aligned}\Delta &= \left[2\mathcal{F}\frac{\partial^2 \xi}{\partial p \partial \psi} - \frac{\partial \mathcal{F}}{\partial p}\frac{\partial \xi}{\partial \psi}\right]_P^2 - \left[-2\mathcal{F}\frac{\partial^2 \xi}{\partial p^2}\right]_P \left[-2\mathcal{F}\frac{\partial^2 \xi}{\partial \psi^2} + \frac{\partial \mathcal{F}}{\partial \psi}\frac{\partial \xi}{\partial \psi}\right]_P \\ &= \left[\mathcal{F} - \left(\frac{\partial \xi}{\partial \psi}\right)^2\right]_P \left[2\mathcal{F}\frac{\partial^2 \mathcal{F}}{\partial p^2} - \left(\frac{\partial \mathcal{F}}{\partial p}\right)^2\right]_P \\ &= \frac{4}{\rho^2 c^2} v_y^2 \left(V^2 - c^2\right) \qquad (A3.4)\end{aligned}$$

cf. [see $(17)_{3,4}$ and $(20)_{1,3}$]

$$V^2 = \mathcal{F} = 2\mathcal{G}(\psi) - 2\int^p \frac{\mathrm{d}p}{\rho} \ ; \quad \frac{\partial \mathcal{F}}{\partial p} = -\frac{2}{\rho(p,\psi)} \ ; \quad \frac{\partial^2 \mathcal{F}}{\partial p^2} = \frac{2}{\rho^2}\frac{\partial \rho}{\partial p} = \frac{2}{\rho^2 c^2} \ . \qquad (A3.5)$$

4. The relations (23) along the characteristics of (21).

We re-arrange $(A3.3)$ into the form

$$2\mathcal{F}\left[\frac{\partial^2 \xi}{\partial p^2}\mathrm{d}p + \frac{\partial^2 \xi}{\partial p \partial \psi}\mathrm{d}\psi\right] - \frac{\partial \mathcal{F}}{\partial p}\frac{\partial \xi}{\partial \psi}\mathrm{d}\psi \pm \sqrt{\Delta}\mathrm{d}\psi = 0$$

which can be transcribed as

$$-2\mathcal{F}\mathrm{d}y - \frac{\partial\mathcal{F}}{\partial p}\frac{\partial\xi}{\partial\psi}\mathrm{d}\psi \pm \sqrt{\Delta}\mathrm{d}\psi = 0, \quad y = -\frac{\partial\xi}{\partial p}\ , \qquad \text{along } \overline{\mathcal{C}}_\pm$$

and results in $(23)_1$ cf. $(17)_{3,4}$, $(A3.4)$, and $(A3.5)$.

Next, in order to obtain $(23)_2$, we compute

$$\frac{2\mathcal{F}\frac{\partial^2\xi}{\partial p\partial\psi} - \frac{\partial\mathcal{F}}{\partial p}\frac{\partial\xi}{\partial\psi} \pm \sqrt{\Delta}}{-2\mathcal{F}\frac{\partial^2\xi}{\partial p^2}} = -\frac{2\mathcal{F}\frac{\partial^2\xi}{\partial\psi^2} - \frac{\partial\mathcal{F}}{\partial\psi}\frac{\partial\xi}{\partial\psi}}{2\mathcal{F}\frac{\partial^2\xi}{\partial p\partial\psi} - \frac{\partial\mathcal{F}}{\partial p}\frac{\partial\xi}{\partial\psi} \mp \sqrt{\Delta}}\ . \tag{A4.1}$$

We carry this into $(A3.3)$ to get

$$\left(\frac{\mathrm{d}p}{\mathrm{d}\psi}\right)_\pm = -\frac{\left[2\mathcal{F}\frac{\partial^2\xi}{\partial\psi^2} - \frac{\partial\mathcal{F}}{\partial\psi}\frac{\partial\xi}{\partial\psi}\right]_P}{\left[2\mathcal{F}\frac{\partial^2\xi}{\partial p\partial\psi} - \frac{\partial\mathcal{F}}{\partial p}\frac{\partial\xi}{\partial\psi}\right]_P \mp \sqrt{\Delta}}\ . \tag{A4.2}$$

