Note on the Poisson structure of the damped oscillator

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Abstract

The damped harmonic oscillator is one of the most studied systems with respect to the problem of quantizing dissipative systems. Recently Chakravarty et al. (J. Math. Phys. 48, 032701 (2007)) applied the Prelle-Singer method to construct conserved quantities and an explicit time-independent Lagrangian and Hamiltonian structure for the damped oscillator. Here we describe the associated Poisson bracket which generates the continuous flow, pointing out that there is a subtle problem of definition on the whole phase space. The action-angle variables for the system are also presented, and we further explain how to extend these considerations to the discrete setting. Some implications for the quantum case are briefly mentioned.

1 Introduction

The damped oscillator, given by the second order differential equation

$$\ddot{x} + \alpha\dot{x} + \beta x = 0, \tag{1}$$

has been extensively studied for over sixty years from the point of view of quantization. The interest in this model, which is elementary from the viewpoint of the classical theory of differential equations, is due to the fact that in the physically interesting case, where $\alpha > 0$ and $\beta > 0$, the dynamics is dissipative. Rewriting the equation (1) as the first order system

$$\dot{x} = y, \qquad \dot{y} = -\beta x - \alpha y, \tag{2}$$

the energy $E(t) = \frac{1}{2}(y^2 + \beta x^2)$ satisfies $\frac{dE}{dt} = -\alpha y^2 < 0$, so the system loses energy ($E(t) \to 0$ as $t \to \infty$) compared with the harmonic oscillator ($\alpha = 0$, $dE/dt = 0$; we set the particle mass to be 1 throughout). The non-conservative and irreversible nature of the system is hard to reconcile with quantum mechanics, where usually the Hamiltonian operator has a time-independent spectrum and generates a reversible unitary evolution on an appropriate Hilbert space of wave functions.

One way to deal with the quantization of the damped oscillator, which goes back to Caldirola and Kanai [1], is to start from a time-dependent Lagrangian for the classical system, namely $L(x, \dot{x}, t) = \frac{1}{2}e^{\alpha t}(\dot{x}^2 - \beta x^2)$. This gives the time-dependent Hamiltonian $H = H(x, p, t) = \frac{1}{2}(p^2 e^{-\alpha t} + x^2 e^{\alpha t})$, whose canonical quantization leads to the conclusion that the Heisenberg uncertainty principle can be violated by the system for sufficiently large t (see chapter 13 in [2]). As the frictional force corresponding to the velocity-dependent damping term $-\alpha\dot{x}$ can be regarded as a macroscopic effect due to the environment, it is perhaps unsurprising that the microscopic rules of quantum theory seem to break down here. Nevertheless, this has not reduced the necessity of finding consistent ways to treat dissipative quantum systems, both for the quantum theory of decoherence and in the description of heavy ion scattering, radiating electrons, and many other physical processes. One way to model dissipation is to couple the harmonic oscillator to a heat bath, while another popular approach (due to Bateman [3]) is to double the number of degrees of freedom by combining

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the damped oscillator with its time-reversed version, or by complexifying the system (see e.g. [4]). For a detailed historical survey of different approaches to the quantum damped oscillator, with extensive references, the reader is referred to [5], while an updated account of classical and quantum dissipative models and their applications is provided by the recent monograph [2].

In recent work with other co-authors [6], one of us applied the Prelle-Singer method in order to find explicit formulae for first integrals and time-independent Lagrangians for the damped oscillator. In particular, for the case of underdamped motion, $\alpha^2/4 < \beta$, an explicit Lagrangian is given by

$$L(x,\dot{x}) = -\frac{1}{2}\log\left(\dot{x}^2 + \alpha x\dot{x} + \beta x^2\right) + \frac{1}{\omega}\left(\frac{\dot{x}}{x} + \frac{\alpha}{2}\right)\arctan\left[\frac{1}{\omega}\left(\frac{\dot{x}}{x} + \frac{\alpha}{2}\right)\right], \tag{3}$$

