ON THE NON-INTEGRABILITY OF THE POPOWICZ PEAKON SYSTEM

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ABSTRACT. We consider a coupled system of Hamiltonian partial differential equations introduced by Popowicz, which has the appearance of a two-field coupling between the Camassa-Holm and Degasperis-Procesi equations. The latter equations are both known to be integrable, and admit peaked soliton (peakon) solutions with discontinuous derivatives at the peaks. A combination of a reciprocal transformation with Painlevé analysis provides strong evidence that the Popowicz system is non-integrable. Nevertheless, we are able to construct exact travelling wave solutions in terms of an elliptic integral, together with a degenerate travelling wave corresponding to a single peakon. We also describe the dynamics of N-peakon solutions, which is given in terms of an Hamiltonian system on a phase space of dimension $3N$.

1. **Introduction.** The members of a one-parameter family of partial differential equations, namely

$$m_t + um_x + bu_x m = 0, \qquad m = u - u_{xx} \tag{1}$$

with parameter b, have been studied recently. The case $b = 2$ is the Camassa-Holm equation [1], while $b = 3$ is the Degasperis-Procesi equation [3], and it is known that (with the possible exception of $b = 0$) these are the only integrable cases [14], while all of these equations (apart from $b = -1$) arise as a shallow water approximation to the Euler equations [6]. All of the equations have at least one Hamiltonian structure [12], this being given by

$$m_t = \mathcal{B}\frac{\delta H}{\delta m}, \quad \mathcal{B} = -b^2 m^{1-1/b} \partial_x m^{1/b} \hat{G} m^{1/b} \partial_x m^{1-1/b}, \tag{2}$$

with $\hat{G} = (\partial_x - \partial_x^3)^{-1}$ and the Hamiltonian $H = (b-1)^{-1} \int m \, dx$ for $b \neq 0, 1$ (and the latter special cases admit a similar expression).

One of the most interesting features of these equations is that their soliton solutions are not smooth, but rather the field u has a discontinuous derivative at one or more peaks (hence the name *peakons*), while the corresponding field m is measure valued. More precisely for the single peakon the solution has the form

$$u = c \exp(-|x - ct - x_0|), \qquad \text{with} \quad m = 2c\,\delta(x - ct - x_0)$$

(with x_0 being an arbitrary constant), while N-peakon solutions are given by

$$u = \sum_{j=1}^{N} p_j(t) \exp(-|x - q_j(t)|), \qquad m = 2 \sum_{j=1}^{N} p_j(t)\delta(x - q_j(t)), \qquad (3)$$

where the amplitudes $p_j(t)$ and peak positions $q_j(t)$ satisfy a Hamiltonian dynamical system for any b. For $b = 2$ the q_j and p_j are canonically conjugate position and momentum variables for an integrable geodesic flow with the co-metric $g^{jk} = \exp(-|q_j - q_k|)$ [1], and for $b = 3$ the peakon motion is again integrable, being described by Hamilton's equations for a different Poisson structure [4, 5], but for arbitrary b the N-peakon dynamics is unlikely to be integrable in general [10].

There is currently much interest in generalisations of the Camassa-Holm equations and its relatives. Qiao has found an integrable equation of this type with cubic nonlinearity [18], and another such example was discovered very recently by Vladimir Novikov [15]; one of the authors spoke at the AIMS meeting in Arlington on this topic (for more details see [13]). An important challenge is to understand the solutions of coupled equations with two or more components, and in higher dimensions. For example the Euler-Poincaré equation on the appropriate diffeomorphism group (EPDiff) can be used to describe fluids in two or more spatial dimensions, as well as appearing in computational anatomy [9, 11]. Chen *et al.* found an integrable two-component analogue of the Camassa-Holm equation [2], which also admits multi-peakon solutions [7]. Popowicz has constructed another two-component Camassa-Holm equation using supersymmetry algebra [16].

The purpose of this short note is to summarise some preliminary results that we have obtained on the two-component system given by

$$\begin{aligned} m_t + m_x(2u + v) + 3m(2u_x + v_x) &= 0, \\ n_t + n_x(2u + v) + 2n(2u_x + v_x) &= 0, \\ m = u - u_{xx}, \qquad n &= v - v_{xx}. \end{aligned} \qquad (4)$$

This system can be considered as a coupling between the Camassa-Holm equation and the Degasperis-Procesi equation (corresponding to (1) for $b = 2, 3$ respectively); it reduces to the former when $u = 0$, and to the latter when $v = 0$. The system (4) was obtained by Popowicz by taking a Dirac reduction of a three-field local Hamiltonian operator [17]. By construction, this system has a (nonlocal) Hamiltonian structure, and due to the existence of conservation laws it was conjectured that it should be integrable (although no second Hamiltonian structure was found).

