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Singular Finite-Gap Operators and Indefinite Metric. I.

Abstract. *Many "real" inverse spectral data for periodic finite-gap operators (consisting of Riemann Surface with marked "infinite point", local parameter and divisors of poles) lead to operators with real but singular coefficients. These operators cannot be considered as self-adjoint in the ordinary (positive) Hilbert spaces of functions of x . In particular, it is true for the special case of Lamé' operators with elliptic potential $n(n+1)\wp(x)$ where eigenfunctions were found in XIX Century by Hermit. However, such Baker-Akhiezer (BA) functions present according to the ideas of works [1, 2], right analog of the Discrete and Continuous Fourier Bases on Riemann Surfaces. It turns out that these operators for the nonzero genus are symmetric in some indefinite inner product, described in this work. The analog of Continuous Fourier Transform is an isometry in this inner product. In the next work with number II we will present exposition of the similar theory for Discrete Fourier Series.*

Introduction

Broad family of the so-called "Baker-Akhiezer" (BA) functions on Riemann surfaces were invented since 1974 when periodic finite gap solutions were found for the famous KdV equation.

They were used for the solution of periodic problems for KdV, KP and other systems of Soliton Theory like NLS, SG, for many Completely Integrable Hamiltonian Systems. The Spectral Theory of "finite-gap" periodic 1D and 2D Schrodinger Operators was developed since 1974 based on the Analysis on Riemann Surfaces. It was found in 1987 that some BA functions generate construction of analogs of the Laurent-Fourier decomposition for functions and tensor fields on Riemann surfaces (the Krichever-Novikov Bases and Algebras [1]). They were used for the multi-loop operator quantization of Closed Bosonic Strings (i.e. for genus more than zero.) Another ideas similar to some sort of Harmonic Analysis with spectral parameter on Riemann Surfaces and useful here, were developed for other goals in the works [5].

The present authors observed in the Appendix to the work [2] that these constructions lead also to the analog of continuous Fourier Transform. The present work is direct continuation of [2]. It was motivated by the following

Problem: Consider one-dimensional Lamé' Operator $L = -\partial_x^2 + u(x)$ whose potential u is equal to the $n(n+1)$ -times Weierstrass elliptic function \wp with poles on real line. **Does it have any reasonable spectral theory on the whole real line?** We need to answer this question because our analog of continuous Fourier Transform is based exactly on the singular Hermit eigenfunctions of this operator in the simplest nontrivial elliptic case.

Let us remind here that 150 years ago Hermit found all family of formal eigenfunction for this operator. In fact it consists of the "Bloch-Floquet" eigenfunctions in modern terminology. However they are singular on the line and do not serve any spectral problem in Hilbert space. Hermit used only those of them who belong to the discrete spectrum on the finite interval $[0T]$ between the neighboring singularities, needed for the Lamé' problem. No spectral interpretation of singular eigenfunctions for the spectral theory on the whole line was known.

We found indefinite inner product associated with this problem. This is our main result but the exposition is more general: We constructed indefinite inner products associated with BA functions and non-selfadjoint "algebraic" periodic operators with Bloch-Floquet function meromorphic on Riemann Surfaces of finite genus like in the finite-gap theory.

In our text we assume, that all finite-gap operators are periodic in the real variable $x \in R$. It is very likely, that the main results of this paper are valid for generic finite-gap quasiperiodic potentials, but this extension may lead to additional analytical difficulties.

Remark 1 *Singular Bloch-Floquet eigenfunctions are known also for $k + 1$ -particle Moser-Calogero operator with Weierstrass elliptic pairwise potential if coupling constant is equal to $n(n + 1)$. They form (in the center of mass variables x) a k -dimensional complex algebraic variety. Hermit-type result is not obtained here yet: no one function was constructed until now serving the discrete spectrum in the bounded domain between the poles. Our case corresponds to $k = 1$. We believe that for all $k > 1$ this algebraic family of eigenfunctions also serves spectral problem in some indefinite inner product in the space of functions in the whole space R^k similar to the case $k = 1$.*

Chapter 1. Canonical contours and Inner Product of BA functions.

Let a nonsingular complex algebraic curve (Riemann surface) Γ be given with selected point $P = \infty \in \Gamma$, local coordinate $z = k^{-1}$ near P such that $z(P) = 0$. We fix also "divisor" $D = \gamma_1 + \dots + \gamma_g$ on Γ and construct standard BA function $\Psi_D(x, z)$, $z \in \Gamma$, meromorphic in the variable z , with first order poles in the points $\gamma_j \in \Gamma$ and with asymptotics $\Psi = \exp\{ikx\}(1 + O(k^{-1}) + \dots)$. We define a differential 1-form $d\mu$ with asymptotics $d\mu = dk + \text{regular}$ near P and divisor of zeroes $(d\mu) = D + D^*$:

$$D + D^* \sim K + 2P$$

Here the sign \sim means the so-called "linear equivalence" of divisors in Algebraic Geometry, K means the divisor of differential forms. So the divisor D^* is completely determined by the divisor D . A "dual" BA function (1-form) $\Psi_D^*(x, z)$ was invented long ago by Krichever. It was actively used in the joint works [1] and has asymptotics $\Psi_D^*(x, z) = \exp\{-ikx\}(1 + O(k^{-1}) + \dots)$ with divisor D^* . A Dual BA form is $\Psi^*d\mu$. So we have

$$\Psi_D^*(x, z) = \Psi_{D^*}(-x, z)$$

as a scalar BA function. One should multiply it by the form $d\mu$ to get a dual 1-form.