where $\omega = \sqrt{\beta - \alpha^2/4}$. Applying a Legendre transformation to this Lagrangian yields the Hamiltonian

$$H(x,p) = \log x - \log\cos(\omega p x) - \frac{\alpha}{2}px, \tag{4}$$

with the canonically conjugate momentum $p = \partial L/\partial\dot{x}$ (not to be confused with the kinetic momentum $y = \dot{x}$). This Hamiltonian was first found by Havas [7], who made a detailed study of possible Lagrangian structures for the damped oscillator. A full review of time-dependent/independent Lagrangians and the transformations between them, including the formula (3), is given in chapter 3 of [2], while in chapter 4 the Hamiltonian (4) is compared with the Caldirola-Kanai formulation.

The purpose of this short note is to present a simple derivation of the possible Poisson brackets for the classical damped oscillator, as well as constructing the action-angle variables for the system and pointing out some problems with the definition of the Hamiltonian. It is also explained how these results extend immediately to the discrete case (that is, linear difference equations of second order), and we conclude with some remarks on the quantum case.

2 Poisson brackets

To begin with we make a simple observation concerning first integrals for linear ordinary differential equations with constant coefficients. A scalar equation of Nth order for $x = x(t)$ can be written as $P(D)x = 0$, where $D = d/dt$ and P is a polynomial of degree N having roots $\lambda_1, \lambda_2, \ldots, \lambda_N$. Such an equation can be rewritten as a first order matrix system $\dot{\mathbf{x}} = M\mathbf{x}$ for an $N \times N$ constant matrix M, and for any such matrix let λ_j, $j = 1, \ldots, N$ be the roots of its characteristic polynomial $P(\lambda) = \det(M - \lambda I)$. Suppose that the matrix M is diagonalizable (which is true, in particular, when the λ_j are distinct), in which case there is a linear transformation L transforming to the eigenvector basis, $\mathbf{z} = L\mathbf{x}$, such that the components of the vector $\mathbf{z}$ satisfy $\dot{z}_j = \lambda_j z_j$, $j = 1, \ldots, N$. Then for $N > 1$ each of the quantities $C_{j,k} = z_j^{\lambda_k} z_k^{-\lambda_j}$ $(j \neq k)$ is a time-independent first integral for the linear equation $\dot{\mathbf{x}} = M\mathbf{x}$, being a function of the phase variables (that is, the components of $\mathbf{x}$) and the constant matrix M only. Moreover, the quantities $C_{j,j+1}$ for $j = 1, \ldots, N-1$ are functionally independent.

For the particular case $N = 2$ corresponding to the equation (1), or equivalently the system (2), the matrix $M = \begin{pmatrix} 0 & 1 \\ -\beta & -\alpha \end{pmatrix}$, with eigenvalues $\lambda_\pm = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - \beta}$, and the system decouples as

$$\dot{z} = \lambda_+ z, \quad \dot{w} = \lambda_- w, \qquad \text{for} \quad z = y - \lambda_- x, \quad w = y - \lambda_+ x.$$

To provide the system with a (time-independent) Hamiltonian structure it is necessary to specify a particular first integral (the Hamiltonian $H = H(z,w)$) together with a Poisson bracket $\{\,,\,\}$ such that the equations of motion in the coordinates z, w take the form $\dot{z} = \{z, H\}$, $\dot{w} = \{w, H\}$. If we take $H = f(C)$ to be a smooth function of the first integral $C = z^{\lambda_-} w^{-\lambda_+}$ then the Poisson bracket is found immediately to be

$$\{z,w\} = -z^{1-\lambda_-} w^{1+\lambda_+}/f'(C) = -\frac{zw}{C\,f'(C)}.$$

Equivalently, the symplectic form is $\frac{Cf'(C)}{zw}\,dz\wedge dw$. Since z and w are functions of x, y, the integral C can be rewritten in terms of them, and then in the original coordinates the bracket is given by $\{x,y\} = (\lambda_- - \lambda_+)^{-1}(y^2+\alpha xy+\beta x^2)/(Cf'(C))$. The freedom in the choice of time-independent Hamiltonian structure for the damped oscillator corresponds to the choice of f.

