

CLOSED ORBITS AND UNIFORM S -INSTABILITY IN INVARIANT THEORY

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ABSTRACT. In this paper we consider various problems involving the action of a reductive group G on an affine variety V . We prove some general rationality results about the G -orbits in V . In addition, we extend fundamental results of Kempf and Hesselink regarding optimal destabilizing parabolic subgroups of G for such general G -actions.

We apply our general rationality results to answer a question of Serre concerning how his notion of G -complete reducibility behaves under separable field extensions. Applications of our new optimality results also include a construction which allows us to associate an optimal destabilizing parabolic subgroup of G to any subgroup of G . Finally, we use these new optimality techniques to provide an answer to Tits' Centre Conjecture in a special case.

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1. INTRODUCTION

There are two main strands concerning basic constructions in geometric invariant theory in this paper. Firstly, we prove some general rationality results about the orbits of a reductive group G acting on an affine variety V . Secondly, we extend fundamental results of Kempf and Hesselink regarding optimal destabilizing parabolic subgroups of G for such general G -actions.

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More precisely, let G be a reductive linear algebraic group, defined over a field k . We denote the algebraic closure of k by \bar{k} and identify varieties over \bar{k} with their sets of \bar{k} -points. Suppose G acts on an affine variety V over k . Let $v \in V(k)$. It is well known that the closure $\overline{G \cdot v}$ of the G -orbit $G \cdot v$ is a union of G -orbits, exactly one of which is closed. Moreover, the Hilbert-Mumford Theorem [12, Thm. 1.4] tells us that there exists a cocharacter λ of G such that $v' := \lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists and $G \cdot v'$ is the unique closed orbit in $\overline{G \cdot v}$. Our first aim is to study the more delicate question: what is the structure of the $G(k)$ -orbits and of their closures? When k is perfect, Theorem 3.4 provides a partial answer. Let λ be a k -defined cocharacter of G such that $v' := \lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists and is $G(k)$ -conjugate to v . Then v' is $R_u(P_\lambda)(k)$ -conjugate to v (see Sec. 2.2 for the definition of P_λ). Theorem 3.4 was first proved in unpublished work of Kraft and Kuttler for k algebraically closed of characteristic zero.

In order to pursue this question further, we require an appropriate extension of the concept of orbit closure to the non-algebraically closed case (see Definition 3.10): We say that the $G(k)$ -orbit $G(k) \cdot v$ is *cocharacter-closed over k* if for any k -defined cocharacter λ of G such that $v' := \lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists, v' is $G(k)$ -conjugate to v .

This is a central new concept of our investigation. Clearly, this notion depends only on the $G(k)$ -orbit $G(k) \cdot v$ of v and not on v itself. It follows from the Hilbert-Mumford Theorem that $G \cdot v$ is closed if and only if $G(\bar{k}) \cdot v$ is cocharacter-closed over \bar{k} . It is sensible, therefore, to consider the $G(k)$ -orbits that are cocharacter-closed over k as a generalization to non-algebraically closed fields k of the closed G -orbits.

Our main result in this subject, Theorem 3.12, illustrates the usefulness of this notion: for G connected we show that $G(k) \cdot v$ is cocharacter-closed over k if and only if for any k -defined cocharacter λ of G if $v' := \lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists, v' is $R_u(P_\lambda)(k)$ -conjugate to v . This allows us to remove the hypothesis that k is perfect in Theorem 3.4 if $G(k) \cdot v$ is cocharacter-closed over k .

In Question 3.15 we pose a rationality question for this notion of cocharacter-closed orbits for field extensions k_1/k . If k is perfect, then the answer to Question 3.15 is yes (Corollary 4.10) and also if k_1/k is separable, one part of Question 3.15 holds (Theorem 3.16). For arbitrary k , however, it can happen that $G(k) \cdot v$ is not cocharacter-closed over k but $G \cdot v$ is closed or vice versa, see Remark 5.8.

Our second main strand combines the instability notions of Kempf and Hesselink. In [12], Kempf shows that if $v \in V$ is a point whose G -orbit $G \cdot v$ is not closed, and S is a G -stable closed subvariety of V which meets the closure of $G \cdot v$, then there is an *optimal class* of cocharacters which move v into S (by taking limits). In a similar vein, in [11] Hesselink develops a notion of *uniform instability*: here the single point $v \in V$ in Kempf's construction is replaced with a subset X of V , but the G -stable subvariety S is taken to be a single point of V . Moreover, Hesselink's results work for non-algebraically closed fields. Our constructions, culminating in Theorem 4.4, combine these two ideas within the single framework of *uniform S -instability*, providing a useful extension of these optimality methods in geometric invariant theory.

As well as being of interest in their own right, both topics of this paper have applications in the theory of G -complete reducibility, introduced by Serre [22] and developed in [1], [2], [4], and [5]. In particular, we are able to use our general results on G -orbits and rationality

to answer a question of Serre about how G -complete reducibility behaves under extensions of fields (Theorem 5.11). Our notion of cocharacter-closure allows us to give a geometric characterization of G -complete reducibility over a field k (Theorem 5.7), thereby extending [1, Cor. 3.7]. Our optimality results allow us to attach to any subgroup H of G a optimal parabolic subgroup of G containing H , which is proper if and only if H is not G -completely reducible (Theorem 5.16). This optimal parabolic subgroup provides a very useful tool in the study of subgroups of reductive groups. As an illustration of its effectiveness, we give short proofs of some existing results, and prove a special case of Tits' Centre Conjecture (Theorem 5.30).

We also refer to [5], where we discuss further consequences of some of our fundamental results of the present paper.

2. NOTATION AND PRELIMINARIES

2.1. Basic notation. Let k be a field, let k_s denote its separable closure, and let \bar{k} denote its algebraic closure. Note that $k_s = \bar{k}$ if k is perfect. We denote the Galois group $\text{Gal}(k_s/k) = \text{Gal}(\bar{k}/k)$ by Γ . We use the notion of a k -scheme from [7, AG.11]: a k -scheme is a \bar{k} -scheme together with a k -structure. So k -schemes are assumed to be of finite type and reduced separated k -schemes are called k -varieties. Furthermore, a subscheme of a scheme V over k or over \bar{k} is always a subscheme of V as a scheme over \bar{k} and points of V are always closed points of V as a scheme over \bar{k} . Non-reduced schemes are only used in Section 4 and there they only play a technical rôle; we always formulate our results for k -varieties.

Now let V be a k -variety. If k_1/k is an algebraic extension, then we write $V(k_1)$ for the set of k_1 -points of V . By a *separable point* we mean a k_s -point. If W is a subvariety of V , then we set $W(k_1) = W \cap V(k_1)$. Here we do not assume that W is k -defined, so $W(k_1)$ can be empty even when $k_1 = k_s$. The Galois group Γ acts on V ; see, e.g., [23, 11.2]. Recall the Galois criterion for a closed subvariety W of V to be k -defined: W is k -defined if and only if it contains a Γ -stable set of separable points of V which is dense in W (see [7, Thm. AG.14.4]).

Let H be a k -defined linear algebraic group. We denote by $\langle S \rangle$ the algebraic subgroup of H generated by a subset S of H . We let $Z(H)$ denote the centre of H and H^0 the connected component of H that contains 1. For K a subgroup of H , we denote the centralizer of K in H by $C_H(K)$ and the normalizer of K in H by $N_H(K)$. We denote the group of algebraic automorphisms of H by $\text{Aut } H$.

For the set of cocharacters (one-parameter subgroups) of H we write $Y(H)$; the elements of $Y(H)$ are the homomorphisms from the multiplicative group \bar{k}^* to H . We denote the set of k -defined cocharacters by $Y_k(H)$. There is a left action of H on $Y(H)$ given by $(h \cdot \lambda)(a) = h\lambda(a)h^{-1}$ for $\lambda \in Y(H)$, $h \in H$ and $a \in \bar{k}^*$. The subset $Y_k(H)$ is stabilized by $H(k)$.

The *unipotent radical* of H is denoted $R_u(H)$; it is the maximal connected normal unipotent subgroup of H . The algebraic group H is called *reductive* if $R_u(H) = \{1\}$; note that we do not insist that a reductive group is connected.

Throughout the paper, G denotes a k -defined reductive algebraic group, possibly disconnected. We say an affine G -variety V is k -defined if both V and the action of G on V are k -defined. By a rational G -module, we mean a finite-dimensional vector space over \bar{k} with a linear G -action. If both V and the action are k -defined, then we say the rational G -module is k -defined.

Suppose T is a maximal torus of G . Let $\Psi = \Psi(G, T)$ be the set of roots of G relative to T . Let $\alpha \in \Psi$. Then U_α denotes the *root subgroup* of G associated to α .

2.2. Non-connected reductive groups. The crucial idea which allows us to deal with non-connected groups is the introduction of so-called *Richardson parabolic subgroups* (*R-parabolic subgroups*) of a reductive group G . We briefly recall the main definitions and results; for more details and further results, the reader is referred to [1, Sec. 6].

Definition 2.1. For each cocharacter $\lambda \in Y(G)$, let $P_\lambda = \{g \in G \mid \lim_{a \rightarrow 0} \lambda(a)g\lambda(a)^{-1} \text{ exists}\}$ (see Sec. 2.3 for the definition of limit). Recall that a subgroup P of G is *parabolic* if G/P is a complete variety. The subgroup P_λ is parabolic in this sense, but the converse is not true: e.g., if G is finite, then every subgroup is parabolic, but the only subgroup of G of the form P_λ is G itself. If we define $L_\lambda = \{g \in G \mid \lim_{a \rightarrow 0} \lambda(a)g\lambda(a)^{-1} = g\}$, then $P_\lambda = L_\lambda \rtimes R_u(P_\lambda)$, and we also have $R_u(P_\lambda) = \{g \in G \mid \lim_{a \rightarrow 0} \lambda(a)g\lambda(a)^{-1} = 1\}$. The map $c_\lambda : P_\lambda \rightarrow L_\lambda$ given by $c_\lambda(g) = \lim_{a \rightarrow 0} \lambda(a)g\lambda(a)^{-1}$ is a surjective homomorphism of algebraic groups with kernel $R_u(P_\lambda)$; it coincides with the usual projection $P_\lambda \rightarrow L_\lambda$. The subgroups P_λ for $\lambda \in Y(G)$ are called the *Richardson-parabolic* (or *R-parabolic*) *subgroups* of G . Given an R-parabolic subgroup P , a *Richardson-Levi* (or *R-Levi*) *subgroup* of P is any subgroup L_λ such that $\lambda \in Y(G)$ and $P = P_\lambda$.

If G is connected, then the R-parabolic subgroups (resp. R-Levi subgroups of R-parabolic subgroups) of G are exactly the parabolic subgroups (resp. Levi subgroups of parabolic subgroups) of G ; indeed, most of the theory of parabolic subgroups and Levi subgroups of connected reductive groups carries over to R-parabolic and R-Levi subgroups of arbitrary reductive groups. In particular, all R-Levi subgroups of an R-parabolic subgroup P are conjugate under the action of $R_u(P)$. If P, Q are R-parabolic subgroups of G and $P^0 = Q^0$, then $R_u(P) = R_u(Q)$.

Lemma 2.2. *Let P, Q be R-parabolic subgroups of G with $P \subseteq Q$ and $P^0 = Q^0$, and let M be an R-Levi subgroup of Q . Then $P \cap M$ is an R-Levi subgroup of P .*

Proof. Fix a maximal torus T of G such that $T \subseteq M$. Then $T \subseteq P$, since $P^0 = Q^0$. There exists a unique R-Levi subgroup L of P such that $T \subseteq L$, [1, Cor. 6.5]. There exists a unique R-Levi subgroup M' of Q such that $L \subseteq M'$, [1, Cor. 6.6]. Since M is the unique R-Levi subgroup of Q that contains T , [1, Cor. 6.5], we must have $M = M'$. Hence $L \subseteq P \cap M$. If this inclusion is proper, then $P \cap M$ meets $R_u(P) = R_u(Q)$ non-trivially, a contradiction. We deduce that $L = P \cap M$. \square

We now consider some rationality issues. The proof of the next lemma follows immediately from the definitions of limit and of the actions of Γ on k_s -points and on k_s -defined morphisms.

Lemma 2.3. *Let $\lambda \in Y_{k_s}(G)$ and let $\gamma \in \Gamma$. Then $P_{\gamma \cdot \lambda} = \gamma \cdot P_\lambda$.*

Remark 2.4. If G is connected, then a parabolic subgroup P of G is k -defined if and only if $P = P_\lambda$ for some $\lambda \in Y_k(G)$, [23, Lem. 15.1.2(ii)]. However, the analogous result for R-parabolic subgroups of a non-connected group G is not true in general. To see this, let T be a non-split one-dimensional torus over k and let F be the group of order 2 acting on T by inversion. Then T is a k -defined R-parabolic subgroup of the reductive group $G := FT$,

but T is not of the form P_λ for any λ over k , because $Y_k(G) = \{0\}$. Our next set of results allow us to deal with this problem.

Lemma 2.5. *Let $\lambda \in Y(G)$.*

- (i) *If P_λ is k -defined, then so is $R_u(P_\lambda)$. Moreover, if λ belongs to $Y_k(G)$, then P_λ , L_λ and the isomorphism $L_\lambda \times R_u(P_\lambda) \rightarrow P_\lambda$ are k -defined.*
- (ii) *Suppose P_λ is k -defined. Then there exists $\mu \in Y_k(G)$ such that $P_\lambda \subseteq P_\mu$ and $P_\lambda^0 = P_\mu^0$.*
- (iii) *Let P be a k -defined R -parabolic subgroup. Then any k -defined maximal torus of P is contained in a unique k -defined R -Levi subgroup of P and any two k -defined R -Levi subgroups of P are conjugate by a unique element of $R_u(P)(k)$.*

Proof. (i). We have that P_λ^0 is k -defined, so $R_u(P_\lambda) = R_u(P_\lambda^0)$ is k -defined, by [7, Prop. V.20.5]. Now assume that $\lambda \in Y_k(G)$. Choose a maximal torus T of G such that T is defined over k and $\lambda \in Y_k(T)$. Then $L_\lambda = C_G(\lambda(\bar{k}^*))$ is defined over k , by [7, Cor. III.9.2]. The rest now follows, because the multiplication map $G \times G \rightarrow G$ is defined over k .