Our functions $\Psi_D(x, z)$, $\Psi_D^*(x, z)$ are also meromorphic in x .

The Canonical Contours κ_c we define by the equation $p_I = c$ where dp is meromorphic (second kind) differential form such that $dp = dk + \text{regular}$ near $P = \infty$, and $\oint_{\gamma} dp \in R$ is purely real for all closed paths $\gamma \subset \Gamma$ avoiding the point P . So the imaginary part of p is an one-valued function p_I . **We choose local parameter z depending on Canonical Contour κ_c** in such a way that $\exp\{ikx\}$, $k \in \kappa_c$, $x \in R$ is bounded for $z \rightarrow 0$ along this contour. This requirement defines completely the local parameter $z = k^{-1}$ modulo terms of the order z^3 , so our BA function is associated with this specific contour (value of $p_I = c$). **Canonical Contour is canonically oriented by the one-valued real function p_I on the oriented manifold Γ .**

Remark 2 *The finite-gap operator, constructed by the curve Γ is x -periodic with the period T if and only if the function $e^{iT p}$ is single-valued in Γ , or equivalently, if all periods of dp have the form:*

$$\oint_s dp = \frac{2\pi}{T} n_s, \quad n_s \in \mathbb{Z},$$

where s is an arbitrary closed contour.

Remark 3 In the work [2] we especially considered the case where our divisor D is equal to $D = gP = g\infty$ where g is genus of Γ . In this case we proved important "Multiplicative Property" of our "Fourier" BA basis:

$$\Psi_{g\infty}(x, z)\Psi_{g\infty}(y, z) = L_g\Psi_{g\infty}(x + y, z)$$

where L is a linear differential operator in x with coefficients independent on $z \in \Gamma$, $L = \partial_x^g + \dots$ - see [2]. This construction extends the construction [1] of the discrete Fourier bases done in the late 1980s for the needs of the Bosonic (closed) String Theory. The multiplicative properties of Fourier type series and transform are important in the Nonlinear Problems like String Theory. This specific case is not much different from others in the purely linear Harmonic Analysis discussed in the present work. Poles of Ψ in the variable x necessary appear in this case, so our inner products are indefinite - see below.

Let us define a **C-linear Inner Product** of smooth functions on the canonical contour $\kappa_c \subset \Gamma$ depending on the choice of divisor D and generated by the basis of functions $\Psi_D(x, z)$ restricted to the canonical contour κ_c .

Statement. For the basic BA functions we have **The Orthogonality Relations on Riemann Surface, i.e. on the contour $\kappa_c \subset \Gamma$:**

$$\begin{aligned} (\Psi_D(x, z), \Psi_D(y, z))_{\kappa_c} &= \int_{\kappa_c} \Psi_D(x, z)\Psi_{D^*}(-y, z)d\mu(z) = \\ &= 2\pi\delta(x - y) \end{aligned}$$

Proof. The form at the right-hand side is holomorphic in the variable z . Therefore this integral does not depend on c . If $x > y$, this integral vanishes as $c \rightarrow +\infty$. Equivalently, for $x < y$ this integral vanishes as $c \rightarrow -\infty$, therefore

$$(\Psi_D(x, z), \Psi_D(y, z))_{\kappa_c} = 0 \text{ for } x \neq y. \quad (1)$$

If we modify the integrand outside a neighbourhood of the point P , the resulting integral is the same up to a regular function of x, y . Let us expand the functions $\Psi_D(x, z)$, $\Psi_D^*(x, z)$ near the point P :

$$\Psi_D(x, z) = e^{ikx} \left[1 + \frac{\phi(x)}{k} + O\left(\frac{1}{k^2}\right) \right]$$

$$\begin{aligned}\Psi_D^*(x, z) &= e^{-ikx} \left[1 + \frac{\phi^*(x)}{k} + O\left(\frac{1}{k^2}\right) \right] \\ (\Psi_D(x, z), \Psi_D(y, z))_{\kappa_c} &= \int_{-\infty}^{+\infty} e^{ik(x-y)} \left[1 + \frac{\phi(x) + \phi^*(y)}{k} \right] dx + \text{regular function} = \\ &= 2\pi\delta(x-y) + \pi i \operatorname{sgn}(x-y)[\phi(x) + \phi^*(y)] + \text{regular function}.\end{aligned}$$

For $x = y$ the integrand has no essential singularities and only one first-order pole at P with the residue $\phi(x) + \phi^*(x)$. Therefore $\phi^*(x) = -\phi(x)$, and

$$(\Psi_D(x, z), \Psi_D(y, z))_{\kappa_c} = 2\pi\delta(x-y) + \text{regular function}.$$

Comparing it with (1) we complete the proof.