Up to an overall constant multiplier, there are two choices of f that are of particular interest here. The first choice is $f(C) = \frac{1}{2}C^{2/(\lambda_- - \lambda_+)}$ (for $\lambda_+ \neq \lambda_-$), which gives

$$H = \frac{1}{2}(y^2+\alpha xy+\beta x^2)\left(\frac{y-\lambda_- x}{y-\lambda_+ x}\right)^{\alpha/(\lambda_+-\lambda_-)}, \qquad \{x,y\} = ((y-\lambda_- x)/(y-\lambda_+ x))^{-\alpha/(\lambda_+-\lambda_-)}. \quad (5)$$

This choice has the advantage that when $\alpha = 0$ the Hamiltonian reduces to the standard expression for the energy of the harmonic oscillator, and in that case x and $y = \dot{x}$ are canonically conjugate variables i.e. $\{x,y\} = 1$. The above formula for H is precisely the form of first integral that was given in [8], where the Lagrangian and an implicit Hamiltonian structure was given only in the underdamped case $\alpha^2/4 < \beta$. The other interesting choice is $f(C) = \log C$, when $Cf'(C) = 1$, in which case (after rescaling) the Poisson bracket for the variables z, w is of log-canonical type $\{z,w\} = zw$ (i.e. canonical in the variables $\log z$, $\log w$), while the first integral $H = \log C$ is basically the same as the quantity I given in [6] for the underdamped and critically damped cases, up to rescaling and shifting by a constant.

At this point it is important to point out the slightly formal nature of the above results. Although essentially correct, the expression for the first integral C suffers from the serious defect that it is not a single-valued real function on the whole phase space: the eigenvalues $\lambda_\pm$ are algebraic functions of the coefficients α, β, and taking them as exponents means that C is a transcendental function of z and w which is only uniquely defined and real when $\lambda_\pm$ are real and $z > 0$, $w > 0$. In order to make precise sense of the foregoing expressions it is necessary to distinguish between the underdamped case $\alpha^2/4 < \beta$, the overdamped case $\alpha^2/4 > \beta$, and the critical case $\alpha^2/4 = \beta$.

2.1 The underdamped case

In the underdamped case $\alpha^2/4 < \beta$ the eigenvalues are a complex conjugate pair, $\lambda_\pm = -\alpha/2 \pm i\omega$, and the variables z, w are also complex conjugates, $w = \overline{z}$. It is convenient to use polar coordinates ρ, θ such that

$$x = \omega^{-1}\rho\sin\theta, \qquad y + \frac{\alpha}{2}x = \rho\cos\theta, \quad (6)$$

so that $z = \rho e^{i\theta}$, $w = \rho e^{-i\theta}$. In these coordinates, we can take the first choice above, namely $H = \frac{1}{2}C^{2/(\lambda_- - \lambda_+)}$, so that the Hamiltonian (5) and the Poisson bracket become

$$H = \frac{1}{2}\rho^2 e^{\frac{\alpha\theta}{\omega}}, \qquad \{\theta,\rho\} = \frac{\omega}{\rho}e^{-\frac{\alpha\theta}{\omega}}. \quad (7)$$

In terms of the original coordinates, $\rho^2 = (y^2+\alpha xy+\beta x^2)$, the polar angle is $\theta = \arg z$, $z = y+\frac{\alpha}{2}x+i\omega x$, and $\{x,y\} = e^{-\frac{\alpha\theta}{\omega}}$. The equations of motion for the radius and angle are $\dot{\rho} = \{\rho, H\} = -\alpha\rho$, $\dot{\theta} = \{\theta, H\} = \omega$, describing spiral orbits.

