COMPLETE REDUCIBILITY AND CONJUGACY CLASSES OF TUPLES IN ALGEBRAIC GROUPS AND LIE ALGEBRAS

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Abstract. Let H be a reductive subgroup of a reductive group G over an algebraically closed field k. We consider the action of H on G^n, the n-fold Cartesian product of G with itself, by simultaneous conjugation. We give a purely algebraic characterization of the closed H-orbits in G^n, generalizing work of Richardson which treats the case $H = G$.

This characterization turns out to be a natural generalization of Serre's notion of G-complete reducibility. This concept appears to be new, even in characteristic zero. We discuss how to extend some key results on G-complete reducibility in this framework. We also consider some rationality questions.

Contents

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1. Introduction

Let G be a reductive linear algebraic group over an algebraically closed field, and suppose G acts on an affine variety V. A fundamental problem in geometric invariant theory is to determine the closed orbits of G in V. These orbits correspond to the points in the quotient variety $V/\!\!/G$, so this is the first step towards understanding the geometry of the quotient. Often it is of particular interest to find the open subset of stable orbits, which consists of points on which the quotient map $\pi\colon V \to V/\!\!/G$ is especially well-behaved. Moreover, once the closed orbits are known, one can study degeneration phenomena: the way in which a point in a non-closed orbit can be brought inside a closed orbit by taking a limit along a cocharacter.

An important family of examples arises as follows. Take G to be a subgroup of $\mathrm{GL}(W)$ for some finite-dimensional vector space W, choose a subvariety C of $\mathrm{End}(W)$ that is stable under conjugation by G, and take V to be C^n for some $n \in \mathbb{N}$, where G acts on C^n by simultaneous conjugation. Typically C carries some algebraic structure: it might be a subgroup of $\mathrm{GL}(W)$, a Lie subalgebra, or an associative subalgebra of $\mathrm{End}(W)$. For instance, if $G = \mathrm{GL}(W)$ and $C = \mathrm{End}(W)$, then, by work of H. Kraft ([10, Prop. 4.4] or [11, II.2.7 Satz 2]), for any $v = (c_1, \dots, c_n) \in C^n = V$, the orbit $G \cdot v$ is closed if and only if W is semisimple as an A-module, where A is the associative subalgebra of $\mathrm{End}(W)$ generated by $c_1, \dots, c_n$; moreover, if $G \cdot v$ is not closed, then the degeneration process referred to above is the "semisimplification", in which one replaces the A-module W with the direct sum of its composition factors.

Now consider the case when $C = G$. In his seminal work [18, Thm. 16.4], Richardson gave an algebraic characterization of the closed G-orbits in G^n. In [2, Thm. 3.1] it was shown that his criterion for an orbit to be closed can be formulated using the representation-theoretic notion of G-complete reducibility, due to Serre, [19]. This concept has been much studied and it has proved a useful tool for exploring the subgroup structure of simple algebraic groups. The approach to G-complete reducibility via geometric invariant theory has proved very fruitful, cf. [2], [3], [4], and [5].

It is natural to extend Richardson's study and determine the closed H-orbits in G^n for an arbitrary closed reductive subgroup H of G. In this paper we show that there is also an algebraic interpretation of the closed orbit condition in this case. We introduce the notion of a *relatively G-completely reducible subgroup of G with respect to H* (Subsection 3.1) and prove the following result.

Theorem 1.1. *Let H be a reductive subgroup of G. Let K be the algebraic subgroup of G generated by elements $x_1, \dots, x_n \in G$. Then $H \cdot (x_1, \dots, x_n)$ is closed in G^n if and only if K is relatively G-completely reducible with respect to H.*

This generalizes Richardson's result [18, Thm. 16.4] which is the special case of Theorem 1.1 when $H = G$.

Note that we can embed G in some $\mathrm{GL}(W)$, so this fits into the general setting discussed above (take $C = G$ and replace G with H). In fact, we can take G to be equal to $\mathrm{GL}(W)$ if we wish (cf. Corollary 3.6).

The definition of G-complete reducibility involves cocharacters of G. Theorems about G-complete reducibility often involve taking not arbitrary cocharacters of G but cocharacters of a proper reductive subgroup H of G [4, Prop. 5.7]. The notion of relative complete

reducibility gives a systematic way to formalise such arguments: hence our results are of interest even if one is concerned mainly with G-complete reducibility.

Armed with Theorem 1.1, we explore some basic properties of relative G-complete reducibility in Section 3. We study the extent to which results about G-complete reducibility extend to relative G-complete reducibility, concentrating on what happens when one varies H or other parameters in the definition. Returning briefly to the more general setting described above, we extend the notion of relative G-complete reducibility to certain Lie algebras and associative algebras associated to G: for instance, we consider the case when C is the associative subalgebra of $\mathrm{End}(W)$ spanned by G. We also characterize the H-stable orbits in G^n in terms of this notion of relative G-complete reducibility (Proposition 3.16); this generalizes Richardson's result for the special case $H = G$, [18, Prop. 16.7].

In Section 4, we define relative G-complete reducibility over an arbitrary field. We answer a generalization of a question due to Serre about the behaviour of G-complete reducibility under separable field extensions (Theorem 4.12). We finish the paper with a section containing a collection of examples. In particular, we study the case of relative $\mathrm{GL}(W)$-complete reducibility and give some characterizations in terms of the natural module W.

2. Notation and preliminaries

2.1. **Basic notation.** Let k be an algebraically closed field, and let H be a linear algebraic group defined over k. We denote by $\overline{\langle S \rangle}$ the algebraic subgroup of H generated by a subset S of H. We let $Z(H)$ denote the centre of H and H^0 the connected component of H that contains 1. If K is a subgroup of H, then $C_H(K)$ is the centralizer of K in H and $N_H(K)$ is the normalizer of K in H. We say that H is *linearly reductive* if every rational representation of H is semisimple.

For the set of cocharacters (one-parameter subgroups) of H we write $Y(H)$; the elements of $Y(H)$ are the homomorphisms from k^* to H. There is an action of H on $Y(H)$ given by $(h \cdot \lambda)(a) = h\lambda(a)h^{-1}$ for $\lambda \in Y(H)$, $h \in H$ and $a \in k^*$.

The *unipotent radical* of H is denoted $R_u(H)$; it is the maximal connected normal unipotent subgroup of H. The algebraic group H is called *reductive* if $R_u(H) = \{1\}$; note that we do not insist that a reductive group is connected.

Throughout the paper, G denotes a reductive algebraic group, possibly non-connected. We use the notation $\mathfrak{g}$ for $\mathrm{Lie}\, G$; likewise for closed subgroups of G.

Frequently, we consider the diagonal action of G on G^n, the n-fold cartesian product of G with itself, by simultaneous conjugation:

$$g \cdot (x_1, \ldots, x_n) := (gx_1g^{-1}, \ldots, gx_ng^{-1}),$$

for all $g \in G$ and $(x_1, \ldots, x_n) \in G^n$. Note that any subgroup H of G acts on G^n in the same way. We also consider the action of G on $\mathfrak{g}^n$ by diagonal simultaneous adjoint action.

If H is a closed subgroup of G and $\mathbf{x} = (x_1, \ldots, x_n) \in G$ for some $n \in \mathbb{N}$, then we say H is *topologically generated by* $\mathbf{x}$ (or *by* $x_1, \ldots, x_n$) if $H = \overline{\langle x_1, \ldots, x_n \rangle}$.

2.2. **Non-connected reductive groups.** Since we want to work with reductive groups which are not necessarily connected, we need to extend several familiar ideas from connected reductive groups. The crucial ingredient of this extension is the introduction of so-called *Richardson parabolic subgroups* (*R-parabolic subgroups*) of the reductive group G. We briefly

recall the main definitions and results; for more details and further results, the reader is referred to [2, Sec. 6].

Definition 2.1. For each cocharacter $\lambda \in Y(G)$, let $P_\lambda = \{g \in G \mid \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1} \text{ exists}\}$ (for the formal definition of such limits, see Definition 2.2). Recall that a subgroup P of G is *parabolic* if G/P is a complete variety. The subgroup P_λ is parabolic in this sense, but the converse is not true: e.g., if G is finite, then every subgroup is parabolic, but the only subgroup of G of the form P_λ is G itself. We define $L_\lambda = \{g \in G \mid \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1} = g\}$. Then L_λ is a reductive subgroup of G and we have $L_\lambda = C_G(\lambda(k^*))$ and $P_\lambda = L_\lambda \ltimes R_u(P_\lambda)$. We also have $R_u(P_\lambda) = \{g \in G \mid \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1} = 1\}$. The map

$$c_\lambda : P_\lambda \to L_\lambda \quad \text{given by} \quad c_\lambda(g) = \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1}$$

is a surjective homomorphism of algebraic groups with kernel $R_u(P_\lambda)$; it coincides with the usual projection $P_\lambda \to L_\lambda$. The subgroups P_λ for $\lambda \in Y(G)$ are called the *R-parabolic subgroups* of G. Given an R-parabolic subgroup P, an *R-Levi subgroup* of P is any subgroup L_λ such that $\lambda \in Y(G)$ and $P = P_\lambda$. Note that if P, Q are R-parabolic subgroups of G with $P^0 = Q^0$ then $R_u(P) = R_u(Q)$. If G is connected, then the R-parabolic subgroups (resp. R-Levi subgroups of R-parabolic subgroups) of G are exactly the parabolic subgroups (resp. Levi subgroups of parabolic subgroups) of G; indeed, most of the theory of parabolic subgroups and Levi subgroups of connected reductive groups carries over to R-parabolic and R-Levi subgroups of arbitrary reductive groups. In particular, $R_u(P)$ acts simply transitively on the set of all R-Levi subgroups of an R-parabolic subgroup P. Also note that $P_\lambda = G$ if and only if λ is central in G [2, Lem. 2.4]. When it does not cause any confusion, we speak of "R-Levi subgroups of G" when we mean "R-Levi subgroups of R-parabolic subgroups of G".

In this paper, we are interested in reductive subgroups of reductive groups. If H is a subgroup of G, then there is an obvious inclusion $Y(H) \subseteq Y(G)$ of the sets of cocharacters. When H is reductive and $\lambda \in Y(H)$, there is then an R-parabolic subgroup of H associated to λ, as well as an R-parabolic subgroup of G. In order to distinguish between R-parabolic subgroups associated to different subgroups of G, we use the notation $P_\lambda(H)$, $L_\lambda(H)$, etc. where necessary, but we write P_λ for $P_\lambda(G)$ and L_λ for $L_\lambda(G)$. Note that $P_\lambda(H) = P_\lambda \cap H$, $L_\lambda(H) = L_\lambda \cap H$ and $R_u(P_\lambda(H)) = R_u(P_\lambda) \cap H$.

2.3. **Groups acting on varieties.** We recall some general results from geometric invariant theory required in the sequel, see [5], [1, §2], [15], and [16, Ch. 3]. Let V be an affine variety.

Definition 2.2. Let $\phi : k^* \to V$ be a morphism of algebraic varieties. We say that $\lim_{t\to 0} \phi(t)$ exists if there exists a morphism $\widehat{\phi} : k \to V$ (necessarily unique) whose restriction to k^* is ϕ; if this limit exists, then we set $\lim_{t\to 0} \phi(t) = \widehat{\phi}(0)$.

Now suppose the reductive group G acts on V. For $v \in V$ let $G \cdot v$ denote the G-orbit of v in V and $C_G(v)$ the stabilizer of v in G. It follows easily from Definition 2.2 that if $\lim_{t\to 0} \lambda(t) \cdot v$ exists for a cocharacter $\lambda \in Y(G)$, then this limit belongs to the closure $\overline{G \cdot v}$ of $G \cdot v$ in V. The well-known Hilbert-Mumford Theorem [9, Thm. 1.4], gives a converse to

this: if $v \in V$ is such that $G \cdot v$ is not closed in V, then there exists a cocharacter $\lambda \in Y(G)$ such that $\lim_{t\to 0} \lambda(t) \cdot v$ exists and lies in the unique closed G-orbit in $\overline{G \cdot v}$.

We often use the following simple lemma in the sequel, [5, Lem. 2.11].

Lemma 2.3. *Suppose G acts on an affine variety V. Let $v \in V$, let $\lambda \in Y(G)$ and let $u \in R_u(P_\lambda)$. Then $\lim_{a\to 0} \lambda(a) \cdot v$ exists and equals $u \cdot v$ if and only if $u^{-1} \cdot \lambda$ centralizes v.*

The next result is [5, Thm. 3.4] in case $k = \overline{k}$.

Theorem 2.4. *Suppose G acts on an affine variety V. Let $v \in V$ and let $\lambda \in Y(G)$ such that $v' := \lim_{a\to 0} \lambda(a) \cdot v$ exists and is G-conjugate to v. Then v' is $R_u(P_\lambda)$-conjugate to v.*

2.4. **Generic tuples.** In order to establish the link between relative G-complete reducibility with respect to H and H-orbits of tuples, needed for Theorem 1.1, we require the following notion of a generic tuple, see [5, Def. 5.4].