Now we consider class of functions $\phi(z)$ on the contour κ_c , such that their "Transform" is well defined. We interpret them simply as "Components" of function $\tilde{\phi}(x)$ in our BA basis $\Psi_{D^*}(-x, z)$ using the integral:

$$\tilde{\phi}(x) = (\sqrt{2\pi})^{-1} \int_{\kappa_c} \phi(z) \Psi_{D^*}(-x, z) d\mu$$

Statement. For the selected BA function with bounded restriction of $\exp\{ikx\}$ to the contour κ_c near $P = \infty$, this integral is well-defined near ∞ if $\phi(k) = o(k^{-1+\epsilon})$, $\epsilon > 0$.

Proof. The **Inverse Transform** is given by the formula

$$\phi(z) = (\sqrt{2\pi})^{-1} \int \tilde{\phi}(x) \Psi_D(x, z) dx$$

It leads to the same inner product of transformed functions in the x -space treated simply as "Collections of Components" in the previous basis of BA functions $\Psi_D(x, z)$, where x is considered as an "index" numerating the basic vector:

$$\int \tilde{\phi}_1(x) \tilde{\phi}_2(x) dx = (\phi_1, \phi_2)_{\kappa_c}$$

For the same BA functions treated as basis in x -space, we obtain a formula

$$(\Psi_D(x, z), \Psi_D(x, w))_x = \int_x \Psi_D(x, z) \Psi_{D^*}(-x, w) d\mu dx = 2\pi\delta(z, w) \quad (2)$$

$= 0$, $z \neq w$. Here both points $z, w \in \Gamma$ belong to our selected contour κ_c . We assume that these points are nonsingular on this contour. We assume that δ is an one-form in the variable w .

The case of critical contour corresponding to the critical values of the real function p_R should be considered separately. **This formula is meaningful locally only if our BA functions do not contain poles for $x \in R$. It is meaningful globally if our picture is periodic in $x \in R$,** so we have no concentration of poles near $x \rightarrow \pm\infty$. We postpone to the next work extension of our results to the quasi-periodic algebraic case.

Let us discuss, for which classes of functions this transform is well-defined. It depends on the divisor D and on the geometry of contour κ_c : **Does our BA function contain poles? Does divisor contain infinite point or not? Is our contour critical?**

We postpone the last question.

For the case of BA function with poles we invent following rule: **All integrals above taken along the line $x \in R$, should be taken avoiding pole x_0 in the upper half-plane $x + i\epsilon, \epsilon > 0$.** In order to prove that our inner products written as integral along the x -axis, are well-defined, we prove following **Main Lemma**:

Lemma 1 *The expression $\Psi_D(x, z)\Psi_{D^*}(-x, w)$ has residue equal to zero in every pole $x_0 \in R$ as a meromorphic function of the complex variable x in the small strip around the real line.*

The proof follows immediately from Lemma 6 and formula (7) below. We added Appendix 1 to make this proof fully rigorous.

So the integral defining inner product does not depend on the contour surrounding pole.

Let our data consisting of algebraic curve Γ with selected point $P = \infty$ and local parameter k^{-1} near P , be real now. It means precisely that an anti-holomorphic involution is defined

$$\tau : \Gamma \rightarrow \Gamma, \tau^2 = 1$$

such that $\tau(P) = P$ and $\tau^*(k) = \bar{k}$.

Our differential dp is such that $\tau^*(dp) = d\bar{p}$. We define p_I such that $\tau^*(p_I) = -p_I$, so the level $\kappa_0 = (p_I = 0)$ is invariant under τ :

$$\tau : \kappa_0 \rightarrow \kappa_0$$

and differentials dk, dp are real on κ_0 .

Let us point out that our contour κ_0 contains all set of fix-points

$$Fix_\tau \subset \kappa_0$$

where $z \in Fix_\tau$ means $\tau(z) = z$. Following simple geometric statement is useful to clarify relationship between our constructions and some results of the late 1980s (see[4]) about nonsingular real solution to the KPI system with Lax operator $i\partial_x + \partial_y^2 + u(x, y)$:

Lemma 2 *For anti-holomorphic involution τ the fix-point set Fix_τ coincides with canonical contour κ_0 if and only if Fix_τ divides Γ into two parts $\Gamma = \Gamma_+ \cup \Gamma_-$.*

Proof of this lemma easily follows from the obvious fact that κ_0 certainly divides Γ but its smaller part never does. We assume that $P \in Fix_\tau$.

We choose divisor D such that $\tau(D) = D^*$ or

$$D + \tau(D) \sim K + 2P$$

where K is divisor of differential forms. So we have $\tau^*(d\mu) = d\bar{\mu}$.

In this case we define a **Hermitian (or sesqui-linear) possibly indefinite Inner Product** on the contour κ_0 by the formula

$$\begin{aligned} \langle \Psi_D(x, z), \Psi_D(y, z) \rangle_{\kappa_0} &= (\Psi_D(x, z), \bar{\Psi}_D(y, \tau(z)))_{\kappa_0} = \\ &= \int_{\kappa_0} \Psi_D(x, z) \bar{\Psi}_{\tau D}(y, \tau z) d\mu(z) \end{aligned}$$

where the integral above is taken with respect to the canonical orientation of the contour κ_0 .