The problem with the above choice of first integral H is that the polar angle θ is not a single-valued function of the phase plane coordinates $(x,y) \in \mathbb{R}^2$, and in the formula (7) for H this angle appears in the argument of a real exponential; the same is true of the Poisson bracket. Thus, H is not a single-valued function on the phase space: upon making a circuit around a loop enclosing the origin, the value of θ increases by 2π and the value of H is multiplied by $e^{\frac{2\pi\alpha}{\omega}}$. There are various ways to resolve this problem. One possible solution is to work in a cut plane, removing the half-line $\theta = 0$; this makes H single-valued, and is equivalent to making a branch cut for the complex function $\log z$. However, this is not a very satisfactory solution because the spiral trajectories generated by the Hamiltonian H cross the branch cut infinitely many times; thus the solutions of the system only remain within the cut plane for a finite time of length $2\pi/\omega$. A more satisfactory possibility is to take an enlarged phase space where both H and the Poisson bracket are single-valued, namely the infinite-sheeted covering of the plane (equivalent to the Riemann

surface of $\log z$) with coordinates $(\rho,\theta) \in \mathbb{R}_{>0} \times \mathbb{R}$ (where it is also convenient to remove the origin $\rho = 0$, as the bracket in (7) is undefined there). The orbits in the enlarged phase space project down onto the original phase plane via the formulae (6). Yet another possibility is to seek a different choice of first integral which is well-defined on the phase plane. One such choice is $H = \sin\varphi$ with $\varphi = \theta + 2\omega\alpha^{-1}\log\rho$; this is a single-valued smooth function on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, for which a suitable coordinate chart is $(\rho,\theta) \in \mathbb{R}_{>0} \times [0, 2\pi)$. However, the associated Poisson bracket is $\{\theta,\rho\} = \frac{1}{2}\alpha\rho\sec\varphi$ which is not defined on each of the spirals $\varphi = \pi/2 \bmod 2\pi$, $\varphi = 3\pi/2 \bmod 2\pi$, so these two orbits should also be removed from the phase space in this case. Another disadvantage of the latter choice of H is that it does not reduce to the harmonic oscillator Hamiltonian as $\alpha \to 0$.

Given one of the above choices of phase space with a single-valued Hamiltonian and Poisson bracket, the underdamped oscillator is a completely integrable system with one degree of freedom. Then $I = H/\omega$, θ are the action-angle variables (see [9], chapter 10), satisfying

$$\{\theta, I\} = 1, \qquad \dot{I} = -\frac{\partial H}{\partial\theta} = 0, \quad \dot{\theta} = \frac{\partial H}{\partial I} = \omega \qquad \text{for} \qquad H(I) = \omega I.$$

However, note that the level sets of H are non-compact: the orbits are the spirals $\varphi = \theta + 2\omega\alpha^{-1}\log\rho = \theta_0 \bmod 2\pi$, which are fixed by a choice of $\theta_0 \in [0, 2\pi)$. Thus in order to interpret the angle θ as a coordinate on a fixed level set, it must range over the whole of $\mathbb{R}$, rather than being measured mod 2π as in the compact case.

We should point out that the Lagrangian (3) suffers from essentially the same problem of definition as highlighted here, because of the multi-valuedness of the arctangent. Hence the Hamiltonian of Havas, obtained from it by a Legendre transformation, is similarly problematic, because the definition of the conjugate momentum p in terms of x and $\dot{x}$ also involves the arctangent. The Hamiltonian (4) is also singular at $x = 0$ and on the hyperbolae $\omega xp = (2k+1)\pi/2$, $k \in \mathbb{Z}$ in the x, p plane; up to a shift and rescaling this Hamiltonian is just the logarithm of H in (7).