After reviewing the Hamiltonian structure for it in the next section, in section 3 we perform a reciprocal transformation on the system (a nonlocal change of independent variables) which transforms it to a third order partial differential equation for a single scalar field. By applying Painlevé analysis of the singularities in solutions of the reciprocally transformed system, we find the presence of logarithmic branching, which is a strong indicator of non-integrability. Nevertheless, in section 4 we find that the system has exact travelling wave solutions given by an elliptic integral, as well as a degenerate travelling wave which is a peakon. In section 5 we present formulae for N-peakon solutions of (4), which are governed by Hamiltonian dynamics on a $3N$-dimensional phase space. The final section is devoted to some conclusions.

2. **Hamiltonian and Poisson structure.** Popowicz constructed the system (4) from the Hamiltonian operator

$$\mathcal{Z} = -\begin{pmatrix} 9m^{2/3}\partial_x m^{1/3}\hat{G}m^{1/3}\partial_x m^{2/3} & 6m^{2/3}\partial_x m^{1/3}\hat{G}n^{1/2}\partial_x n^{1/2} \\ 6n^{1/2}\partial_x n^{1/2}\hat{G}m^{1/3}\partial_x m^{2/3} & 4n^{1/2}\partial_x n^{1/2}\hat{G}n^{1/2}\partial_x n^{1/2} \end{pmatrix}, \quad (5)$$

where $\hat{G} = (\partial_x - \partial_x^3)^{-1}$. With the Hamiltonian

$$H_0 = \int (m+n)\,\mathrm{d}x, \tag{6}$$

the system can be written as

$$\begin{pmatrix} m_t \\ n_t \end{pmatrix} = \mathcal{Z}\begin{pmatrix} \frac{\delta H_0}{\delta m} \\ \frac{\delta H_0}{\delta n} \end{pmatrix} \equiv \{m, H_0\}.$$

For $\mathcal{Z}$ as in (5), the Poisson bracket between two functionals A, B is given by the standard formula

$$\{A, B\} = \int \begin{pmatrix} \frac{\delta A}{\delta m(z)} & \frac{\delta A}{\delta n(z)} \end{pmatrix} \mathcal{Z}\begin{pmatrix} \frac{\delta B}{\delta m(z)} \\ \frac{\delta B}{\delta n(z)} \end{pmatrix}\,\mathrm{d}z,$$

which is equivalent to specifying the local Poisson brackets between the fields m and n as

$$\begin{aligned}
\{m(x), m(y)\} &= m_x(x)m_x(y)G(x-y) \\
&\quad + 3(m(x)m_x(y) - m_x(x)m(y))G'(x-y) \\
&\quad - 9m(x)m(y)G''(x-y), \\
\{m(x), n(y)\} &= m_x(x)n_x(y)G(x-y) \\
&\quad + (3m(x)n_x(y) - 2m_x(x)n(y))G'(x-y) \qquad (7) \\
&\quad - 6m(x)n(y)G''(x-y), \\
\{n(x), n(y)\} &= n_x(x)n_x(y)G(x-y) \\
&\quad + 2(n(x)n_x(y) - n_x(x)n(y))G'(x-y) \\
&\quad - 4n(x)n(y)G''(x-y),
\end{aligned}$$

where

$$G(x) = \frac{1}{2}\mathrm{sgn}(x)\left(1 - e^{-|x|}\right) \tag{8}$$

is the Green's function of the operator $\hat{G}$. This G satisfies the functional equation

$$G'(\alpha)\Big(G(\beta) + G(\gamma)\Big) + \text{cyclic} = 0 \quad \text{for } \alpha + \beta + \gamma = 0,$$

which is a sufficient condition for the operator (2) to satisfy the Jacobi identity; the general solution to the functional equation was found by Braden and Byatt-Smith in the appendix of [10].

