Integrability, analyticity, isochrony, equilibria, small oscillations, and Diophantine relations

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Summary

New Diophantine relations are obtained, in the guise of matrices having *integer* eigenvalues, or equivalently of polynomials, defined via three-terms recursion relations, having *integer* zeros. The basic idea to arrive at such relations is not new, but the specific application reported in this paper is new, and it is likely to open the way to several analogous findings.

1 Introduction

The general approach to arrive at the findings reported in this paper can be described as follows (see for instance [1]). One starts from an *integrable* dynamical system, namely a system of first-order nonlinear ODEs. One then modifies it so that—thanks to the analyticity properties in *complex* time of the solutions of the original *integrable* system (i. e., its Painlevé property)—the modified system becomes *entirely isochronous*: its solutions, in its *entire* phase space, are *all* periodic with the same *fixed* period in *all* degrees of freedom. (Indeed, the modification entails, essentially, that the time-evolution of the modified system corresponds to the evolution of the original system when the "time" variable of the latter rotates uniformly on a circle in the *complex* plane: the *isochrony* of the modified system is therefore a consequence of the *meromorphic* character of the dependence of the original system is its *complex* "time" variable.The possibility to perform such a modification, transforming via this technique—as described below—an *autonomous* dynamical system into a modified system which is also *autonomous,* requires that the original system satisfy a *grading* property, which is often featured by *integrable* systems analogous to that treated herein: see [1]). One then identifies the *equilibria* of the *isochronous* system (in some cases this can be done explicitly) and investigates, close to these equilibria, its (infinitesimally small) oscillations. Their frequencies are given by the *eigenvalues* of the *matrices* characterizing the *linearized* system near its equilibria (these matrices are of course given, in terms of the values of the dynamical variables at equilibrium, by expressions easily obtainable from the equations of motion of the dynamical system). The fact that the system is *isochronous* entails that, around each equilibrium, these frequencies must *all* be *integer* multiples of a basic period. In this manner one arrives at *Diophantine relations*, namely at the identification of matrices whose *eigenvalues* are *integer* numbers; equivalently, *polynomials defined by three-term recursion relations* are identified, yielding polynomials which factorize in terms of *integer* zeros. These are the final findings produced by this approach.

This route to arrive at these findings is not new, and it might appear contrived: indeed, its formulation via *isochronous* dynamical systems could certainly be replaced by other, equivalent approaches of a more algebraico-geometrical character. Its significance seems to us rather transparent, and its application has yielded interesting findings (for a review see Appendix C of [1], entitled "Diophantine findings and conjectures"). The application of this approach to the class of *integrable* systems treated herein is new, hence the corresponding findings are new. And it appears that analogous results could be obtained by applying the same approach to other classes of *integrable* systems, this being perhaps the most interesting aspect of the findings reported herein.

2 Results

It is well-known that the following nonlinear ODE, of order

$$N = 2M + 1 \ , \tag{1}$$

is *integrable*, and in particular that *all* its solutions $\zeta(\tau)$ possess the ("Painlevé") property to be *meromorphic* functions of the independent variable τ, considered as a *complex* variable:

$$L^M(\zeta) \cdot \zeta' = 0 \ , \tag{2a}$$

where the integro-differential operator $L(\zeta)$ acts as follows on functions $\varphi(\tau)$:

$$L(\zeta) \cdot \varphi(\tau) = \varphi''(\tau) - 4\zeta(\tau)\,\varphi(\tau) - 2\zeta'(\tau) \int^{\tau} d\sigma \ \varphi(\sigma) \ . \tag{2b}$$

The integration in this definition of the operator $L(\zeta)$ is meant to be performed *omitting* the contribution from the lower end of the integration range, and—here and throughout—appended primes denote differentiations with respect to the independent variable τ. The notation $L^M(\zeta)\cdot$ indicates of course the iterated application M times of the operator $L(\zeta)$; and hereafter M is a fixed positive integer, and N the corresponding *odd* positive integer, see (1). In the following we will freely use N and M (sometimes even in the same formula, to write it in neater form), on the understanding that they are always related by (1).