2.2 The overdamped case

In the overdamped case, the variables z, w are real, and then by defining $\zeta^2 = \frac{\alpha^2}{4} - \beta$, $\lambda_\pm = -\alpha/2 \mp \zeta$ the Hamiltonian

$$H = \frac{1}{2}|z|^{1-\alpha/(2\zeta)}|w|^{1+\alpha/(2\zeta)} = \frac{1}{2}\left|\left(y + \frac{\alpha}{2}x\right)^2 - \zeta^2 x^2\right| \exp\left[\frac{\alpha}{2\zeta}\log\left|\frac{y + (\alpha/2+\zeta)}{y + (\alpha/2-\zeta)x}\right|\right]$$

is well-defined and smooth on $\mathbb{R}^2 \setminus \{y + (\alpha/2-\zeta)x = 0\} \setminus \{y + (\alpha/2+\zeta)x = 0\}$ (that is, the plane with the lines $z = 0$ and $w = 0$ removed). In the pair of disconnected regions where $zw > 0$ we can set $z = \rho e^{-\xi}$, $w = \rho e^{\xi}$ for $\rho > 0$ in one component, $\rho < 0$ in the other and $\xi \in \mathbb{R}$. Similarly for $zw < 0$ we can take $z = -\rho e^{-\xi}$, $w = \rho e^{\xi}$ with the same ranges for ρ and ξ. Thus we have

$$x = \begin{cases} \rho\zeta^{-1}\sinh\xi, & zw > 0, \\ \rho\zeta^{-1}\cosh\xi, & zw < 0, \end{cases} \quad \text{and} \quad y + \frac{\alpha}{2}x = \begin{cases} \rho\cosh\xi, & zw > 0, \\ \rho\sinh\xi, & zw < 0, \end{cases} \tag{8}$$

while $H = \frac{1}{2}\rho^2 e^{\alpha\xi/\zeta}$ throughout. In the coordinates (ρ,ξ) the Poisson bracket is $\{\xi,\rho\} = \zeta\rho^{-1}e^{-\alpha\xi/\zeta}$, which is essentially just an analytic continuation of the bracket in the underdamped case, setting $\theta = i\xi$, $\omega = i\zeta$. The action-angle variables are obtained similarly.

2.3 The critically damped case

The critically damped case corresponds to repeated roots of the characteristic polynomial, $\lambda_+ = \lambda_-$, and hence $z = w$ and the formula for C becomes trivial. Instead, one can take the conserved quantity $H = \frac{1}{2}(y + \alpha x/2)^2 \exp[2\alpha x/(2y + \alpha x)]$ as the Hamiltonian, with the Poisson bracket $\{x, y\} = \exp[-2\alpha x/(2y + \alpha x)]$, which correctly reproduces the Hamiltonian and canonical bracket for the free particle in the limit $\alpha \to 0$. However, the line $y + \frac{\alpha}{2}x = 0$ must be removed from the phase space.

3 The discrete case

The discrete analogue of the above model is the second order linear difference equation

$$x_{n+1} = ax_n + bx_{n-1} \tag{9}$$

with constant coefficients a, b. The general solution of the recurrence is $x_n = K_1\mu_+^n + K_2\mu_-^n$ where $\mu_\pm$ are the roots of $\mu^2 = a\mu + b$, and K_1, K_2 are arbitrary constants. For our purposes it is handy to rewrite this solution as

$$x_n = A\exp(nw)\sinh(v_0 + nv), \qquad \text{for} \quad a = 2e^w\cosh v, \quad b = -e^{2w},$$

where A, v_0 are two arbitrary constants; the roots of the characteristic quadratic are given in terms of the parameters w, v by $\mu_\pm = \exp(w \pm v)$. With this parametrization, w, v and the constants A, v_0 must be allowed to be complex, even for real coefficients a, b. For example, the Fibonacci sequence ($a = b = 1$ with $x_1 = x_2 = 1$) arises from $w = i\pi/2$, $v = \operatorname{arcsinh}(1/2) - i\pi/2$ with $A = 2/\sqrt{5}$, $v_0 = 0$.

