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Singular Finite-Gap Operators and Indefinite Metric. II [3]

Abstract. *Many "real" inverse spectral data for periodic finite-gap operators (consisting of Riemann Surface with marked "infinite point", local parameter and divisors of poles) lead to operators with real but singular coefficients. These operators cannot be considered as self-adjoint in the ordinary (positive) Hilbert spaces of functions of x. In particular, it is true for the special case of Lame' operators with elliptic potential $n(n+1)\wp(x)$ where eigenfunctions were found in XIX Century by Hermit. However, such Baker-Akhiezer (BA) functions present according to the ideas of works [1, 2], right analog of the Discrete and Continuous Fourier Bases on Riemann Surfaces. It turns out that these operators for the nonzero genus are symmetric in the indefinite inner product, described in this work. The analog of Continuous Fourier Transform is an isometry in this inner product. Its image in the space of functions of the variable $x \in R$ is described.*

Introduction

Broad family of the so-called "Baker-Akhiezer" (BA) functions on Riemann surfaces were invented since 1974 when periodic finite gap solutions were found for the famous KdV equation.

They were used for the solution of periodic problems for KdV, KP and other systems of Soliton Theory like NLS, SG, for many Completely Integrable Hamiltonian Systems. The Spectral Theory of "finite-gap" periodic 1D and 2D Schrodinger Operators was developed since 1974 based on the Analysis on Riemann Surfaces. It was found in 1987 that some BA functions generate construction of analogs of the Laurent-Fourier decomposition for functions and tensor fields on Riemann surfaces (the Krichever-Novikov Bases and Algebras [1]). They were used for the multi-loop operator quan-

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[3]First version of this work can be found on the website http://arxiv.org/abs/0903.3976 New results are added at the end of the article. New references are included after discussion with several colleagues. Especially fruitful discussion with I.Krichever helped us to improve our article.

tization of Closed Bosonic Strings (i.e. for genus more than zero.) Another ideas similar to some sort of Harmonic Analysis with spectral parameter on Riemann Surfaces and useful here, were developed for other goals in the works [6].

The present authors observed in the Appendix to the work [2] that these constructions lead also to the analog of continuous Fourier Transform. The present work is direct continuation of [2]. It was motivated by the following

Problem: Consider one-dimensional Lame' Operator $L = -\partial_x^2 + u(x)$ whose potential u is equal to the $n(n+1)$-times Weierstrass elliptic function $\wp$ with poles on real line. **Does it have any reasonable spectral theory on the whole real line?** We need to answer this question because our analog of continuous Fourier Transform is based exactly on the singular Hermit eigenfunctions of this operator in the simplest nontrivial elliptic case.

Let us remind here that 150 years ago Hermit found all family of formal eigenfunction for this operator. In fact it consists of the "Bloch-Floquet" eigenfunctions in modern terminology. However they are singular on the line and do not serve any spectral problem in Hilbert space. Hermit used only those of them who belong to the discrete spectrum on the finite interval $[0T]$ between the neighboring singularities, needed for the Lame' problem. No spectral interpretation of singular eigenfunctions for the spectral theory on the whole line was known.

We found indefinite inner product associated with this problem. This is our main result but the exposition is more general: We constructed indefinite inner products associated with BA functions and non-selfadjoint "algebraic" periodic operators with Bloch-Floquet function meromorphic on Riemann Surfaces of finite genus like in the finite-gap theory.

We assume in this work, that all finite-gap operators are periodic in the real variable $x \in R$. It is very likely, that our main results are valid for generic finite-gap x-quasiperiodic potentials, but this extension may lead to additional analytical difficulties.

Remark 1 *Singular Bloch-Floquet eigenfunctions are known also for the $k + 1$-particle Moser-Calogero operator with Weierstrass elliptic pairwise potential if coupling constant is equal to $n(n+1)$. They form (if center of mass is staying) a k-dimensional complex algebraic variety. The Hermit-type result is not obtained here yet for $k > 1$: no one function was constructed until now serving the discrete spectrum in the bounded domain inside of the poles. Our case corresponds to $k = 1$. We believe that for all $k > 1$ this algebraic*

family of eigenfunctions also serves spectral problem in some indefinite inner product in the proper space of functions defined in the whole space R^k, similar to the case $k = 1$.

Chapter 1. Canonical contours and Inner Product of BA functions.