(ii). After conjugating λ by an element of P_λ , we may assume that $\lambda \in Y(T)$ for some k -defined maximal torus T of P_λ . Note that such a torus exists, by [7, Thm. V.18.2(i)]. Since T splits over a finite Galois extension of k , λ has only finitely many Γ -conjugates. Let $\mu \in Y(T)$ be their sum. Since P_λ is k -defined, we have $P_{\gamma \cdot \lambda} = P_\lambda$ for all $\gamma \in \Gamma$. By considering the pairings of λ and μ with the coroots of G relative to T , we deduce that $P_\mu^0 = P_\lambda^0$. Using a G -equivariant embedding of G acting on itself by conjugation into a finite-dimensional G -module, we deduce that $\lim_{a \rightarrow 0} \mu(a) \cdot g$ exists if $\lim_{a \rightarrow 0} (\gamma \cdot \lambda)(a) \cdot g$ exists for all $\gamma \in \Gamma$. So $P_\lambda \subseteq P_\mu$.

(iii). Because of [7, Prop. V.20.5] and [1, Cors. 6.5, 6.6, 6.7], it is enough to show that the unique R -Levi subgroup of P containing a given k -defined maximal torus of P is k -defined. Let T be a k -defined maximal torus of P . By the proof of (ii), there exists $\mu \in Y_k(T)$ such that $P \subseteq P_\mu$ and $P^0 = P_\mu^0$. Clearly, L_μ is the R -Levi subgroup of P_μ containing T , and it is k -defined by (i). The unique R -Levi subgroup of P containing T is $P \cap L_\mu$, by Lemma 2.2. Since $P \cap G(k_s)$ and $L_\mu \cap G(k_s)$ are Γ -stable, the same holds for $P \cap L_\mu \cap G(k_s)$. So it suffices to show that this set is dense in $P \cap L_\mu$. This follows, because the components of $P \cap L_\mu$ are components of L_μ and the separable points are dense in each component of L_μ . \square

Corollary 2.6. *Let $\lambda \in Y_k(G)$ and let $\mu \in Y(G)$ such that $P_\lambda = P_\mu$ and L_μ is k -defined. Then there exists $\nu \in Y_k(G)$ such that $P_\lambda = P_\nu$ and $L_\mu = L_\nu$.*

Proof. By Lemma 2.5(iii), there exists $u \in R_u(P_\lambda)(k)$ such that $L_{u \cdot \lambda} = u L_\lambda u^{-1} = L_\mu$, so we can take $\nu = u \cdot \lambda$. \square

2.3. G -varieties. If G acts on a set V , then we denote for a subset S of V , the pointwise stabilizer $\{g \in G \mid g \cdot s = s \text{ for all } s \in S\}$ of S in G by $C_G(S)$ and the setwise stabilizer $\{g \in G \mid g \cdot S = S\}$ of S in V by $N_G(S)$.

Now suppose G acts on a variety V and let $v \in V$. Then for each cocharacter $\lambda \in Y(G)$, we can define a morphism of varieties $\phi_{v,\lambda} : \bar{k}^* \rightarrow V$ via the formula $\phi_{v,\lambda}(a) = \lambda(a) \cdot v$. If this morphism extends to a morphism $\widehat{\phi}_{v,\lambda} : \bar{k} \rightarrow V$, then we say that $\lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists, and set this limit equal to $\widehat{\phi}_{v,\lambda}(0)$; note that such an extension, if it exists, is necessarily unique.

If $x \in P_\lambda$, then we have

$$(2.7) \quad \lim_{a \rightarrow 0} \lambda(a) \cdot (x \cdot v) = c_\lambda(x) \cdot \left(\lim_{a \rightarrow 0} \lambda(a) \cdot v \right).$$

Suppose that the G -variety V is k -defined. If $\phi_{v,\lambda}$ is k -defined, then $\widehat{\phi}_{v,\lambda}$ is k -defined and $\lim_{a \rightarrow 0} \lambda(a) \cdot v \in V(k)$; in particular, this is the case if $\lambda \in Y_k(G)$ and $v \in V(k)$. Note that the set of $v \in V$ such that $\lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists is P_λ -stable by Eqn. (2.7).

In many of our proofs, we want to reduce the case of a general (k -defined) affine G -variety to the case of a (k -defined) rational G -module. Such a reduction is possible, thanks to [12, Lem. 1.1(a)], for example. As this situation arises many times in the sequel, we now set up some standard notation which will be in force throughout the paper.

Let V be a rational G -module. Given $\lambda \in Y(G)$ and $n \in \mathbb{Z}$, we define

$$(2.8) \quad \begin{aligned} V_{\lambda,n} &:= \{v \in V \mid \lambda(a) \cdot v = a^n v \text{ for all } a \in \overline{k}^*\}, \\ V_{\lambda,\geq 0} &:= \sum_{n \geq 0} V_{\lambda,n} \quad \text{and} \quad V_{\lambda,>0} := \sum_{n > 0} V_{\lambda,n}. \end{aligned}$$

Then $V_{\lambda,\geq 0}$ consists of the vectors $v \in V$ such that $\lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists, $V_{\lambda,>0}$ is the subset of vectors $v \in V$ such that $\lim_{a \rightarrow 0} \lambda(a) \cdot v = 0$, and $V_{\lambda,0}$ is the subset of vectors $v \in V$ such that $\lim_{a \rightarrow 0} \lambda(a) \cdot v = v$. Note that if $v \in V_{\lambda,\geq 0}$, then $\lim_{a \rightarrow 0} \lambda(a) \cdot v \in V_{\lambda,0}$. Of course, similar remarks apply to $-\lambda$, $V_{\lambda,\leq 0} = V_{-\lambda,\geq 0}$, $V_{\lambda,0} := V_{-\lambda,0}$ and $V_{\lambda,<0} := V_{-\lambda,>0}$.

Now let T be a torus in G with $\lambda \in Y(T)$. For $\chi \in X(T)$, let V_χ denote the corresponding weight space of T in V . If $v \in V$, then we denote by v_χ the component of v in the weight space V_χ and we put $\text{supp}_T(v) = \{\chi \in X(T) \mid v_\chi \neq 0\}$, called the *support of v with respect to T* . Then $V_{\lambda,0}$, $V_{\lambda,\geq 0}$ and $V_{\lambda,>0}$ are the direct sums of the subspaces $V_{\lambda,\langle \lambda, \chi \rangle}$, where $\chi \in X(T)$ is such that $\langle \lambda, \chi \rangle = 0, \geq 0$ and > 0 , respectively. Furthermore, $v \in V_{\lambda,\geq 0}$ if and only if $\langle \lambda, \chi \rangle \geq 0$ for all $\chi \in \text{supp}_T(v)$.

Finally, we recall a standard result [6, Lem. 5.2]. Suppose T is a maximal torus of G with $\lambda \in Y(T)$. Let $\alpha \in \Psi = \Psi(G, T)$, $v \in V_{\lambda,n}$ and $u \in U_\alpha$. Then

$$(2.9) \quad u \cdot v - v \in \sum_{m \geq 1} V_{\lambda, n+m \langle \lambda, \alpha \rangle}.$$

Hence for any $v \in V_{\lambda,\geq 0}$, we have

$$(2.10) \quad u \cdot v - v \in V_{\lambda,>0}.$$

We continue with some further preliminary results used in the proofs below.

Lemma 2.11. *Suppose G acts on an affine variety V . Let $v \in V$, let $\lambda \in Y(G)$ and let $u \in R_u(P_\lambda)$. Then $\lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists and equals $u \cdot v$ if and only if $u^{-1} \cdot \lambda$ centralizes v .*

Proof. If $\lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists and equals $u \cdot v$, then λ fixes $u \cdot v$ and therefore $u^{-1} \cdot \lambda$ centralizes v . Now assume that the latter is the case. Then $\lim_{a \rightarrow 0} \lambda(a) u^{-1} \lambda(a)^{-1} = 1$ and $u^{-1} \lambda(a)^{-1} u$ fixes v for all $a \in \overline{k}^*$, so $u \cdot v = \left(\lim_{a \rightarrow 0} \lambda(a) u^{-1} \lambda(a)^{-1} \right) \cdot u \cdot v = \lim_{a \rightarrow 0} \lambda(a) \cdot (u^{-1} \lambda(a)^{-1} u) \cdot v = \lim_{a \rightarrow 0} \lambda(a) \cdot v$. \square

Lemma 2.12. *Suppose G acts on an affine variety V . Let $v \in V$, $\lambda \in Y(G)$, such that $v' := \lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists. Furthermore, let $x \in P_{-\lambda}$ and $u \in R_u(P_\lambda)$ be such that $xu \cdot v$ is $\lambda(\bar{k}^*)$ -fixed. Then $v' = u \cdot v$.*

Proof. Without loss, we may assume that V is a finite-dimensional vector space and that the G -action is linear. Write $x = yl$, where $y \in R_u(P_{-\lambda})$ and $l \in L_\lambda$. Since $V_{\lambda, \leq 0}$ is $P_{-\lambda}$ -stable and $ylu \cdot v \in V_{\lambda, 0}$, we have that $lu \cdot v = y^{-1}ylu \cdot v \in V_{\lambda, \leq 0}$. On the other hand, $lu \cdot v \in V_{\lambda, \geq 0}$, since $v \in V_{\lambda, \geq 0}$ and $V_{\lambda, \geq 0}$ is P_λ -stable. So $lu \cdot v \in V_{\lambda, 0}$. It follows that

$$lu \cdot v = \lim_{a \rightarrow 0} \lambda(a) \cdot lu \cdot v = \lim_{a \rightarrow 0} \lambda(a) lu \lambda(a)^{-1} \cdot \lim_{a \rightarrow 0} \lambda(a) \cdot v = l \cdot v'.$$

So $v' = u \cdot v$. □

Remark 2.13. The proof of Lemma 2.12 also works if we replace the assumption that $xu \cdot v$ is $\lambda(\bar{k}^*)$ -fixed by the weaker assumption that $\lim_{a \rightarrow 0} \lambda(a)^{-1} \cdot (xu \cdot v)$ exists. If $xu \cdot v$ is $\lambda(\bar{k}^*)$ -fixed, then we can draw the additional conclusion that $ylu \cdot v = lu \cdot v$, since $R_u(P_{-\lambda})$ acts trivially on $V_{\lambda, \leq 0}/V_{\lambda, < 0}$, by (2.9).

Lemma 2.14. *Let V be a rational G -module. Let $\lambda, \mu \in Y(G)$ such that $\lambda(\bar{k}^*)$ and $\mu(\bar{k}^*)$ commute. Then for $t \in \mathbb{N}$ sufficiently large, the following hold:*

- (i) $V_{t\lambda + \mu, \geq 0} \subseteq V_{\lambda, \geq 0}$, $V_{t\lambda + \mu, > 0} \supseteq V_{\lambda, > 0}$ and $V_{t\lambda + \mu, 0} = V_{\lambda, 0} \cap V_{\mu, 0}$;
- (ii) $P_{t\lambda + \mu} \subseteq P_\lambda$ (hence $R_u(P_{t\lambda + \mu}) \supseteq R_u(P_\lambda)$) and $L_{t\lambda + \mu} = L_\lambda \cap L_\mu$.

Furthermore, if $t \in \mathbb{N}$ such that property (i) holds and $v \in V$ is such that $v' := \lim_{a \rightarrow 0} \lambda(a) \cdot v$ and $v'' := \lim_{a \rightarrow 0} \mu(a) \cdot v'$ exist, then $\lim_{a \rightarrow 0} (t\lambda + \mu)(a) \cdot v$ exists and equals v'' .

Proof. Choose a maximal torus T of G such that $\lambda, \mu \in Y(T)$. Let Φ be the set of weights of T on V . Choose $t \in \mathbb{N}$ large enough such that for any $\chi \in \Phi$ with $\langle \lambda, \chi \rangle \neq 0$, we have that $\langle t\lambda + \mu, \chi \rangle \neq 0$ has the same sign as $\langle \lambda, \chi \rangle$. Then (i) follows. Part (ii) follows from the argument of the proof of [13, Prop. 6.7] (increasing t if necessary). Alternatively, it can be deduced from part (i) by embedding G with the conjugation action G -equivariantly in a rational G -module W and observing that $P_\nu = W_{\nu, \geq 0} \cap G$ and $L_\nu = W_{\nu, 0} \cap G$.

Now assume that $t \in \mathbb{N}$ is such that (i) holds and let $v \in V$ be such that the limits v' and v'' above exist. Since (i) holds, we have for all $\chi \in \Phi$ that $\langle t\lambda + \mu, \chi \rangle = 0$ if and only if $\langle \lambda, \chi \rangle = 0$ and $\langle \mu, \chi \rangle = 0$. For $\nu \in Y(T)$, let $\Phi_{\nu, \geq 0}$ and $\Phi_{\nu, 0}$ be the sets of weights $\chi \in \Phi$ such that $\langle \nu, \chi \rangle \geq 0$ and $\langle \nu, \chi \rangle = 0$, respectively. Then $\text{supp}_T(v) \subseteq \Phi_{\lambda, \geq 0}$ and $\text{supp}_T(v) \cap \Phi_{\lambda, 0} \subset \Phi_{\mu, \geq 0}$. It follows that $\lim_{a \rightarrow 0} (t\lambda + \mu)(a) \cdot v$ exists and equals v'' . □

We finish the section with a result that lets us pass from k -points to arbitrary points. Let V be a k -defined finite-dimensional rational G -module and let k_1/k be a field extension. Let $v \in V(k_1)$ and let $\lambda \in Y_k(G)$. Pick a basis $(\alpha_i)_{i \in I}$ for k_1 over k ; then we can write $v = \sum_{i \in J} \alpha_i v_i$ for some finite subset J of I and certain (unique) $v_i \in V(k)$. Clearly, we may assume that $J = \{1, \dots, n\}$ for some $n \in \mathbb{N}$. Set $\mathbf{v} = (v_1, \dots, v_n) \in V^n$ and let G act diagonally on V^n .

Lemma 2.15. *With the notation as above, the following hold:*

- (i) $\lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists if and only if $\lim_{a \rightarrow 0} \lambda(a) \cdot \mathbf{v}$ exists if and only if $\lim_{a \rightarrow 0} \lambda(a) \cdot v_i$ exists for each i .

- (ii) Suppose the limits in (i) exist. Then for any $g \in G(k)$, we have $v' = g \cdot v$ if and only if $\mathbf{v}' = g \cdot \mathbf{v}$ if and only if $v'_i = g \cdot v_i$ for each i .