The linear recurrence can be reinterpreted as an iterated map in the plane, $\mathbf{v} \mapsto M\mathbf{v}$, with $\mathbf{v} = (x, y)^T$ being a vector and the matrix $M = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}$. This is equivalent to rewriting the recurrence as the first order system $x_{n+1} = y_n$, $y_{n+1} = bx_n + ay_n$. Now $\mu_\pm$ are the eigenvalues of M, and (assuming they are distinct) the matrix can be diagonalized to rewrite the system in decoupled form as $z_{n+1} = \mu_+ z_n$, $w_{n+1} = \mu_- w_n$, where $z_n = y_n - \mu_- x_n$, $w_n = y_n - \mu_+ x_n$. Then $C_n = (z_n)^{\log\mu_-}(w_n)^{-\log\mu_+}$ is independent of n, i.e. $C_{n+1} = C_n = C$, so this is a conserved quantity of the map on the phase space (the plane). To write a conserved quantity in terms of the original (x, y) coordinates in the plane it is more convenient to take $H = \frac{1}{2}C^{-1/v}$, which yields

$$H = \frac{1}{2}(y^2 - axy - bx^2)[(y - \mu_- x)/(y - \mu_+ x)]^{-w/v}. \tag{10}$$

The same issues of multi-valuedness arise for this conserved quantity as in the continuous case.

For this map of the plane to be regarded as a completely integrable map in the Liouville-Arnold sense [10], it must preserve a symplectic structure in addition to having the conserved quantity H. If we parametrize

$$x = r\sinh u, \qquad y = r\exp(w)\sinh(u + v)$$

(corresponding to how the solution for x_n was written to begin with) then in the (potentially complex) coordinates (u, r) the map is $r \mapsto e^w r$, $u \mapsto u + v$, we have $H = \frac{r^2}{2}\exp(2w(1 - u/v))\sinh^2 v$, and the two-form $r^{-1}dr \wedge du$ is preserved by the map. This symplectic form corresponds to the Poisson bracket $\{u, r\} = r$, which implies that

$$\{x, y\} = r^2 e^w \sinh v = (y^2 - axy - bx^2)/(e^w \sinh v). \tag{11}$$

So this is the sense in which the map (or equivalently the recurrence) is a completely integrable system.

The conserved quantity H is clearly a transcendental (and multi-valued) function of (x, y) for a generic choice of coefficients a, b. However, for $w/v = p/q \in \mathbb{Q}$ (written as a fraction in lowest terms) the qth power of H is a rational function of x and y. In particular, when $w = 0$, which corresponds to $b = -\mu_+\mu_- = -1$, we have the recurrence

$$x_{n+1} - ax_n + x_{n-1} = 0, \tag{12}$$

which is satisfied by the Chebyshev polynomials $P_n(a/2) = \cos n\vartheta$ with $a = 2\cos\vartheta$. This particular recurrence is equivalent to the map $x \mapsto y$, $y \mapsto ay - x$, which has a polynomial conserved quantity and an invariant Poisson bracket given by[1]

$$H = H(x, y) = \frac{1}{2}(x^2 - axy + y^2), \qquad \{x, y\} = 1.$$

[1]Note that in order to get the canonical bracket, we have rescaled the $b = -1$ case of the Poisson bracket (11) by the factor $2H(x, y)/\sinh v$, which is constant along each orbit of the map.

This special case is also "dual" to the nonlinear recurrence $x_{n+1}x_{n-1} = x_n^2 - H$, in the sense that the orbit of the recurrence (12) starting from the initial data $(x, y) = (x_0, x_1)$ is the same as the orbit for the nonlinear one provided that $a = x/y + y/x - H/(xy)$; see [11] for the case $H = -1$. The nonlinear map preserves the log-canonical Poisson bracket $\{x, y\} = xy$ [12].