The fact that the ODE (2a) is *integrable*—as well as the very fact that it is indeed an ODE rather than an integro-differential equation, as it might at first sight appear to be, see (2b)—is of course well-known: this equation is just the stationary version of the Mth PDE of the KdV class (with the "spatial" independent variable denoted here as τ), see for instance [2] [3]. More explicit expressions of this hierarchy of integrable ODEs are available (see for instance [4], and references therein), but they are too complicated to be displayed here.

On the other hand obtaining the first few of these ODEs by iteration is a straightforward exercise. For instance for $M=1$ (2a), reads

$$\zeta'''(\tau) = 6\zeta'(\tau)\,\zeta(\tau) \ , \tag{3a}$$

and for $M=2$ it reads

$$\zeta'''''(\tau) = 10\zeta'''(\tau)\,\zeta(\tau) + 20\zeta''(\tau)\,\zeta'(\tau) - 30\zeta'(\tau)\,\zeta^2(\tau) \ . \tag{3b}$$

Via the definition (entailing $\zeta_1(\tau) = \zeta(\tau)$)

$$\left(\frac{d}{d\tau}\right)^{n-1} \zeta(\tau) = \zeta_n(\tau) \ , \tag{4}$$

with, here and hereafter (unless otherwise indicated), $n = 1, 2, 3, ..., N$, the single third-order ODE (3a) is seen to be equivalent to the system of 3 first-order ODEs

$$\zeta_1' = \zeta_2 \ , \quad \zeta_2' = \zeta_3 \ , \quad \zeta_3' = 6\zeta_2\zeta_1 \ , \tag{5a}$$

and the single fifth-order ODE (3b) is seen to be equivalent to the system of 5 first-order ODEs

$$\zeta_n' = \zeta_{n+1} \ , \quad n = 1, 2, 3, 4 \ ; \quad \zeta_5' = 10\zeta_4\zeta_1 + 20\zeta_3\zeta_2 - 30\zeta_2\zeta_1^2 \ . \tag{5b}$$

Likewise, the single ODE, of order N, satisfied by the dependent variable $\zeta(\tau)$, reading

$$\zeta_N' = f_N\left(\zeta_{N-1}, \zeta_{N-2}, ..., \zeta_1\right) \tag{6}$$

with the polynomial function $f_N\left(\zeta_{N-1}, \zeta_{N-2}, ..., \zeta_1\right)$ defined by identifying via (4) this ODE with the Nth order ODE (2), is seen to be equivalent to the system of N first order ODEs

$$\zeta_n' = \zeta_{n+1} \ , \quad n = 1, 2, ..., N-1 \ ; \quad \zeta_N' = f_N\left(\zeta_{N-1}, \zeta_{N-2}, ..., \zeta_1\right) \ . \tag{7}$$

For instance this definition of $f_N\left(\zeta_{N-1}, \zeta_{N-2}, ..., \zeta_1\right)$ entails (see (3a) or (5a))

$$f_3(\zeta_2, \zeta_1) = 6\zeta_2\zeta_1 \tag{8a}$$

and (see (3b) or (5b))

$$f_5(\zeta_4, \zeta_3, \zeta_2, \zeta_1) = 10\zeta_4\zeta_1 + 20\zeta_3\zeta_2 - 30\zeta_2\zeta_1^2 \ . \tag{8b}$$

The *integrable* dynamical system (7)—with the N functions $\zeta_n \equiv \zeta_n(\tau)$ considered as N dependent variables—is our starting point. This choice represents the main novelty of our treatment; the possibility of analogous developments, using the same methodology, see below, but with different points of departure (say, other hierarchies of integrable nonlinear PDEs), is obvious.

The fact that this system of ODEs, (7), is *integrable* entails that it possesses the Painlevé property: *all* its solutions $\zeta_n(\tau)$ are *meromorphic* functions of the complex variable τ (see for instance [3], and in particular the relevant papers by S. P. Novikov and others referred to there).