Proof. Part (ii) is obvious. In part (i), it follows easily from the definitions of limit and direct product that the second limit exists if and only if the third limit exists. Since λ is k -defined, we have that $V_{\lambda, \geq 0}$ is k -defined, that is, $V_{\lambda, \geq 0} = \bigoplus_{i \in I} \alpha_i(V_{\lambda, \geq 0} \cap V(k))$. So $v \in V_{\lambda, \geq 0}$ if and only if $v_i \in V_{\lambda, \geq 0}$ for all $i \in J$. Hence the first limit exists if and only if the third limit exists. This completes the proof. \square

3. ORBITS AND RATIONALITY

In this section we prove the rationality results indicated in the Introduction. We maintain the notation from the previous sections.

Even when one is interested mainly in rationality questions, one must sometimes consider points that are not k -points. For instance, we want to work with a k -defined subgroup H of G by taking a finite generating tuple $\mathbf{h} = (h_1, \dots, h_n)$ for some $n \in \mathbb{N}$, but H need not admit such a tuple where the h_i are all k -points. The following definition deals with this problem.

Definition 3.1. Let V be a k -defined affine G -variety and let $v \in V$. We say that v is a *weak k -point* or is *weakly k -defined* if $C_{G(k_s)}(v)$ is Γ -stable. It is immediate that a k -point is a weak k -point. We do not require $C_G(v)$ to be k -defined in this definition; note that even when v is a k -point, $C_G(v)$ is k -closed but need not be k -defined.

Theorem 3.2. *Suppose V is a k -defined affine G -variety. Let $v \in V$ and let $\lambda \in Y_k(G)$ be such that $v' := \lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists. If v' is $R_u(P_\lambda)(k_s)$ -conjugate to v and v is weakly k -defined, then v' is $R_u(P_\lambda)(k)$ -conjugate to v .*

Proof. Set $P = P_\lambda$. By hypothesis, there exists $u \in R_u(P)(k_s)$ such that $v' = u \cdot v$. By Lemma 2.11, $\mu := u^{-1} \cdot \lambda \in Y_{k_s}(G)$ centralizes v , so $\mu(\bar{k}^*) \subseteq C_P(v)$, and $\mu(k_s^*) \subseteq C_{G(k_s)}(v) \cap P$. Note that since $u \in P$, we have $P_\mu = P$. Let H be the subgroup of G generated by the Γ -conjugates of $\mu(\bar{k}^*)$; then the union of the Γ -conjugates of $\mu(k_s^*)$ is dense in H , so H is closed, connected and k_s -defined, by [7, AG.14.5, I.2.2], and $H \subseteq P$, since $\mu(\bar{k}^*) \subseteq P$ and P is Γ -stable. Moreover, since $C_{G(k_s)}(v)$ is Γ -stable, we can conclude that $H \subseteq C_P(v)$. Since H has a Γ -stable dense set of separable points, H is k -defined, and hence contains a k -defined maximal torus S . There exists $h \in H$ such that $\mu' := h \cdot \mu$ belongs to $Y(S)$; note that μ' centralizes v , and since $h \in P$, we deduce that $P_{\mu'} = P$.

By [7, Cor. III.9.2], $C_P(S)$ is k -defined, so it has a k -defined maximal torus T . Note that $S \subseteq T$, since S commutes with T and T is maximal. There exists a unique k -defined R-Levi subgroup L of P containing T , by Lemma 2.5(iii). But $L_{\mu'}$ is an R-Levi subgroup of P containing T , so $L_{\mu'} = L$. Thus we have two R-Levi subgroups L_λ and $L_{\mu'}$ of P , both k -defined. By Lemma 2.5(iii), there exists a unique $u_0 \in R_u(P)(k)$ such that $L_\lambda = u_0 L_{\mu'} u_0^{-1}$. We also have $\mu' = hu^{-1} \cdot \lambda$, and since $hu^{-1} \in P$, we can write $hu^{-1} = u_1 l$ with $u_1 \in R_u(P)$ and $l \in L_\lambda$. But L_λ centralizes λ , so $\mu' = u_1 l \cdot \lambda = u_1 \cdot \lambda$. So $u_0^{-1} L_\lambda u_0 = L_{\mu'} = L_{u_1 \cdot \lambda} = u_1 L_\lambda u_1^{-1}$. Since $R_u(P)$ acts simply transitively on the set of R-Levi subgroups of P , we must have $u_1 = u_0^{-1}$, and hence $\mu' = u_0^{-1} \cdot \lambda$. Applying Lemma 2.11 again, we see that $v' = u_0 \cdot v$, because μ' centralizes v . This proves the theorem. \square

Example 3.3. The assumption that v is weakly k -defined in Theorem 3.2 is necessary. For instance, let $G = \mathrm{SL}_2(k)$ acting on $V = G$ by conjugation. Choose $y \in k_s \setminus k$ and

$x \in k^* \setminus \{\pm 1\}$. Let $v' = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ and $v = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$, and define $\lambda \in Y_k(G)$ by $\lambda(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$. It is easily seen that $v' = \lim_{a \rightarrow 0} \lambda(a) \cdot v$ and that v' is $R_u(P_\lambda)(k_s)$ -conjugate to v but not $R_u(P_\lambda)(k)$ -conjugate to v .

We can now state our first main result.

Theorem 3.4. *Suppose k is perfect. Suppose V is a k -defined affine G -variety and let $v \in V$. Let $\lambda \in Y_k(G)$ such that $v' := \lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists and is $G(k)$ -conjugate to v . Then v' is $R_u(P_\lambda)(k)$ -conjugate to v .*

Proof. Fix a maximal torus T of P_λ such that $\lambda \in Y(T)$ and a Borel subgroup B of P_λ^0 such that $T \subseteq B$. Let B^- be the Borel subgroup of G opposite to B with respect to T ; note that $B^- \subseteq P_{-\lambda}$.

We begin with the case that k is algebraically closed. We can assume that $v \neq v'$. For $a \in k^*$, set $v_a = \lambda(a) \cdot v$; then $v_a \neq v'$ for all $a \in k^*$. We show that $v \in P_{-\lambda}^0 R_u(P_\lambda) \cdot v'$. Let $\varphi : G \rightarrow G \cdot v'$ be the orbit map of v' . Then φ is open, by [7, AG Cor. 18.4]. The set $P_{-\lambda}^0 R_u(P_\lambda)$ contains the big cell $B^- B \subseteq G^0$, which is an open neighbourhood of 1 in G , so $\varphi(P_{-\lambda}^0 R_u(P_\lambda))$ is an open neighbourhood of $\varphi(1) = v'$ in $G \cdot v'$. The image of k^* under the limit morphism $\hat{\phi}_{v,\lambda} : k \rightarrow V$ meets this neighbourhood, so there exists $a \in k^*$ such that $v_a \in P_{-\lambda}^0 R_u(P_\lambda) \cdot v'$. But then also $v \in P_{-\lambda}^0 R_u(P_\lambda) \cdot v'$, since $P_{-\lambda}^0 R_u(P_\lambda)$ is stable under left multiplication by elements of T . Lemmas 2.5 and 2.12 now imply that $v' = u \cdot v$ for some $u \in R_u(P_\lambda)$. This completes the proof when k is algebraically closed.

Now assume k is perfect. First assume $v \in V(k)$. By the algebraically closed case, we know that v and v' are $R_u(P_\lambda)$ -conjugate. Since v is a weak k -point, we can apply Theorem 3.2 to deduce that v and v' are $R_u(P_\lambda)(k)$ -conjugate. Now let v be arbitrary. Let $\mathbf{v}, \mathbf{v}' \in V^n$ be as in Lemma 2.15. Then $\lim_{a \rightarrow 0} \lambda(a) \cdot \mathbf{v} = \mathbf{v}'$ and \mathbf{v} and \mathbf{v}' are $G(k)$ -conjugate, so they are $R_u(P_\lambda(k))$ -conjugate by the argument above. Lemma 2.15 implies that v and v' are $R_u(P_\lambda)(k)$ -conjugate, as required. \square

Remark 3.5. For k algebraically closed and of characteristic zero, Theorem 3.4 was proved in unpublished work of Kraft and Kuttler (by a method different from ours). We do not know whether this theorem holds for arbitrary k .

The following consequence of Theorem 3.4 is used in the proof of [5, Prop. 3.36].

Corollary 3.6. *Let G_1 and G_2 be reductive groups and let V be an affine $(G_1 \times G_2)$ -variety. Let $v \in V$ and $\lambda_1 \in Y(G_1)$ and assume that $v' := \lim_{a \rightarrow 0} \lambda_1(a) \cdot v$ exists. Then the following hold:*

- (i) *If v' is $(G_1 \times G_2)$ -conjugate to v , then it is G_1 -conjugate to v . In particular, $G_1 \cdot v$ is closed if $(G_1 \times G_2) \cdot v$ is.*
- (ii) *Let $\pi : V \rightarrow V//G_2$ be the canonical projection and assume that $\pi^{-1}(\pi(v)) = G_2 \cdot v$. If $\pi(v')$ is G_1 -conjugate to $\pi(v)$, then v' is G_1 -conjugate to v .*

Proof. (i). By Theorem 3.4, there exists $u \in R_u(P_{\lambda_1}(G_1 \times G_2))$ such that $v' = u \cdot v$. But $R_u(P_{\lambda_1}(G_1 \times G_2)) = R_u(P_{\lambda_1}(G_1)) \times \{1\}$, so v' is G_1 -conjugate to v , as required. The second assertion follows immediately from the Hilbert-Mumford Theorem.

(ii). This follows immediately from (i). \square

The following example shows that the converse of Corollary 3.6(i) does not hold in general.

Example 3.7. Let $G = G_1 \times G_2$, where $G_i = k^*$ for $i = 1, 2$ (here k is assumed algebraically closed). Set $V = k^2$, and let G act on V as follows:

$$(t_1, t_2) \cdot (x_1, x_2) := (t_1^2 t_2^{-1} x_1, t_2^2 t_1^{-1} x_2),$$

for $t_i \in G_i$ and $(x_1, x_2) \in V$. Consider the point $(1, 1) \in V$. Then the G_i -orbits of $(1, 1)$ are closed (no cocharacter of G_i can move $(1, 1)$ outside its G_i -orbit), but the G -orbit of $(1, 1)$ is not closed (if $\lambda \in Y(G)$ is given by $\lambda(a) = (a, a)$, then $\lim_{a \rightarrow 0} \lambda(a) \cdot (1, 1) = (0, 0)$).

Remark 3.8. Suppose G acts on an affine variety V . Let $v \in V$ and let $\lambda \in Y(Z(G^0))$. If $v' := \lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists and is G -conjugate to v , then $v' = v$: this follows immediately from Theorem 3.4, since $R_u(P_\lambda) = R_u(P_\lambda(G^0)) = R_u(G^0) = \{1\}$. It is also easy to supply a direct proof.

Here is a further consequence of Theorem 3.4: it gives a criterion for determining whether an orbit is closed when k is perfect.

Corollary 3.9. *Assume k is perfect. Let V be a k -defined affine G -variety and let $v \in V$ be a weak k -point. Suppose there exists $\lambda \in Y_k(G)$ such that $v' := \lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists and is not $R_u(P_\lambda)(k)$ -conjugate to v . Then $v' \notin G \cdot v$. In particular, $G \cdot v$ is not closed.*

Proof. Suppose v' is G -conjugate to v . Then v' is $R_u(P_\lambda)$ -conjugate to v , by Theorem 3.4. Since v is a weak k -point and k is perfect, v' is $R_u(P_\lambda)(k)$ -conjugate to v by Theorem 3.2, a contradiction. Hence $v' \notin G \cdot v$, and thus this orbit is not closed. \square

In order to state our next main result, we need an appropriate extension of the concept of orbit closure to the non-algebraically closed case.

Definition 3.10. Let V be a k -defined affine G -variety. Let $v \in V$. We say that the $G(k)$ -orbit $G(k) \cdot v$ is *cocharacter-closed over k* if for any $\lambda \in Y_k(G)$ such that $v' := \lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists, v' is $G(k)$ -conjugate to v . Note that we do not require v to be a k -point.

Remark 3.11. Clearly, whether or not an orbit $G(k) \cdot v$ is cocharacter closed depends only on the $G(k)$ -orbit $G(k) \cdot v$ of v and not on v itself. It follows from the Hilbert-Mumford Theorem that $G \cdot v$ is closed if and only if $G(\bar{k}) \cdot v$ is cocharacter-closed over \bar{k} . Therefore, it is natural to consider the $G(k)$ -orbits that are cocharacter-closed over k as a generalization to non-algebraically closed fields k of the closed G -orbits.

Our next result says that we can remove the hypothesis that k is perfect in Theorem 3.4 if we assume that G is connected and $G(k) \cdot v$ is cocharacter-closed over k .

Theorem 3.12. *Suppose V is a k -defined affine G -variety. Assume that G is connected. Let $v \in V$. Then the following are equivalent:*

- (i) $G(k) \cdot v$ is cocharacter-closed over k ;
- (ii) for all $\lambda \in Y_k(G)$, if $v' := \lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists, then v' is $R_u(P_\lambda)(k)$ -conjugate to v .

Proof. It is immediate that (ii) implies (i), so we need to prove that (i) implies (ii). Assume $G(k) \cdot v$ is cocharacter-closed over k . Without loss of generality we can assume that V is a k -defined rational G -module (cf. Subsection 2.3). We argue by induction on $\dim V_{\lambda,0}$ for $\lambda \in Y_k(G)$. Suppose $\lambda \in Y_k(G)$ and let $v \in V$ such that $v' := \lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists. Let S

be a maximal k -split torus of G with $\lambda \in Y_k(S)$, let ${}_k\Psi$ be the set of roots of G relative to S and let ${}_kW = N_G(S)/C_G(S)$ be the Weyl group over k . Any $w \in {}_kW$ has a representative in $N_G(S)(k)$, see [7, V.21.2]. We have $C_G(S) \subseteq P_\lambda$. Fix a minimal k -defined parabolic subgroup P of G with $C_G(S) \subseteq P \subseteq P_\lambda$. Using the notation of [7, V.21.11], the choice of P corresponds to a choice of simple roots ${}_k\Delta \subseteq {}_k\Psi$ and then, $P_\lambda = {}_kP_J$ for a unique subset J of ${}_k\Delta$. Define the subset ${}_kW^J$ of ${}_kW$ as in [7, V.21.21], and for each $w \in {}_kW^J$ define the subgroup U'_w of $R_u(P_\lambda)$, as in [7, V.21.14]. For each $w \in {}_kW^J$, let \dot{w} be a representative of w in $N_G(S)(k)$. Then, by [7, V.21.16 and V.21.29] or [9, 3.16 proof], we have

$$(3.13) \quad G(k) = \bigcup_{w \in {}_kW^J} U'_w(k)\dot{w}P_\lambda(k)$$

and

$$(3.14) \quad \dot{w}^{-1}U'_w\dot{w} \subseteq R_u(P_{-\lambda}) \quad \text{for each } w \in {}_kW^J.$$

Since $G(k) \cdot v$ is cocharacter-closed over k , there exists $g \in G(k)$ such that $v' = g \cdot v$. By (3.13) and Lemma 2.5, we have $g = u'\dot{w}lu$ for some $w \in {}_kW^J$, $u' \in U'_w(k)$, $l \in L_\lambda(k)$ and $u \in R_u(P_\lambda)(k)$. Now the argument splits in two cases. Put $n = \dot{w}$.