Definition 2.5. Let K be a closed subgroup of G and let $G \hookrightarrow \mathrm{GL}_m$ be an embedding of algebraic groups. Then $\mathbf{k} = (k_1, \ldots, k_n) \in K^n$ is called a *generic tuple of K for the embedding $G \hookrightarrow \mathrm{GL}_m$* if the k_i generate the associative subalgebra of Mat_m spanned by K. (Frequently we simply say that $\mathbf{k}$ *generates* the associative subalgebra of Mat_m spanned by K.) We call $\mathbf{k} \in K^n$ a *generic tuple of K* if it is a generic tuple of K for some embedding $G \hookrightarrow \mathrm{GL}_m$.

Clearly, generic tuples exist for any embedding $G \hookrightarrow \mathrm{GL}_m$ for n sufficiently large. The main properties of generic tuples are given by the next lemma, which is [5, Lem. 5.5].

Lemma 2.6. *Let K be a closed subgroup of G, let $\mathbf{k} \in K^n$ be a generic tuple of K for some embedding $G \hookrightarrow \mathrm{GL}_m$ and let K' be the closed subgroup of G generated by (the components of) $\mathbf{k}$. Then we have:*

(i) *$C_H(\mathbf{k}) = C_H(K') = C_H(K)$ for any subgroup H of G;*
(ii) *K' is contained in the same R-parabolic and the same R-Levi subgroups of G as K;*
(iii) *If $K \subseteq P_\lambda$ for some $\lambda \in Y(G)$, then $c_\lambda(\mathbf{k})$ is a generic tuple of $c_\lambda(K)$ for the given embedding $G \hookrightarrow \mathrm{GL}_m$.*

Note that in Lemma 2.6(iii), $c_\lambda(\mathbf{k})$ is the tuple obtained by applying c_λ to each entry in $\mathbf{k}$.

Remark 2.7. If K is a closed subgroup of G which is topologically generated by a tuple $\mathbf{k} = (k_1, ..., k_n) \in K^n$, then $\mathbf{k}$ is a generic tuple for K in the sense of Definition 2.5. To see this, consider an embedding $G \hookrightarrow \mathrm{GL}_m$. Since the minimal polynomial of each k_i has non-zero constant term, we can express k_i^{-1} as a polynomial in k_i. Hence, if A is the associative subalgebra of Mat_m generated by $\mathbf{k}$, then A contains the inverses of each of the k_i, so it contains the subgroup of GL_m generated by $\mathbf{k}$. But A is closed, so it contains K.

3. Relative G-complete reducibility

3.1. **Relative G-complete reducibility.** The key idea for the proof of Theorem 1.1, which is proved in the next subsection, is the notion of relative G-complete reducibility, defined below.

Definition 3.1. Let K be a subgroup of G and let H be any reductive subgroup of G. We say that K is *relatively G-completely reducible with respect to H* if for every $\lambda \in Y(H)$ such that K is contained in P_λ, there exists $\mu \in Y(H)$ such that $P_\lambda = P_\mu$ and $K \subseteq L_\mu$. We sometimes use the shorthand *relatively G-cr with respect to H*.

Remarks 3.2. (i). In the case $H = G$, Definition 3.1 coincides with the usual definition of G-complete reducibility [2, Sec. 1], [19, 3.2]; we sometimes refer to this as the *absolute case*. Note that a subgroup of G is relatively G-completely reducible with respect to H if and only if it is relatively G-completely reducible with respect to H^0. Thus we may assume without loss that H is connected. If H^0 is central in G, then every subgroup of G is relatively G-completely reducible with respect to H.

(ii). If $K \subseteq H$, then K is relatively G-cr with respect to H if and only if K is H-cr.

(iii). Recall that in characteristic zero a subgroup of G is G-completely reducible if and only if it is reductive. We don't know of any simple characterization of relative G-complete reducibility in this case. So this appears to be a new notion even in characteristic zero.

(iv). We note that one of the basic properties [19, Prop. 4.1] of G-cr subgroups is not inherited by relatively G-cr subgroups in general: if we take $H \subseteq Z(G)$, then all subgroups of G are relatively G-cr with respect to H. In particular, it is possible for non-reductive (even unipotent) subgroups to be relatively G-cr with respect to a subgroup H.

(v). As noted in (iv), in general a relatively G-cr subgroup of G need not be G-cr. Also a G-cr subgroup need not be relatively G-cr. For instance, let L be an R-Levi subgroup of some R-parabolic subgroup P of G. Then L is G-cr by [19, Prop. 3.2], [2, §6.3]. Let M be any other R-Levi subgroup of P. Then for any maximal torus T of G that lies in M we have that L is not relatively G-cr with respect to T. For there is a $\lambda \in Y(T)$ with $P = P_\lambda$ and for any such λ, we have $L_\lambda = M$ by [2, Cor. 6.5]. Hence $L \not\subseteq L_\lambda$. Here is another example: there exists reductive G with a reductive subgroup H and a subgroup K of H such that K is G-cr but not H-cr [4, Prop. 7.7]; then K is not relatively G-cr with respect to H by (ii) above.

For examples of relatively G-cr subgroups, see Section 5 where we specifically study the case when $G = \mathrm{GL}(V)$.

The following lemma gives some detailed information about conjugacy of R-Levi subgroups in the subgroups P_λ and $P_\lambda(H)$ for $\lambda \in Y(H)$.

Lemma 3.3. *Let H be a reductive subgroup of G.*

(i) *Let $\lambda, \mu \in Y(H)$ such that $P_\lambda = P_\mu$ and let u be the element of $R_u(P_\lambda(H))$ such that $uL_\lambda(H)u^{-1} = L_\mu(H)$. Then $uL_\lambda u^{-1} = L_\mu$.*

(ii) *Let K be a subgroup of G. The K is relatively G-completely reducible with respect to H if and only if for every $\lambda \in Y(H)$ such that $K \subseteq P_\lambda$ there exists $u \in R_u(P_\lambda(H))$ such that $K \subseteq L_{u\cdot\lambda}$.*

Proof. (i). Set $\mu' = u \cdot \lambda$; then $\mu' \in Y(H)$, since $u \in R_u(P_\lambda(H))$. We have $L_{\mu'}(H) = uL_\lambda u^{-1} = L_\mu(H)$, so $\mu, \mu' \in Y(Z(L_\mu(H))^0)$. Let T be a maximal torus of L_μ containing $Z(L_\mu(H))^0$. Then since $\mu' \in Y(T)$, we have $T \subseteq L_{\mu'}$, and $L_{\mu'}$ is an R-Levi subgroup of P_λ. Thus L_μ and $L_{\mu'}$ are R-Levi subgroups of P_λ containing a common maximal torus, so they are equal, by [2, Cor. 6.5].

(ii). This is clear from part (i) and Definition 3.1. □

Remark 3.4. Let $H \leq G$ be reductive. Let K be a closed subgroup of G, let $\mathbf{k} \in K^n$ be a generic tuple of K and let K' be the closed subgroup of G generated by $\mathbf{k}$. Then it follows from Lemma 2.6(ii) that K is relatively G-cr with respect to H if and only if K' is.

3.2. **Relative G-complete reducibility and closed orbits.** Throughout the rest of this section, H denotes a reductive subgroup of G. The following result is [5, Thm. 5.9] generalized to our relative setting. In view of Remark 2.7, the final assertion in part (iii) gives Theorem 1.1. The proof of [5, Thm. 5.9] goes through here with the obvious modifications and alterations of notation. For the convenience of the reader, we include a proof.

Theorem 3.5. *Let H be a connected reductive subgroup of G.*

(i) *Let $n \in \mathbb{N}$, let $\mathbf{k} \in G^n$ and let $\lambda \in Y(H)$ such that $\mathbf{m} := \lim_{a\to 0} \lambda(a) \cdot \mathbf{k}$ exists. Then the following are equivalent:*
   (a) *$\mathbf{m}$ is H-conjugate to $\mathbf{k}$;*
   (b) *$\mathbf{m}$ is $R_u(P_\lambda(H))$-conjugate to $\mathbf{k}$;*
   (c) $\dim H \cdot \mathbf{m} = \dim H \cdot \mathbf{k}$.

(ii) *Let K be a subgroup of G and let $\lambda \in Y(H)$. Suppose $K \subseteq P_\lambda$ and set $M = c_\lambda(K)$. Then $\dim C_H(M) \geq \dim C_H(K)$ and the following are equivalent:*
   (a) *M is H-conjugate to K;*
   (b) *M is $R_u(P_\lambda(H))$-conjugate to K;*
   (c) *$K \subseteq L_\mu$ for some $\mu \in Y(H)$ such that $P_\lambda = P_\mu$;*
   (d) $\dim C_H(M) = \dim C_H(K)$.

(iii) *Let K, λ and M be as in (ii) and let $\mathbf{k} \in K^n$ be a generic tuple of K. Then the assertions in (i) are equivalent to those in (ii). In particular, K is relatively G-completely reducible with respect to H if and only if $H \cdot \mathbf{k}$ is closed in G^n.*

Proof. (i). This follows immediately from Theorem 2.4 and [6, Prop. I.1.8].

(ii) and (iii). Let $\mathbf{k} \in K^n$, let K' be the closed subgroup of G generated by $\mathbf{k}$ and let $\lambda \in Y(H)$. Then $\lim_{a\to 0} \lambda(a) \cdot \mathbf{k}$ exists if and only if $K' \subseteq P_\lambda$. Now assume that $\mathbf{m} = \lim_{a\to 0} \lambda(a) \cdot \mathbf{k}$ exists. Let $u \in R_u(P_\lambda(H))$. Then $\mathbf{k} = u \cdot \mathbf{m}$ if and only if $u \cdot \lambda$ fixes $\mathbf{k}$ (Lemma 2.3) if and only if $K' \subseteq L_{u\cdot\lambda} = uL_\lambda u^{-1}$. Since $R_u(P_\lambda(H))$ acts simply transitively on the R-Levi subgroups of P_λ of the form L_μ with $\mu \in Y(H)$ (Lemma 3.3), the first assertion of (iii) follows once we have proved (ii). For this purpose we pick a generic tuple $\mathbf{k} \in K^n$ of K. Then $\mathbf{m} = c_\lambda(\mathbf{k})$ is a generic tuple of M, by Lemma 2.6(iii). Now the first assertion of (ii) follows from Lemma 2.6(i) and the fact that $\dim H \cdot \mathbf{m} \leq \dim H \cdot \mathbf{k}$ (see [6, Prop. I.1.8]), since $\dim H \cdot \mathbf{k} = \dim H - \dim C_H(\mathbf{k})$ and likewise for $\mathbf{m}$. Now we prove the equivalences in (ii). Clearly, (b) implies (a) and (a) implies (d). Furthermore, we have for $u \in R_u(P_\lambda(H))$ that $K \subseteq L_{u\cdot\lambda}$ if and only if $K = c_{u\cdot\lambda}(K) = uMu^{-1}$. So (b) is equivalent to (c). Now assume that (d) holds. Then, thanks to Lemma 2.6(i) again, we have $\dim H \cdot \mathbf{m} = \dim H \cdot \mathbf{k}$. So $\mathbf{m}$ is $R_u(P_\lambda(H))$-conjugate to $\mathbf{k}$ by (i). We have seen above that this means that K' is contained in an R-Levi subgroup of P_λ. Since $\mathbf{k}$ is generic the same must then hold for K, by Lemma 2.6(ii), that is, (c) holds. The final assertion of (iii) follows from the first and the Hilbert-Mumford Theorem. □

Corollary 3.6. *Let M be a reductive subgroup of G. Let K and H be subgroups of M and assume that H is reductive. Then K is relatively G-completely reducible with respect to H if and only if it is relatively M-completely reducible with respect to H.*

Proof. Since a subset of M^n is closed if and only if it is closed in G^n, the result follows immediately from Theorem 3.5(iii). □

We have an analogue of part of [2, Prop. 3.12]:

Corollary 3.7. *Let H be a reductive subgroup of G, and let K be a subgroup of G which is relatively G-completely reducible with respect to H. Then $C_H(K)$ is reductive.*

Proof. Let $\mathbf{k}$ be a generic tuple for K. Since K is relatively G-cr with respect to H, the orbit $H \cdot \mathbf{k}$ is closed in G^n, by Theorem 3.5(iii), and therefore affine. Hence $C_H(\mathbf{k}) = C_H(K)$ is reductive, by [17, Lem. 10.1.3]. □

3.3. **Relative complete reducibility for Lie subalgebras of $\mathfrak{g}$.** It is straightforward to extend our definitions and results to Lie subalgebras of $\mathfrak{g}$. We first record a standard result which gives some properties of the Lie algebras of R-parabolic and R-Levi subgroups of G (cf. [18, §2.1]).