It is well-known—and in any case clear from its definition, see (4), (6) and (7)—that the function $f_N(\zeta_{N-1}, \zeta_{N-2}, ..., \zeta_1)$ features the following scaling property:

$$f_N\left(\alpha^N \zeta_{N-1}, \alpha^{N-1} \zeta_{N-2}, ..., \alpha^2 \zeta_1\right) = \alpha^{N+2}\ f_N\left(\zeta_{N-1}, \zeta_{N-2}, ..., \zeta_1\right)\ . \quad (9)$$

It is therefore possible (see for instance [1]), via the following change of dependent and independent variables,

$$z_n(t) = \exp\left[i\ \ (n+1)\ \ t\right]\ \zeta_n(\tau)\ \ , \quad \tau = i\ \left[1 - \exp(i\ t)\right]\ , \quad (10)$$

to transform the (*autonomous* and *integrable*) dynamical system (7) into the following system,

$$\dot{z}_n - i\ \ (n+1)\ \ z_n = z_{n+1}\ , \quad n = 1, ..., N-1\ , \quad (11a)$$

$$\dot{z}_N - i\ \ (N+1)\ \ z_N = f_N(z_{N-1}, z_{N-2}, ..., z_1)\ , \quad (11b)$$

which is of course as well *autonomous* and *integrable*, and is moreover *isochronous*, so that *all* its solutions feature the periodicity property

$$z_n(t + 2\pi) = z_n(t)\ . \quad (12)$$

Here and below i is of course the imaginary unit, $i^2 = -1$, and a superimposed dot denotes differentiation with respect to the independent variable t, so that in particular $\dot{\tau}(t) = \exp(i\ t)$ (see (10), and note that it also entails $\tau(0) = 0$ hence $z_n(0) = \zeta_n(0)$; this simplifies the relation among the initial data of the two dynamical systems (7) and (11), but in fact plays no relevant role in the following developments).

The fact that the system (11) is *isochronous*, see (12), is an obvious consequence [1] of the change of variables (10) together with the *meromorphic* character of all the solutions $\zeta_n(\tau)$ of the integrable dynamical system (7).

Let now

$$z_n(t) = \bar{z}_n\ , \quad (13)$$

denote an equilibrium configuration of the dynamical system (11), so that the N numbers $\bar{z}_n$ satisfy the set of N algebraic equations

$$-i\ \ (n+1)\ \ \bar{z}_n = \bar{z}_{n+1}\ , \quad n = 1, ..., N-1\ , \quad (14a)$$

$$-i\ \ (N+1)\ \ \bar{z}_N = f_N(\bar{z}_{N-1}, \bar{z}_{N-2}, ..., \bar{z}_1)\ . \quad (14b)$$

It is then clearly convenient to set

$$\bar{z}_n = n!\ \ (-i)^{n+1}\ \ y\ , \quad (15)$$

which guarantees that the $N-1$ equations (14a) are all automatically satisfied, while, to also satisfy the remaining equation (14b), the number y is then required to satisfy the following polynomial equation of order $M+1=(N+1)/2$:

$$(N+1)!\ y=f_N\left((N-1)!\ y,(N-2)!\ y,...,3!\ y,2\ y,y\right)\ . \tag{16}$$

Note that, to write this equation in a neater way, we took advantage of the scaling property (9). (The fact that this is a polynomial equation of degree $M+1=(N+1)/2$ in the unknown y is a clear consequence of the definition of the function $f_N\left(z_{N-1},z_{N-2},...,z_1\right)$, as given above: see in particular (8)). Hence this polynomial equation has $M=(N-1)/2$ solutions, in addition to the trivial solution $y=0$ (which clearly is always featured by this equation: see in particular (8)).

For instance a simple calculation shows that for $M=1$, $N=3$ the nonvanishing value of y is $y=2$, while for $M=2$, $N=5$ the 2 nonvanishing values of y are $y=2$ and $y=6$. The fact that these values of y are *integer* numbers was not a *priori* expected; it remains to be checked whether this property persists for larger values of N, and if so to understand why.