Case 1: n normalizes $V_{\lambda,0}$. Then $n^{-1}u'nlu \cdot v = n^{-1} \cdot v' \in V_{\lambda,0}$. Furthermore, $n^{-1}u'n \in R_u(P_{-\lambda})$ by (3.14), so $n^{-1}u'nl \in P_{-\lambda}$. The desired conclusion follows from Lemma 2.12.

Case 2: n does not normalize $V_{\lambda,0}$. Let Φ be the set of weights of S on V and for $\nu \in \Phi$ let $\Phi_{\nu, \geq 0}$ be the set of weights $\chi \in \Phi$ such that $\langle \nu, \chi \rangle \geq 0$. We have $v' = u'nlu \cdot v$ and therefore $n^{-1}u'^{-1} \cdot v' = lu \cdot v$. Furthermore, $u'^{-1} \cdot v' - v' \in V_{\lambda, > 0}$ by (2.10), whence $\text{supp}_S(v') \subseteq \text{supp}_S(u'^{-1} \cdot v')$. Now n^{-1} normalizes S , so

$$n^{-1} \cdot \text{supp}_S(v') \subseteq n^{-1} \cdot \text{supp}_S(u'^{-1} \cdot v') = \text{supp}_S(n^{-1}u'^{-1} \cdot v') \subseteq \Phi_{\lambda, \geq 0},$$

since $n^{-1}u'^{-1} \cdot v' = lu \cdot v \in V_{\lambda, \geq 0}$. It follows that $\text{supp}_S(v') \subseteq n \cdot \Phi_{\lambda, \geq 0} = \Phi_{n \cdot \lambda, \geq 0}$. So $v'' := \lim_{a \rightarrow 0} (n \cdot \lambda)(a) \cdot v'$ exists. We can choose $\gamma \in Y_k(G)$ of the form $\gamma = t\lambda + n \cdot \lambda$ for $t \in \mathbb{N}$ sufficiently large such that the following hold:

- (1) $V_{\gamma,0} \subseteq V_{\lambda,0}$;
- (2) $v'' = \lim_{a \rightarrow 0} \gamma(a) \cdot v$;
- (3) $v'' = \lim_{a \rightarrow 0} \gamma(a) \cdot v'$;
- (4) $P_\gamma \subseteq P_\lambda$.

Properties (1), (2) and (4) follow immediately from Lemma 2.14, while (3) follows from (2) since $\lim_{a \rightarrow 0} (n \cdot \lambda)(a) \cdot v' = v''$ and $\lambda(k^*)$ fixes v' . Equality in (1) implies that $V_{\lambda,0} = V_{\gamma,0} \subseteq V_{n \cdot \lambda,0} = n \cdot V_{\lambda,0}$ and therefore that $V_{\lambda,0} = n \cdot V_{\lambda,0}$, contradicting the fact that n does not normalize $V_{\lambda,0}$. So the inclusion in (1) is proper and $\dim V_{\gamma,0} < \dim V_{\lambda,0}$. Now $G(k) \cdot v' = G(k) \cdot v$ is cocharacter-closed over k . So, by the induction hypothesis, v and v' are both $R_u(P_\gamma)(k)$ -conjugate — and hence $P_\gamma(k)$ -conjugate — to v'' , and hence v and v' are $P_\gamma(k)$ -conjugate. By (4), v and v' are $P_\lambda(k)$ -conjugate. But then they are $R_u(P_\lambda)(k)$ -conjugate, by Lemmas 2.5 and 2.12. \square

In view of Theorems 3.2, 3.4 and 3.12, it is natural to ask the following rationality question.

Question 3.15. Let V be an affine k -defined G -variety. Let $v \in V$ be weakly k -defined. Suppose k_1/k is an algebraic extension. Is it true that $G(k_1) \cdot v$ is cocharacter-closed over k_1 if and only if $G(k) \cdot v$ is cocharacter-closed over k ?

Our final result in this section gives an affirmative answer to the forward implication of Question 3.15 in one case.

Theorem 3.16. *Let k_1/k be an algebraic extension of fields and let V be a k -defined affine G -variety. Let $v \in V$ be weakly k -defined. Suppose that: (a) k_1/k is algebraic and G is connected; or (b) k is perfect. If $G(k_1) \cdot v$ is cocharacter-closed over k_1 , then $G(k) \cdot v$ is cocharacter-closed over k .*

Proof. Suppose $\lambda \in Y_k(G)$ such that $v' = \lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists. Then $\lambda \in Y_{k_1}(G)$. Since $G(k_1) \cdot v$ is cocharacter-closed over k_1 , v' is $R_u(P_\lambda)(k_1)$ -conjugate to v , by Theorem 3.12 in case (a) and Theorem 3.4 in case (b). Since k_1/k is separable, Theorem 3.2 implies that v' is $R_u(P_\lambda)(k)$ -conjugate to v . Hence $G(k) \cdot v$ is cocharacter-closed over k , as required. \square

Remarks 3.17. (i). If $v \in V(k)$, then the reverse direction holds for k perfect in Theorem 3.16 and the answer to Question 3.15 is yes: this follows from Corollary 4.10 below.

(ii). For arbitrary k it can happen that $G(k) \cdot v$ is not cocharacter-closed over k but $G \cdot v$ is closed or vice versa, even when $v \in V(k)$ (see Remark 5.8).

4. UNIFORM S -INSTABILITY

In this section we show that the results of Kempf in [12] extend to *uniform instability* as defined by W. Hesselink in [11]. Since this is a straightforward modification of Kempf's arguments, we only indicate the relevant changes. We point out here that the extension to non-connected G is inessential for the results in this section, since they follow immediately from the corresponding statements for G connected. We state the results for G non-connected, because this is more convenient for our applications. As our field k is not necessarily algebraically closed, we restrict to k -defined cocharacters of G in Kempf's optimization procedure, cf. [11].

Throughout this section, G is a reductive k -defined normal subgroup of a k -group G' which acts on an affine \bar{k} -variety V , and S is a non-empty G -stable closed subvariety of V .

Definition 4.1. A $G'(k)$ -invariant norm $\|\cdot\|$ on $Y_k(G)$ is a non-negative real-valued function $\|\cdot\|$ on $Y(G)$ such that

- (a) $\|g \cdot \lambda\| = \|\lambda\|$ for any $g \in G'(k)$ and any $\lambda \in Y_k(G)$,
- (b) for any k -split k -defined torus T of G , there is a positive definite integer-valued form (\cdot, \cdot) on $Y_k(T)$ such that $(\lambda, \lambda) = \|\lambda\|^2$ for any $\lambda \in Y_k(T)$.

Note that a G' -invariant norm $\|\cdot\|$ on $Y(G)$ determines a $G'(k)$ -invariant norm on $Y_k(G)$. We say that the norm $\|\cdot\|$ is k -defined if it is Γ -invariant (see [12]). A k -defined G' -invariant norm on $Y(G)$ always exists, thanks to [19, Sec. 7] and [11, Sect. 1].

Definition 4.2. For each non-empty subset X of V , define $\Lambda(X)$ as the set of $\lambda \in Y(G)$ such that $\lim_{a \rightarrow 0} \lambda(a) \cdot x$ exists for all $x \in X$, and put $\Lambda(X, k) = \Lambda(X) \cap Y_k(G)$. Extending Hesselink [11], we call X *uniformly S -unstable* if there exists $\lambda \in \Lambda(X)$ such that $\lim_{a \rightarrow 0} \lambda(a) \cdot x \in S$ for all $x \in X$, and we say that such a cocharacter *destabilizes X into S* or is a *destabilizing cocharacter for X with respect to S* . We call X *uniformly S -unstable over k* if there exists such a λ in $\Lambda(X, k)$. We say that $x \in V$ is *S -unstable over k* if $\{x\}$ is uniformly S -unstable over k . Finally, (uniformly S -) unstable without specifying a field always means (uniformly

S -) unstable over \bar{k} . By the Hilbert-Mumford Theorem, $x \in V$ is S -unstable if and only if $\overline{G \cdot x} \cap S \neq \emptyset$.

Let $x \in V$ and let $\lambda \in \Lambda(x)$. Let $\varphi : \bar{k} \rightarrow V$ be the morphism $\widehat{\phi}_{x,\lambda}$ from Section 2.3. If $x \notin S$, then the scheme-theoretic inverse image $\varphi^{-1}(S)$ has affine ring $\bar{k}[T]/(T^m)$ for a unique integer $m \geq 0$ ($T^0 = 1$), and we define $a_{S,x}(\lambda) := m$. If $x \in S$, then we define $a_{S,x}(\lambda) := \infty$. For a non-empty subset X of V and $\lambda \in \Lambda(X)$, we define $a_{S,X}(\lambda) := \min_{x \in X} a_{S,x}(\lambda)$. Note that $a_{S,X}(\lambda) > 0$ if and only if $\lim_{a \rightarrow 0} \lambda(a) \cdot x \in S$ for all $x \in X$, $a_{S,X}(\lambda) = 0$ if and only if $\lim_{a \rightarrow 0} \lambda(a) \cdot x \notin S$ for all $x \in X$, and $a_{S,X}(\lambda) = \infty$ if and only if $X \subseteq S$.

Now we choose a $G'(k)$ -invariant norm $\|\cdot\|$ on $Y_k(G)$.

Definition 4.3. Let X be a non-empty subset of V . If $X \subseteq S$, we put $\Omega(X, S, k) = \{0\}$, where 0 denotes the trivial cocharacter of G . Now assume $X \not\subseteq S$. If the function $\lambda \mapsto a_{S,X}(\lambda)/\|\lambda\|$ has a finite strictly positive maximum value on $\Lambda(X, k) \setminus \{0\}$, then we define $\Omega(X, S, k)$ as the set of indivisible cocharacters in $\Lambda(X, k) \setminus \{0\}$ on which this function takes its maximum value. Otherwise we define $\Omega(X, S, k) = \emptyset$. Note that X is uniformly S -unstable over k (in the sense of Definition 4.2) provided $\Omega(X, S, k) \neq \emptyset$. The set $\Omega(X, S, k)$ is called the *optimal class of X with respect to S over k* .

We are now able to state and prove the analogue of Kempf's instability theorem ([12, Thm. 4.2]) in this setting.

Theorem 4.4. *Let X be a non-empty subset of V which is uniformly S -unstable over k . Then $\Omega(X, S, k)$ is non-empty and has the following properties:*

- (i) $\lim_{a \rightarrow 0} \lambda(a) \cdot x \in S$ for all $\lambda \in \Omega(X, S, k)$ and any $x \in X$.
- (ii) For all $\lambda, \mu \in \Omega(X, S, k)$, we have $P_\lambda = P_\mu$. Let $P(X, S, k)$ denote the unique R -parabolic subgroup of G so defined. (Note that $P(X, S, k)$ is k -defined by Lemma 2.5.)
- (iii) If $g \in G'(k)$, then $\Omega(g \cdot X, g \cdot S, k) = g \cdot \Omega(X, S, k)$.
- (iv) $R_u(P(X, S, k))(k)$ acts simply transitively on $\Omega(X, S, k)$: that is, for each k -defined R -Levi subgroup L of $P(X, S, k)$, there exists one and only one $\lambda \in \Omega(X, S, k)$ with $L = L_\lambda$. Hence $N_{G(k)}(X) \subseteq P(X, S, k)(k)$.

Proof. If $X \subseteq S$, then $\Omega(X, S, k) = \{0\}$ and $P(X, S, k) = G$, so all the statements are trivial in this case. Hence we may assume that $X \not\subseteq S$. We have that G^0 is k -defined and, clearly, $Y_k(G) = Y_k(G^0)$ and $R_u(P_\lambda) = R_u(P_\lambda(G^0))$. So we may assume that G is connected. We use Kempf's "state formalism", [12, Sec. 2]. Actually we may consider states as only defined on k -split subtori of G . First we need an analogue over k of [12, Thm. 2.2]. This is completely straightforward: we simply work with $Y_k(G)$ instead of $Y(G)$ and use the conjugacy of the maximal k -split tori of G under $G(k)$, [7, V.20.9(ii)], as in [11].

Next we need a way to associate to a non-empty finite subset $X_0 \neq \{0\}$ of a k -defined rational G -module V_0 a bounded admissible state. This is done as in [11, 2.4]. Then [12, Lem. 3.2] holds with V and v replaced by V_0 and X_0 , respectively.

Finally, we need to construct two bounded admissible states as in [12, Lem. 3.3]. This is done precisely as in the proof of *loc. cit.* The embedding of V in a k -defined G -module V_0 (denoted V in *loc. cit.*) and the morphism $f : V \rightarrow W$, W a rational G -module, with $f^{-1}(0) = S$ can be chosen as in [12]. Let Ξ and Υ be the state of X in V_0 and the state of

$f(X)$ in W , respectively. Then assertions (i), (ii), (iii) and the first assertion of (iv) follow as in [12].

The final assertion of (iv) is proved as follows. Fix $\lambda \in \Omega(X, S, k)$. Let $g \in N_{G(k)}(X)$. Then $g \cdot \Omega(X, S, k) = \Omega(g \cdot X, g \cdot S, k) = \Omega(X, S, k)$, by (iii). So $g \cdot \lambda \in \Omega(X, S, k)$. By the first assertion of (iv) $g \cdot \lambda = u \cdot \lambda$ for some $u \in R_u(P(X, S, k))(k)$. So $u^{-1}g \in C_G(\lambda(\bar{k}^*)) = L_\lambda \subseteq P(X, S, k)$ and therefore $g \in P(X, S, k) \cap G(k) = P(X, S, k)(k)$. \square

Definition 4.5. We call $P(X, S, k)$ from Theorem 4.4 the *optimal destabilizing parabolic subgroup for X with respect to S over k* .

Next we discuss rationality properties of this construction. If X is uniformly S -unstable over k and k_1/k is a field extension, then X is uniformly S -unstable over k_1 . We want to investigate the relationship between $P(X, S, k)$ and $P(X, S, k_1)$. It appears that one can say little in general if k_1/k is not separable, so we consider the special case when $k_1 = k_s$. We denote the k -closure of X by X^k , cf. [7, AG 11.3]. We obtain a rationality result as in [11, Thm. 5.5].