Lemma 3.8. *For $\lambda \in Y(G)$, put $\mathfrak{p}_\lambda = \mathrm{Lie}(P_\lambda)$ and $\mathfrak{l}_\lambda = \mathrm{Lie}(L_\lambda)$. Let $x \in \mathfrak{g}$. Then*

(i) *$x \in \mathfrak{p}_\lambda$ if and only if $\lim_{a\to 0} \lambda(a) \cdot x$ exists;*

(ii) *$x \in \mathfrak{l}_\lambda$ if and only if $\lim_{a\to 0} \lambda(a) \cdot x$ exists and equals x if and only if $\lambda(k^*)$ centralizes x;*

(iii) *$x \in \mathrm{Lie}(R_u(P_\lambda))$ if and only if $\lim_{a\to 0} \lambda(a) \cdot x$ exists and equals 0.*

Definition 3.9. For $\lambda \in Y(G)$ define the subalgebras $\mathfrak{p}_\lambda$ and $\mathfrak{l}_\lambda$ of $\mathfrak{g}$ as in Lemma 3.8. Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ and let H be any reductive subgroup of G. We call $\mathfrak{k}$ *relatively G-completely reducible with respect to H* if for every $\lambda \in Y(H)$ such that $\mathfrak{k} \subseteq \mathfrak{p}_\lambda$, there exists $\mu \in Y(H)$ such that $P_\lambda = P_\mu$ and $\mathfrak{k} \subseteq \mathfrak{l}_\mu$. In case $H = G$, we say $\mathfrak{k}$ is *G-completely reducible*.

The following can be shown with the same arguments as Theorem 3.5. We emphasize that Theorem 3.10(iii) characterizes the closed H-orbits in $\mathfrak{g}^n$.

Theorem 3.10. *Let $H \leq G$ be reductive.*

(i) *Let $n \in \mathbb{N}$, let $\mathbf{k} \in \mathfrak{g}^n$ and let $\lambda \in Y(H)$ such that $\mathbf{m} := \lim_{a\to 0} \lambda(a) \cdot \mathbf{k}$ exists. Then the following are equivalent:*
   (a) *$\mathbf{m}$ is H-conjugate to $\mathbf{k}$;*
   (b) *$\mathbf{m}$ is $R_u(P_\lambda(H))$-conjugate to $\mathbf{k}$;*
   (c) $\dim H \cdot \mathbf{m} = \dim H \cdot \mathbf{k}$.

(ii) *Let $\mathfrak{k}$ be a Lie subalgebra of $\mathfrak{g}$ and let $\lambda \in Y(H)$. Suppose $\mathfrak{k} \subseteq \mathfrak{p}_\lambda$ and set $\mathfrak{m} = c_\lambda(\mathfrak{k})$. Then $\dim C_H(\mathfrak{m}) \geq \dim C_H(\mathfrak{k})$ and the following are equivalent:*
   (a) *$\mathfrak{m}$ is H-conjugate to $\mathfrak{k}$;*
   (b) *$\mathfrak{m}$ is $R_u(P_\lambda(H))$-conjugate to $\mathfrak{k}$;*
   (c) *$\mathfrak{k} \subseteq \mathfrak{l}_\mu$ for some $\mu \in Y(H)$ such that $P_\lambda = P_\mu$;*
   (d) $\dim C_H(\mathfrak{m}) = \dim C_H(\mathfrak{k})$.

(iii) *Let $\mathfrak{k}$, λ and $\mathfrak{m}$ be as in (ii), and suppose $\mathbf{k} \in \mathfrak{k}^n$ for some n is a generating tuple for $\mathfrak{k}$ (as a Lie algebra). Put $\mathbf{m} = c_\lambda(\mathbf{k})$. Then the assertions in (i) are equivalent to those in (ii). In particular, $\mathfrak{k}$ is relatively G-completely reducible with respect to H if and only if $H \cdot \mathbf{k}$ is closed in $\mathfrak{g}^n$.*

Remark 3.11. For $H = G$, Definition 3.9 is due to G. McNinch, [14]. The final statement of Theorem 3.10(iii) generalizes [14, Thm. 1(1)] which is the case $H = G$. Note also that Theorem 3.10 generalizes [5, Thm. 5.25] which is the special case $H = G$.

3.4. **Relative complete reducibility for associative subalgebras of** $\mathrm{End}(V)$**.** Let $G = \mathrm{GL}(V)$ and let H be any reductive subgroup of G. Using the characterization of parabolic and Levi subgroups of G in terms of flags in V, we see that $\mathfrak{p}_\lambda$ and $\mathfrak{l}_\lambda$, for $\lambda \in Y(H)$, are associative subalgebras of $\mathrm{End}(V)$. This means that one can define the notion of *relative G-complete reducibility with respect to H* for associative subalgebras of $\mathrm{End}(V)$ in the obvious way. Observe that in the absolute case, i.e., when $H = \mathrm{GL}(V)$, we obtain that an associative subalgebra A of $\mathrm{End}(V)$ is G-completely reducible precisely when V is a semisimple A-module.

Obviously, a closed subgroup of G or a subalgebra of $\mathfrak{g} = \mathfrak{gl}(V)$ is relatively G-cr with respect to H if and only if this holds for the associative subalgebra of $\mathrm{End}(V)$ that it generates. If K is a closed subgroup of G which is topologically generated by a tuple $\mathbf{k} \in K^n$, then $\mathbf{k}$ generates the associative subalgebra of $\mathrm{End}(V)$ generated by K. Similarly, if $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$ which is generated by a tuple $\mathbf{k} \in \mathfrak{k}^n$, then $\mathbf{k}$ generates the associative subalgebra of $\mathrm{End}(V)$ generated by $\mathfrak{k}$ (cf. Remark 2.7).

If a tuple $\mathbf{a} \in \mathrm{End}(V)^n$ generates the associative subalgebra A of $\mathrm{End}(V)$, then the analogue of Theorem 3.5 holds; in particular, A is relatively G-cr with respect to H if and only if $H \cdot \mathbf{a}$ is closed. Therefore, in the absolute case $H = \mathrm{GL}(V)$, the final statement of the analogue of Theorem 3.5(iii) in this setting recovers a fundamental result in the representation theory of associative algebras due to H. Kraft: V is a semisimple A-module if and only if the G-orbit $G \cdot \mathbf{a}$ is closed in $\mathrm{End}(V)^n$, see [10, Prop. 4.4] or [11, II.2.7 Satz 2]. In that sense, this concept of relative $\mathrm{GL}(V)$-complete reducible associative subalgebras of $\mathrm{End}(V)$ with respect to H generalizes this fundamental work of Kraft. We hope to investigate this concept in a future study.

3.5. H**-stable points in** G^n **and relative** G**-irreducibility with respect to** H**.** Recall the notion of a *stable point* for the action of a reductive group G on a variety V [17, 1.4]:

Definition 3.12. Let $Z = \bigcap_{v \in V} C_G(v)$ be the kernel of the action of G on V. We say that $v \in V$ is a *stable point* for the action of G or a *G-stable point* provided the orbit $G \cdot v$ is closed in V and $C_G(v)/Z$ is finite.

Remark 3.13. Let $V/\!\!/G$ be the variety corresponding to the k-algebra $k[V]^G$ and let $\pi : V \to V/\!\!/G$ be the morphism corresponding to the inclusion $k[V]^G \to k[V]$. In general, π is not a quotient morphism in the sense of [6, §6]. Let V^s denote the set of G-stable points in V. Then V^s is a (possibly empty) G-stable open subset of V and $V^s = \pi^{-1}(\pi(V^s))$. Further, $\pi(V^s)$ is an open subset in $V/\!\!/G$ and is a geometric quotient of G. For $v \in V^s$, we have $G \cdot v = \pi^{-1}(\pi(v))$ (see [12, Sec. 4]).

In [18, Prop. 16.7], Richardson characterizes the G-stable points in G^n. We can easily extend this result to the H-stable points in G^n for reductive subgroups H of G. To do this, we first extend the notion of G-irreducibility from [19, §3.2] to the relative setting:

Definition 3.14. Let H and K be subgroups of G with H reductive. We say that K *is relatively G-irreducible (G-ir) with respect to H* if whenever $\lambda \in Y(H)$ and $K \subseteq P_\lambda$, we have $P_\lambda = G$. For $H = G$, this relative notion agrees with that of G-irreducibility, cf. [19, §3.2], [2, §2.4].

Remark 3.15. Obviously, relative G-irreducibility with respect to H implies relative G-complete reducibility with respect to H. It is clear that for subgroups H, K and M of

G with H and M reductive and $H, K \subseteq M$, if K is relatively G-irreducible with respect to H, then K is relatively M-irreducible with respect to H, cf. [2, Cor. 2.7].

The notion of relative G-irreducibility is exactly what we need to characterize H-stability in G^n. In view of Remark 2.7, our next result generalizes [18, Prop. 16.7]; see also [2, Prop. 2.13]. Since Richardson's proof applies mutatis mutandis, we do not include it.

Proposition 3.16. *Let H be a reductive subgroup of G. Let K be a subgroup of G and let $\mathbf{k} \in K^n$ be a generic tuple for K. Then K is relatively G-irreducible with respect to H if and only if $\mathbf{k}$ is an H-stable point in G^n.*

The following provides analogues of Corollaries 3.22 and 3.5 in [2].

Proposition 3.17. *Let H and K be subgroups of G and suppose that H is reductive.*

(i) *Let S be a torus of $C_H(K)$ and set $L = C_G(S)$. Then K is relatively G-completely reducible with respect to H if and only if K is relatively L-completely reducible with respect to $H \cap L$.*

(ii) *The R-Levi subgroups L_μ of G for $\mu \in Y(H)$ that are minimal with respect to containing K are precisely the subgroups of the form $C_G(S)$ where S is a maximal torus of $C_H(K)$. If L is such an R-Levi subgroup of G, then K is relatively G-completely reducible with respect to H if and only if K is relatively L-irreducible with respect to $H \cap L$.*

Proof. First we prove the first assertion of (ii). If S is a torus in G, then $C_G(S) = L_\lambda$ for some $\lambda \in Y(S)$ by the arguments of the proof of [2, Cor. 6.10]. Now assume that S is a maximal torus of $C_H(K)$. If $\mu \in Y(H)$ such that $K \subseteq L_\mu \subseteq C_G(S)$, then $\mu(k^*)$ commutes with S and is contained in $C_H(K)$. So, by the maximality of S, $\mu(k^*) \subseteq S$ and $L_\mu = C_G(S)$.

Now assume that $\lambda \in Y(H)$ and L_λ is minimal among the R-Levi subgroups L_μ of G with $\mu \in Y(H)$ and $K \subseteq L_\mu$. Put $S = (Z(L_\lambda) \cap H)^0$. Then one easily checks that $L_\lambda = C_G(S)$. Let T be a torus of $C_H(K)$ with $S \subseteq T$. Then $K \subseteq C_G(T) \subseteq C_G(S) = L_\lambda$. As we have seen above $C_G(T) = L_\mu$ for some $\mu \in Y(T) \subseteq Y(H)$. So, by the minimality of L_λ, $C_G(T) = L_\lambda$ and $T \subseteq S$. So S is a maximal torus of $C_H(K)$.

To prove (i) and the second assertion of (ii) let $\mathbf{k}$ be a generic tuple for K. Let S be a torus of $C_H(K)$ and let $L = C_G(S)$. Note that $H \cap L = C_H(S)$ is reductive. By Theorem 3.5(iii), K is relatively G-completely reducible with respect to H if and only if $H \cdot \mathbf{k}$ is closed in G^n and likewise if we replace G and H by L and $H \cap L$, respectively. So (i) now follows from [17, Thm. C].

Now assume that S is a maximal torus of $C_H(K)$. Then $H \cdot \mathbf{k}$ is closed in G^n if and only if $\mathbf{k}$ is a stable point for the $(H \cap L)$-action on L^n by [18, Lem. 16.6]. By Proposition 3.16, this is equivalent to K being relatively L-irreducible with respect to $H \cap L$. □

Lemma 3.18. *Let H be a reductive subgroup of G and let $\lambda \in Y(H)$. Then the R-parabolic subgroups P_μ of G contained in P_λ with $\mu \in Y(H)$ are precisely the subgroups of the form $Q \ltimes R_u(P_\lambda)$, where $Q = P_\nu(L_\lambda)$ and $\nu \in Y(L_\lambda(H))$.*

Proof. If $\mu \in Y(H)$ such that $P_\mu \subseteq P_\lambda$, then P_μ is of the stated form if some $R_u(P_\lambda(H))$-conjugate of P_μ is of this form. Now we can replace μ by an $R_u(P_\lambda(H))$-conjugate which lies in $Y(L_\lambda(H))$. The rest of the proof is completely analogous to that of [2, Lem. 6.2(ii)]. □

We can now generalize [5, Prop. 5.10], showing that we can associate to each H-conjugacy class of subgroups of G a unique H-conjugacy class of subgroups which are relatively G-cr

with respect to H; Theorem 3.19 below is the group-theoretic analogue of the statement that the closure of each H-orbit in G^n contains a unique closed H-orbit.