We take into account here that $\bar{\Psi}_{\tau D}(y, \tau z)$ is meromorphic in the variable z , has poles in τD and asymptotics $\exp\{-iky\}(1 + O(k^{-1} + \dots))$ near P for $y, k \in \mathbb{R}$. So for the "real" variables it coincides with our C -linear expression above.

In the x -space we have following inner product:

$$\langle \Psi_D(x, z), \Psi_D(x, w) \rangle_x = \int_x [\Psi_D(x, z) \bar{\Psi}_{\tau D}(\bar{x}, w) d\mu] dx$$

Let us point out that

$$\bar{\Phi}_{\tau D}(\bar{x}, w) = \Psi_{\tau D}(-x, w)$$

for the real values of the variables k, x . It is meromorphic in x . So the residue in the x -pole is equal to zero for the product $\Psi_D(x, z)\bar{\Psi}_{\tau D}(\bar{x}, \tau(z))$ under the sign of integral because it is the same as in the Lemma 1 above.

We are coming to the following

Lemma 3 *1. The hermitian inner product above on the contour κ_0 is positively defined if and only if $\kappa_0 = \text{Fix}_\tau$, and the form $d\mu$ is positive on the contour κ_0 . 2. The hermitian inner product in the x -space is well-defined avoiding every pole of Ψ in the upper half-plane in x . It is positive if and only if our BA function $\Psi_D(x, z)$ does not have poles on the real line x .*

The statement 1 makes sense because our form $d\mu$ is real on this contour. We have $\tau(z) = z$ for $z \in \text{Fix}_\tau$, and upper part Γ^+ of Γ induces natural orientation of the contour κ_0 . It is interesting to compare this result with [4]. The statement 2 is crucial for our work, so we present a full proof in the Appendix 1 using what we call The Cauchy-Baker-Akhiezer Kernel. This quantity is borrowed from the work [5], but some additional improvements are needed here. Besides that, no full proof was presented in the work [5].

One can see following sources for the violation of positivity of the inner product on the contour κ_0 :

1. $\kappa_0 \neq \text{Fix}_\tau$. We have here $\tau(z) \neq z$ for z outside of fix-point set. Such inner product is always indefinite.

2. $\text{Fix}_\tau = \kappa_0$ but the divisor D is chosen such that $d\mu$ has different signs on some components (see chapter 2).

Only poles of Ψ on the real line x are responsible for the non-positivity of the inner product in the x -space. This is central part of our work.

We are going to consider this picture in more details for the important hyperelliptic case in the next chapter.

Chapter 2. The indefinite Inner Product for Hyperelliptic Riemann Surfaces. Schrodinger Operators with singular potential.

Consider now the most important case of nonsingular Hyperelliptic Riemann Surfaces Γ associated with second order periodic operators L : Let Γ is presented in the form

$$w^2 = (z - z_0) \times \dots (z - z_{2g}) = R(z)$$

where typical point (except branching points) is written as $\gamma = (z, \pm)$. We take branching point P with $z = \infty$ as our "infinity" with local coordinate $k^{-1} = z^{-1/2} = u$. Every generic divisor $D = \gamma_1 + \dots + \gamma_g$ defines a Baker-Akhiezer function $\Psi_D(x, z)$ with standard analytic properties described above. It satisfies to the equation

$$L\Psi = (-\partial_x^2 + U(x))\Psi(x, z) = z\Psi(x, z)$$

Our requirement is that the potential $U(x)$ is periodic $U(x + T) = U(x)$ for real x . From the finite-gap theory we know that necessary and sufficient condition to have real nonsingular potential $U(x)$ (we call it a **Canonical Inverse Spectral Conditions**) consists of two parts:

- 1.The Strong Reality Condition for Γ : all branching points z_j are real and distinct. Let $z_0 < z_1 < \dots < z_{2g}$.
- 2.The divisor D is **Proper** i.e. such that $\gamma_k = (\alpha_k, +)$ or $\gamma_k = (\alpha_k, -)$ where $z_{2k-1} \leq \alpha_k \leq z_{2k}, k = 1, 2, \dots, g$ (exactly one divisor point is located in every a -cycle).

There are two commuting anti-holomorphic involutions τ_{\pm} of the Riemann Surface Γ where $\tau_{\pm}(z, +) = (\bar{z}, \pm)$. Let $\tau_+ = \sigma, \tau_- = \tau$. Our contour κ_0 is equal to Fix_{τ} . It coincides with union of spectral zones. The set Fix_{σ} coincides with union of spectral gaps:

The union of our a -cycles a_k form finite part of the fix-point set for the anti-involution $\sigma(p) = p$. Their projection on the z -line coincide with finite "Gaps" $[z_{2k-1}, z_{2k}], k = 1, 2, \dots, g$, in the Spectral Theory of operator L in the Hilbert Space $L_2(\mathbb{R})$ of the square-integrable complex-valued functions on the real line. So we have $\sigma(D) = D$ and $\tau D = D^*$ where $D + D^* = K + 2P$.

We know however that inverse spectral data lead to the real operators L in other **Non-Canonical Real Cases**:

- 1.The Riemann Surface Γ is Real. It simply means that the set of branching points $z_j, j = 0, 1, \dots, 2g$, is invariant under the anti-involution $z \rightarrow \bar{z}$.