We now choose a G' -invariant norm $\|\cdot\|$ on $Y(G)$. Note that this determines a $G(k_1)$ -invariant norm on $Y_{k_1}(G)$ for any subfield k_1 of \bar{k} .

Theorem 4.6. *Assume that V is an affine k -variety and that S and the action of G on V are k -defined. Let X be a non-empty subset of V .*

- (i) *X is uniformly S -unstable over k if and only if X^k is uniformly S -unstable over k_s .*
- (ii) *Assume that X is uniformly S -unstable over k and that the norm $\|\cdot\|$ on $Y(G)$ is k -defined. Then $\Omega(X, S, k)$ consists of the k -defined cocharacters in $\Omega(X^k, S, k_s)$. In particular, the cocharacters in $\Omega(X, S, k)$ are optimal for X^k over k_s .*

Proof. The embedding $V \hookrightarrow V_0$ and the morphism $f : V \rightarrow W$ of the proof of Theorem 4.4 can be chosen to be defined over k , see [7, I.1.9]. One can then easily check that for $\lambda \in Y_k(G)$ and any integer r ,

$$(4.7) \quad \text{the set } \{x \in V \mid \lambda \in \Lambda(x), a_{S,x}(\lambda) \geq r\} \text{ is } k\text{-closed,}$$

cf. proof of [11, Thm. 5.5]. It follows that $\Lambda(X, k) = \Lambda(X^k, k)$ and that $a_{S,X}(\lambda) = a_{S,X^k}(\lambda)$ for all $\lambda \in \Lambda(X, k)$.

So we may assume that X is k -closed and we have to show that $\Lambda(X, k_s)$ contains a k -defined cocharacter. If Z is a k -variety (over \bar{k}), then $\Gamma = \text{Gal}(k_s/k)$ acts on the set Z and the k -closed subsets of Z are the Γ -stable closed subsets of Z ; see [23, 11.2.8(ii)]. Furthermore, if Z_1 and Z_2 are k -varieties, then Γ acts on the k_s -defined morphisms from Z_1 to Z_2 and such a morphism is k -defined if and only if it is fixed by Γ ; see [23, 11.2.9]. So in our case Γ acts on the sets G and V and X is Γ -stable. Now we can finish the proof as in [12, Thm. 4.2] or [11, Thm. 5.5]. \square

Corollary 4.8. *Suppose the hypotheses of Theorem 4.6 hold and that $\|\cdot\|$ is k -defined. Let k_1/k be a separable algebraic extension. Denote the k -closure of X by X^k . Then X is uniformly S -unstable over k if and only if X^k is uniformly S -unstable over k_1 , and in this case we have $\Omega(X, S, k) = \Omega(X^k, S, k_1) \cap Y_k(G)$ and $P(X, S, k) = P(X^k, S, k_1)$.*

Remarks 4.9. (i). Hesselink's optimal class consists in general of virtual cocharacters, since, essentially, he requires $a_{S,X}(\lambda) = 1$ (he minimizes the norm). We work with Kempf's optimal

class which consists of indivisible cocharacters. There is an obvious bijection between the two optimal classes.

(ii). If k is not perfect, then X can be S -unstable over \bar{k} but need not be S -unstable over k (cf. [1, Ex. 5.11]).

(iii). Assume that k is perfect and that $X = \{v\}$ with v a k -point of V outside S whose G -orbit closure meets S . Then Corollary 4.8 gives the existence of a k -defined destabilizing cocharacter for v and S which is optimal over \bar{k} . This was first proved by Kempf in [12, Thm. 4.2].

Corollary 4.10 below and Corollary 4.8 answer Question 3.15 for perfect k .

Corollary 4.10. *Suppose that k is perfect. Let V be an affine G -variety over k . Let $v \in V(k)$. Then $G \cdot v$ is closed if and only if $G(k) \cdot v$ is cocharacter-closed over k .*

Proof. If $G(k) \cdot v$ is not cocharacter-closed over k , then $G \cdot v$ is not closed, by Corollary 3.9. Conversely, suppose $G \cdot v$ is not closed. Let S be the unique closed G -orbit in $\overline{G \cdot v}$. Clearly, $\overline{G \cdot v}$ is k -defined (see e.g. [23, 1.9.1]). Let $\gamma \in \Gamma$. Then $\gamma(S)$ is a closed G -orbit which is contained in $\overline{G \cdot v}$, so it is equal to S . It follows that S is Γ -stable and therefore k -defined, since k is perfect. Now v is S -unstable by the Hilbert-Mumford Theorem and therefore S -unstable over k , by Theorem 4.6(i). Since $S \cap G \cdot v = \emptyset$, it is clear that $G(k) \cdot v$ is not cocharacter-closed over k . \square

Remark 4.11. We finish the section by pointing out an open problem. Let V be a k -defined affine G -variety and suppose the norm $\| \cdot \|$ is k -defined. If X is uniformly S -unstable over k , then we may construct the optimal destabilizing parabolic subgroup $P(X, S, k)$ for X . In this case X is also uniformly S -unstable over \bar{k} — that is, X is uniformly S -unstable — so we may construct the optimal destabilizing parabolic subgroup $P(X, S) = P(X, S, \bar{k})$ for X . If k is not perfect, our methods do not tell us whether or not $P(X, S, k) = P(X, S)$.

Now suppose moreover for simplicity that $X = \{v\}$ where $v \in V(k)$. Let $\lambda \in Y_k(G)$ such that $v' := \lim_{a \rightarrow 0} \lambda(a) \cdot v$ exists. If v' is not G -conjugate to v , then we can take S to be $\overline{G \cdot v'}$ and construct the set of optimal destabilizing cocharacters $\Omega(X, S, k)$ over k ; roughly speaking, cocharacters in $\Omega(X, S, k)$ are the cocharacters in $Y_k(G)$ that take v to S “as quickly as possible”. Now suppose instead that v' is G -conjugate to v but not $G(k)$ -conjugate to v . One may hope to find a cocharacter that takes v into $G(k) \cdot v'$ as quickly as possible, and associate to this an “optimal destabilizing” R-parabolic subgroup of G . The formalism above does not permit us to do this: for if S is a closed G -stable subset of V that contains $G \cdot v'$, then S contains $G \cdot v$ and $\Omega(X, S, k)$ consists only of the zero cocharacter. It is natural to ask whether such a cocharacter can be found, but this question does not appear to have been raised in the literature before; it arises naturally when one considers G -complete reducibility (see Section 5.2). We hope to return to this in future work.

5. APPLICATIONS TO G -COMPLETE REDUCIBILITY

In this section we discuss some applications of the theory developed in this paper, with particular reference to Serre’s concept of G -complete reducibility. We briefly recall the definitions here; for more details, see [1], [22].

Definition 5.1. A subgroup H of G is said to be *G -completely reducible* (G -cr) if whenever H is contained in an R-parabolic subgroup P of G , there exists an R-Levi subgroup L of

P containing H . Similarly, a subgroup H of G is said to be G -completely reducible over k if whenever H is contained in a k -defined R-parabolic subgroup P of G , there exists a k -defined R-Levi subgroup L of P containing H .

We have noted (Remark 2.4) that not every k -defined R-parabolic subgroup of G need stem from a cocharacter in $Y_k(G)$. However, our next result shows that when considering questions of G -complete reducibility over k , it suffices to just look at k -defined R-parabolic subgroups of G of the form P_λ with $\lambda \in Y_k(G)$.

Lemma 5.2. *Let H be a subgroup of G . Then H is G -completely reducible over k if and only if for every $\lambda \in Y_k(G)$ such that H is contained in P_λ , there exists $\mu \in Y_k(G)$ such that $P_\lambda = P_\mu$ and $H \subseteq L_\mu$.*

Proof. Assume that for every $\lambda \in Y_k(G)$ such that H is contained in P_λ , there exists $\mu \in Y_k(G)$ such that $P_\lambda = P_\mu$ and $H \subseteq L_\mu$. Let $\lambda \in Y(G)$ such that P_λ is k -defined and $H \subseteq P_\lambda$. After conjugating λ by an element of P_λ , we may assume that $\lambda \in Y(T)$ for some k -defined maximal torus T of P_λ . By Lemma 2.5(ii), there exists $\mu \in Y_k(T)$ such that $P_\lambda \subseteq P_\mu$ and $P_\lambda^0 = P_\mu^0$. Note that $L_\lambda = L_\mu \cap P_\lambda$, by Lemma 2.2. By assumption, there exists $\nu \in Y_k(G)$ such that $P_\mu = P_\nu$ and $H \subseteq L_\nu$. There exists $u \in R_u(P_\mu) = R_u(P_\lambda)$ such that $uL_\mu u^{-1} = L_\nu$. But then $L_{u\cdot\lambda} = uL_\lambda u^{-1} = u(L_\mu \cap P_\lambda)u^{-1} = L_\nu \cap P_\lambda$ contains H . By Lemma 2.5(iii), $L_{u\cdot\lambda}$ is k -defined, since $L_{u\cdot\lambda}^0 = L_\nu^0$ is k -defined. The other implication follows from Corollary 2.6. \square

Remark 5.3. If k is algebraically closed (or even perfect, see [1, Thm. 5.8]) and H is k -defined, then H is G -cr over k if and only if H is G -cr.

5.1. Geometric criteria for G -complete reducibility. In [1], we show that the G -cr concept has a geometric interpretation in terms of the action of G on G^n , the n -fold Cartesian product of G with itself, by simultaneous conjugation. Let $\mathbf{h} = (h_1, \dots, h_n) \in G^n$ and let H be the closed subgroup of G generated by the h_i . Then $G \cdot \mathbf{h}$ is closed in G^n if and only if H is G -cr [1, Cor. 3.7]. To generalize this to subgroups that are not topologically finitely generated, we need the following concept.

Definition 5.4. Let H be a closed subgroup of G and let $G \hookrightarrow \mathrm{GL}_m$ be an embedding of algebraic groups. Then $\mathbf{h} = (h_1, \dots, h_n) \in H^n$ is called a *generic tuple of H for the embedding $G \hookrightarrow \mathrm{GL}_m$* if the h_i generate the associative subalgebra of Mat_m spanned by H . We call $\mathbf{h} \in H^n$ a *generic tuple of H* if it is a generic tuple of H for some embedding $G \hookrightarrow \mathrm{GL}_m$.

Clearly, generic tuples exist for any embedding $G \hookrightarrow \mathrm{GL}_m$ and n sufficiently large. The next lemma gives the main properties of generic tuples.

Lemma 5.5. *Let H be a closed subgroup of G , let $\mathbf{h} \in H^n$ be a generic tuple of H for some embedding $G \hookrightarrow \mathrm{GL}_m$ and let H' be the closed subgroup of G generated by \mathbf{h} . Then we have:*

- (i) $C_G(\mathbf{h}) = C_G(H') = C_G(H)$;
- (ii) H' is contained in the same R-parabolic and the same R-Levi subgroups of G as H ;
- (iii) If $H \subseteq P_\lambda$ for some $\lambda \in Y(G)$, then $c_\lambda(\mathbf{h})$ is a generic tuple of $c_\lambda(H)$ for the given embedding $G \hookrightarrow \mathrm{GL}_m$.

Proof. Let $\mathbf{h} = (h_1, \dots, h_n)$. By assumption, the h_i generate the associative subalgebra A of Mat_m spanned by H . For $\lambda \in Y(\mathrm{GL}_m)$ let \mathcal{P}_λ be the subset of elements $x \in \mathrm{Mat}_m$ such that

$\lim_{a \rightarrow 0} \lambda(a) \cdot x$ exists and let \mathcal{L}_λ be the centralizer of $\lambda(k^*)$ in Mat_m . Denote the limit morphism $\mathcal{P}_\lambda \rightarrow \mathcal{L}_\lambda$ by c_λ . The well-known characterization of \mathcal{P}_λ and \mathcal{L}_λ in terms of flags shows that they are subalgebras of Mat_m and that c_λ is a homomorphism of algebras. For $\lambda \in Y(G)$ we have $P_\lambda(G) = G \cap \mathcal{P}_\lambda$ and $L_\lambda(G) = G \cap \mathcal{L}_\lambda$.

(i). If a subset S of Mat_m generates the associative subalgebra E of Mat_m , then $C_G(S) = G \cap C_{\text{Mat}_m}(E)$. So $C_G(H) = G \cap C_{\text{Mat}_m}(A) = C_G(\mathbf{h})$.

(ii). If a subset S of G generates the associative subalgebra E of Mat_m , then $S \subseteq P_\lambda(G)$ if and only if $E \subseteq \mathcal{P}_\lambda$, and $S \subseteq L_\lambda(G)$ if and only if $E \subseteq \mathcal{L}_\lambda$. This implies the assertion.

(iii). Since $c_\lambda : \mathcal{P}_\lambda \rightarrow \mathcal{L}_\lambda$ is a homomorphism of associative algebras, $c_\lambda(\mathbf{h})$ generates the associative subalgebra $c_\lambda(A)$ and this is also the associative subalgebra of Mat_m generated by $c_\lambda(H)$. \square

Remark 5.6. If H is a closed subgroup of G which is topologically generated by a tuple $\mathbf{h} = (h_1, \dots, h_n) \in H^n$, then \mathbf{h} is a generic tuple for H in the sense of Definition 5.4. To see this, consider an embedding $G \hookrightarrow \text{GL}_m$. Since the minimal polynomial of each h_i has non-zero constant term, we can express h_i^{-1} as a polynomial in h_i . Hence, if A is the associative subalgebra of Mat_m generated by \mathbf{h} , then A contains the inverses of each of the h_i , so it contains the subgroup of GL_m generated by \mathbf{h} . But A is closed, so it contains H .

We can translate Theorem 3.12 to the setting of G -complete reducibility using the concept of generic tuples. Note that, in view of Remarks 3.11 and 5.6, Theorem 5.7 generalizes [1, Cor. 3.7], which is just Theorem 5.7 in case $k = \bar{k}$.