Theorem 3.19. *Let H and K be subgroups of G and suppose that H is reductive.*

(i) *There exists $\lambda \in Y(H)$ and a subgroup M of G which is relatively G-completely reducible with respect to H such that $K \subseteq P_\lambda$ and $c_\lambda(K) = M$. Moreover, M is unique up to H-conjugacy and its H-conjugacy class only depends on the H-conjugacy class of K.*

(ii) *Any automorphism of the algebraic group G that normalizes H and stabilizes the H-conjugacy class of K, stabilizes the H-conjugacy class of M.*

(iii) *If $K \subseteq P_\mu$ for $\mu \in Y(H)$, then the above H-conjugacy class associated to $c_\mu(K)$ is the same as that associated to K.*

Proof. Let $\lambda \in Y(H)$ be such that P_λ is minimal among the R-parabolic subgroups P_μ of G with $\mu \in Y(H)$ and $K \subseteq P_\mu$. Then P_λ is also minimal among the R-parabolic subgroups P_μ of G with $\mu \in Y(H)$ and $c_\lambda(K) \subseteq P_\mu$ by the same arguments as in the proof of [5, Prop. 5.10]. It now follows from Lemma 3.18 that $c_\lambda(K)$ is relatively L_λ-irreducible with respect to $L_\lambda(H)$. But then $c_\lambda(K)$ is relatively G-cr with respect to H by Proposition 3.17(i).

The rest of the proof is completely analogous to that of [5, Prop. 5.10]. One has to conjugate with elements from $R_u(P_\lambda(H))$ rather than $R_u(P_\lambda)$. The cocharacters λ and μ in the proof of [5, Prop. 5.10] can now be put in a common maximal torus of $P_\lambda(H)$ and $P_\mu(H)$. □

Remark 3.20. A statement analogous to Theorem 3.19 holds for Lie algebras: that is, given any Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$, we can find a uniquely defined H-conjugacy class of subalgebras of $\mathfrak{g}$ containing $c_\lambda(\mathfrak{k})$ for some $\lambda \in Y(H)$, each member of which is relatively G-cr with respect to H.

3.6. **Optimal parabolic subgroups.** Let K be a subgroup of G. If K is not relatively G-cr with respect to H, then there exists at least one cocharacter $\lambda \in Y(H)$ such that $K \subseteq P_\lambda$, but $K \not\subseteq L_{u\cdot\lambda}$ for any $u \in R_u(P_\lambda(H))$. Following work in [5, Sec. 5], we now show how to make a *canonical* choice for this $\lambda \in Y(H)$; being able to make such a choice has many advantages and shortens some of the proofs which follow. Since the constructions we are going to discuss are very similar to those in [5, Sec. 5], where the case of a non-G-cr subgroup of G is addressed, we omit some details and content ourselves with pointing out the necessary modifications to allow the results to go through here.

We first need to adapt some of the notation from [5, Sec. 4, Sec. 5] to the relative setting. Suppose K is a subgroup of G and suppose $\lambda \in Y(H)$ is such that $K \subseteq P_\lambda$. Set $M := c_\lambda(K)$ and let $S_n(M) = \overline{H \cdot M^n}$ (note that this depends on H as well as M and n). Then K^n is a uniformly $S_n(M)$-unstable set for the action of H on G^n in the sense of [5, Sec. 4]. Any G-invariant norm on $Y(G)$ restricts to an $N_G(H)$-invariant norm on $Y(H)$; let $\| \ \|$ be such a norm. Then [5, Sec. 4] provides a set $\Omega(K^n, S_n(M))$ of cocharacters of H, the so-called *optimal class*. Similarly, if $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and $\lambda \in Y(H)$ is such that $\mathfrak{k} \subseteq \mathfrak{p}_\lambda$, then setting $\mathfrak{m} := c_\lambda(\mathfrak{k})$ and $S_n(\mathfrak{m}) = \overline{H \cdot \mathfrak{m}^n}$, we get an optimal class $\Omega(\mathfrak{k}^n, S_n(\mathfrak{m})) \subseteq Y(H)$.

We have the following analogue of [5, Thm. 5.16] in the relative setting:

Theorem 3.21. *Let K be a subgroup of G and let n be minimal such that K^n contains a generic tuple for K. Let M be a subgroup of G and suppose that $M = c_\lambda(K)$ for some $\lambda \in Y(H)$ with $K \subseteq P_\lambda$. Put $\Omega(K, M) := \Omega(K^n, S_n(M))$. Then the following hold:*

(i) $P_\mu = P_\nu$ *for all* $\mu, \nu \in \Omega(K, M)$. *Let* $P(K, M)$ *denote the unique R-parabolic subgroup of* G *so defined. Then* $K \subseteq P(K, M)$ *and* $R_u(P(K, M) \cap H)$ *acts simply transitively on* $\Omega(K, M)$.

(ii) *For* $g \in N_G(H)$ *we have* $\Omega(gKg^{-1}, gMg^{-1}) = g \cdot \Omega(K, M)$ *and* $P(gKg^{-1}, gMg^{-1}) = gP(K, M)g^{-1}$. *If* $g \in G$ *normalizes* H, K *and* $S_n(M)$, *then* $g \in P(K, M)$.

(iii) *If* $\mu \in \Omega(K, M)$, *then* $\dim C_H(c_\mu(K)) \geq \dim C_H(M)$. *If* M *is* H*-conjugate to* K, *then, trivially,* $\Omega(K, M) = \{0\}$ *and* $P(K, M) = G$. *If* M *is not* H*-conjugate to* K, *then* K *is not contained in any R-Levi subgroup of* $P(K, M)$ *of the form* L_μ, *where* $\mu \in Y(H)$ *is such that* $P_\mu = P(K, M)$.

Proof. We apply [5, Thm. 4.4] with $(G', G, V, X, S) = (N_G(H), H, G^n, K^n, S_n(M))$. Since we associate to K and M an R-parabolic subgroup of G rather than H in *loc. cit.*, we need to give some more arguments.

(i) and (ii). The statements about $\Omega(K, M)$ follow immediately from [5, Thm. 4.4]. Let $\mu \in \Omega(K, M)$. By [5, Thm. 4.4(iv)], $R_u(P_\mu(H)) = R_u(P_\mu) \cap H$ acts simply transitively on $\Omega(K, M)$. Hence $P_\mu = P_\nu$ for all $\mu, \nu \in \Omega(K, M)$. The final assertion in (ii) is proved in the same way as the final assertion of [5, Thm. 4.4(iv)].

(iii). The proof of this is completely analogous to that of [5, Thm. 5.16(iii)]. Note that $P(K, M) = G$ implies that $\Omega(K, M)$ consists of the trivial cocharacter of H only. □

Remark 3.22. Note that Theorem 3.19 provides an obvious choice for the subgroup M in Theorem 3.21: for by Theorem 3.19, if we are given a subgroup K then there is a unique conjugacy class of subgroups of the form $M = c_\lambda(K)$ for $\lambda \in Y(H)$ which are relatively G-cr with respect to H. Since these subgroups are all H-conjugate, the set $S_n(M)$ does not depend on which representative M we choose from this conjugacy class, and hence the optimal destabilizing R-parabolic subgroup $P(K, M)$ of G also does not depend on the choice of M from this class. This leads to the following definition.

Definition 3.23. Let K be any subgroup of G, and let M be a representative from the H-conjugacy class attached to K of subgroups which are relatively G-cr with respect to H, provided by Theorem 3.19. Define $\Omega(K) = \Omega(K, M)$ and $P(K) = P(K, M)$. By Theorem 3.19 and Theorem 3.21, K and $N_{N_G(H)}(K)$ are contained in $P(K)$ and for $\mu \in \Omega(K)$, $c_\mu(K)$ is relatively G-completely reducible with respect to H. So, by Theorem 3.5(ii), if K is not relatively G-completely reducible with respect to H, then K is not contained in any R-Levi subgroup of $P(K)$ of the form L_μ with $\mu \in Y(H)$ and $P_\mu = P(K)$. Note that, *trivially*, $P(K) = G$ if K is relatively G-completely reducible with respect to H. We call $\Omega(K)$ the *optimal class of cocharacters for* K *in* H and $P(K)$ the *optimal destabilizing R-parabolic subgroup for* K *with respect to* H. In case $H = G$, we call $P(K)$ the *optimal destabilizing R-parabolic subgroup for* K.

We have a result analogous to Theorem 3.21 for Lie algebras with a similar proof.

Theorem 3.24. *Let* $\mathfrak{k}$ *be a subalgebra of* $\mathfrak{g}$ *and let* n *be minimal such that* $\mathfrak{k}^n$ *contains a generating tuple for* $\mathfrak{k}$. *Let* $\mathfrak{m}$ *be a subalgebra of* $\mathfrak{g}$ *and suppose that* $\mathfrak{m} = c_\lambda(\mathfrak{k})$ *for some* $\lambda \in Y(H)$ *with* $\mathfrak{k} \subseteq \mathfrak{p}_\lambda$. *Put* $\Omega(\mathfrak{k}, \mathfrak{m}) := \Omega(\mathfrak{k}^n, S_n(\mathfrak{m}))$. *Then the following hold:*

(i) $P_\mu = P_\nu$ *for all* $\mu, \nu \in \Omega(\mathfrak{k}, \mathfrak{m})$. *Let* $P(\mathfrak{k}, \mathfrak{m})$ *denote the unique R-parabolic subgroup of* G *so defined. Then* $\mathfrak{k} \subseteq \operatorname{Lie} P(\mathfrak{k}, \mathfrak{m})$ *and* $R_u(P(\mathfrak{k}, \mathfrak{m}) \cap H)$ *acts simply transitively on* $\Omega(\mathfrak{k}, \mathfrak{m})$.

(ii) *For $g \in N_G(H)$ we have $\Omega(g \cdot \mathfrak{k}, g \cdot \mathfrak{m}) = g \cdot \Omega(\mathfrak{k}, \mathfrak{m})$ and $P(g \cdot \mathfrak{k}, g \cdot \mathfrak{m}) = gP(\mathfrak{k}, \mathfrak{m})g^{-1}$. If $g \in G$ normalizes H, $\mathfrak{k}$ and $S_n(\mathfrak{m})$, then $g \in P(\mathfrak{k}, \mathfrak{m})$.*

(iii) *If $\mu \in \Omega(\mathfrak{k}, \mathfrak{m})$, then $\dim C_H(c_\mu(\mathfrak{k})) \geq \dim C_H(\mathfrak{m})$. If $\mathfrak{m}$ is H-conjugate to $\mathfrak{k}$, then, trivially, $\Omega(\mathfrak{k}, \mathfrak{m}) = \{0\}$ and $P(\mathfrak{k}, \mathfrak{m}) = G$. If $\mathfrak{m}$ is not H-conjugate to $\mathfrak{k}$, then $\mathfrak{k}$ is not contained in $\mathfrak{l}_\mu$ for any $\mu \in Y(H)$ with $P_\mu = P(K, M)$.*

Definition 3.25. Let $\mathfrak{k}$ be any subalgebra of $\mathfrak{g}$, and let $\mathfrak{m}$ be a representative from the H-conjugacy class attached to $\mathfrak{k}$ of subalgebras which are relatively G-cr with respect to H, provided by Remark 3.20. Define $\Omega(\mathfrak{k}) = \Omega(\mathfrak{k}, \mathfrak{m})$ and $P(\mathfrak{k}) = P(\mathfrak{k}, \mathfrak{m})$. We call $\Omega(\mathfrak{k})$ the *optimal class of cocharacters for $\mathfrak{k}$ in H* and $P(\mathfrak{k})$ the *optimal destabilizing R-parabolic subgroup for $\mathfrak{k}$ with respect to H*. In case $H = G$, we call $P(\mathfrak{k})$ the *optimal destabilizing R-parabolic subgroup for K*.

Armed with Theorems 1.1, 3.5 and 3.21, we can extend many results about G-complete reducibility within the framework of relative G-complete reducibility. In the following subsections we aim to illustrate interesting points of our new construction by looking at a selection of results, mainly from [2]; some of these results generalize immediately, while others are more subtle. We observe that all the results below have interpretations in terms of closedness of orbits in G^n in view of Theorem 1.1.

3.7. **New relatively G-completely reducible subgroups from old.** In this section we explore how to generate new relatively G-cr subgroups from a given relatively G-cr subgroup M; for example, by taking suitable normal subgroups of M or looking at suitable subgroups of $N_G(M)$. Our first result generalizes [2, Thm. 3.10], as the latter is simply the special case $H = G$ of Theorem 3.26. The apparent direct analogue of [2, Thm. 3.10] in this context, namely that a normal subgroup N of a subgroup M of G is relatively G-cr with respect to some reductive subgroup H of G provided M is, fails in general; see Examples 5.6 and 5.7 below.

Theorem 3.26. *Let H be a reductive subgroup of G and let K and M be subgroups of G such that $K \subseteq M \subseteq KN_{N_G(H)}(K)$. If M is relatively G-completely reducible with respect to H, then so is K.*

Proof. We prove the contrapositive. So suppose K is not relatively G-cr with respect to H. Let $P(K)$ be the optimal R-parabolic destabilizing subgroup for K with respect to H (Definition 3.23). Then by Definition 3.23, $M \subseteq KN_{N_G(H)}(K) \subseteq P(K)$ and K is not contained in any R-Levi subgroup of $P(K)$ of the form L_μ with $\mu \in Y(H)$ and $P_\mu = P(K)$. Hence M cannot be contained in such an R-Levi subgroup of $P(K)$, as $K \subseteq M$. Thus M is not relatively G-cr with respect to H. □

The following result generalizes the second statement of [2, Prop. 3.19].