We re-arrange $(A4.2)$ into the form

$$2\mathcal{F}\left[\frac{\partial^2\xi}{\partial p\partial\psi}\mathrm{d}p + \frac{\partial^2\xi}{\partial\psi^2}\mathrm{d}\psi\right] - \frac{\partial\xi}{\partial\psi}\left[\frac{\partial\mathcal{F}}{\partial p}\mathrm{d}p + \frac{\partial\mathcal{F}}{\partial\psi}\mathrm{d}\psi\right] \mp \sqrt{\Delta}\mathrm{d}p = 0$$

which can be transcribed as

$$2V^2\mathrm{d}v_x - v_x\mathrm{d}V^2 \mp \sqrt{\Delta} = 0\ , \qquad \text{along } \overline{\mathcal{C}}_\pm$$

and results in $(23)_2$ cf. $(17)_{3,4}$, $(A3.4)$, and $(A3.5)$.

5. Nature of the streamlines of (9)

• We consider first [cf. $(9)_4$] the relation

$$S(p,\rho) = F(\psi) \quad \text{along each streamline of (9)} \tag{A5.1}$$

with F known [as determined by the shock conditions]. Prescription of F as a function of ψ provides a first algebraic relation among p, ρ, V^2, ψ throughout the mentioned anisentropic flow region behind the curved shock.

• Next, we notice that the equations $(9)_{2,3}$ can be re-arranged into the steady two-dimensional version of Crocco's form

$$(\mathrm{curl}\,\vec{\mathbf{v}}) \times \vec{\mathbf{v}} = T\,\mathrm{grad}S - \mathrm{grad}\left(e + \frac{p}{\rho} + \frac{1}{2}V^2\right). \tag{A5.2}$$

From $(A5.1)$, $(A5.2)$ we then get

$$e + \frac{p}{\rho} + \frac{1}{2}V^2 = H(\psi) \quad \text{along each streamline of (6.24).} \tag{A5.3}$$

with H known [as determined by the shock conditions]. Prescription of H as a function of ψ provides a second algebraic relation among p, ρ, V^2, ψ throughout the mentioned anisentropic flow region behind the curved shock.

• ***We verify that, for the particular anisentropic description in sections 4.2, 5.2, the streamlines are not characteristic.*** We consider a local system of coordinates (τ, ν) where $\frac{\partial}{\partial\tau}$ denotes differentiation in the streamline direction and $\frac{\partial}{\partial\nu}$ denotes differentiation in the direction which makes an angle of $+90^0$ with the first. We use

$$\frac{\partial}{\partial x} = \frac{1}{V}\left(v_x\frac{\partial}{\partial\tau} - v_y\frac{\partial}{\partial\nu}\right), \qquad \frac{\partial}{\partial y} = \frac{1}{V}\left(v_y\frac{\partial}{\partial\tau} + v_x\frac{\partial}{\partial\nu}\right) \tag{A5.4}$$

to find from (9) the (τ, ν) form

$$\begin{cases} \rho v_y \dfrac{\partial v_x}{\partial \nu} - \rho v_x \dfrac{\partial v_y}{\partial \nu} = V^2 \dfrac{\partial \rho}{\partial \tau} + \dfrac{1}{2}\rho \dfrac{\partial V^2}{\partial \tau} \\ v_y \dfrac{\partial p}{\partial \nu} = v_x \dfrac{\partial p}{\partial \tau} + \rho V^2 \dfrac{\partial v_x}{\partial \tau} \\ v_x \dfrac{\partial p}{\partial \nu} = -v_y \dfrac{\partial p}{\partial \tau} - \rho V^2 \dfrac{\partial v_y}{\partial \tau} \\ \dfrac{\partial S}{\partial \tau} = 0, \qquad S = S(p, \rho) \ . \end{cases} \tag{A5.5}$$

We notice that $(A5.5)_2$ and $(A5.5)_3$ are coincident cf. $(20)_{1,3}$.