Theorem 5.7. *Suppose that G is connected. Let H be a subgroup of G and $\mathbf{h} \in H^n$ be a generic tuple for H . Then H is G -completely reducible over k if and only if $G(k) \cdot \mathbf{h}$ is cocharacter-closed over k .*

Proof. Suppose that $G(k) \cdot \mathbf{h}$ is cocharacter-closed over k . In order to show that H is G -cr over k , we just need to consider \mathbb{R} -parabolic subgroups of G containing H of the form P_λ with $\lambda \in Y_k(G)$, by Lemma 5.2. Let $\lambda \in Y_k(G)$ be such that P_λ contains H . Then $\mathbf{h}' = c_\lambda(\mathbf{h})$ exists. Since $G(k) \cdot \mathbf{h}$ is cocharacter-closed over k , there exists $u \in R_u(P_\lambda)(k)$, such that $\mathbf{h}' = u \cdot \mathbf{h}$, by Theorem 3.12. By Lemma 2.11, $u^{-1} \cdot \lambda$ centralizes \mathbf{h} . Hence $H \subseteq L_{u^{-1}, \lambda}$. Since $L_{u^{-1}, \lambda}$ is k -defined, H is G -completely reducible over k .

Now assume that H is G -completely reducible over k . Let $\lambda \in Y_k(G)$ such that $\mathbf{h}' = c_\lambda(\mathbf{h})$ exists. Then $H \subseteq P_\lambda$. So, by hypothesis, there exists a k -defined \mathbb{R} -Levi subgroup L of P_λ with $H \subseteq L$. By Lemma 2.5(iii), there exists $u \in R_u(P_\lambda)(k)$ such that $L = u^{-1} L_\lambda u = L_{u^{-1}, \lambda}$. Hence $u^{-1} \cdot \lambda$ centralizes H and so $u^{-1} \cdot \lambda$ centralizes \mathbf{h} . Thus, by Lemma 2.11, we have $\mathbf{h}' = u \cdot \mathbf{h}$. Consequently, $G(k) \cdot \mathbf{h}$ is cocharacter-closed over k , by Theorem 3.12. \square

Remark 5.8. We now provide examples for the failure of Question 3.15 in general. In [4, Ex. 7.22], we give an example of a reductive group G and a subgroup H , both k -defined, such that H is G -completely reducible but not G -completely reducible over k . Let $\mathbf{h} \in H^n$ be a generic tuple for H . Then, by Theorem 5.7, $G \cdot \mathbf{h}$ is closed in G^n but $G(k) \cdot \mathbf{h}$ is not cocharacter-closed over k . Conversely, an example due to McNinch, [1, Ex. 5.11], gives a reductive group G and a subgroup H , both k -defined, such that H is G -completely reducible over k but not G -completely reducible, and this implies that there exists a generic tuple $\mathbf{h} \in H^n$ for some $n \in \mathbb{N}$ such that $G(k) \cdot \mathbf{h}$ is cocharacter-closed over k but $G \cdot \mathbf{h}$ is not

closed. Note that in both of the counterexamples to Question 3.15, the extension \bar{k}/k is not separable.

Even if one is interested only in separable field extensions k_1/k , problems with inseparability can arise. This is the case for the question of Serre's that we discuss below (see the paragraph before Theorem 5.11). Serre's question concerns G -completely reducible subgroups of G , but it can be expressed in terms of whether certain orbits in a certain variety are cocharacter-closed over k , by Theorem 5.7.

The connection between G -complete reducibility and G -orbits of tuples is made transparent by part (iii) of the following theorem which is, essentially, a consequence of Theorem 3.4. It also shows how statements about generic tuples can be translated back into subgroups of G . Note that, in view of Remark 5.6, the final statement of Theorem 5.9(iii) recovers [1, Cor. 3.7].

Theorem 5.9.

- (i) *Let $n \in \mathbb{N}$, let $\mathbf{h} \in G^n$ and let $\lambda \in Y(G)$ such that $\mathbf{m} = \lim_{a \rightarrow 0} \lambda(a) \cdot \mathbf{h}$ exists. Then the following are equivalent:*
 - (a) \mathbf{m} is G -conjugate to \mathbf{h} ;
 - (b) \mathbf{m} is $R_u(P_\lambda)$ -conjugate to \mathbf{h} ;
 - (c) $\dim G \cdot \mathbf{m} = \dim G \cdot \mathbf{h}$.
- (ii) *Let H be a subgroup of G and let $\lambda \in Y(G)$. Suppose $H \subseteq P_\lambda$ and set $M = c_\lambda(H)$. Then $\dim C_G(M) \geq \dim C_G(H)$ and the following are equivalent:*
 - (a) M is G -conjugate to H ;
 - (b) M is $R_u(P_\lambda)$ -conjugate to H ;
 - (c) H is contained in an R-Levi subgroup of P_λ ;
 - (d) $\dim C_G(M) = \dim C_G(H)$.
- (iii) *Let H , λ and M be as in (ii) and let $\mathbf{h} \in H^n$ be a generic tuple of H . Then the assertions in (i) are equivalent to those in (ii). In particular, H is G -completely reducible if and only if $G \cdot \mathbf{h}$ is closed in G^n .*

Proof. (i). This follows immediately from Theorem 3.4 and [7, Prop. I.1.8].

(ii) and (iii). Let $\mathbf{h} \in H^n$, let H' be the closed subgroup of G generated by the members of \mathbf{h} and let $\lambda \in Y(G)$. Then $\lim_{a \rightarrow 0} \lambda(a) \cdot \mathbf{h}$ exists if and only if $H' \subseteq P_\lambda$. Now assume that $\mathbf{m} = \lim_{a \rightarrow 0} \lambda(a) \cdot \mathbf{h}$ exists. Let $u \in R_u(P_\lambda)$. Then $\mathbf{h} = u \cdot \mathbf{m}$ if and only if $u \cdot \lambda$ fixes \mathbf{h} (Lemma 2.11) if and only if $H' \subseteq L_{u \cdot \lambda} = uL_\lambda u^{-1}$. Since $R_u(P_\lambda)$ acts simply transitively on the R-Levi subgroups of P_λ , the first assertion of (iii) follows once we have proved (ii). For this purpose we pick a generic tuple $\mathbf{h} \in H^n$ of H . Then $\mathbf{m} = c_\lambda(\mathbf{h})$ is a generic tuple of M , by Lemma 5.5(iii). Now the first assertion of (ii) follows from the fact that $\dim G \cdot \mathbf{m} \leq \dim G \cdot \mathbf{h}$ (see [7, Prop. I.1.8]), since $\dim G \cdot \mathbf{h} = \dim G - \dim C_G(\mathbf{h})$ and likewise for \mathbf{m} . Now we prove the equivalences. Clearly, (b) implies (a) and (a) implies (d). Furthermore, we have for $u \in R_u(P_\lambda)$ that $H \subseteq L_{u \cdot \lambda}$ if and only if $H = c_{u \cdot \lambda}(H) = uMu^{-1}$. So (b) is equivalent to (c). Now assume that (d) holds. Then $\dim G \cdot \mathbf{m} = \dim G \cdot \mathbf{h}$. So \mathbf{m} is $R_u(P_\lambda)$ -conjugate to \mathbf{h} , by (i). We have seen above that this means that H' is contained in an R-Levi subgroup of P_λ . Since \mathbf{h} is generic for H , the same must then hold for H , that is, (c) holds. The final assertion of (iii) follows from the first and the Hilbert-Mumford Theorem. \square

The G -conjugacy class of the group M in Proposition 5.10 corresponds to the unique closed G -orbit in the G -orbit closure of H .

Proposition 5.10. *Let H be a subgroup of G .*

- (i) *There exists $\lambda \in Y(G)$ and a G -completely reducible subgroup M of G such that $H \subseteq P_\lambda$ and $c_\lambda(H) = M$. Moreover, M is unique up to G -conjugacy and its G -conjugacy class only depends on the G -conjugacy class of H .*
- (ii) *Any automorphism of the algebraic group G that stabilizes the G -conjugacy class of H , stabilizes the G -conjugacy class of M .*
- (iii) *If $H \subseteq P_\mu$, then the procedure described in (i) associates the same G -conjugacy class of subgroups to H and $c_\mu(H)$.*

Proof. (i). Let P_λ be an R-parabolic subgroup of G which is minimal with respect to containing H . Since $H \subseteq c_\lambda(H)R_u(P_\lambda)$ and $R_u(P_\lambda) \subseteq R_u(Q)$ for every R-parabolic subgroup Q of G with $Q \subseteq P_\lambda$, we have that P_λ is also minimal with respect to containing $M = c_\lambda(H)$. So M is L_λ -irreducible and therefore G -cr, see [1, Cor. 6.4, Cor. 3.22].

Now suppose $\lambda, \mu \in Y(G)$ such that $H \subseteq P_\lambda$, $H \subseteq P_\mu$ and such that $M_1 = c_\lambda(H)$ and $M_2 = c_\mu(H)$ are G -cr. Since P_λ and P_μ have a maximal torus in common (see e.g. [7, Cor. IV.14.13]), after possibly replacing M_1 by an $R_u(P_\lambda)$ -conjugate and M_2 by an $R_u(P_\mu)$ -conjugate, we may assume that $\lambda(k^*)$ and $\mu(k^*)$ commute. Clearly, $P_\lambda \cap P_\mu$ is stable under c_λ and c_μ . It follows from [1, Lem. 6.2(iii)] that, on $P_\lambda \cap P_\mu$, the composition $c_\lambda \circ c_\mu = c_\mu \circ c_\lambda$ is the projection $P_\lambda \cap P_\mu \rightarrow L_\lambda \cap L_\mu$ with kernel $R_u(P_\lambda \cap P_\mu)$. So $c_\lambda(M_2) = c_\mu(M_1)$. Now M_1 is G -cr, so, by Theorem 5.9(ii), M_1 is $R_u(P_\mu)$ -conjugate to $c_\mu(M_1)$. Similarly, M_2 is $R_u(P_\lambda)$ -conjugate to $c_\lambda(M_2)$. So M_1 and M_2 are G -conjugate. Finally, we observe that if $H \subseteq P_\lambda$ and $g \in G$, then $gHg^{-1} \subseteq P_{g\lambda}$ and $c_{g\lambda}(gHg^{-1}) = gc_\lambda(H)g^{-1}$, so the G -conjugacy class of M only depends on that of H .

(ii). Let φ be an automorphism of the algebraic group G that stabilizes the G -conjugacy class of H and let $\lambda \in Y(G)$ such that $H \subseteq P_\lambda$ and $c_\lambda(H)$ is G -cr. Then $\varphi(H)$ is G -conjugate to H and $\varphi(H) \subseteq P_{\varphi\circ\lambda}$. Now $\varphi(c_\lambda(H)) = c_{\varphi\circ\lambda}(\varphi(H))$ is G -conjugate to $c_\lambda(H)$ by (i), since $c_\lambda(H)$ is G -cr.

(iii). Assume that $H \subseteq P_\mu$ and let $\lambda \in Y(G)$ such that $H \subseteq P_\lambda$ and $c_\lambda(H)$ is G -cr. After replacing λ by a P_λ -conjugate and μ by a P_μ -conjugate, we may assume that λ and μ commute. As in (i), $P_\lambda \cap P_\mu$ is stable under c_λ and c_μ and $c_\lambda(c_\mu(H)) = c_\mu(c_\lambda(H))$ is $R_u(P_\mu)$ -conjugate to $c_\lambda(H)$ and G -cr, since $c_\lambda(H)$ is G -cr. \square

The interpretation of G -complete reducibility in terms of orbits allows us to provide a partial answer to a question of Serre; for a more general result, see [5, Thm. 4.12]. Let k_1/k be a separable extension of fields. Serre has asked whether it is the case that a k -defined subgroup H of G is G -completely reducible over k if and only if it is G -completely reducible over k_1 . We can now answer one direction of this question:

Theorem 5.11. *Suppose k_1/k is a separable extension of fields. Let H be a k -defined subgroup of G . If H is G -completely reducible over k_1 , then H is G -completely reducible over k .*

Proof. Let $\mathbf{h} \in H^n$ be a generic tuple for H for some n . Suppose $\lambda \in Y_k(G)$ is such that $H \subseteq P_\lambda$. Then since H is G -cr over k_1 , there exists $u_1 \in R_u(P_\lambda)(k_1) \subseteq R_u(P_\lambda)(k_s)$ such that $H \subseteq L_{u_1^{-1}\lambda}$. Thus, $u_1^{-1} \cdot \lambda$ centralizes H and so $u_1^{-1} \cdot \lambda$ centralizes \mathbf{h} . It thus follows from Lemma 2.11 that $\lim_{a \rightarrow 0} \lambda(a) \cdot \mathbf{h} = u_1 \cdot \mathbf{h}$. Since H is k -defined and $C_G(H) = C_G(\mathbf{h})$, by Lemma 5.5(i), we see that $C_{G(k_s)}(\mathbf{h})$ is Γ -stable. Hence we can apply Theorem 3.2 to conclude that there exists $u \in R_u(P_\lambda)(k)$ such that $\lim_{a \rightarrow 0} \lambda(a) \cdot \mathbf{h} = u \cdot \mathbf{h}$. Thus $u^{-1} \cdot \lambda \in Y_k(G)$

centralizes \mathbf{h} , by Lemma 2.11, hence H , and we have $H \subseteq L_{u^{-1}, \lambda}$, a k -defined R-Levi subgroup of P_λ , as required. \square

Remark 5.12. Theorem 5.11 is a group-theoretic analogue of Theorem 3.16. Indeed, in view of Theorem 5.7, in the special case of Theorem 5.11 when there is a generic tuple $\mathbf{h} \in H^n$ of H which is a k -point in H^n , Theorem 5.11 follows from Theorem 3.16.

Example 5.13. We show that the answer to Serre's question is yes when $G = \mathrm{GL}(V)$, where $V = \bar{k}^n$ with the standard k -structure k^n on V . This of course determines the usual k -structure on $\mathrm{GL}(V)$. Let H be a subgroup of G and let A be its enveloping algebra, i.e. the \bar{k} -span of H in $\mathrm{End}_{\bar{k}}(V)$. Then A is k -defined provided H is. To see this, we exhibit a Γ -stable, dense subset of separable points in A and for this set we simply take the k_s -span of $H(k_s)$ in $\mathrm{End}_{\bar{k}}(V)$. As a consequence, we obtain the following characterization of $\mathrm{GL}(V)$ -cr over k under the assumption that H is k -defined: H is $\mathrm{GL}(V)$ -cr over k if and only if $V(k) = k^n$ is a semisimple $A(k)$ -module (if and only if $A(k)$ is a semisimple algebra). Here $A(k)$ denotes the algebra of k -points of A (this is a k -structure on A : $A = \bar{k} \otimes_k A(k)$).