2.The Divisor of Poles D should be such that $\sigma(D) = D$ but not necessarily like in the Canonical Case.

If these conditions are satisfied, then the potential $U(x)$ is real. However, this potential is singular . Otherwise, it would be self-adjoint in the positive Hilbert Space which is impossible. So it is singular in all non-canonical real cases. We call our data **Real Semi-Canonical** if Riemann Surface satisfies to the Strong Reality Condition but the divisor D is nor Proper. In particular,our contour κ_0 coincides with fixpoint set Fix_τ . The potential $U(x)$ has poles in that case.

Orientation of κ_0 in the Real Semi-Canonical case is defined by $dp > 0$. For such spectral curves

$$dp = (z - p_1) \times \dots(z - p_g)dz/\sqrt{(z - z_0) \times \dots(z - z_{2g})}, \quad (3)$$

where all p_k are real and $p_k \in (z_{2k-1}, z_{2k})$.

Another possibility is that not all branching points are real: there are complex adjoint pairs between them. In this case we have Fix_τ essentially smaller than the contour κ_0 . So our operator L is singular.

According to the previous chapter, such operators are symmetric in the Indefinite Inner Product given by the formulas presented there.

Using previous results, we are coming to the following

Theorem: 1.Let Riemann Surface and divisor D are real and finite. In the case $Fix_\tau = \kappa_0$, the form

$$d\mu = (z - \gamma_1) \times \dots(z - \gamma_g)dz/\sqrt{R(z)}$$

is real, nonzero and has a well-defined sign in every component. The set Fix_τ is the spectrum of operator L . 2.Define an Indefinite Hilbert Space as a direct sum of spaces of functions in the components of Fix_τ , which is the standard one for the functions on every component but taken with sign provided by the form $d\mu$ and orientation of the contour κ_0 . Our linear operator $L = -\partial_x^2 + U(x)$ is symmetric, and corresponding "Fourier Transform" defined in the previous paragraph, is isometric in this indefinite inner product. 3.In the case if one (or more) divisor points are infinite $\gamma_g = \infty$, the form $d\mu$ is holomorphic.

Remark 4 From (3) it follows that the sign of $d\mu$ on real ovals of τ with respect to the orientation of κ_0 coincides with the sign of $dp/d\mu$, or, equivalently, with the sign of the ratio:

$$(z - p_1) \dots (z - p_g)/(z - \alpha_1) \dots (z - \alpha_g).$$

Remark 5 We do not describe the exact completion of this space. So our result is incomplete in terms of Modern Functional Analysis.

Remark 6 If r point of divisor D are equal to ∞ (i.e. $D = r\infty + (\alpha_1, \pm) + \dots + (\alpha_{g-r}, \pm)$), we have

$$d\mu = (z - \alpha_1) \times \dots (z - \alpha_{g-r}) dz / \sqrt{R(z)}$$

The special case $r = g$ all divisor is concentrated in the point ∞ . This case was especially considered as a right analog of Fourier Transform: It has Remarkable Multiplicative Properties.

Proof. Our Theorem immediately follows from the results of Chapter 1 and Appendix 1.

Example. Consider the case of real elliptic curve Γ with genus $g = 1$ and real branching points z_0, z_1, z_2, ∞ . We assume that our divisor $D = \gamma$ coincides with $P = \infty$. The Baker-Akhiezer Function Ψ here was found by Hermit, and singular operator L is the Lamé'Operator; it has the form $U(x) = 2\wp(x)$. Our Hilbert Space is a direct sum of 2 spaces

$$H = H_0 \oplus H_\infty$$

Here H_0 consists of functions on the compact circle $c_1 \subset \Gamma$ located over the spectral zone $[z_0, z_1]$ (the finite zone of spectrum). The second subspace H_∞ consists of functions on $R \subset \Gamma$ located over the zone of spectrum $[z_2, \infty]$ and homeomorphic to $R = S^1 \setminus \infty$. They have specific asymptotic at infinity indicated above in the chapter 1.

Statement: Our inner product is positive at H_∞ and negative at H_0 . Proof. We have in this case $d\mu = dz / \sqrt{(z - z_0)(z - z_1)(z - z_2)}$, and orientation of the contour $\kappa_0 = c_1 \cup c_\infty$ is such that $d\mu|_{c_1} < 0, d\mu|_{c_\infty} > 0$. This statement is proved.

For comparison good to consider the "selfadjoint" case such that $\gamma' = (\alpha', \pm)$ where $\alpha' \in [z_1, z_2]$ is located in the finite gap. In this case we have

$$d\mu' = (z - \alpha') dz / \sqrt{(z - z_0)(z - z_1)(z - z_2)}$$

So we have $d\mu' = (z - \alpha') d\mu$. Taking into account that the function p_I is the same in both cases, we see that the factor $(z - \alpha')$ has opposite signs in the gaps c_1 and c_∞ . Therefore in this case the Inner Product is positive (as we knew before).