Proposition 3.27. *Suppose H is a reductive subgroup of G. Let K be a subgroup of G which is relatively G-completely reducible with respect to H, and suppose M is a reductive subgroup of G which contains K and is normalized by a maximal torus of $C_H(K)$. Then M is also relatively G-completely reducible with respect to H.*

Proof. Choose a maximal torus S of $C_H(K)$ which normalizes M. Since MS is reductive and $M \subseteq MS \subseteq MN_H(M)$, we may assume that $S \subseteq M$, by Theorem 3.26. Let $\lambda \in Y(H)$ be such that $M \subseteq P_\lambda$. Since M contains K and K is relatively G-cr with respect to H, we

may assume that $K \subseteq L_\lambda$. Then we have $\lambda(k^*) \subseteq C_H(K) \cap P_\lambda$. But S is a maximal torus of $C_H(K) \cap P_\lambda$, so there exists $g \in C_H(K) \cap P_\lambda$ such that for $\mu := g \cdot \lambda$ we have $\mu(k^*) \subseteq S \subseteq M$. Clearly, we also have $\mu \in Y(H)$ and $P_\mu = P_\lambda$. Since $M \subseteq P_\lambda = P_\mu$, we have $P_\mu(M) = M$. Since M is reductive, this means that $\mu(k^*)$ is central in M, by Definition 2.1. So $M \subseteq L_\mu$, as required. □

Remark 3.28. Note that Proposition 3.27 applies in particular to reductive subgroups of G which contain all of $KC_H(K)$. In particular, if K is relatively G-completely reducible with respect to H, then, provided they are reductive, $N_G(K)$ and $KC_G(K)$ are relatively G-completely reducible with respect to H. This generalizes [2, Cor. 3.16]; recall that the reductivity conditions for $N_G(K)$ and $KC_G(K)$ are always satisfied for a G-cr subgroup K of G, cf. [2, Prop. 3.12].

The following result generalizes [2, Prop. 3.20].

Corollary 3.29. *Suppose H is a reductive subgroup of G. Then any reductive subgroup of G which is normalized by a maximal torus of H is relatively G-completely reducible with respect to H.*

Proof. Let K be a reductive subgroup of G which is normalized by a maximal torus of H. By Theorem 3.26, we may assume that K contains a maximal torus S of H. Now S is H-cr [2, Lem. 2.6], so S is relatively G-cr with respect to H. The result follows from Proposition 3.27 applied to the inclusion $S \subseteq K$. □

3.8. **Relative complete reducibility with respect to different subgroups of G.** In our next set of results we explore what happens when we vary the reductive subgroup H, rather than K. The first result generalizes one direction of [3, Prop. 2.8]. Note that the converse in this case is not true; just take any example of G, H and K where K is not relatively G-cr with respect to H, and let $N = \{1\}$.

Proposition 3.30. *Suppose H is a reductive subgroup of G and N is a normal subgroup of H. For any subgroup K of G, if K is relatively G-completely reducible with respect to H, then K is relatively G-completely reducible with respect to N.*

Proof. First note that since H is reductive and N is normal in H, N is reductive. Suppose $\lambda \in Y(N) \subseteq Y(H)$ is such that $K \subseteq P_\lambda$. Then, as K is relatively G-cr with respect to H, there exists $u \in R_u(P_\lambda(H))$ such that $K \subseteq uL_\lambda u^{-1} = L_{u\cdot\lambda}$, by Lemma 3.3. But $u \in H$, so u normalizes N, and thus $u \cdot \lambda \in Y(N)$, and we are done. □

Corollary 3.31. *Suppose H is a reductive subgroup of G and K is relatively G-completely reducible with respect to H. Suppose M is a reductive subgroup of H which is normalized by $C_H(K)$. Then K is relatively G-completely reducible with respect to M.*

Proof. Since $C_H(K)$ normalizes M, M is normal in $MC_H(K)$. By Proposition 3.30, if K is relatively G-cr with respect to $MC_H(K)$, then K is relatively G-cr with respect to M, so we may assume that $C_H(K) \subseteq M$. Let $\lambda \in Y(M)$ such that $K \subseteq P_\lambda$. Then, as K is relatively G-cr with respect to H, there exists $u \in R_u(P_\lambda(H))$ such that $K \subseteq L_{u\cdot\lambda}$, by Lemma 3.3. But then $u \cdot \lambda \in Y(C_H(K)) \subseteq Y(M)$, and we are done. □

Our final proposition in this subsection is a strengthening of Proposition 3.17(i) and [4, Prop. 5.7]. Before we begin the build-up to the result, we note that it would be possible to

adapt the proof of [4, Prop. 5.7] to prove our new result in positive characteristic. However, there are problems in characteristic zero: [4, Prop. 5.7] is easy to prove in characteristic zero because of the nice characterization of G-complete reducibility in this case (a subgroup is G-cr if and only if it is reductive), but we have no such characterization of relative G-complete reducibility in characteristic zero, so we need a new proof. The following preparatory work, which is motivated by the notion of a generic tuple from Subection 2.4, allows us to present a proof which is valid in any characteristic.

Let W be a finite-dimensional vector space over k and let $n \in \mathbb{N}$. We have an action of $\mathrm{GL}_n(k)$ on W^n given by

$$A \cdot (v_1, \dots, v_n) = (v_1', \dots, v_n'),$$

where

$$v_i' = \sum_{j=1}^{n} a_{ij} v_j$$

and a_{ij} is the ij-component of A. This action of $\mathrm{GL}_n(k)$ commutes with the diagonal $\mathrm{GL}(W)$-action on W^n.

Recall the notion of a stable point from Definition 3.12; and recall from Remark 3.13 that if v is G-stable, then

$$\pi^{-1}(\pi(v)) = G \cdot v, \tag{3.32}$$

where $\pi \colon V \to V/\!\!/G$ is the canonical morphism.

Lemma 3.33. *Let $\mathbf{w} = (w_1, \dots, w_n) \in W^n$ such that the w_i are linearly independent. Let F be a subgroup of $\mathrm{GL}_n(k)$ such that F is a finite extension of $\mathrm{SL}_n(k)$. Then $\mathbf{w}$ is a stable point for the action of F on W^n.*

Proof. Since the w_i are linearly independent, $C_F(\mathbf{w})$ is trivial. It remains to check that $F \cdot \mathbf{w}$ is closed. Let $\lambda \in Y(F)$ such that $\lim_{x \to 0} \lambda(x) \cdot \mathbf{w}$ exists. Choose $A \in \mathrm{GL}_n(k)$ such that $A \cdot \lambda$ takes values in the subgroup of F of diagonal matrices. Set $\mathbf{w}' = A \cdot \mathbf{w} = (w_1', \dots, w_n')$. Then $\lim_{x \to 0} (A \cdot \lambda)(x) \cdot \mathbf{w}'$ exists. There exist $m_1, \dots, m_n \in \mathbb{Z}$ such that for all $x \in k^*$, we have

$$(A \cdot \lambda)(x) \cdot (w_1', \dots, w_n') = (x^{m_1} w_1', \dots, x^{m_n} w_n').$$

Since $F^0 = \mathrm{SL}_n(k)$, we must have $\sum_{i=1}^n m_i = 0$. As $w_i' \neq 0$ and $\lim_{x \to 0} x^{m_i} w_i'$ exists for each i, we must have $m_i \geq 0$ for each i. This forces all the m_i to be zero, and so $A \cdot \lambda$ is trivial, hence λ is trivial. We conclude that $F \cdot \mathbf{w}$ is closed, by the Hilbert-Mumford Theorem. $\square$

Now suppose that $\dim W = n$ and $\mathbf{w} \in W^n$ consists of a basis for W. Let $g \in \mathrm{GL}(W)$. Then $g \cdot \mathbf{w}$ also consists of a basis for W. Hence there exists a unique element $A(g, \mathbf{w}) \in \mathrm{GL}_n(k)$ such that

$$g \cdot \mathbf{w} = A(g, \mathbf{w}) \cdot \mathbf{w}. \tag{3.34}$$

Let $\mathbf{w}' \in W^n$ be another basis for W. We can write $\mathbf{w}' = A \cdot \mathbf{w}$ for some $A \in \mathrm{GL}_n(k)$. Using the fact that the actions of $\mathrm{GL}(W)$ and $\mathrm{GL}_n(k)$ commute, a straightforward calculation shows that

$$A(g, \mathbf{w}') = A A(g, \mathbf{w}) A^{-1}.$$

Hence $\det A(g, \mathbf{w})$ is independent of $\mathbf{w}$. Moreover, if $g' \in \mathrm{GL}(W)$, then we have

$$A(g'g, \mathbf{w}) = A(g', g \cdot \mathbf{w}) A(g, \mathbf{w}).$$

Whence the map

$$\mathrm{GL}(W) \to k^*,\ g \mapsto \det A(g, \mathbf{w}) \tag{3.35}$$

is a homomorphism and is independent of $\mathbf{w}$.

We can now give our strengthening of Proposition 3.17(i) and [4, Prop. 5.7]; note our new result is significantly stronger, since we do not require K to be a subgroup of $C_G(S)$. In the case $H = G$, Proposition 3.36 and Corollary 3.37 identify instances where G-complete reducibility implies or is implied by relative G-complete reducibility with respect to a proper subgroup.

Proposition 3.36. *Let K, H and S be closed subgroups of G such that H is reductive and S normalizes H and K. Put $M = C_H(S)^0$.*

(i) *Suppose that S is reductive and HS-completely reducible. Then M is reductive. Moreover, K is relatively G-completely reducible with respect to H if it is relatively G-completely reducible with respect to M.*

(ii) *Suppose that*
   (a) *$\mathfrak{h}/\mathfrak{c}_{\mathfrak{h}}(S)$ does not have any trivial S-composition factors;*
   (b) *$\mathfrak{c}_{\mathfrak{h}}(S) = \mathrm{Lie}\, C_H(S)$;*
   (c) *M is reductive.*
   Then K is relatively G-completely reducible with respect to M if it is relatively G-completely reducible with respect to H.

Proof. (i). Suppose that S is reductive and HS-cr (note that HS is reductive because H and S are). Then M is reductive, since $C_H(S)$ is H-cr, by [4, Thm. 5.4(a)]. Suppose that K is not relatively G-cr with respect to H. We show that K is not relatively G-cr with respect to M. Let $\Omega(K)$ be the class of optimal cocharacters for K in H and let $P(K)$ be the corresponding optimal destabilizing R-parabolic subgroup for K with respect to H (Definition 3.23). By Theorem 3.21(ii), since $S \subseteq N_G(H) \cap N_G(K)$, we have $S \subseteq P(K)$, and hence S normalizes $P(K) \cap H$. By [4, Lem. 5.1] (applied to the reductive group HS), S normalizes an R-Levi subgroup of $P(K) \cap H$, and by [5, Thm. 4.4(iv)], this subgroup is of the form $L_\lambda(H)$ for a unique $\lambda \in \Omega(K)$. But S acts on $\Omega(K)$, by Theorem 3.21(ii), so S must fix λ, and we have $S \subseteq L_\lambda = C_G(\lambda(k^*))$. Hence $\lambda \in Y(M)$. If K is contained in an R-Levi subgroup of $P_\lambda(M)$ then $K \subseteq L_{u\cdot\lambda}(M)$ for some $u \in R_u(P_\lambda(M))$. But then $u \in R_u(P_\lambda(H))$, so K is contained in $L_{u\cdot\lambda}(H)$, an R-Levi subgroup of M, which is impossible. We conclude that K is not relatively G-cr with respect to M either.

(ii). After embedding G in some GL_m, we may assume that $G = \mathrm{GL}_m$. Let E be the linear span of K in Mat_m. Let $\mathbf{e} \in K^n$ for some n be a basis for the associative algebra E. Then it follows from Theorem 3.10 that K is relatively G-cr with respect to H if and only if $H \cdot \mathbf{e}$ is closed, and likewise if we replace H by M. Before we proceed further, we briefly give the main idea of the proof. We want to apply [4, Thm. 5.4(b)] to $\mathbf{e}$. We cannot do this directly because S does not centralize K — it only normalizes K — and hence $\mathbf{e}$ need not be centralized by S. The point of the argument below, and of Lemma 3.33, is to allow us to replace $\mathbf{e}$ with $\pi(\mathbf{e})$, which **is** centralized by S.