Again, we may transcribe $(A5.5)_4$ by

$$\frac{\partial S}{\partial p}\frac{\partial p}{\partial \tau} + \frac{\partial S}{\partial \rho}\frac{\partial \rho}{\partial \tau} = 0 \quad \text{for each given } \psi \text{ [or } S\text{]} \tag{A5.6}$$

and we identify

$$c^2 = -\frac{\left(\frac{\partial S}{\partial \rho}\right)_p}{\left(\frac{\partial S}{\partial p}\right)_\rho} = \left(\frac{\partial \rho}{\partial p}\right)_S^{-1} \quad \text{for each given } \psi \text{ [or } S\text{]} \tag{A5.7}$$

as an ad hoc anisentropic sound speed. We notice that c^2 considered in $(A5.7)$ corresponds to c^2 previously taken into account in (21).

We use $(A5.1)$ to determine $\frac{\partial \rho}{\partial \nu}$ at the points of a streamline $\mathcal{C}_0$. Precisely, we compute

$$\frac{\partial S}{\partial y} = \frac{\partial S}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial S}{\partial \rho}\frac{\partial \rho}{\partial y} = F'(\psi)\frac{\partial \psi}{\partial y} = \rho v_x F'(\psi)$$

which could be transcribed cf. $(A5.4)$, $(A5.6)$ and $(A5.7)$ by:

$$v_x \frac{\partial \rho}{\partial \nu} = -v_y \frac{\partial \rho}{\partial \tau} - \frac{\rho V^2}{c^2}\frac{\partial v_y}{\partial \tau} - \frac{\rho v_x V F'(\psi)}{\left(\frac{\partial S}{\partial \rho}\right)_p} \ . \tag{A5.8}$$

Next, we use $(A5.3)$ to compute

$$\frac{\partial}{\partial y}\left[e(p, \rho) + \frac{p}{\rho} + \frac{1}{2}V^2\right] = \rho v_x H'(\psi)$$

which could be transcribed cf. $(A5.4)$, by:

$$\begin{aligned} &v_x^2 \frac{\partial v_x}{\partial \nu} + v_x v_y \frac{\partial v_y}{\partial \nu} \\ &= \rho v_x V H'(\psi) - v_x v_y \frac{\partial v_x}{\partial \tau} + \frac{\partial v_y}{\partial \tau}\left[\frac{\rho V^2}{c^2}\left(\frac{\partial e}{\partial \rho} - \frac{p}{\rho^2}\right) + \rho V^2\left(\frac{\partial e}{\partial \rho} + \frac{1}{\rho}\right) - v_y^2\right] + \left(\frac{\partial e}{\partial \rho} - \frac{p}{\rho^2}\right)\frac{\rho v_x V F'(\psi)}{\left(\frac{\partial S}{\partial \rho}\right)_p} \end{aligned} \tag{A5.9}$$

We add this relation to $(A5.5)_1$ in order to determine $\frac{\partial v_x}{\partial \nu}$, $\frac{\partial v_y}{\partial \nu}$ at the points of a streamline $\mathcal{C}_0$.

Equations $(9)_{2,3}$ contribute in this computation with the relations $(A5.3)$ [leading to $(A5.9)$] and $(A5.5)_2$ [as $(A5.5)_2$ and $(A5.5)_3$ are coincident]. Equation $(9)_4$ contributes in this computation with the relation $(A5.1)$ [leading to $(A5.8)$]

We have $(A5.5)_1$, $(A5.9)$, $(A5.5)_2$ and $(A5.8)$ at our disposal to determine $\frac{\partial \rho}{\partial \nu}$, $\frac{\partial p}{\partial \nu}$, $\frac{\partial v_x}{\partial \nu}$, $\frac{\partial v_y}{\partial \nu}$ at the points of a streamline. ***The verification is complete.***