The Standard Fourier Transform we have for the case of genus zero: The Riemann Surface Γ has 2 branching points $z_0 = 0, z_1 = \infty$. The spectral zone in Γ has one component c_∞ only located over $[0, \infty]$. It is isomorphic to $R = S^1 \setminus \infty$. The measure $d\mu$ coincides with standard measure. So our Hilbert Space is exactly $H = H_\infty = L_2(R)$, and inner product is positive. For genus more than zero we can have positive inner product only for smooth potentials where divisor points are located in the finite gaps (one gap—one point).

In all cases where our divisor contains infinite point or any point located in the infinite gap, we have indefinite inner product. We can always move this point by some time shift to infinity.

In all cases where our divisor contains two (or more) points located in the same finite gap, we have indefinite inner product.

We can easily describe the sign corresponding to the cycle $c_j \subset \Gamma$ located over the zone $[z_{2j-1}, z_{2j}]$, i.e. how it enters the Indefinite Hilbert Space:

Take divisor points $\gamma_s = (\alpha_s, \pm)$, where $\alpha_s \in [z_{2q_s-1}, z_{2q_s}]$, $s = 1, \dots, r$, and $\gamma_{r+k} = \infty$ for all $k > r$. As we know, it simply coincides with sign of the expression $dp/d\mu$ where:

$$d\mu = (z - \alpha_1) \times \dots (z - \alpha_r) dz / \sqrt{(z - z_0) \times \dots (z - z_{2g})}$$

on the cycle c_k , taking into account the orientation of the contour κ_0 provided by the function p_I as it was explained in the Chapter 1. **For example, for $r = 0$ (the case of Fourier Transform with important Multiplicative Properties extending the Hermit-Lame' potentials $n(n + 1)\wp(x) = U(x)$), the signs corresponding to c_k , are alternating.** Here we have

$$d\mu = dz / \sqrt{R(z)}$$

Appendix 1. The Cauchy-Baker-Akhiezer Kernel

Following [5], let us define the Cauchy-Baker-Akhiezer Kernel $\omega(x, z, w)$, $x \in \mathbb{C}$, $z \in \Gamma \setminus P$, $w \in \Gamma \setminus P$ by the following analytic properties:

1. For a fixed x the kernel $\omega(x, z, w)$ is a meromorphic function in z and a meromorphic 1-form in w .
2. For fixed x, w the kernel $\omega(x, z, w)$ has exactly $g + 1$ simple poles in z at the points $\gamma_1, \dots, \gamma_g, w$.
3. For fixed x, z the kernel $\omega(x, z, w)$ has simple zeroes in w at the points $\gamma_1, \dots, \gamma_g$, and a simple pole with residue 1 at the point z . In local coordinates we have

$$\omega(x, z, w) = \frac{dw}{w - z} + \text{regular terms as } w \rightarrow z. \quad (4)$$

4. For fixed x, w the kernel $\omega(x, z, w)$ has an essential singularity in the variable z at the point $P = \infty$:

$$\omega(x, z, w) = e^{ik(z)x} \left(O \left(\frac{1}{k(z)} \right) \right). \quad (5)$$

5. For fixed x, z the kernel $\omega(x, z, w)$ has an essential singularity in the variable w at the point $P = \infty$:

$$\omega(x, z, w) = e^{-ik(w)x} \left(O \left(\frac{1}{k(w)} \right) \right) dk(w). \quad (6)$$

For generic spectral data the kernel $\omega(x, z, w)$ exists and is unique, by the Riemann-Roch Theorem. The proof is analogous to the proof of existence and uniqueness for the Baker-Akhiezer function. Following the idea of the work [5], we prove one of the most important properties of this Kernel:

Fundamental Lemma. Following Formula is Valid:

$$\partial_x \omega(x, z, w) = -i \Psi(x, z) \Psi^*(x, w) d\mu(w). \quad (7)$$

Proof. For a fixed w the right-hand side of (7) has the following analytic properties:

1. $\partial_x \omega(x, z, w)$ is meromorphic in the variable z on $\Gamma \setminus P$ and has exactly g simple poles at the points $\gamma_1, \dots, \gamma_g$.
2. $\partial_x \omega(x, z, w) = O(1)e^{ik(z)x}$ as $z \rightarrow P$.

Therefore for a fixed w the expression $\partial_x \omega(x, z, w)$ is proportional to $\Psi(x, z)$. Similarly for a fixed z the expression $\partial_x \omega(x, z, w)$ is proportional to $\Psi^*(x, w)d\mu(w)$. Therefore $\partial_x \omega(x, z, w) = c\Psi(x, z)\Psi^*(x, w)d\mu(w)$. Assuming z and w are both close to P , we obtain $c = -i$. Our Fundamental Lemma is proved.

Remark 7 *Let $x = 0$. Then the kernel $\omega(0, z, w)$ coincides with the standard meromorphic analog of Cauchy kernel on the closed Riemann surfaces (see [7], [8]).*

Remark 8 *It is natural to define Cauchy-Baker-Akhiezer kernel $\omega(\vec{t}, z, w)$ depending on all KP times $\vec{t} = (t_1, t_2, t_3, \dots)$, $x = t_1$, $y = t_2$, $t = t_3$ (see [5]). Essential singularities for the kernel $\omega(\vec{t}, z, w)$ have the following form:*

$$\omega(\vec{t}, z, w) = e^{i \sum_{j=1}^{\infty} t_j k^j(z)} \left(O \left(\frac{1}{k(z)} \right) \right), \quad z \rightarrow P, \quad (8)$$

$$\omega(\vec{t}, z, w) = e^{-i \sum_{j=1}^{\infty} t_j k^j(w)} \left(O \left(\frac{1}{k(w)} \right) \right) dk(w), \quad w \rightarrow P. \quad (9)$$

Here we assume that only finite number of variables t_n are different from 0.