Since S normalizes K, we have a homomorphism $\varphi : S \to \mathrm{GL}(E)$ and E^n is an S-stable subset of $(\mathrm{Mat}_m)^n$ under the diagonal action. Define $\psi : S \to k^*$ by $\psi(s) = \det A(\varphi(s), \mathbf{e})$, where $A(g, \mathbf{e})$ is as in Eqn. (3.34). If $\psi(S)$ is finite, then we define $F \leq \mathrm{GL}_n$ by $F = \det^{-1}(\psi(S))$. Now assume that $\psi(S) = k^*$. Let $\Lambda \leq k^*$ be the subgroup of all roots of unity

and for a positive integer l let $\Lambda_l \leq k^*$ be the group of lth roots of unity. Since Λ is dense in k^* and the inverse image of a dense subgroup under a surjective homomorphism of algebraic groups is dense, we have that the ascending chain $\psi^{-1}(\Lambda_{l!})_{l\geq 1}$ of subgroups of S is dense in S. By similar arguments as in [4, Prop. 3.7], we can now replace S by some $\psi^{-1}(\Lambda_{l!})$ without changing M and such that assumptions (a), (b) and (c) still hold. Now we are again in the situation that $\psi(S)$ is finite and again we define $F \leq \mathrm{GL}_n$ by $F = \det^{-1}(\varphi(S))$. Let $\pi : (\mathrm{Mat}_m)^n \to (\mathrm{Mat}_m)^n /\!\!/ F$ be the canonical projection.

Now assume that K is relatively G-cr with respect to H. Let $\lambda \in Y(M)$ such that $\mathbf{e}' := \lim_{a\to 0} \lambda(a)\cdot\mathbf{e}$ exists. Since $H\cdot\mathbf{e}$ is closed, we have $\mathbf{e}' \in H\cdot\mathbf{e}$. So $\pi(\mathbf{e}') = \lim_{a\to 0} \lambda(a)\cdot\pi(\mathbf{e}) \in H\cdot\pi(\mathbf{e})$. By our choice of F we have that for every $s \in S$, $A(\varphi(s), \mathbf{e}) \in F$. So $\pi(\mathbf{e})$ is S-fixed. Now $M \cdot \pi(\mathbf{e})$ is closed in $H \cdot \pi(\mathbf{e})$, by [4, Thm. 5.4(b)], so $\pi(\mathbf{e}') \in M \cdot \pi(\mathbf{e})$. But then $\mathbf{e}' \in M \cdot \mathbf{e}$ by Lemma 3.33 and [5, Cor. 3.6(ii)]. So $M \cdot \mathbf{e}$ is closed and thus K is relatively G-completely reducible with respect to M. □

We get the following analogue of [2, Cor. 3.21].

Corollary 3.37. *Let G, H, K, S, and $M = C_H(S)^0$ be as in Proposition 3.36. Suppose that S is linearly reductive. Then K is relatively G-completely reducible with respect to H if and only if it is relatively G-completely reducible with respect to M.*

Proof. Since S is linearly reductive, S is HS-cr [2, Lem. 2.6] and conditions (a)–(c) of Proposition 3.36(ii) all hold [17, Lem. 4.1, Prop. 10.1.5]. The result now follows from Proposition 3.36. □

4. Rationality questions

There is an obvious way to extend the notion of relative G-complete reducibility by considering non-algebraically closed fields. Throughout this section, k denotes any field and we assume that G is a reductive k-group. Furthermore, we assume that H is a reductive k-defined subgroup of G. We let k_s denote the separable closure of k, and $\overline{k}$ the algebraic closure of k. We denote the Galois group $\mathrm{Gal}(k_s/k) = \mathrm{Gal}(\overline{k}/k)$ by Γ. Algebraic groups and varieties will always be defined over $\overline{k}$ and points will always be $\overline{k}$-defined points. If V is a k-variety and k_1/k is an algebraic extension, then we denote the set of k_1-points of V by $V(k_1)$. We call elements of $V(k_s)$ *separable points*. Note that Γ acts on $V = V(\overline{k})$. A closed subvariety W of V is defined over k if and only if it contains a Γ-stable set of separable points of V which is dense in W; see [6, Thm. A.14.4]. The set of k-defined cocharacters of a k-group K is denoted $Y_k(K)$.

We begin with the definition of relative G-complete reducibility over k:

Definition 4.1. Let K be a subgroup of G. We say that K is *relatively G-completely reducible over k with respect to H* if for every $\lambda \in Y(H)$ such that P_λ is k-defined and $K \subseteq P_\lambda$, there exists $\mu \in Y(H)$ such that $P_\lambda = P_\mu$, L_μ is k-defined and $K \subseteq L_\mu$.

In order to deal with the definition of relative G-complete reducibility over k, we need some more detailed information about R-parabolic subgroups defined over k.

Lemma 4.2. *Let $\lambda \in Y(H)$. If P_λ is k-defined, then $P_\lambda(H)$ is k-defined. The analogous assertions hold for L_λ and $R_u(P_\lambda)$.*

Proof. Thanks to [20, Prop. 12.1.5], $P_\lambda(H) = P_\lambda \cap H$ is k-defined if $\operatorname{Lie} P_\lambda(H) = \operatorname{Lie} P_\lambda \cap \operatorname{Lie} H$; similarly for L_λ and $R_u(P_\lambda)$. The result now follows from Lemma 3.8 applied to G and H. □

Lemma 4.3. *Let H be a k-defined reductive subgroup of G, let $\lambda \in Y(H)$ and assume that P_λ is k-defined. Then there exists $\mu \in Y_k(H)$ such that $P_\lambda \subseteq P_\mu$ and $P_\lambda^0 = P_\mu^0$.*

Proof. This follows from the proof of [5, Lem. 2.5(ii)], replacing the maximal torus of P_λ with a maximal torus of $P_\lambda(H)$. □

Remark 4.4. It is not true that any k-defined R-parabolic subgroup of G stems from a cocharacter defined over k; see [5, Rem. 2.4].

The next lemma gives a version of Lemma 3.3(i) over k. Note that any two k-defined R-Levi subgroups of a k-defined R-parabolic subgroup are $R_u(P)(k)$-conjugate [5, Lem. 2.5(iii)].

Lemma 4.5. *Let $\lambda, \mu \in Y(H)$ such that $P_\lambda = P_\mu$ and let u be the element of $R_u(P_\lambda(H))$ such that $uL_\lambda(H)u^{-1} = L_\mu(H)$. Then $uL_\lambda u^{-1} = L_\mu$. Furthermore, if G, H, P_λ, L_λ and L_μ are k-defined, then $u \in R_u(P_\lambda(H))(k)$.*

Proof. The first part of this is Lemma 3.3. The final assertion follows from [5, Lem. 2.5(iii)] and the fact that $R_u(P_\lambda(H))(k) = R_u(P_\lambda(H)) \cap R_u(P_\lambda)(k)$. □

Corollary 4.6. *Let $\lambda \in Y_k(H)$ and let $\mu \in Y(H)$ such that $P_\lambda = P_\mu$ and L_μ is k-defined. Then there exists $\nu \in Y_k(H)$ such that $P_\lambda = P_\nu$ and $L_\mu = L_\nu$.*

Proof. By Lemma 4.5, there exists $u \in R_u(P_\lambda(H))(k)$ such that $L_{u\cdot\lambda} = uL_\lambda u^{-1} = L_\mu$, so we can take $\nu = u \cdot \lambda$. □

We can now show that when discussing relative G-complete reducibility over k, it suffices to consider R-parabolic subgroups of the form P_λ with $\lambda \in Y_k(H)$, rather than all k-defined R-parabolic subgroups stemming from a cocharacter of H.

Lemma 4.7. *Let K be a subgroup of G. Then K is relatively G-completely reducible over k with respect to H if and only if for every $\lambda \in Y_k(H)$ such that K is contained in P_λ, there exists $\mu \in Y_k(H)$ such that $P_\lambda = P_\mu$ and $K \subseteq L_\mu$.*

Proof. Assume that for every $\lambda \in Y_k(H)$ such that K is contained in P_λ, there exists $\mu \in Y_k(H)$ such that $P_\lambda = P_\mu$ and $K \subseteq L_\mu$. Let $\lambda \in Y(H)$ such that P_λ is k-defined and $K \subseteq P_\lambda$. Note that $P_\lambda(H)$ is k-defined by Lemma 4.2. After conjugating λ by an element of $P_\lambda(H)$ we may assume that $\lambda \in Y(T)$ for some k-defined maximal torus T of $P_\lambda(H)$. By Lemma 4.3 (with $H = T$) there exists $\mu \in Y_k(T)$ such that $P_\lambda \subseteq P_\mu$ and $P_\lambda^0 = P_\mu^0$. By [5, Lem. 2.2], we have $L_\lambda = L_\mu \cap P_\lambda$. By assumption there exists $\nu \in Y_k(H)$ such that $P_\mu = P_\nu$ and $K \subseteq L_\nu$. By Lemma 3.3 there exists $u \in R_u(P_\mu(H)) = R_u(P_\lambda(H))$ such that $uL_\mu u^{-1} = L_\nu$. But then $u \cdot \lambda \in Y(H)$ and $L_{u\cdot\lambda} = uL_\lambda u^{-1} = u(L_\mu \cap P_\lambda)u^{-1} = L_\nu \cap P_\lambda$ contains K.

By [5, Lem. 2.5(iii)], $L_{u\cdot\lambda}$ is k-defined, since $L^0_{u\cdot\lambda} = L^0_\nu$ is k-defined. Hence K is relatively G-cr over k with respect to H. The other implication follows from Corollary 4.6. □

In order to generalize Theorem 1.1 to arbitrary fields, we need a notion of a "closed orbit" for a group $H(k)$ of k points of a reductive k-group H acting on a k-variety. As we shall show, the correct notion for us is given by the following definition, see [5, Def. 3.10]:

Definition 4.8. Let H be a reductive k-group and let V be an affine H-variety over k. Let $v \in V$. We say that the $H(k)$-orbit $H(k) \cdot v$ is *cocharacter-closed over k* if for any $\lambda \in Y_k(H)$ such that $v' := \lim_{a \to 0} \lambda(a) \cdot v$ exists, v' is $H(k)$-conjugate to v.

For the proof of Theorem 4.11 we need the following two extensions of Theorem 2.4 to non-algebraically closed fields. The first result is [5, Thm. 3.4]. Here we require the field to be perfect.

Theorem 4.9. *Suppose k is a perfect field. Let H be a reductive k-group and let V be an affine H-variety defined over k. Let $v \in V$ and let $\lambda \in Y(H)$ such that $v' := \lim\limits_{a \to 0} \lambda(a) \cdot v$ exists and is $H(k)$-conjugate to v. Then v' is $R_u(P_\lambda(H))(k)$-conjugate to v.*

The second result is [5, Thm. 3.12]. Here we require H to be connected and the two assertions need to be quantified over all k-defined cocharacters from H.

Theorem 4.10. *Let H be a connected reductive k-group and let V be an affine H-variety defined over k. Let $v \in V$. Then the following are equivalent:*

(i) *$H(k) \cdot v$ is cocharacter-closed over k;*
(ii) *for all $\lambda \in Y_k(H)$ such that $v' := \lim_{a \to 0} \lambda(a) \cdot v$ exists, v' is $R_u(P_\lambda(H))(k)$-conjugate to v.*

Using the preceding discussion, we can now extend parts of Theorem 3.5 to non-algebraically closed fields. The final assertion in part (iii) is the desired extension of Theorem 1.1 to arbitrary fields.

Theorem 4.11. *Assume that $H \leq G$ is connected reductive.*

(i) *Let $n \in \mathbb{N}$, let $\mathbf{k} \in G^n$ and let $\lambda \in Y_k(H)$ such that $\mathbf{m} = \lim_{a \to 0} \lambda(a) \cdot \mathbf{k}$ exists. Then the following are equivalent:*
  (a) *$\mathbf{m}$ is $R_u(P_\lambda(H))(k)$-conjugate to $\mathbf{k}$;*
  (b) *there exists $\mu \in Y_k(H)$ such that $P_\lambda = P_\mu$ and $\mu(k^*)$ fixes $\mathbf{k}$.*
  If k is perfect, then (a) and (b) are also equivalent to the following:
  (c) *$\mathbf{m}$ is $H(k)$-conjugate to $\mathbf{k}$.*
(ii) *Let K be a subgroup of G and let $\lambda \in Y_k(H)$. Suppose $K \subseteq P_\lambda$ and set $M = c_\lambda(K)$. Then the following are equivalent:*
  (a) *M is $R_u(P_\lambda(H))(k)$-conjugate to K;*
  (b) *$K \subseteq L_\mu$ for some $\mu \in Y_k(H)$ such that $P_\lambda = P_\mu$.*
(iii) *Let K, λ and M be as in (ii) and let $\mathbf{k} \in K^n$ be a generic tuple of K. Then the assertions in (i) are equivalent to those in (ii). Furthermore, K is relatively G-completely reducible over k with respect to H if and only if $H(k) \cdot \mathbf{k}$ is cocharacter closed over k.*

Proof. (i) and (ii). The proofs of the equivalence of (a) and (b) in both cases follow from the fact that $R_u(P_\lambda(H))(k)$ acts simply transitively on the set of R-Levi subgroups of P_λ that are of the form L_μ, where $\mu \in Y_k(H)$ is such that $P_\mu = P_\lambda$, by Lemma 4.5. The equivalence of part (a) and part (c) in (i) follows from Theorem 4.9.