Finally, if H and A are as above and $k_1 \subseteq \bar{k}$ is an algebraic extension of k , then $A(k_1) = k_1 \otimes_k A(k)$. It follows from [10, Cor. 69.8 and Cor. 69.10] that $A(k)$ is semisimple if and only if $A(k_1)$ is semisimple, provided k_1 is a separable extension of k . By the above this means that H is $\mathrm{GL}(V)$ -cr over k if and only if H is $\mathrm{GL}(V)$ -cr over k_1 .

5.2. Optimal destabilizing parabolic subgroups for subgroups of G . In this section we assume G is a normal k -subgroup of a k -group G' . We fix a G' -invariant norm $\| \cdot \|$ on $Y(G)$, see Definition 4.1.

Let H be a subgroup of G such that H is not G -completely reducible. Suppose there exist $\mathbf{h} := (h_1, \dots, h_n) \in H^n$ such that H is generated by \mathbf{h} . Then $G \cdot \mathbf{h}$ is not closed in G^n , so there is an optimal destabilizing parabolic subgroup $P_{\mathbf{h}}$ of G for \mathbf{h} . Several recent results involving G -complete reducibility have rested on this construction [14], [1, Sec. 3, Thm. 5.8], [4, Thm. 5.4(a)]. There are some technical problems in applying it. For instance, not every H is topologically finitely generated [13, Sec. 9]; sometimes one wishes to have the h_i to be k -rational (see the proof of [1, Thm. 5.8], for example). Moreover, if $g \in G$ normalizes H , then g need not centralize \mathbf{h} .

We now show how to associate an optimal destabilizing R-parabolic subgroup $P(H)$ to H using uniform S -instability. This avoids the above problems and yields shorter, cleaner proofs, because we need not deal explicitly with a generating tuple for H .

Remark 5.14. We can regard the following construction as a generalization of the Borel-Tits construction [8], which associates to a non-trivial unipotent element $u \in G$ a parabolic subgroup P_{BT} of G such that $u \in R_u(P)$. More generally, this construction associates to a non-reductive subgroup H of G a parabolic subgroup P of G such that $R_u(H) \subseteq R_u(P)$. Our construction works for any non- G -completely reducible H , including the case when H is reductive. Note, however, that if H is non-reductive, then P_{BT} does not necessarily coincide with $P(H)$, [11, Rem. 8.4].

Definition 5.15. Let M be a subgroup of G . Given $n \in \mathbb{N}$, set $S_n(M) := \overline{G \cdot M^n}$, a closed G -stable subset of G^n . Note that $S_n(M)$ only depends on the G -conjugacy class of M . Now suppose there exists $\lambda \in Y_k(G)$ such that $H \subseteq P_\lambda$ and $M = c_\lambda(H)$. Note that if k is algebraically closed, then some subgroup of G in the G -conjugacy class attached to

H of G -cr subgroups of G , provided by Proposition 5.10, satisfies this hypothesis. Then we have $c_\lambda(H^n) \subseteq M^n \subseteq S_n(M)$, so H^n is uniformly $S_n(M)$ -unstable over k (in the sense of Definition 4.2).

Theorem 5.16. *Let G, G' and $\|\cdot\|$ be as above. Let H be any subgroup of G and let n be minimal such that H^n contains a generic tuple of H (cf. Definition 5.4). Let M be a subgroup of G and suppose that $M = c_\lambda(H)$ for some $\lambda \in Y_k(G)$ with $H \subseteq P_\lambda$. Put $\Omega(H, M, k) := \Omega(H^n, S_n(M), k)$. Then the following hold:*

- (i) $P_\mu = P_\nu$ for all $\mu, \nu \in \Omega(H, M, k)$. Let $P(H, M, k)$ denote the unique R -parabolic subgroup of G so defined. Then $H \subseteq P(H, M, k)$ and $R_u(P(H, M, k))(k)$ acts simply transitively on $\Omega(H, M, k)$.
- (ii) For $g \in G'(k)$ we have $\Omega(gHg^{-1}, gMg^{-1}, k) = g \cdot \Omega(H, M, k)$ and $P(gHg^{-1}, gMg^{-1}, k) = gP(H, M, k)g^{-1}$. If $g \in G(k)$ normalizes H and $S_n(M)$, then $g \in P(H, M, k)$.
- (iii) If $\mu \in \Omega(H, M, k)$, then $\dim C_G(c_\mu(H)) \geq \dim C_G(M)$. If M is G -conjugate to H , then, trivially, $\Omega(H, M, k) = \{0\}$ and $P(H, M, k) = G$. If M is not G -conjugate to H , then H is not contained in any R -Levi subgroup of $P(H, M, k)$.

Proof. (i) and (ii). This follows immediately from Theorem 4.4.

(iii). We have $\dim C_G(\mathbf{m}) \geq \dim C_G(M)$ for all $\mathbf{m} \in G \cdot M^n$. Since $\mathbf{m} \mapsto \dim C_G(\mathbf{m})$ is upper semi-continuous, cf. [18, Lem. 3.7(c)], this inequality holds for all $\mathbf{m} \in S_n(M)$.

Let $\mu \in \Omega(H, M, k)$. Let $\mathbf{h} \in H^n$ be a generic tuple of H . Then $c_\mu(\mathbf{h})$ is a generic tuple of $c_\mu(H)$, by Lemma 5.5(iii). So $\dim C_G(c_\mu(H)) = \dim C_G(c_\mu(\mathbf{h})) \geq \dim C_G(M)$, since $c_\mu(\mathbf{h}) \in S_n(M)$.

It follows easily from the definitions that $P(H, M, k) = G$ if and only if $\Omega(H, M, k) = \{0\}$ if and only if $H^n \subseteq S_n(M)$. Clearly, the latter is the case if M is G -conjugate to H . Now assume that M is not G -conjugate to H and pick $\mu \in \Omega(H, M, k)$. Then $\dim C_G(M) > \dim C_G(H)$, by Theorem 5.9(ii) (applied to λ). So $\dim C_G(c_\mu(H)) > \dim C_G(H)$, by the above and H is not contained in any R -Levi subgroup of $P(H, M, k)$, by Theorem 5.9(ii) (applied to μ). \square

Definition 5.17. We call $\Omega(H, M, k)$ the *optimal class of k -defined cocharacters for H and M* and we call $P(H, M, k)$ the *optimal destabilizing R -parabolic subgroup for H and M over k* . Assume the G -conjugacy class given by Proposition 5.10 contains a group M of the form $c_\lambda(H)$ for some $\lambda \in Y_k(G)$. Then we set $\Omega(H, k) := \Omega(H, M, k)$ and $P(H, k) := P(H, M, k)$. Under this assumption we have, by Proposition 5.10 and Theorem 5.16, that $N_{G(k)}(H)$ is contained in $P(H, k)$ and that for $\mu \in \Omega(H, k)$, $c_\mu(H)$ is G -completely reducible. So, by Theorem 5.9(ii), if H is not G -completely reducible, then it is not contained in any R -Levi subgroup of $P(H, k)$. Note that, trivially, $P(H, k) = G$ if H is G -completely reducible. We call $\Omega(H, k)$ the *optimal class of k -defined cocharacters for H* and we call $P(H, k)$ the *optimal destabilizing R -parabolic subgroup for H over k* .

Note that the above assumption on M is satisfied if k is algebraically closed. In that case we usually suppress the k argument and write simply $\Omega(H)$ and $P(H)$ instead and refer to these as the *optimal class of cocharacters for H* and the *optimal destabilizing R -parabolic subgroup for H* , respectively.

We now suppose that the fixed norm $\|\cdot\|$ on $Y(G)$ is k -defined, cf. Definition 4.1. We get the following rationality result.

Theorem 5.18. *Let G, G', H and n be as in Theorem 5.16 and assume that H is k -closed. Then the following hold:*

- (i) Suppose that M is a subgroup of G such that $M = c_\lambda(H)$ for some $\lambda \in Y_{k_s}(G)$ with $H \subseteq P_\lambda$ and such that $S_n(M)$ is k -defined (this is the case in particular if M is k -defined). Then $\Omega(H, M, k_s)$ is well-defined and contains a k -defined element. Moreover, $P(H, M, k_s)$ is k -defined.
- (ii) If k is perfect, then $\Omega(H)$ is well-defined and contains a k -defined element. Moreover, $P(H)$ is k -defined.

Proof. (i). This follows immediately from Theorem 4.6.

(ii). Since k is perfect, $k_s = \bar{k}$. By (i) it suffices to show that $S_n(M)$ is k -defined where M is as in Proposition 5.10. For any $\gamma \in \Gamma$, $\gamma \cdot M$ is G -cr, since Γ permutes the R-parabolic subgroups and R-Levi subgroups of G , and we have $\gamma \cdot M = \gamma \cdot c_\lambda(H) = c_{\gamma \cdot \lambda}(H)$. It follows from Proposition 5.10 that the subgroups $\gamma \cdot M$ for $\gamma \in \Gamma$ are pairwise G -conjugate, so $S_n(M)$ is Γ -stable and hence is k -defined, since k is perfect. \square

Remarks 5.19. (i). Let M be as in Theorem 5.16 and let M_0 be a G -cr subgroup from the G -conjugacy class associated to H by Proposition 5.10. Then we have $S_n(M_0) \subseteq S_n(M)$ for any n . To prove this we may, by the final assertion in Proposition 5.10, assume that $M = H$. Furthermore, we may assume that $M_0 = c_\lambda(H)$ for some $\lambda \in Y(G)$ with $H \subseteq P_\lambda$. Since $\lambda(a) \cdot H^n \subseteq G \cdot H^n$ for all $a \in k^*$, we have $M_0^n = c_\lambda(H^n) \subseteq S_n(H)$. So $S_n(M_0) \subseteq S_n(H)$.

(ii). Note that $G \cdot M^n$ need not be closed: e.g., take G non-abelian, n to be 1 and $H = M$ to be a maximal torus of G .

Example 5.20. We give an example of the usefulness of this construction (cf. [14] and [1, Thm. 3.10]). Let H be a G -completely reducible subgroup of G and let N be a normal subgroup of H . We prove that N is G -completely reducible. Suppose not. Then $H \subseteq N_G(N) \subseteq P$, where P is the R-parabolic subgroup given in Definition 5.17 for the non- G -completely reducible subgroup N , where $G' = G$. Since N is not contained in an R-Levi subgroup of P , neither is H . But H is assumed to be G -completely reducible, so this is impossible. We deduce that N is G -completely reducible after all.

Here is a second example. It also helps to illustrate the gap in the theory pointed out in Remark 4.11.

Example 5.21. Assume k is perfect and H is a k -defined subgroup of G . Suppose H is not G -completely reducible. Then H is not contained in any R-Levi subgroup of the optimal destabilizing R-parabolic subgroup $P(H)$ of G . Now $P(H) = P_\lambda$ for some $\lambda \in Y_k(G)$, by Theorem 5.18(ii), so H is not G -completely reducible over k . This proves the forward direction of [1, Thm. 5.8]. The proof of the reverse direction given in [1] is essentially just a special case of the proof of Theorem 3.2.

One deduces from the above as in [1, Thm. 5.8] that if k_1/k is a separable algebraic extension of fields and G and H are k -defined, then, under the hypothesis that k is perfect, H is G -completely reducible over k_1 if and only if H is G -completely reducible over k . We proved the forward direction in Theorem 5.11 without the hypothesis that k is perfect. We cannot prove the reverse direction by passing to \bar{k} using the argument in the previous paragraph: for H can be G -completely reducible over k (or k_1) and yet not G -completely reducible over \bar{k} , or vice versa (see Section 1). To give a direct proof, one would like to be able to associate an “optimal destabilizing R-parabolic subgroup” P to H having the property that P is defined over k_1 and no R-Levi k_1 -subgroup of P contains H ; optimality

should imply that P is $\text{Gal}(k_1/k)$ -stable and hence k -defined, which would show that H is not G -completely reducible over k . We cannot take P to be $P(H, c_\mu(H), k_1)$ for any $\mu \in Y_{k_1}(H)$, because if H is G -completely reducible, then $P(H, c_\mu(H), k_1)$ is just G .

Remark 5.22. It seems plausible that the optimal destabilizing R-parabolic subgroup $P(H)$ from Definition 4.5 is independent of the norm on $Y(G)$, but it is not yet clear how the arguments of [11, Sect. 7] can be adjusted to our setting. One could even ask for conditions for independence of the norm in the general setting of Section 4.

5.3. Counterparts for Lie subalgebras. There are counterparts to our results for Lie subalgebras of the Lie algebra $\mathfrak{g} = \text{Lie } G$ of G . All of our results carry over with obvious modifications. For instance, if \mathfrak{h} is not G -completely reducible, then there is an optimal destabilizing parabolic subgroup P of G such that $\mathfrak{h} \subseteq \mathfrak{p}$ but $\mathfrak{h} \not\subseteq \mathfrak{l}$ for any R-Levi subgroup L of P , see Theorem 5.26 below. Many of the proofs are actually easier in the Lie algebra case: for example, it often suffices to work in connected G . We just state the counterparts of Theorems 5.9 and 5.16 in this Lie algebra setting. We leave the details of the proofs to the reader.

For a subgroup H of G we denote its Lie algebra $\text{Lie } H$ by \mathfrak{h} . We start with the analogue of Definition 5.1 in this setting, cf. [15]; see also [5, Sec. 3.3].

Definition 5.23. A subalgebra \mathfrak{h} of \mathfrak{g} is *G -completely reducible* if for any R-parabolic subgroup P of G such that $\mathfrak{h} \subseteq \mathfrak{p}$, there is an R-Levi subgroup L of P such that $\mathfrak{h} \subseteq \mathfrak{l}$.

We require some standard facts concerning Lie algebras of R-parabolic and R-Levi subgroups of G (cf. [20, §2.1]).

Lemma 5.24. For $\lambda \in Y(G)$, put $\mathfrak{p}_\lambda = \text{Lie}(P_\lambda)$ and $\mathfrak{l}_\lambda = \text{Lie}(L_\lambda)$. Let $x \in \mathfrak{g}$. Then

- (i) $x \in \mathfrak{p}_\lambda$ if and only if $\lim_{a \rightarrow 0} \lambda(a) \cdot x$ exists;
- (ii) $x \in \mathfrak{l}_\lambda$ if and only if $\lim_{a \rightarrow 0} \lambda(a) \cdot x$ exists and equals x ;
- (iii) $x \in \text{Lie}(R_u(P_\lambda))$ if and only if $\lim_{a \rightarrow 0} \lambda(a) \cdot x$ exists and equals 0.

The map $c_\lambda : \mathfrak{p}_\lambda \rightarrow \mathfrak{l}_\lambda$, given by $x \mapsto \lim_{a \rightarrow 0} \lambda(a) \cdot x$ coincides with the usual projection of \mathfrak{p}_λ onto \mathfrak{l}_λ . In analogy to the construction for subgroups of G , we consider the action of G on \mathfrak{g}^n by simultaneous adjoint action.