To stress the dependence of $\omega(\vec{t}, z, w)$ on the divisor $D = \gamma_1 + \dots + \gamma_g$, we shall write $\omega_D(\vec{t}, z, w)$ if necessary.

Lemma 4 *Denote by $D(\vec{t})$ the divisor of zeros for the function $\Psi_D(\vec{t}, z)$. Then for any \vec{t}' we have following transformation law:*

$$\omega_D(\vec{t}, z, w) = \frac{\Psi_D(\vec{t}', z)}{\Psi_D(\vec{t}', w)} \omega_{D(\vec{t}')}(\vec{t} - \vec{t}', z, w) \quad (10)$$

The next special case plays leading role in our investigation because it is the most natural source for the singular operators:

Lemma 5 *Assume, that exactly one point of the divisor D lies at the infinite points $P = \infty$: $D = \gamma_1 + \gamma_2 + \dots + \gamma_{g-1} + P$. Then for generic $\gamma_1, \dots, \gamma_{g-1}$ one can write an especially simple formula for the kernel $\omega(\vec{t}, z, w)$:*

$$\omega(\vec{t}, z, w) = \frac{\theta[\sum_j \vec{U}_j t_j + \vec{A}(z) - \vec{A}(w) - \vec{A}(\gamma_1) - \dots - \vec{A}(\gamma_{g-1}) - \vec{K}]}{\theta[\sum_j \vec{U}_j t_j - \vec{A}(\gamma_1) - \dots - \vec{A}(\gamma_{g-1}) - \vec{K}]} \times \\ \times \frac{C \cdot d\mu(w)}{\theta[\vec{A}(z) - \vec{A}(w) - \vec{A}(\gamma_1) - \dots - \vec{A}(\gamma_{g-1}) - \vec{K}]} \cdot \exp \left[i \sum_j t_j \int_w^z \Omega_j \right] \quad (11)$$

Here Ω_j are meromorphic differentials with an unique pole at the point P ,

$$\Omega_j = d(k^j) + \text{regular terms}$$

and zero a -periods, U_j denotes the normalized vector of b -periods for Ω_j :

$$U_j^k = \frac{1}{2\pi} \oint_{b_k} \Omega_j,$$

$\vec{A}(\gamma)$ denotes the Abel transform with the starting point P , \vec{K} is the vector of Riemann constants, $d\mu$ is the holomorphic differential with the zeroes $\gamma_1, \dots, \gamma_{g-1}$. Let ν be a local coordinate near P such, that $d\mu = d\nu(1 + o(1))$. Then the normalization constant C is defined by:

$$C = \partial_\nu \Big|_{\nu=0} \theta[-\vec{A}(v) - \vec{A}(\gamma_1) - \dots - \vec{A}(\gamma_{g-1}) - \vec{K}]. \quad (12)$$

Using standard arguments, one can easily check, that the expression (11) is single-valued in Γ , and for generic \vec{t} it has the proper poles in z and w . Let $z \rightarrow w$. Then

$$\omega(\vec{t}, z, w) \sim \frac{C \cdot d\mu(w)}{\theta[\vec{A}(z) - \vec{A}(w) - \vec{A}(\gamma_1) - \dots - \vec{A}(\gamma_{g-1}) - \vec{K}]} \quad (13)$$

For z different from $\gamma_1, \dots, \gamma_g, \gamma_1^*, \dots, \gamma_g^*$ denominator of (13) has a first-order zero at $w = z$. If $z = \gamma_j, j = 1, \dots, g$, then the denominator vanishes identically. If $z = \gamma_j^*, j = 1, \dots, g$, then denominator has a second-order pole. Therefore the zeroes of the numerator coincides with the zeroes of the denominator's differential, and the residue of (13) at $w = z$ is regular in

Γ . Therefore this residue is constant. Normalization (12) means, that the residue is equal to 1 at $z = \infty$. It completes the proof.

It follows from (11) that the Cauchy-Baker-Akhizer Kernel $\omega(\vec{t}, z, w)$ is meromorphic in all t_j . Combining (11) with (10) we obtain following:

Lemma 6 *For any divisor D such, that $\Psi_D(x, z)$ is defined for generic x , the kernel $\omega_D(x, z, w)$ is meromorphic in x .*