It was observed by Popowicz that, apart from the Hamiltonian H_0, the system (4) has additional conserved quantities that can be written as

$$\begin{aligned}
H_1 &= \int (nm^{-2/3})^\lambda m^{1/3}\,\mathrm{d}x, \\
H_2 &= \int \left(-9n_x^2 n^{-2}m^{-1/3} + 12n_x m_x n^{-1}m^{-4/3} + -4m_x^2 m^{-7/3}\right)(nm^{-2/3})^\lambda\,\mathrm{d}x,
\end{aligned} \tag{9}$$

where in each case the parameter λ is arbitrary. In [17] it is remarked that, having three conserved quantities, the system is likely to be integrable. The existence of a mere three (or a few) conservation laws does not guarantee integrability, and a more

precise requirement (or better, a definition of integrability) in infinite dimensions is that an integrable system should have infinitely many commuting symmetries [14]. In fact, since they contain an arbitrary parameter, each of H_1 and H_2 provide infinitely many independent conservation laws for the system. However, a brief calculation shows that the gradient of each functional appearing in (9) is in the kernel of the Hamiltonian operator $\mathcal{Z}$ for all λ, so that all of these conserved quantities are Casimirs for the associated Poisson bracket. Hence, regardless of the choice of λ, neither H_1 nor H_2 can generate a non-trivial flow that commutes with the time evolution ∂_t.

The fact that the combination $w = nm^{-2/3}$ appears in the conserved functionals (9) suggests that it is worthwhile to eliminate either m or n and use this as a dependent variable. Also, as noted by Popowicz, the conservation laws corresponding to H_1 are reminiscent of analogous ones for the Camassa-Holm/Degasperis-Procesi equations, which provide a reciprocal transformation to an equivalent system with different independent variables. We now make use of these observations.

3. **Reciprocal transformation and Painlevé analysis.** In order to eliminate n we can rewrite (4) as

$$\begin{aligned} (m^{1/3})_t &= -(m^{1/3}C)_x, \\ w_t &= -Cw_x, \end{aligned} \tag{10}$$

with

$$C = 2u + v, \quad m = u - u_{xx}, \quad wm^{2/3} = v - v_{xx}. \tag{11}$$

The first equation is in conservation form, and from our previous experience with the Degasperis-Procesi equation [4] we are led to take the reciprocal transformation

$$\begin{aligned} \mathrm{d}X &= p\,\mathrm{d}x - C\,p\,\mathrm{d}t, \qquad p = m^{1/3}, \\ \mathrm{d}T &= \mathrm{d}t, \end{aligned} \tag{12}$$

so that derivatives transform as $\partial_x = p\partial_X$, $\partial_t = \partial_T - Cp\partial_X$.

In terms of the new independent variables X, T and the dependent variables p, w, the system (10) becomes

$$\begin{aligned} (p^{-1})_T &= C_X, \\ w_T &= 0, \end{aligned} \tag{13}$$

and solving the latter two equations in (11) for u, v we can write

$$C = 2u + v = 2m + wm^{2/3} + (p\partial_X)^2(2u + v) = 2p^3 + wp^2 + p(pC_X)_X.$$

Substituting back for C_X from the first of (13) gives $C = p^3 + wp^2 + p(p(p^{-1})_T))_X$, and differentiating both sides of the latter with respect to X and substituting for C_X once more produces a single equation of third order for p, namely

$$p_{XXT} = \frac{p_X p_{XT}}{p} + \frac{(1 - p_X^2)p_T}{p^2} + 2wpp_X + (w_X + 6p_X)p^2. \tag{14}$$

From the second equation (13), the coefficient $w = w(X)$ is an arbitary function of X (independent of T). It turns out that the presence of this arbitrary function provides an obstruction to integrability, from the point of view of the Painlevé analysis of the partial differential equation (14). It is also easy to calculate the images under this reciprocal transformation of the conserved densities corresponding to (9): the density for H_1 is transformed to w^λ, and that for H_2 becomes $-9w^{\lambda-2}w_X^2$, both of which are trivial (since w is no longer a dynamical variable).

To analyse the singularities of the equation (14) we apply the Weiss-Tabor-Carnevale test. The details of the analysis are almost identical to that for the equation obtained from (1) by an analogous reciprocal transformation, so we will only give a brief description of the results and refer the reader to [8] for details of a similar calculation. There are two types of local expansion around an arbitrary singular manifold $\phi = \phi(X, T) = 0$ corresponding to singularities on the right hand side of equation (14). For simplicity we can take the Kruskal reduced ansatz $\phi = X - f(T)$ with f an arbitrary function of T, and it is sufficient to take w to be a non-zero constant. For the first type of expansion, p is regular as $\phi \to 0$, and we have $p = \pm\phi + \alpha_2\phi^2 + \alpha_3\phi^3 + \ldots$, where $\alpha_1(T)$ and $\alpha_3(T)$ are arbitrary. The resonances are at $-1, 1, 2$, corresponding to the arbitrariness of f, α_2, α_3 respectively, and all resonance conditions are satisfied, so that this defines the leading part of a local expansion that is analytic around $\phi = X - f(T) = 0$. However, for the other type of expansion, we have $p = \sum_{j \geq -1} \beta_j\phi^j$ which gives resonances $-1, 2, 3$. The leading coefficient satisfies $\beta_{-1}^2 = -\dot{f}/2$, while at the next order $\beta_0 = -w/4$. The resonance -1 corresponds to f, and the resonance condition for the value 3 (corresponding to β_2) is satisfied automatically, but the resonance 2 corresponding to β_1 gives the additional condition

