On a nonlinear integrable difference equation on the square 3D-inconsistent

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Abstract

We present a nonlinear partial difference equation defined on a square which is obtained by combining the Miura transformations between the Volterra and the modified Volterra differential-difference equations. This equation is not symmetric with respect to the exchange of the two discrete variables and does not satisfy the 3D-consistency condition necessary to belong to the Adler-Bobenko-Suris classification. Its integrability is proved by constructing its Lax pair.

The uncovery of new nonlinear integrable completely discrete equations is always a very challenging problem as, by proper continuous limits, many other results on differential-difference and partial differential equations can be obtained. In the case of differential equations by now a lot is known starting from the pioneering works by Gardner, Green, Kruskal and Miura. A summary of these results is already of public domain and presented for example in the Encyclopedia of Mathematical Physics [5] or in the Encyclopedia of Nonlinear Science [6]. Among those results let us mention the classification scheme of nonlinear integrable partial differential equations introduced by Shabat using the formal symmetry approach, see [11] for a review. The classification of differential-difference equations has also been carried out using the formal symmetry approach by Yamilov [17] and it is a well defined procedure which can be easily computerized for many families of equations [10, 18].

In the completely discrete case the situation is different. Many researchers have tried to carry out the approach of formal symmetries introduced by Shabat, without any success up to now. One of the first exhaustive results in this context, based on completely different ideas, is given by the Adler-Bobenko-Suris (ABS) classification of $\mathbb{Z}^2$-lattice equations defined on the square lattice [2]. By now many results are known on the ABS equations, see for instance [7, 8, 13, 14]. However the analysis of the transformation properties of these lattice equations cannot be considered yet complete and new results which help the understanding of the interrelations between them and some differential-difference equations can still be found [9].

A two-dimensional partial difference equation is a functional relation among the values of a function $u : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ at different points of the lattice of indices i, j. It involves the independent variables i, j and the lattice parameters $\alpha, \beta \in \mathbb{C}$:

$$\mathcal{E}(i, j, u_{i,j}, u_{i+1,j}, u_{i,j+1}, ...; \alpha, \beta) = 0.$$

The so-called ABS list of integrable lattice equations is given by those affine linear (i.e. polynomial of degree one in each argument) partial difference equations of the form

$$\mathcal{E}(i, j, u_{i,j}, u_{i+1,j}, u_{i,j+1}, u_{i+1,j+1}; \alpha, \beta) = 0, \tag{1}$$

whose integrability is based on the *consistency around a cube* (or 3D-consistency) [2].

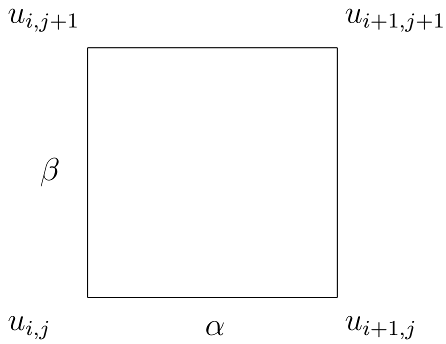

Figure 1: A square lattice

The main idea of the consistency method is the following. One starts from a square lattice, defines the variables on the vertices $u_{i,j}, u_{i+1,j}, u_{i,j+1}, u_{i+1,j+1}$ (see Figure 1) and considers the multilinear equation relating these variables, namely eq. (1). By solving it for $u_{i+1,j+1}$ one obtains a rational expression and the same holds for any field variable. One then adjoins a third direction and imagines the map giving $u_{i+1,j+1,k+1}$ as being the composition of maps on the various planes (see Figure 2). There exist three different ways to obtain $u_{i+1,j+1,k+1}$ and the consistency constraint is that they all lead to the same result. This gives strict conditions on the nonlinear equation, but they are not sufficient to determine it completely. Two further constraints have been introduced by Adler, Bobenko and Suris. They are:

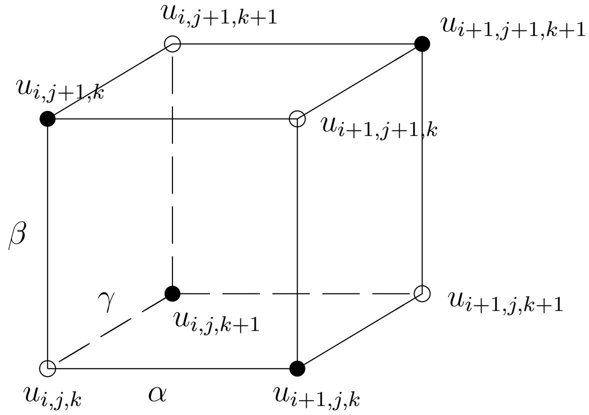

Figure 2: Three-dimensional consistency

- *D_4-symmetry.* $\mathcal{E}$ is invariant under the group of the square symmetries:

$$\mathcal{E}(u_{i,j}, u_{i+1,j}, u_{i,j+1}, u_{i+1,j+1}; \alpha, \beta) = \pm\mathcal{E}(u_{i,j}, u_{i,j+1}, u_{i+1,j}, u_{i+1,j+1}; \beta, \alpha) =$$

$$\pm\mathcal{E}(u_{i+1,j}, u_{i,j}, u_{i+1,j+1}, u_{i,j+1}; \alpha, \beta).$$

- *Tetrahedron property.* The function $u_{i+1,j+1,k+1}$ is independent of $u_{i,j,k}$.

The following transformations, which do not violate the two constraints listed above, are assumed to identify equivalence classes:

- Action on all field variables by one and the same (independent of lattice parameter) Möbius transformation.

- Simultaneous point change of all parameters.

Under the above constraints Adler, Bobenko and Suris obtained a complete classification of $\mathbb{Z}^2$-lattice systems, whose integrability is ensured as the *consistency around a cube* also furnishes their Lax pairs [2, 4, 12].

As it is known [16], the modified Volterra equation

$$u_{i,t} = (u_i^2 - 1)(u_{i+1} - u_{i-1}) \tag{2}$$

is transformed into the Volterra equation $v_{i,t} = v_i(v_{i+1} - v_{i-1})$ by two discrete Miura transformations:

$$v_i^{\pm} = (u_{i+1} \pm 1)(u_i \mp 1). \tag{3}$$

For any solution u_i of eq. (2), one obtains by the transformations (3) two solutions v_i^+, v_i^- of the Volterra equation. From a solution of the Volterra equation v_i we obtain two solutions of the modified Volterra equation $u_{i,0}$ and $u_{i,1}$. The composition of the Miura transformations (3)

$$v_i = (u_{i+1,0} + 1)(u_{i,0} - 1) = (u_{i+1,1} - 1)(u_{i,1} + 1) \tag{4}$$

provides a Bäcklund transformation for eq. (2). Eq. (4) provides a way to construct from a solution $u_{i,0}$ of eq. (2) a new solution $u_{i,1}$. Iterating eq. (4), one can construct infinitely many solutions:

$$\ldots \leftarrow u_{i,-2} \leftarrow u_{i,-1} \leftarrow u_{i,0} \rightarrow u_{i,1} \rightarrow u_{i,2} \rightarrow \ldots$$

Rewriting eq. (4) as a chain of equations relating the solutions $u_{i,j}$, we obtain the following completely discrete equation on the square:

$$(u_{i+1,j} + 1)(u_{i,j} - 1) = (u_{i+1,j+1} - 1)(u_{i,j+1} + 1). \tag{5}$$

This equation does not belong to the ABS classification, as it is not invariant under the exchange of i and j. However eq. (5) is invariant under a rotational symmetry of π. By a straightforward calculation, using a symbolic computation program like Maple, one can easily show its 3D-inconsistency. Recently Adler, Bobenko and Suris [3] extended the previous definition to systems of equations 3D-consistent on a cube. Eq. (5) can be embedded into such a 3D-consistent system [15].