Theorem 5.25.

- (i) Let $n \in \mathbb{N}$, let $\mathfrak{h} \in \mathfrak{g}^n$ and let $\lambda \in Y(G)$ such that $\mathfrak{m} = \lim_{a \rightarrow 0} \lambda(a) \cdot \mathfrak{h}$ exists. Then the following are equivalent:
 - (a) \mathfrak{m} is G -conjugate to \mathfrak{h} ;
 - (b) \mathfrak{m} is $R_u(P_\lambda)$ -conjugate to \mathfrak{h} ;
 - (c) $\dim G \cdot \mathfrak{m} = \dim G \cdot \mathfrak{h}$.
- (ii) Let \mathfrak{h} be a subalgebra of \mathfrak{g} and let $\lambda \in Y(G)$. Suppose $\mathfrak{h} \subseteq \mathfrak{p}_\lambda$ and set $\mathfrak{m} = c_\lambda(\mathfrak{h})$. Then $\dim C_G(\mathfrak{m}) \geq \dim C_G(\mathfrak{h})$ and the following are equivalent:
 - (a) \mathfrak{m} is G -conjugate to \mathfrak{h} ;
 - (b) \mathfrak{m} is $R_u(P_\lambda)$ -conjugate to \mathfrak{h} ;
 - (c) \mathfrak{h} is contained in the Lie algebra of an R-Levi subgroup of P_λ ;
 - (d) $\dim C_G(\mathfrak{m}) = \dim C_G(\mathfrak{h})$.

- (iii) Let \mathfrak{h} , λ and \mathfrak{m} be as in (ii) and let $\mathbf{h} \in \mathfrak{h}^n$ be a generating tuple of \mathfrak{h} . Then the assertions in (i) are equivalent to those in (ii). In particular, \mathfrak{h} is G -completely reducible if and only if $G \cdot \mathbf{h}$ is closed in \mathfrak{g}^n .

Note that the final statement of Theorem 5.25(iii) is [15, Thm. 1(1)].

If \mathfrak{h} is a Lie subalgebra of \mathfrak{g} and $\mathfrak{h} \subseteq \mathfrak{p}_\lambda$ for $\lambda \in Y(G)$, then setting $\mathfrak{m} := c_\lambda(\mathfrak{h})$ and $S_n(\mathfrak{m}) := \overline{G \cdot \mathfrak{m}^n}$, we get an optimal class $\Omega(\mathfrak{h}^n, S_n(\mathfrak{m}))$ of cocharacters, as in Definition 5.15.

Theorem 5.26. *Let G , G' and $\|\cdot\|$ be as in Theorem 5.16. Let \mathfrak{h} be any subalgebra of \mathfrak{g} and let n be minimal such that \mathfrak{h}^n contains a generating tuple of \mathfrak{h} . Let \mathfrak{m} be a subalgebra of \mathfrak{g} and suppose that $\mathfrak{m} = c_\lambda(\mathfrak{h})$ for some $\lambda \in Y_k(G)$ with $\mathfrak{h} \subseteq \mathfrak{p}_\lambda$. Put $\Omega(\mathfrak{h}, \mathfrak{m}, k) := \Omega(\mathfrak{h}^n, S_n(\mathfrak{m}), k)$. Then the following hold:*

- (i) $P_\mu = P_\nu$ for all $\mu, \nu \in \Omega(\mathfrak{h}, \mathfrak{m}, k)$. Let $P(\mathfrak{h}, \mathfrak{m}, k)$ denote the unique R -parabolic subgroup of G so defined. Then $\mathfrak{h} \subseteq \text{Lie}(P(\mathfrak{h}, \mathfrak{m}, k))$ and $R_u(P(\mathfrak{h}, \mathfrak{m}, k))(k)$ acts simply transitively on $\Omega(\mathfrak{h}, \mathfrak{m}, k)$.
- (ii) For $g \in G'(k)$ we have $\Omega(g \cdot \mathfrak{h}, g \cdot \mathfrak{m}, k) = g \cdot \Omega(\mathfrak{h}, \mathfrak{m}, k)$ and $P(g \cdot \mathfrak{h}, g \cdot \mathfrak{m}, k) = gP(\mathfrak{h}, \mathfrak{m}, k)g^{-1}$. If $g \in G(k)$ normalizes \mathfrak{h} and $S_n(\mathfrak{m})$, then $g \in P(\mathfrak{h}, \mathfrak{m}, k)$.
- (iii) If $\mu \in \Omega(\mathfrak{h}, \mathfrak{m}, k)$, then $\dim C_G(c_\mu(\mathfrak{h})) \geq \dim C_G(\mathfrak{m})$. If \mathfrak{m} is G -conjugate to \mathfrak{h} , then, trivially, $\Omega(\mathfrak{h}, \mathfrak{m}, k) = \{0\}$ and $P(\mathfrak{h}, \mathfrak{m}, k) = G$. If \mathfrak{m} is not G -conjugate to \mathfrak{h} , then \mathfrak{h} is not contained in the Lie algebra of any R -Levi subgroup of $P(\mathfrak{h}, \mathfrak{m}, k)$.

Definition 5.27. We call $\Omega(\mathfrak{h}, \mathfrak{m}, k)$ the *optimal class of k -defined cocharacters for \mathfrak{h} and \mathfrak{m}* and we call $P(\mathfrak{h}, \mathfrak{m}, k)$ the *optimal destabilizing R -parabolic subgroup for \mathfrak{h} and \mathfrak{m} over k* .

Assume the G -conjugacy class given by the Lie algebra analogue of Proposition 5.10 contains a subalgebra \mathfrak{m} of the form $c_\lambda(\mathfrak{h})$ for some $\lambda \in Y_k(G)$. Then we set $\Omega(\mathfrak{h}, k) := \Omega(\mathfrak{h}, \mathfrak{m}, k)$ and $P(\mathfrak{h}, k) := P(\mathfrak{h}, \mathfrak{m}, k)$. Under this assumption we have, by the Lie algebra analogue of Proposition 5.10 and Theorem 5.26, that $N_G(\mathfrak{h})$ is contained in $P(\mathfrak{h}, k)$ and that for $\mu \in \Omega(\mathfrak{h}, k)$, $c_\mu(\mathfrak{h})$ is G -completely reducible. So, by Theorem 5.25(ii), if \mathfrak{h} is not G -completely reducible, then \mathfrak{h} is not contained in the Lie algebra of any R -Levi subgroup of $P(\mathfrak{h}, k)$. Note that, trivially, $P(\mathfrak{h}, k) = G$ if \mathfrak{h} is G -completely reducible. We call $\Omega(\mathfrak{h}, k)$ the *optimal class of k -defined cocharacters for \mathfrak{h}* and we call $P(\mathfrak{h}, k)$ the *optimal destabilizing R -parabolic subgroup for \mathfrak{h} over k* .

Note that the above assumption on \mathfrak{m} is satisfied if k is algebraically closed. In that case we usually suppress the k argument and write simply $\Omega(\mathfrak{h})$ and $P(\mathfrak{h})$ instead and refer to these as the *optimal class of cocharacters for \mathfrak{h}* and the *optimal destabilizing R -parabolic subgroup for \mathfrak{h}* , respectively.

Example 5.28. As a further illustration of the power of our construction, using Theorem 5.26, we give a short, alternate proof of [15, Thm. 1(2)] which states that $\mathfrak{h} = \text{Lie}(H)$ is G -completely reducible if H is G -completely reducible.

Let H be a subgroup of G . Assume that \mathfrak{h} is not G -cr. Let $P(\mathfrak{h})$ be the optimal destabilizing R -parabolic subgroup for \mathfrak{h} . By Theorem 5.26(ii), $N_G(\mathfrak{h}) \subseteq P(\mathfrak{h})$. Clearly, $H \subseteq N_G(\mathfrak{h})$. Moreover, $H \subseteq L_\mu$ implies $\mathfrak{h} \subseteq \mathfrak{l}_\mu$ for any $\mu \in Y(G)$. As this cannot happen for $P(\mathfrak{h})$, by Theorem 5.26(iii), H is not G -cr. Thus we can conclude that if H is G -cr, then so is \mathfrak{h} .

5.4. A special case of the Centre Conjecture. In this final section we describe an application of optimal destabilizing parabolic subgroups to the theory of spherical buildings [24]. Suppose from now on that G is connected. Let $X = X(G, k)$ be the spherical Tits

building of G over k ; then X is a simplicial complex whose simplices correspond to the k -defined parabolic subgroups of G . The conjugation action of $G(k)$ on itself naturally induces an action of $G(k)$ on X . We identify X with its geometric realization. A subcomplex Y of X is *convex* if whenever two points of Y are not opposite in X , then Y contains the unique geodesic joining these points, and Y is *contractible* if it has the homotopy type of a point. The following is a version due to Serre of the so-called Centre Conjecture of J. Tits [22, §2.4]. This has been proved by B. Mühlherr and J. Tits for spherical buildings of classical type [16].

Conjecture 5.29. *Let Y be a convex and contractible subcomplex of X . Then there is a point $y \in Y$ such that y is fixed by any automorphism of X that stabilizes Y .*

A point $y \in Y$ whose existence is asserted in Conjecture 5.29 is frequently referred to as a “natural centre” of Y . Our idea is to take the barycentre of the simplex corresponding to the optimal destabilizing parabolic subgroup in an appropriate sense as the centre of Y . This approach is not new; indeed, it was part of the motivation for Kempf’s paper [12] on optimality (cf. [17, §2, p. 64]). We show how to make this work to prove the Centre Conjecture in the case that Y is the *fixed point subcomplex* X^H for some subgroup H of G , where X^H consists of all the simplices in X corresponding to parabolic subgroups containing H .

Theorem 5.30. *Suppose G is semisimple and k is a perfect field. Let H be a subgroup of G and suppose that $Y := X^H$ is contractible. Then there is a point $y \in Y$ which is fixed by any element of $(\text{Aut } G)(k)$ that stabilizes Y .*

Proof. Since we are assuming that G is semisimple and defined over k , $\text{Aut } G$ is an algebraic group also defined over k [24, 5.7.2]. Let K be the intersection of all the k -defined parabolic subgroups of G that contain H . Then K is k -defined because k is perfect, and $X^K = Y$, cf. proof of [3, Thm. 3.1]. Since Y is contractible, K is not G -cr over k , by a result of Serre [22, §3], and hence K is not G -cr, by [1, Thm. 5.8].

Now let M be a representative of the unique G -conjugacy class of G -cr subgroups attached to K given by Proposition 5.10. Let $P = P(K)$ be the optimal destabilizing parabolic subgroup for K (over \bar{k}), Definition 5.17. Then P is a parabolic subgroup of G containing K , by Theorem 5.16(i), and P is defined over k , by Theorem 5.18(ii), so P corresponds to a simplex of Y . Moreover, any element of $(\text{Aut } G)(k)$ that stabilizes Y also normalizes K , and hence stabilizes the G -conjugacy class of M , by Proposition 5.10(ii). So any such automorphism normalizes P , by Theorem 5.16(ii), with $G' = \text{Aut } G$. We can therefore take y to be the barycentre of the simplex corresponding to P . \square

Remark 5.31. The assumption that G is semisimple in Theorem 5.30 allows us to apply our optimality results because it ensures that $\text{Aut } G$ is an algebraic group. In the context of buildings, however, this assumption is no loss: given any connected reductive G and any subgroup H of G , the building of G is isomorphic to the building of the semisimple group $G/Z(G)$, and the subgroup H is G -cr if and only if the image of H in $G/Z(G)$ is $G/Z(G)$ -cr. Moreover, all automorphisms of $X(G)$ which come from $\text{Aut } G$ survive this transition from G to $G/Z(G)$.

To establish that the Centre Conjecture holds for subcomplexes of the form $Y = X^H$, we need to find a centre $y \in Y$ which is fixed by all building automorphisms of X that stabilize

X^H , not just the building automorphisms that arise from algebraic automorphisms of G . For most G , however, $\text{Aut } X$ is generated by $\text{Aut } G$ together with field automorphisms: see [24, Cor. 5.9] for more details. We finish by showing how to deal with field automorphisms in some cases.

Recall that Γ denotes the group $\text{Gal}(k_s/k)$. Following [24, 5.7.1], any $\gamma \in \Gamma$ induces an automorphism of the building $X = X(G, \bar{k})$, which we also denote by γ . Recall that Γ also acts on the set of cocharacters $Y(G)$ and we can ensure that the norm is invariant under this action (i.e., the norm is k -defined, see also [12, Sec. 4]).

Theorem 5.32. *Suppose G is connected. Let $X = X(G, \bar{k})$ be the building of G over the algebraic closure of k . Let H be a subgroup of G and suppose that $Y := X^H$ is contractible. Let Γ_Y denote the subgroup of Γ which stabilizes Y . Then there is a point $y \in Y$ which is fixed by any element of Γ_Y .*

Proof. As in the proof of Theorem 5.30, let K be the intersection of the parabolic subgroups corresponding to simplices in Y . Then $Y = X^K$, and since Y is stabilized by all $\gamma \in \Gamma_Y$, we have $\gamma \cdot K = K$ for all $\gamma \in \Gamma_Y$. Let λ and $M = c_\lambda(K)$ be as in Proposition 5.10. Then $\gamma \cdot M = \gamma \cdot c_\lambda(K) = c_{\gamma \cdot \lambda}(K)$ is also G -cr for all $\gamma \in \Gamma_Y$. Since M is unique up to G -conjugacy, by Proposition 5.10, we have that the $\gamma \cdot M$ with $\gamma \in \Gamma_Y$ are all G -conjugate. Because the norm is Γ -invariant, if we let $P = P(K)$ be the optimal parabolic subgroup for K , then P is stabilized by Γ_Y . We can therefore take y to be the barycentre of the simplex corresponding to P . \square

Remarks 5.33. (i). Combining Theorem 5.30 and Theorem 5.32 goes a long way towards proving the full version of Tits' Centre Conjecture for subcomplexes of the form X^H in many cases. For example, if G is a split simple group of adjoint type defined over a finite field $k = \mathbb{F}_p$ then, with a few exceptions, the automorphism group of $X(G, \bar{k})$ is a split extension of $\text{Aut } G$ by the automorphism group of the field \bar{k} (see [24, Cor. 5.10]), and the results above show how to deal with many of these automorphisms.

(ii). Theorems 5.30 and 5.32 improve on [3, Thm. 3.1].

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