$$\dot{\beta}_{-1}w = -\frac{\ddot{f}w}{4\beta_{-1}} = 0 \tag{15}$$

which is not satisfied unless $w \equiv 0$ (since f is arbitrary). When $w \equiv 0$ the equation (14) corresponds to the integrable Degasperis-Procesi equation (see [4]; this is also clear from the fact that $n \equiv 0$ in that case). The failure of the resonance condition for non-vanishing w means that the local expansion around a pole must be augmented with infinitely many terms in $\log \phi$, so that the Painlevé property does not hold. These logarithmic terms are an indicator that (14) is not integrable, and hence the system (4) cannot be.

Before concluding this section, we should like to make a remark on the form of the Popowicz peakon system[1]. Although it appears to consist of four equations for the fields u, v, m, n, the system (4) really only involves relations between m, n and the quantity $C = 2u + v$. Indeed, from (10) and (11), or from the original form of the system, it is easy to see that it can be rewritten in a more elegant form, namely as the three equations

$$\begin{aligned} m_t + Cm_x + 3C_x m &= 0, \\ n_t + Cn_x + 2C_x n &= 0, \\ 2m + n &= C - C_{xx}. \end{aligned} \tag{16}$$

Much of the analysis of the solutions can be simplified by using this form of the system. From the Hamiltonian point of view, the system really consists of a pair of coupled integro-differential Hamiltonian equations for the fields m, n, namely the first two equations in (16) with the nonlocal quantity

$$C = \frac{1}{2} \int e^{-|x-y|} \Big(2m(y) + n(y) \Big) \, \mathrm{d}y.$$

4. **Travelling waves.** Travelling wave solutions of (4) arise by putting $u(x, t) = U(s), v(x, t) = V(s)$, with $s = x - ct$, to get

$$M(2U + V - c)^3 = K_1, \quad N(2U + V - c)^2 = K_2, \tag{17}$$

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[1]We are grateful to one of the referees for making this point.

with $M = U - U''$, $N = V - V''$, where K_1, K_2 are constants, and $C(s) = 2U + V = c - kM^{-1/3}$ where $k = -K_1^{1/3}$. (The primes denote s derivatives.) From this it is also apparent that $w = NM^{-2/3} = K_2 K_1^{-2/3} = \ell =$ constant. Comparing with (12) and (13) is clear that travelling waves of (4) are transformed to travelling waves $p(X, T) = P(S)$ of (14) moving with speed k, with

$$P(S) = M^{1/3}(s), \quad S = X - kT, \quad \mathrm{d}S = M^{1/3}(s)\,\mathrm{d}s.$$

The ordinary differential equation for travelling waves of (14) (with constant $w = \ell$) can be integrated twice to yield

$$\left(\frac{dP}{dS}\right)^2 = -\frac{2}{k}\left(P^4 + \ell P^3 + mP^2 + cP\right) + 1 \equiv Q(P),$$

for $k \neq 0$, where m is another integration constant. This reduces to an elliptic integral of the first kind,

$$S + \mathrm{const} = \int \frac{\mathrm{d}P}{\sqrt{Q(P)}},$$

so that P is an elliptic function of S. Note that these travelling waves provide meromorphic solutions of (14), but this does not contradict the Painlevé analysis in the previous section, because for travelling waves the singular manifold is of the form $\phi = X - f(T) = S - S_0$ for constant S_0, so that $f(T) = kT + S_0$, implying $\ddot{f} = 0$ which removes the obstruction to the Painlevé property in (15).

In the original variable s, we have a third kind differential

$$\mathrm{d}s = \frac{\mathrm{d}P}{P\sqrt{Q(P)}}$$

(so that $M(s) = P^3(S)$ has algebraic branch points as a function of s). For particular choices of constants, when the quartic Q has a double root, the elliptic integral reduces to an elementary one in terms of hyperbolic functions, corresponding to smooth solitary wave solutions with the characteristic soliton shape.