(iii). The equivalence of (i)(b) and (ii)(b) in this situation is obvious, so (i) and (ii) are equivalent. For the final assertion of (iii) use Theorem 4.10 and the equivalence of (i) and (ii). □

The following result extends [5, Thm. 5.11] to our setting of relative G-complete reducibility. The proof is completely analogous to the proof given in loc. cit.

Theorem 4.12. *Suppose k_1/k is a separable extension of fields. Let K be a k-defined subgroup of G. If K is relatively G-completely reducible over k_1 with respect to H, then K is relatively G-completely reducible over k with respect to H.*

We end this section with a converse of Theorem 4.12 in case k is perfect. For this we need a version over k of the results on optimal parabolic subgroups from Section 3.6.

We adapt some of the notation from [5, Sec. 4, Sec. 5] to the relative setting over k. Suppose K is a subgroup of G and suppose $\lambda \in Y_k(H)$ is such that $K \subseteq P_\lambda$. Set $M := c_\lambda(K)$ and, as in Section 3.6, let $S_n(M) = \overline{H \cdot M^n}$. Then K^n is uniformly $S_n(M)$-unstable over k for the action of H on G^n in the sense of [5, Def. 4.2]. Any $G(k)$-invariant norm on $Y_k(G)$ restricts to an $N_{G(k)}(H)$-invariant norm on $Y_k(H)$; let $\| \ \|$ be such a norm. Then [5, Def. 4.3] provides an *optimal class* $\Omega(K^n, S_n(M), k) \subseteq Y_k(H)$ of k-defined cocharacters of H.

Theorem 4.13. *Let K be a subgroup of G and let n be minimal such that K^n contains a generic tuple for K. Let M be a subgroup of G and suppose that $M = c_\lambda(K)$ for some $\lambda \in Y_k(H)$ with $K \subseteq P_\lambda$. Put $\Omega(K, M, k) := \Omega(K^n, S_n(M), k)$. Then the following hold:*

(i) *$P_\mu = P_\nu$ for all $\mu, \nu \in \Omega(K, M, k)$. Let $P(K, M, k)$ denote the unique R-parabolic subgroup of G so defined. Then $K \subseteq P(K, M, k)$ and $R_u(P(K, M, k) \cap H)(k)$ acts simply transitively on $\Omega(K, M, k)$.*

(ii) *$\Omega(gKg^{-1}, gMg^{-1}, k) = g \cdot \Omega(K, M, k)$ and $P(gKg^{-1}, gMg^{-1}, k) = gP(K, M, k)g^{-1}$, for $g \in N_{G(k)}(H)$. If $g \in G(k)$ normalizes H, K and $S_n(M)$, then $g \in P(K, M, k)$.*

(iii) *If $\mu \in \Omega(K, M, k)$, then $\dim C_H(c_\mu(K)) \geq \dim C_H(M)$. If M is H-conjugate to K, then, trivially, $\Omega(K, M, k) = \{0\}$ and $P(K, M, k) = G$. If M is not H-conjugate to K, then K is not contained in any R-Levi subgroup of $P(K, M, k)$ of the form L_μ with $\mu \in Y(H)$ and $P_\mu = P(K, M, k)$.*

Proof. The group $N_{G(k_s)}(H)$ is Γ-stable, since H is k-defined. Let N be the closure of this group. Then N is a k-defined closed subgroup of G, H is normal in N, and $N(k) = N_{G(k)}(H)$. Now we apply [5, Thm. 4.4] with $(G', G, V, X, S) = (N, H, G^n, K^n, S_n(M))$ with the modifications indicated in the proof of Theorem 3.21. □

Definition 4.14. Maintain the notation as in Theorem 4.13. We call $\Omega(K, M, k)$ the *optimal class of k-defined cocharacters for K and M with respect to H* and we call $P(K, M, k)$ the *optimal destabilizing R-parabolic subgroup for K and M over k with respect to H*. If the G-conjugacy class given by Theorem 3.19 contains a group M of the form $M = c_\mu(K)$ for some $\mu \in Y_k(G)$, then we set $\Omega(K, k) := \Omega(K, M, k)$ and $P(K, k) := P(K, M, k)$. We call $\Omega(K, k)$ the *optimal class of k-defined cocharacters for K with respect to H* and we call $P(K, k)$ the *optimal destabilizing R-parabolic subgroup for K over k with respect to H*. If k is algebraically closed, then we usually suppress the k argument and write simply $\Omega(K, M)$, etc.

Remarks 4.15. (i). Suppose the G-conjugacy class given by Theorem 3.19 contains a group M of the form $M = c_\mu(K)$ for some $\mu \in Y_k(G)$ (this is automatic if k is algebraically closed). By Theorems 3.19 and 4.13, K and $N_{N_{G(k)}(H)}(K)$ are contained in $P(K, k)$ and for any $\mu \in \Omega(K, k)$, $c_\mu(K)$ is relatively G-completely reducible with respect to H.

(ii). If K is relatively G-completely reducible with respect to H, then, trivially, $P(K, k) = G$. If K is not relatively G-completely reducible with respect to H, then K is not contained in any R-Levi subgroup of $P(K, k)$ of the form L_μ with $\mu \in Y(H)$ and $P_\mu = P(K, k)$.

Now choose an $N_G(H)$-invariant k-defined norm $\|\ \|$ on $Y(H)$ (such norms exist, see [5, Def. 4.1]). We get the following rationality result. The proof is completely analogous to that of [5, Thm. 5.18].

Proposition 4.16. *Let K and n be as in Theorem 4.13 and assume that K is k-closed.*

(i) *Suppose that M is a subgroup of G such that $M = c_\lambda(K)$ for some $\lambda \in Y_{k_s}(G)$ and such that $S_n(M)$ is k-defined (this is the case in particular if M is k-defined). Then $\Omega(K, M, k_s)$ is well-defined and contains an element defined over k and $P(K, M, k_s)$ is defined over k.*

(ii) *If k is perfect, then $\Omega(K)$ is well-defined and contains an element defined over k and $P(K)$ is defined over k.*

The following is a converse to Theorem 4.12 which generalizes [2, Thm. 5.8].

Theorem 4.17. *Suppose k_1/k is an extension of perfect fields. Let K be a k-defined subgroup of G. If K is relatively G-completely reducible over k with respect to H, then K is relatively G-completely reducible over k_1 with respect to H.*

Proof. By Theorem 4.12, we may assume that $k_1 = \overline{k}$. Suppose K is not relatively G-completely reducible with respect to H. Then H is not contained in any R-Levi subgroup of the optimal destabilizing R-parabolic subgroup $P(K)$ with respect to H. Now $P(K)$ is k-defined by Proposition 4.16(ii), so H is not G-completely reducible over k with respect to H. □

5. Examples and counterexamples

5.1. **Relative $\mathrm{GL}(V)$-complete reducibility.** In this subsection we investigate the concept of relative complete reducibility in case the ambient reductive group is a general linear group.

Let V be a finite-dimensional k-vector space and set $G = \mathrm{GL}(V)$. Recall that if K is a subgroup of G, then K is G-cr if and only if V is a semisimple module for K. In this section we give an analogous interpretation for relative G-complete reducibility with respect to a smaller general linear group H inside G.

Proposition 5.1. *Let U be a subspace of V, and pick a direct complement $\tilde{U}$ to U. Let $H = \mathrm{GL}(U) \subseteq G$ embedded in the obvious way. Let K be a subgroup of $G = \mathrm{GL}(V)$. Then K is relatively G-completely reducible with respect to $H = \mathrm{GL}(U)$ if and only if the following two conditions hold:*

(i) *every K-submodule of V contained in U has a K-complement in V containing $\tilde{U}$;*

(ii) *every K-submodule of V containing $\tilde{U}$ has a K-complement in V contained in U.*

Proof. First suppose K is relatively G-cr with respect to H. For condition (i), let $W \subseteq U$ be a K-submodule of V contained in U, and suppose that the dimension of W is r. Choose a basis $\{e_1, \ldots, e_r\}$ for W and extend it to a basis $\{e_1, \ldots, e_m\}$ for U; together with our fixed basis for $\tilde{U}$, this gives us a basis $\{e_1, \ldots, e_n\}$ for all of V. Let $\lambda \in Y(G)$ be the cocharacter whose matrix has block form

$$\lambda(a) = \begin{pmatrix} aI_r & 0 \\ 0 & I_{n-r} \end{pmatrix}, \qquad \text{for } a \in k^*,$$

with respect to this basis. Then we can view λ as a cocharacter of H and of G, and P_λ is precisely the stabilizer of W in G, so $K \subseteq P_\lambda$. Since K is relatively G-cr with respect to H, there exists $u \in R_u(P_\lambda(H))$ such that $K \subseteq L_{u\cdot\lambda}$, by Lemma 3.3. Now L_λ corresponds to a decomposition $V = W \oplus W'$, for some subspace W' containing $\tilde{U}$, and thus $L_{u\cdot\lambda} = uL_\lambda u^{-1}$ corresponds to a decomposition $V = W \oplus \tilde{W}$, where $\tilde{W} = uWu^{-1}$ is now a K-submodule. Also, since $\tilde{U} \subseteq W'$, $u \in H$, and H centralizes $\tilde{U}$, we have $\tilde{U} \subseteq \tilde{W}$, as required.

For condition (ii), suppose W is a K-submodule of V such that $\tilde{U} \subseteq W$, and let the dimension of W be r. Let W' be any subspace of U such that $V = W' \oplus W$ as a vector space. Choose a basis $\{e_1, \ldots, e_{n-r}\}$ for W' and a basis $\{e_{n-r+1}, \ldots, e_m\}$ for $U \cap W$; together with our fixed basis for $\tilde{U}$, this gives a basis for all of V. Now let $\lambda \in Y(G)$ be the cocharacter whose matrix has block form

$$\lambda(a) = \begin{pmatrix} a^{-1}I_{n-r} & 0 \\ 0 & I_r \end{pmatrix} \qquad \text{for } a \in k^*,$$

with respect to this basis. Then we can view λ as a cocharacter of H and of G, and $K \subseteq P_\lambda$. Similarly to the previous paragraph, we can now conjugate the subspace W' to a K-submodule $\tilde{W} \subseteq U$, as required.

Conversely, suppose conditions (i) and (ii) hold, and let $\lambda \in Y(H)$ be a cocharacter with $K \subseteq P_\lambda$. We can choose a basis of U so that λ has the following block form

$$\lambda(a) = \begin{pmatrix} a^{n_1}I_{r_1} & 0 & \cdots & 0 & 0 \\ 0 & a^{n_2}I_{r_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a^{n_s}I_{r_s} & 0 \\ 0 & 0 & \cdots & 0 & I_{n-m} \end{pmatrix} \qquad \text{for } a \in k^*,$$

where each $n_i \in \mathbb{Z}$ with $n_1 > n_2 > \ldots > n_s$, and $\sum_{i=1}^{s} r_i = m$. Then $P_\lambda(H)$ is the stabilizer of a flag

$$0 = U_0 \subset U_1 \subset U_2 \subset \ldots \subset U_s = U$$

of subspaces in U. To see how this flag gives rise to a flag in V corresponding to P_λ, there are two cases to consider: firstly, when $n_i = 0$ for some i, and secondly, when we have a sequence $n_1 > \ldots > n_{i-1} > 0 > n_i > \ldots > n_s$ (here we include the extreme cases $0 > n_i$ and $0 < n_i$ for all i).

In the first case, P_λ is the stabilizer of a flag

$$U_0 \subset U_1 \subset \ldots \subset U_{i-1} \subset U_i \oplus \tilde{U} \subset U_{i+1} \oplus \tilde{U} \subset \ldots \subset U_s \oplus \tilde{U} = V$$

of subspaces of V. Since $K \subseteq P_\lambda$, each entry in this flag is a K-submodule (here we mean, for example, that $U_{i+1} \oplus \tilde{U}$ is K-stable, not that U_{i+1} and $\tilde{U}$ are K-stable individually); note also that each entry is either contained in U or contains $\tilde{U}$. Thus we can apply (i) and (ii) to successively find complements which provide a flag of K-submodules

$$U'_s \subset U'_{r-1} \subset \ldots \subset U'_i \subset U'_i \oplus \tilde{U} \subset \ldots \subset U'_1 \oplus \tilde{U} \subset U'_0 \oplus \tilde{U} = V,$$

where each U'_j is a subspace of U with $U = U_j \oplus U'_j$ (here, for each $1 \leq j \leq i-1$, we find a K-complement W_j to U_j containing $\tilde{U}$, and then we write $W_j = U'_j \oplus \tilde{U}$ as a vector space, where $U'_j = U \cap W_j$). This flag corresponds to a parabolic subgroup Q of G opposite to P_λ; i.e. $L := P_\lambda \cap Q$ is a Levi subgroup of P_λ. Since the flag defining Q consists of K-submodules, we have $K \subseteq Q$, hence $K \subseteq L$.