The next step is to linearize the *isochronous* system of ODEs (11) near its equilibria (see (15) with (16)). Hence we set

$$z_n\left(t\right)=\bar{z}_n+\varepsilon\ w_n\left(t\right)\ , \tag{17}$$

in (11), and in the limit of infinitesimal ε we obtain, for the N dependent variables $w_n\left(t\right)$, the linear system of ODEs

$$\dot{w}_n-i\ \ (n+1)\ \ w_n=w_{n+1}\ ,\quad n=1,...,N-1\ , \tag{18a}$$

$$\dot{w}_N-i\ \ (N+1)\ \ w_N=\sum_{n=1}^{N-1}f_{N,n}\ w_n\ , \tag{18b}$$

where clearly

$$f_{N,n}=\frac{\partial\ f_N}{\partial\ z_n}\left(\bar{z}_{N-1},\bar{z}_{N-2},...,\bar{z}_2,\bar{z}_1\right)\ ,\quad n=1,...,N-1\ , \tag{19a}$$

or equivalently, via (15),

$$\begin{aligned} f_{N,n} &= \frac{\partial\ f_N}{\partial\ z_n}\Big((N-1)!\ \ (-i)^N\ \ y,(N-2)!\ \ (-i)^{N-1}\ \ y,...,-2iy,-y\Big)\ ,\\ n &= 1,...,N-1\ . \end{aligned} \tag{19b}$$

We also set, for notational convenience (see below),

$$f_{N,n}=(i)^n\ \ g_{N,n}\ . \tag{20}$$

So, for instance, for $N=3$, from (8a) one easily gets

$$g_{3,1}=12\ y\ ,\quad g_{3,2}=6\ y\ , \tag{21a}$$

hence, corresponding to the root $y = 2$,

$$g_{3,1} = 24 \ , \quad g_{3,2} = 12 \ , \tag{21b}$$

and likewise, for $N = 5$, from (8b) one easily gets

$$\begin{aligned} g_{5,1} &= 120\ y\ (y-2) \ , \quad g_{5,2} = 30\ y\ (y-4) \ , \\ g_{5,3} &= -40\ y \ , \quad g_{5,4} = -10\ y \ , \end{aligned} \tag{22a}$$

hence, corresponding to the root $y = 2$,

$$g_{5,1} = 0 \ , \quad g_{5,2} = -120 \ , \quad g_{5,3} = -80 \ , \quad g_{5,4} = -20 \ , \tag{22b}$$

and corresponding to the root $y = 6$,

$$g_{5,1} = 2880 \ , \quad g_{5,2} = 360 \ , \quad g_{5,3} = -240 \ , \quad g_{5,4} = -60 \ . \tag{22c}$$

The N basic solutions—with $m = 1, ..., N$—of the linear system of ODEs (18) (with (19)) read of course

$$\underline{w}^{(m)}(t) = \underline{\bar{w}}^{(m)}\ \exp(-i\ x_n\ t) \ , \tag{23}$$

with $\underline{w}^{(m)}(t)$ indicating the N-vector of components $w_n^{(m)}(t)$ and the N numbers x_n, respectively the N constant N-vectors $\underline{\bar{w}}^{(m)}$, being the N eigenvalues, respectively the N corresponding eigenvectors, of the $N \times N$ matrix $\underline{A}$ defined componentwise as follows:

$$A_{n,n} = -(n+1) \ , \quad A_{n,n+1} = i \ , \quad n = 1, ..., N-1 \ ; \tag{24a}$$

$$A_{N,n} = (i)^{n+1}\ g_{N,n} \ , \quad n = 1, ..., N-1 \ ; \ A_{N,N} = -(N+1) \ , \tag{24b}$$

with all other matrix elements vanishing. Equivalently, the N numbers x_n are the N roots of the following Nth degree monic polynomial in x:

$$P_N(x) = \det[x\ \underline{I} - \underline{A}] \ , \tag{25}$$

where of course $\underline{I}$ is the $N \times N$ unit matrix. It is convenient to reformulate this equation as follows:

$$P_N(x) = \det[\underline{B}(x)] \ , \tag{26}$$

with the $N \times N$ matrix $\underline{B}(x)$ defined componentwise as follows:

$$\begin{aligned} B_{n,n}(x) &= x + n + 1 + \frac{g_{N,n}}{g_{N,n+1}} \ , \quad n = 1, ..., N-2 \ ; \\ B_{N-1,N-1}(x) &= x + N + (-1)^M\ \frac{g_{N,N-1}}{x+N+1} \ , \\ B_{N,N}(x) &= x + N + 1; \end{aligned} \tag{27a}$$

$$B_{n,n+1}(x) = 1 \ , \quad n = 1, ..., N-1 \ ; \tag{27b}$$

$$B_{n,n-1}(x) = (x+n+1)\,\frac{g_{N,n}}{g_{N,n+1}}\ , \quad n = 2, ..., N-1\ , \tag{27c}$$

with all other matrix elements vanishing. The equality of the two expressions (25) with (24) and (26) with (27) of the polynomial $P_N(x)$ is guaranteed by the fact that the matrix $\underline{B}(x)$ is obtained from the matrix $x\,\underline{I} - \underline{A}$ by subtracting from each column of this matrix (except, of course, from the last column) the subsequent column multiplied by a coefficient adjusted so as to yield a vanishing value for the bottom term of the resulting column, an operation that does not change the value of the determinant but has the merit of making the matrix $\underline{B}(x)$ *tridiagonal*. And we also multiplied (for cosmetic reasons) the nm-th matrix element by i^{m-n}, another operation that does not affect the determinant (it amounts to multiplying the matrix from the right by $\underline{J}$ =diag(i^n) and from the left by $\underline{J}^{-1}$).

And since the last line of the $N \times N$ matrix $\underline{B}(x)$ has all elements vanishing except for the last (diagonal) one reading $B_{N,N}(x) = x + N + 1$, one sees that another equivalent expression of the polynomial $P_N(x)$ is provided by the following formula,

$$P_N(x) = \det\left[\underline{C}(x)\right]\ , \tag{28}$$

where $\underline{C}(x)$ is now the following tridiagonal $(N-1)\times(N-1)$ matrix (obtained from $\underline{B}(x)$ by multiplying its next-to-last line by $B_{N,N}(x) = x + N + 1$ and by eliminating its last line and column):

$$C_{n,n}(x) = x + n + 1 + \frac{g_{N,n}}{g_{N,n+1}}\ , \quad n = 1, ..., N-2\ , \tag{29a}$$

$$C_{N-1,N-1}(x) = (x+N)(x+N+1) + (-1)^M g_{N,N-1}\ ; \tag{29b}$$

$$C_{n,n+1}(x) = 1\ , \quad n = 1, ..., N-2\ ; \tag{29c}$$

$$C_{n,n-1}(x) = (x+n+1)\,\frac{g_{N,n}}{g_{N,n+1}}\ , \quad n = 2, ..., N-2\ , \tag{29d}$$

$$C_{N-1,N-2}(x) = (x+N)(x+N+1)\,\frac{g_{N,N-2}}{g_{N,N-1}}\ , \tag{30}$$

again with all other matrix elements vanishing.

The structure of the tridiagonal $(N-1)\times(N-1)$ matrix $\underline{C}(x)$—see in particular (29b), the last element listed in (29c) and (30)—suggests setting

$$\begin{aligned} P_N(x) \quad = \quad & \left[(x+N)(x+N+1) + (-)^M g_{N,N-1}\right]\ p^{(N)}_{N-2}(x) \\ & -(x+N)(x+N+1)\ \frac{g_{N,N-2}}{g_{N,N-1}}\ p^{(N)}_{N-3}(x)\ , \end{aligned} \tag{31}$$

with the monic polynomials $p^{(N)}_m(x)$, of degree m in x, defined as follows:

$$p^{(N)}_m(x) = \det\left[\underline{c}(m;x)\right]\ , \tag{32}$$

where the $m \times m$ tridiagonal matrix $\underline{c}(m;x)$ is defined componentwise as follows:

$$c_{n,n}(m;x) = x + n + 1 + \frac{g_{N,n}}{g_{N,n+1}} \ , \quad n = 1, ..., m \ ; \tag{33a}$$