Soliton-type travelling wave solutions must have a constant non-zero background, since the requirement that U and V tend to zero as $s \to \pm\infty$ implies $K_1 = 0 = K_2$ in (17), hence $k = 0$ and the above analysis does not apply. However, in this case we can have a weak solution of (17) which is the peakon solution

$$u(x, t) = a\, e^{-|x - ct|}, \quad v(x, t) = b\, e^{-|x - ct|},$$

and

$$m(x, t) = 2a\delta(x - ct), \quad n(x, t) = 2b\delta(x - ct),$$

where a is an arbitrary constant and $c = 2a + b$ is the wave speed. In the next section we extend this to multi-peakon solutions.

5. **Hamiltonian dynamics of peakons.** The N-peakon solutions have the appearance of a simple sum of N single peakons but with both the amplitudes and positions of the peaks being time-dependent, like so:

$$u(x, t) = \sum_{j=1}^{N} a_j(t)e^{-|x - q_j(t)|}, \qquad v(x, t) = \sum_{j=1}^{N} b_j(t)e^{-|x - q_j(t)|}. \tag{18}$$

In the above expressions, $a_j(t)$ and $b_j(t)$ are the amplitudes of the waves and $q_j(t)$ is the position of the peak of both waves. The main result is as follows.

Theorem 5.1. *The Popowicz system (4) admits N-peakon solutions of the form (18), where the amplitudes a_j, b_j and positions q_j satisfy the dynamical system*

$$\dot{a}_j = 2a_j \sum_{k=1}^{N} (2a_k + b_k)\, \mathrm{sgn}(q_j - q_k)\, e^{-|q_j - q_k|},$$

$$\dot{b}_j = b_j \sum_{k=1}^{N} (2a_k + b_k)\, \mathrm{sgn}(q_j - q_k)\, e^{-|q_j - q_k|},$$

$$\dot{q}_j = \sum_{k=1}^{N} (2a_k + b_k)\, e^{-|q_j - q_k|}$$

for $j = 1, \ldots, N$. These equations are an Hamiltonian system

$$\dot{a}_j = \{a_j, h\}, \quad \dot{b}_j = \{b_j, h\}, \quad \dot{q}_j = \{q_j, h\}$$

with the Hamiltonian $h = 2\sum_{j=1}^{N}(a_j + b_j)$, and the Poisson bracket

$$\{a_j, a_k\} = 2a_j a_k \mathrm{sgn}(q_j - q_k) e^{-|q_j - q_k|},$$

$$\{b_j, b_k\} = \frac{1}{2} b_j b_k \mathrm{sgn}(q_j - q_k) e^{-|q_j - q_k|},$$

$$\{q_j, q_k\} = \frac{1}{2} \mathrm{sgn}(q_j - q_k)\left(1 - e^{-|q_j - q_k|}\right),$$

$$\{q_j, a_k\} = a_k e^{-|q_j - q_k|},$$

$$\{q_j, b_k\} = \frac{1}{2} b_k e^{-|q_j - q_k|},$$

$$\{a_j, b_k\} = a_j b_k \mathrm{sgn}(q_j - q_k) e^{-|q_j - q_k|}.$$

This Poisson bracket has N Casimirs $\mathcal{C}_j = a_j/b_j^2$ for $j = 1, \ldots, N$.

The proof of the above result, which will be presented elsewhere, is based on integration of the equations (4) and the brackets (7) against suitable test functions with support around each of the peaks. Here it is worth remarking that although the phase space has dimension $3N$, fixing the values of the N Casimirs reduces the motion onto $2N$-dimensional symplectic leaves. Once this has been done, one can eliminate the a_j, say, and solve $2N$ equations for b_j, q_j.

6. Concluding remarks. Painlevé analysis provides very strong evidence that the system (4) is not integrable. This raises the question of whether the N-peakon system can be integrable for $N > 1$. The Liouville-Arnold theorem requires the existence of a further $N - 1$ independent conserved quantities in involution (in addition to h and the Casimirs $\mathcal{C}_j$ which satisfy $\{\mathcal{C}_j, F\} = 0$ for any function F on phase space). The first interesting case is the 2-peakon problem, which requires just one additional conserved quantity. In fact, a direct calculation shows that the independent quantity

$$J = b_1 b_2 \left(1 - \exp(-|q_1 - q_2|)\right)$$

is in involution with the Hamiltonian, $\{J, h\} = 0$, so that the $N = 2$ peakon system is completely integrable.

For the EPDiff equation in multi-dimensions, and for the Camassa-Holm equation in particular, it is known that the map to a submanifold of measure-valued solutions is a momentum map [11], so one might expect that the same would be true for the

peakon solutions in the Popowicz system. There is also the question of whether these peakons are stable solutions. We propose to address these issues in future.

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