Now the flags $U_0 \subset \ldots \subset U_s$ and $U_0' \supset \ldots \supset U_s'$ are opposite flags in U, and thus correspond to opposite parabolic subgroups of H. The common Levi subgroup of these parabolics gives a decomposition of U into direct summands. Pick a new basis of U compatible with this decomposition, and define a new cocharacter μ of H which has the same block form with respect to this new basis as λ had with respect to the old basis. Then $P_\mu = P_\lambda$, $P_{-\mu} = Q$, and $L_\mu = L$.

In the second case, we get a flag

$$U_1 \subset \ldots \subset U_i \subset U_i \oplus \tilde{U} \subset U_{i+1} \oplus \tilde{U} \subset \ldots \subset U_s \oplus \tilde{U} = V,$$

and the same construction works. In both cases, we have shown that given $\lambda \in Y(H)$ with $K \subseteq P_\lambda$, there exists $\mu \in Y(H)$ with $P_\lambda = P_\mu$ and $K \subseteq L_\mu$; i.e., K is relatively G-cr with respect to H, as required. □

Remarks 5.2. (i). Note that conditions (i) and (ii) in Proposition 5.1 are "dual" to each other in the sense that the complement in (i) is a submodule of the form in (ii), and vice versa. One considers only those decompositions of V as a direct sum of K-submodules that are compatible with the fixed decomposition $V = U \oplus \tilde{U}$ (even though the latter is not required to be K-stable!). Even in the very special case considered in Proposition 5.1 the concept of a relatively completely reducible subgroup of $\mathrm{GL}(V)$ appears to be new.

(ii). Note that for $H = G$ in Proposition 5.1 we recover the fact that K is $\mathrm{GL}(V)$-completely reducible if and only if V is a semisimple K-module.

(iii). Fixing the complementary subspace $\tilde{U}$ to U in Proposition 5.1 fixes the embedding of H in G. Note that this is crucial, as the result depends on the choice of $\tilde{U}$. To see this, consider the special case that K acts completely reducibly on V and U is a K-submodule of V. Then it follows from Proposition 5.1 that K is relatively G-cr with respect to H if and only if $\tilde{U}$ is also a K-submodule.

We can refine Proposition 5.1 to show more accurately how the conclusion depends on the structure of V as a K-module, and how this interacts with the subspace U. To do this, we first define two operations on the set of subspaces of V. Firstly, for any subspace $W \subseteq V$, let $\sigma_K(W)$ be the K-submodule of V generated by the submodules contained in W; note $\sigma_K(W) \subseteq W$. Secondly, let $\iota_K(W)$ denote the smallest K-submodule of V containing W (i.e., the intersection of all such submodules). Then we have the following result.

Proposition 5.3. *Let G, U, $\tilde{U}$, H and K be as in Proposition 5.1. Then K is relatively G-completely reducible with respect to H if and only if the following two conditions hold:*

(i) *$\sigma_K(U)$ is a completely reducible K-module;*
(ii) *$V = \sigma_K(U) \oplus \iota_K(\tilde{U})$.*

Proof. Suppose K is relatively G-cr with respect to H. We first show (i). Let W be a K-submodule of $\sigma_K(U)$. Then $W \subseteq U$, so there exists a K-submodule $\tilde{W} \supseteq \tilde{U}$ with $V = W \oplus \tilde{W}$, by condition (i) of Proposition 5.1. Then $\tilde{W} \cap \sigma_K(U)$ is a K-complement to W in $\sigma_K(U)$.

We now show (ii). By condition (i) of Proposition 5.1, since $\sigma_K(U)$ is a K-submodule contained in U, there exists a K-complement to $\sigma_K(U)$ containing $\tilde{U}$. Since $\iota_K(\tilde{U})$ is the smallest such K-submodule, we must have $\sigma_K(U) \cap \iota_K(\tilde{U}) = \{0\}$. By condition (ii) of Proposition 5.1, since $\iota_K(\tilde{U})$ is a K-submodule containing $\tilde{U}$, there exists a K-complement

to $\iota_K(\tilde{U})$ contained in U. Since $\sigma_K(U)$ is the largest such K-submodule, we must have $\sigma_K(U) + \iota_K(\tilde{U}) = V$.

Now suppose (i) and (ii) hold. If W is a K-submodule of V contained in U, then $W \subseteq \sigma_K(U)$, by definition. By (i), there exists a K-complement W' to W in $\sigma_K(U)$. Then $W' \oplus \iota_K(\tilde{U})$ is a K-complement to W in V containing $\tilde{U}$, by (ii), giving condition (i) of Proposition 5.1. If W is a K-submodule of V containing $\tilde{U}$, then $W \supseteq \iota_K(U)$, by definition. Let W' be a K-complement to $W \cap \sigma_K(W)$ in $\sigma_K(W)$. Then W' is a K-complement to W in V contained in U, giving condition (ii) of Proposition 5.1. □

The following corollary gives a necessary condition for a subgroup to be relatively G-cr with respect to a Levi subgroup L of G in terms of the corresponding direct sum decomposition of V:

Corollary 5.4. *Suppose L is a Levi subgroup of $G = \mathrm{GL}(V)$, with corresponding decomposition $V = U_1 \oplus \cdots \oplus U_s$. Let K be a subgroup of G. Then if K is relatively G-completely reducible with respect to L, the following two conditions hold for each i:*

(i) *every K-submodule of V contained in U_i has a K-complement containing $\bigoplus_{j\neq i} U_j$;*
(ii) *every K-submodule of V containing $\bigoplus_{j\neq i} U_j$ has a K-complement contained in U_i.*

Proof. We have $L = \mathrm{GL}(U_1) \times \cdots \times \mathrm{GL}(U_s)$. If K is relatively G-cr with respect to L then Proposition 3.30 implies that K is relatively G-cr with respect to $\mathrm{GL}(U_i)$ for each i. The result now follows from Proposition 5.1. □

5.2. **More examples and counterexamples.**

Remark 5.5. We noted in Remark 3.2(iv) that in general a relative G-cr subgroup need not be reductive. However, one can ensure that it is reductive under suitable conditions. For example, if H is a maximal rank reductive subgroup of G and K is a subgroup of G which is relatively G-completely reducible with respect to H and which is normalized by a maximal torus of H, then K is reductive.

In order to show this we closely follow Serre's original argument [19, Prop. 4.1]. Suppose that $R_u(K) \neq \{1\}$. By [7, Prop. 3.1] (which extends to the case when G is not connected, thanks to [12, Prop. 5.4]), there exists an R-parabolic subgroup P of G such that $N_G(R_u(K)) \subseteq P$ and $R_u(K) \subseteq R_u(P)$. In particular, $T, K \subseteq P$ for some maximal torus T of H. Since H has maximal rank, T is also a maximal torus of G, so we can find $\lambda \in Y(T) \subseteq Y(H)$ such that $P_\lambda = P$. Since K is relatively G-cr with respect to H, there exists $\mu \in Y(H)$ such that $K \subseteq L_\mu$ and $P_\lambda = P_\mu$. It follows that $R_u(K) \subseteq L_\mu \cap R_u(P_\mu) = \{1\}$, a contradiction. Consequently, K is reductive, as claimed.

We noted at the beginning of subsection 3.7 that the direct analogue of [2, Thm. 3.10], namely that a normal subgroup N of a subgroup M of G is relatively G-cr with respect to some reductive subgroup H of G provided M is, fails in general. We now present two examples which demonstrate this failure.

Example 5.6. Let $\mathrm{char}\, k$ be arbitrary. Let $G = \mathrm{GL}_3(k)$ and let H be the image of $\mathrm{SL}_2(k)$ embedded in G by $A \mapsto \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array}\right)$. Let T be the standard maximal torus of G consisting of the diagonal matrices in G. Let α and β be the standard simple roots of G with respect to T (corresponding respectively to the $(2,3)$- and $(1,2)$-entries of matrices in G). Let B be

the Borel subgroup of G consisting of upper triangular matrices in G. Put $K = U_\alpha U_{\alpha+\beta}$. Then K is relatively G-cr with respect to H, but the normal subgroups U_α and $U_{\alpha+\beta}$ of K are not.

Take for example $N = U_{\alpha+\beta}$ and define $\lambda \in Y(H)$ by $\lambda(a) = \mathrm{diag}(1, a, a^{-1})$. Then $N \subseteq P_\lambda = s_\alpha B s_\alpha$ which is a Borel subgroup of G. Clearly, N is not in any Levi subgroup of P_λ, since $N \subseteq R_u(P_\lambda)$. So $U_{\alpha+\beta}$ is not relatively G-cr with respect to H. The argument for U_α is similar; take the same $\lambda \in Y(H)$ and replace P_λ by $P_{-\lambda}$.

Now we show that K is relatively G-cr with respect to H by showing that $\lambda \in Y(H)$ and $H \subseteq P_\lambda$ implies that $\lambda = 1$. Let $\lambda \in Y(H)$. We can find $h \in H$ such that $\mu := h \cdot \lambda$ is in diagonal form. Then we have for $k \in K$ that $\lim_{a\to 0} \lambda(a)k\lambda(a)^{-1}$ exists if and only if $\lim_{a\to 0} \mu(a)hkh^{-1}\mu(a)^{-1}$ exists. Since K is stable under conjugation by H we may now assume that λ is in diagonal form, that is, $\lambda(a) = \mathrm{diag}(1, a^n, a^{-n})$ for some integer n. It is now straightforward to show that $n = 0$.

The next example shows that a connected reductive normal subgroup of a connected reductive group that is relatively G-cr with respect to H need not be relatively G-cr with respect to H.

Example 5.7. Suppose $p = 2$. Let V_1 and V_2 be copies of the vector space k^2 and let $H_1 = H_2 = k^* \times \mathrm{SL}_2(k)$. Define $\delta_i \in Y(H_i)$ by $\delta_i(x) = (x, I_i)$, where I_i is the identity matrix in H_i. Define an action of $H_1 \times H_2$ on $V_1 \oplus V_2$ by

$$((x_1, A_1), (x_2, A_2)) \cdot (v_1, v_2) = (x_2^{-1} A_1 v_1, x_1^{-1} A_2 v_2),$$

where $A_i v_i$ denotes the usual matrix product. We choose an embedding of $(H_1 \times H_2) \ltimes (V_1 \oplus V_2)$ inside a reductive group G. Let $\pi_i \colon H_i \times V_i \to V_i$ be the canonical projection. Let M_i be the copy of $\mathrm{SL}_2(k)$ inside H_i. We can choose a copy N_i of $\mathrm{PGL}_2(k)$ inside $M_i \ltimes V_i$ such that $\pi_i(N_i) = M_i$ but N_i is not $(H_i \ltimes V_i)$-conjugate to a subgroup of H_i; to see this, note that the image of the adjoint representation of $\mathrm{SL}_2(k)$ in $\mathrm{GL}_3(k)$ lies in $[P, P]$, where P is a maximal parabolic subgroup of $\mathrm{GL}_3(k)$, and $[P, P]$ is isomorphic to $M_i \ltimes V_i$, so we can take N_i to be this image.

Let $H = \{((x_1, A_1), (x_2, A_2)) \in H_1 \times H_2 \,|\, x_2 = x_1^{-1}\}$ and set $K = N_1 N_2$. Then K is isomorphic to $N_1 \times N_2$, since N_1 and N_2 commute with each other and have disjoint Lie algebras. So K is a connected reductive group and N_1, N_2 are connected normal subgroups of K. We show that K is relatively G-cr with respect to H but N_1 and N_2 are not.

Let $\lambda \in Y(H)$ such that $K \subseteq P_\lambda$. We can write $\lambda = \lambda_1 + \lambda_2$ where $\lambda_i \in Y(H_i)$. We have $N_i \subseteq P_\lambda$. Now $Z(H_i) \subseteq P_\lambda$ because $Z(H_i)$ centralizes the image of λ. Since $Z(H_i)$ acts trivially on M_i and acts non-trivially as multiplication by scalars on V_i, it is clear that M_i and the non-trivial subgroup U_i of V_i generated by the set $\{\pi_i(n)n^{-1} \,|\, n \in N_i\}$ are both contained in P_λ. Because M_i does not lie in a proper parabolic subgroup of H_i, we must have $\lambda = n_1\delta_1 + n_2\delta_2$ for some $n_1, n_2 \in \mathbb{Z}$. We must have $n_1 + n_2 = 0$ by our choice of H. We have $\lambda(x)v\lambda(x)^{-1} = x^{n_1 - n_2}v$ for $v \in V_1$ and $\lambda(x)v\lambda(x)^{-1} = x^{n_2 - n_1}v$ for $v \in V_2$. Since the non-trivial subgroups U_1 and U_2 lie in P_λ, this forces $n_1 = n_2 = 0$. Hence λ is the trivial cocharacter and we conclude that K is relatively G-cr with respect to H.

A similar argument shows that $N_1 \subseteq P_\lambda$, where $\lambda = \delta_1 - \delta_2$. Now N_i is not H-conjugate to a subgroup of H_i, and this implies that N_1 is not relatively G-cr with respect to H. Similarly N_2 is not relatively G-cr with respect to H.

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