The construction of the Lax pair can be done in a way that is parallel to the derivation of the nonlinear difference equation done above. Let us consider the spectral problem for the modified Volterra equation (2)

$$L_i = \begin{pmatrix} -\lambda^{-1} & u_i \\ -u_i & \lambda \end{pmatrix}, \tag{6}$$

found in [1], and the standard scalar spectral problem of the Volterra equation, written in matrix form,

$$M_i = \begin{pmatrix} \lambda - \lambda^{-1} & -v_i \\ 1 & 0 \end{pmatrix}. \tag{7}$$

The existence of the two Miura transformations (3) between the two equations imply the existence of two nonsingular Darboux matrices $E_i^{(+)}, E_i^{(-)}$ between the spectral problems:

$$E_i^{(+)} = \begin{pmatrix} 1 & \lambda v_i(u_{i,0} + 1) \\ \lambda & -v_i(1 + u_{i,0}) \end{pmatrix}, \qquad E_i^{(-)} = \begin{pmatrix} -1 & \lambda v_i(u_{i,1} - 1) \\ \lambda & -v_i(1 - u_{i,1}) \end{pmatrix}. \tag{8}$$

The matrix $E_i^{(+)}$ will provide a solution $u_{i,0}$ of the modified Volterra equation, while the matrix $E_i^{(-)}$ will provide a different solution, $u_{i,1}$. So, the two solutions $u_{i,0}$ and $u_{i,1}$ are given by the two Lax equations

$$E_{i+1}^{(+)} M_i = L_{i,0} E_i^{(+)}, \qquad E_{i+1}^{(-)} M_i = L_{i,1} E_i^{(-)}, \tag{9}$$

where

$$L_{i,j} = \left(\begin{array}{cc} -\lambda^{-1} & u_{i,j} \\ -u_{i,j} & \lambda \end{array} \right). \tag{10}$$

The equation (4), relating the two solutions $u_{i,0}$ and $u_{i,1}$, is obtained by eliminating from eqs. (9) the matrix M_i and the dependence of v_i. So its Lax equation is given by

$$N_{i+1,0} L_{i,0} = L_{i,1} N_{i,0}, \tag{11}$$

where $N_{i,0} = E_i^{(-)} (E_i^{(+)})^{-1}$. Taking into account the definition (8), formulae (4) for v_i, the discrete equation (5), and introducing as before the chain of equations for any j, we get that the Lax equation associated to eq. (5) is given by

$$N_{i+1,j} L_{i,j} = L_{i,j+1} N_{i,j},$$

with $L_{i,j}$ given by eq. (10) and

$$N_{i,j} = \left(\begin{array}{cc} \lambda w_{i,j} - \lambda^{-1} & -(w_{i,j}+1) \\ w_{i,j}+1 & \lambda - \lambda^{-1} w_{i,j} \end{array} \right), \qquad w_{i,j} = \frac{u_{i,j}+1}{u_{i,j+1}-1}.$$

This is not the only case when we can encounter 3D–inconsistent integrable equations. For example, the modified–modified Volterra equation will provide in the same way a discrete equation on the square

$$(1 + u_{i,j} u_{i+1,j})(\mu u_{i+1,j+1} + \mu^{-1} u_{i,j+1}) = (1 + u_{i,j+1} u_{i+1,j+1})(\mu u_{i,j} + \mu^{-1} u_{i+1,j}), \tag{12}$$

where μ is an arbitrary non–zero constant. This equation has the same symmetry properties as eq. (5) and is also 3D–inconsistent when $\mu^4 \neq 1$. For $\mu^4 = 1$ eq. (12) is 3D–consistent, but in this case the equation is degenerate and can be written as $(T_j \pm 1) \frac{\mu u_{i,j} + \mu^{-1} u_{i+1,j}}{1 + u_{i,j} u_{i+1,j}} = 0$, where T_j is the shift operator for the j index. Also eq. (12) can be embedded into a system 3D-consistent on a cube [15].

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