$$c_{n,n+1}(m;x) = 1 \ , \quad n = 1, ..., m-1 \ ; \tag{33b}$$

$$c_{n,n-1}(m;x) = (x + n + 1) \frac{g_{N,n}}{g_{N,n+1}} \ , \quad n = 2, ..., m \ , \tag{33c}$$

with all other matrix elements vanishing: hence it coincides with the first (upper-left) minor, of order $m \times m$, of the matrix $\underline{C}(x)$. Here of course the positive integer m is restricted to be less than $N-1$,

$$m \leq N - 2 \ . \tag{34}$$

This definition of the monic polynomials $p_m^{(N)}(x)$ clearly entails that they satisfy the three-term recursion relation

$$\begin{aligned} p_m^{(N)}(x) \quad &= \quad \left(x + m + 1 + \frac{g_{N,m}}{g_{N,m+1}} \right) p_{m-1}^{(N)}(x) \\ &\quad - (x + m + 1) \frac{g_{N,m}}{g_{N,m+1}} \, p_{m-2}^{(N)}(x) \ , \end{aligned} \tag{35}$$

indeed they are defined by this recursion relation together with the initial conditions

$$p_{-1}^{(N)}(x) = 0, \quad p_0^{(N)}(x) = 1 \ , \tag{36a}$$

entailing

$$p_1^{(N)}(x) = x + 2 + \frac{g_{N,1}}{g_{N,2}} \ , \tag{36b}$$

$$p_2^{(N)}(x) = \left(x + 3 + \frac{g_{N,2}}{g_{N,3}} \right) \left(x + 2 + \frac{g_{N,1}}{g_{N,2}} \right) - (x+3) \frac{g_{N,2}}{g_{N,3}} \ , \tag{36c}$$

and so on up to $m = N - 2$.

Hence in particular, for $N = 3$ (via (21b))

$$p_1^{(3)}(x) = x + 4 \ , \tag{37a}$$

entailing, via (21b) and (31),

$$P_3(x) = (x-1)(x+4)(x+6) \ ; \tag{37b}$$

and for $N = 5$ and $y = 2$ (via (22b))

$$\begin{aligned} p_1^{(5)}(x) \quad &= \quad x + 2 \ , \quad p_2^{(5)}(x) = x^2 + \frac{13}{2} x + 9 = (x+2)(x + \frac{9}{2}) \ , \\ p_3^{(5)}(x) \quad &= \quad x^3 + 13x^2 + 52x + 60 = (x+2)(x+5)(x+6) \ , \end{aligned} \tag{38a}$$

entailing, via (22b) and (31),

$$P_5(x) = (x-1)(x+2)(x+5)(x+6)(x+8) \ , \tag{38b}$$

while for $N=5$ and $y=6$ (via (22b))

$$\begin{aligned} p_1^{(5)}(x) &= x+10 \ , \quad p_2^{(5)}(x) = x^2 + \frac{7}{2}x - 9 \ , \\ p_3^{(5)}(x) &= x^3 + 13x^2 + 40x - 12 = (x+6)\left(7x + x^2 - 2\right) \ , \end{aligned} \tag{39a}$$

entailing, via (22c) and (31),

$$P_5(x) = (x-3)(x-1)(x+6)(x+8)(x+10) \ . \tag{40a}$$

The main result of this paper is the *Diophantine* observation that the N zeros x_n of the polynomial $P_N(x)$, see (25) or (26) or (28) or (31), must all be *integers*, and all different among themselves, for all the definitions of the quantities $g_{N,m}$ entailed by the above developments; let us re-emphasize that there will generally be $M+1 = (N+1)/2$ different definitions of these quantities, corresponding to the $M+1 = (N+1)/2$, generally different, roots of the polynomial (16) (but the root $y=0$ clearly yields trivially the results $x_n = n+1$, since the corresponding matrix is *triangular*). This conclusion is implied by the fact that all the solutions (23) must satisfy the *isochrony* property (12), and it is of course verified by the examples displayed above corresponding to $M=1$, see ([**?**]), and to $M=2$, see ([**?**]) and ([**?**]).

The next task shall be to display these findings for larger values of M, and especially for arbitrary M.

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