Remark 9 *Assume, that operators, associated with the curve Γ are strictly periodic in x with period T . Then following formula is true:*

$$\omega(x + T, z, w) = \omega(x, z, w) \cdot e^{i[p(z)-p(w)]x}. \quad (14)$$

Let us derive the orthogonality relation for BA functions treated as basis in the space of functions in x -space (2). Their inner products already were discussed in the work [6]. We have

$$\begin{aligned} \int_{-nT}^{nT} \Psi(x, z) \Psi^*(x, w) d\mu dx &= i\omega(x, z, w) \Big|_{-nT}^{nT} = \\ &= i\omega(0, z, w) [e^{i[p(z)-p(w)]nT} - e^{-i[p(z)-p(w)]nT}] = \\ &= [p(w) - p(z)]\omega(0, z, w) \int_{-nT}^{nT} e^{i[p(z)-p(w)]x} dx. \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} \int_{-nT}^{nT} \Psi(x, z) \Psi^*(x, w) d\mu dx = 2\pi[p(w) - p(z)]\omega(0, z, w)\delta(p(z) - p(w)),$$

and

$$\lim_{n \rightarrow \infty} \int_{-nT}^{nT} \Psi(x, z) \Psi^*(x, w) d\mu dx = 0 \text{ for } z \neq w, \quad z, w \in \kappa_c.$$

Let $w \rightarrow z$. Substituting (4) and taking into account, that the orientation on the canonical contour κ_c is defined by dp , we obtain our final result:

Following Orthogonality Relations for BA functions as a basis in the x -space, are true:

$$\begin{aligned} & \left(\int_{-\infty}^{\infty} \Psi(x, z) \Psi^*(x, w) dx \right) d\mu = \\ &= 2\pi(p(w) - p(z)) \left[\frac{dp(w)}{p(w) - p(z)} + \text{regular terms} \right] \delta(p(z) - p(w)) = 2\pi\delta(z, w) \end{aligned}$$

Appendix 2. The Hyperelliptic Case. The Periodic Boundary Conditions.

The Cauchy-Baker-Akhiezer Kernel.

In the hyperelliptic case there exists a simple explicit formula for the Cauchy-Baker-Akhiezer kernel:

$$\omega(x, z, w) = i \frac{\Psi(x, z)\Psi_x^*(x, w) - \Psi_x(x, z)\Psi^*(x, w)}{w - z} d\mu(w)$$

It is easy to check, that all analytic properties are fulfilled. Moreover,

$$\begin{aligned} \partial_x \omega(x, z, w) &= [\Psi(x, z)\Psi_{xx}^*(x, w) - \Psi_{xx}(x, z)\Psi^*(x, w)] \frac{i}{w - z} d\mu = \\ &= [(-z - U(x))\Psi(x, z)\Psi^*(x, w) - (-w - U(w))\Psi(x, z)\Psi^*(x, w)] \frac{i}{w - z} d\mu = \\ &= -i\Psi(x, z)\Psi^*(x, w)d\mu. \end{aligned}$$

The periodic boundary conditions.

We assume that our finite-gap operators are periodic with the period T . In addition to the spectral problem in the whole line one can consider the periodic boundary problem with an fixed unitary multiplier:

$$\Psi(x + T, z) = \varkappa \Psi(x, z), \quad |\varkappa| = 1. \quad (15)$$

For regular potentials this problem is self-adjoint and has only discrete spectrum. Let us enumerate the points z_j in Γ , $j = 1, 2, \dots, \infty$ such, that

$$e^{iT p(z_j)} = \varkappa.$$

All these points lie in the canonical contour κ_0 . Each finite oval contains only finite number of points z_j .

Lemma 7 *The scalar product for the basic eigenfunctions is given by:*

$$\int_0^T \Psi(x, z_j)\Psi^*(x, z_k)dx = \delta_{jk} \frac{dp(z_j)}{d\mu(z_j)}. \quad (16)$$

As above we deform the integration contour in the x -plane to avoid singularities.

Proof. For $j \neq k$

$$\int_0^T \Psi(x, z_j) \Psi^*(x, z_k) dx = \frac{i\omega(x, z_j, z_k) \Big|_0^T}{d\mu(z_k)} = \frac{i\omega(0, z_j, z_k)}{d\mu(z_k)} [e^{iT[p(z_j)-p(z_k)]} - 1] = 0.$$

Let $k = j$.

$$\begin{aligned} \int_0^T \Psi(x, z_j) \Psi^*(x, z_j) dx &= \lim_{w \rightarrow z_j} \frac{i\omega(x, z_j, w) \Big|_0^T}{d\mu(w)} = \\ &= \lim_{w \rightarrow z_j} \left[\frac{idp(w)}{d\mu(w)[p(w) - p(z_j)]} + \text{regular terms} \right] [e^{iT[p(z_j)-p(w)]} - 1] = \frac{dp(z_j)}{d\mu(z_j)}. \end{aligned}$$

Assume now, that the spectral data Γ, D satisfy the reality constraints. Taking into account that for real spectral curves $\Psi^*(x, z) = \bar{\Psi}(x, \tau z)$ we obtain following:

Theorem. *Let us define the scalar product by*

$$(\Psi(x, z_j), \Psi(x, z_k))_x = \int_0^T \Psi(x, z_j) \bar{\Psi}(\bar{x}, z_k) . dx \quad (17)$$

*For singular potentials this scalar product is not positive defined. The dimension of negative subspace is finite and coincides with the number of points z_j such, that $dp(z_j)/d\mu(z_j) < 0$. Therefore we have a **Pontryagin-Sobolev** space of functions where our singular finite-gap operator is symmetric.*

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