It is straightforward to show that the Hamiltonian (10) generates a linear flow $\dot{x} = \{x, H\}$, $\dot{y} = \{y, H\}$ with the bracket (11), which implies the following second order equation:

$$\ddot{x} - \frac{4wH}{v}\dot{x} + 4H^2\left(\frac{w^2}{v^2} - 1\right)x = 0.$$

Since the orbit of the map and the continuous flow (starting from the same point in phase space) both lie on the same one-dimensional level set $H = \text{const}$, this means that the map given by (9) is an exact discretization of the damped oscillator, in the sense that it interpolates the flow. A more direct way to see this is to start from the explicit formula for the general solution $x(t)$ for (1), and then upon setting $x_n = x(nv)$ it is simple to derive an addition formula of the form (9) that connects x_n to $x_{n\pm1}$. In particular, the recurrence (12) is the exact discretization of the harmonic oscillator. This is a special case of a result in [13] which says that an ordinary differential equation whose general solution is known admits an exact discretization scheme.

4 Conclusions

Although there is a considerable amount of literature devoted to the damped oscillator, as far as we are aware the problem of multi-valuedness of the Hamiltonian and Poisson structure has not been discussed before. The problem of the correct choice of definition for the Hamiltonian in the classical case is the origin of some of the difficulties in finding a consistent quantization of the system, since the choice of time-independent H determines the Poisson bracket, which in turn determines pairs of canonically conjugate coordinates and momenta. However, if it is required that the Hamiltonian for the underdamped case should reduce to the harmonic oscillator in the limit $\alpha \to 0$, then the choice of time-independent H is unique up to the addition of terms of order α. As emphasized by Razavy [14], different choices of Hamiltonian structure and canonically conjugate variables produce different wave functions and energy levels upon quantization. Moreover, since the classical Hamiltonian for the damped oscillator is always a transcendental function of x and/or its conjugate momentum p (cf. equation (4) for instance), attempting to canonically quantize in these variables produces serious problems of operator ordering, and generally produces inconsistent results. Indeed, due to the Grünwald-van Hove Theorem (see the first appendix in [15]), even canonical quantization of *polynomial* Hamiltonians of degree three or higher can produce ambiguous results.

As an example of a naive attempt to quantize the Hamiltonian (5), we note that the variables x, y (which are not canonically conjugate) are not suitable due to the transcendental factor. However, using polar coordinates we can rewrite the Hamiltonian as $H = \alpha R\Theta$ where $R = \rho^2/2$ and $\Theta = \alpha^{-1}e^{\alpha\theta/\omega}$. These variables are canonically conjugate, $\{\Theta, R\} = 1$, so we wish to replace them by self-adjoint operators $\hat{\Theta}$, $\hat{R}$ satisfying $[\hat{\Theta}, \hat{R}] = i\hbar$. Upon symmetrizing, the quantum Hamiltonian should be $\hat{H} = \frac{\alpha}{2}(\hat{\Theta}\hat{R} + \hat{R}\hat{\Theta})$, which is self-adjoint provided that $\hat{\Theta}$ and $\hat{R}$ are. Taking a wave function Ψ on coordinate (Θ) space, regarded as $\Psi = \Psi(\Theta)$, the operator $\hat{R} = -i\hbar\partial/\partial\Theta$ is formally self-adjoint and satisfies the correct canonical commutation relation with $\hat{\Theta}$. If we define $\psi(\theta) = \Psi(\Theta)$ then the time-independent Schrödinger equation $\hat{H}\Psi = E\Psi$ becomes $-i\hbar\omega\partial\psi/\partial\theta = (E + i\alpha\hbar/2)\psi$. For discrete energy levels it is sufficient to require that ψ is periodic with period 2π, which immediately implies $E = n\hbar\omega - i\alpha\hbar/2$ for integer n. Apart from the lack of ground state, this reproduces the semi-classical spectrum of the harmonic oscillator in the limit $\alpha \to 0$. However, the fact that the spectrum is complex for $\alpha \neq 0$ means that $\hat{H}$ cannot be self-adjoint, which contradicts the original assumptions about $\hat{\Theta}$, $\hat{R}$.

Many of the above considerations should also apply to other damped *nonlinear* oscillators, such as the modified Emden oscillator [16].

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