Let a nonsingular complex algebraic curve (Riemann surface) Γ be given with selected point $P = \infty \in \Gamma$, local coordinate $z = k^{-1}$ near P such that $z(P) = 0$. We fix also "divisor" $D = \gamma_1 + ... + \gamma_g$ on Γ and construct standard BA function $\Psi_D(x, z), z \in \Gamma$, meromorphic in the variable z, with first order poles in the points $\gamma_j \in \Gamma$ and with asymptotics $\Psi = \exp\{ikx\}(1 + O(k^{-1}))$. We define a differential 1-form $d\mu$ with asymptotics $d\mu = dk + regular$ near the infinity P, and zeroes of the form $(d\mu) = D + D^*$ such that

$$D + D^* \sim K + 2P$$

Here the sign $\sim$ means the so-called "linear equivalence" of divisors in Algebraic Geometry, K means the divisor of differential forms. So the divisor D^* is completely determined by the divisor D. A "dual" BA function (1-form) $\Psi_D^*(x, z)$ was invented long ago by Krichever. It was actively used in the joint works [1] and has asymptotics $\Psi_D^*(x, z) = \exp\{-ikx\}(1 + O(k^{-1}))$ with divisor D^*. A Dual BA form is $\Psi^* d\mu$. So we have

$$\Psi_D^*(x, z) = \Psi_{D^*}(-x, z)$$

as a scalar BA function. One should multiply it by the form $d\mu$ to get a dual 1-form.

Both functions $\Psi_D(x, z)$, $\Psi_D^*(x, z)$ are meromorphic in x.

The Canonical Contours κ_c we define by the equation $p_I = c$ where dp is meromorphic (second kind) differential form such that $dp = dk + regular$ near $P = \infty$, and $\oint_\gamma dp \in R$ is purely real for all closed paths $\gamma \subset \Gamma$ avoiding the point P. So the imaginary part of p is an one-valued function p_I. **The Canonical Contour is canonically oriented by the one-valued real function p_I on the oriented manifold Γ. The Special Canonical Contour corresponds to the case $c = 0$ such that the expression $\exp\{ikx\}$ is bounded for real $x \in R$ and $k \to \infty$ along the contour.**

Remark 2 *The finite-gap operator, constructed by the curve Γ is x-periodic with the period T if and only if the function e^{iTp} is single-valued in Γ, or equivalently, if all periods of dp have the form:*

$$\oint_s dp = \frac{2\pi}{T} n_s, \quad n_s \in \mathbb{Z},$$

where s in an arbitrary closed contour.

Remark 3 *In the work [2] we especially considered the case where our divisor D is equal to $D = gP = g\infty$ where g is genus of Γ. In this case we proved important "Multiplicative Property" of our "Fourier" BA basis:*

$$\Psi_{g\infty}(x, z)\Psi_{g\infty}(y, z) = L_g \Psi_{g\infty}(x + y, z)$$

where L is a linear differential operator in x with coefficients independent on $z \in \Gamma, L = \partial_x^g + ... -$ see[2]. This construction extends the construction [1] of the discrete Fourier bases done in the late 1980s for the needs of the Bosonic (closed) String Theory. The multiplicative properties of Fourier type series and transform are important in the Nonlinear Problems like String Theory. This specific case is not much different from others in the purely linear Harmonic Analysis discussed in the present work. Poles of Ψ in the variable x necessary appear in this case, so our inner products are indefinite– see below. Especially effective formulas for operator L_g were obtained in the work [10].

Let us define **a C-linear Inner Product** of smooth functions on the canonical contour $\kappa_c \subset \Gamma$ depending on the choice of divisor D and generated by the basis of functions $\Psi_D(x, z)$ restricted to the canonical contour κ_c.

Statement. For the basic BA functions we have **The Orthogonality Relations on Riemann Surface, i.e. on the contour $\kappa_c \subset \Gamma$:**

$$(\Psi_D(x, z), \Psi_D(y, z))_{\kappa_c} = \int_{\kappa_c} \Psi_D(x, z)\Psi_{D^*}(-y, z)d\mu(z) =$$

$$= 2\pi\delta(x - y)$$

The exact meaning of this statement and of the δ-function depends on the functional classes where this product is well-defined. For the specific real nonsingular "self-adjoint" case with positive inner product such relation was first found in the work [4]–see formula 12. **We are going to use this type**

of relations for the singular operators where inner product is not positive. The most important case for us is the case of Special Canonical Contours $c = 0$. Let us present sketch of the proof:

The form at the right-hand side is holomorphic in the variable z. Therefore this integral does not depend on c. If $x > y$, this integral vanishes as $c \to +\infty$. Equivalently, for $x < y$ this integral vanishes as $c \to -\infty$, therefore

$$(\Psi_D(x, z), \Psi_D(y, z))_{\kappa_c} = 0 \quad \text{for} \quad x \neq y. \tag{1}$$

If we modify the integrand outside a neighbourhood of the point P, the resulting integral is the same up to a regular function of x, y. Let us expand the functions $\Psi_D(x, z)$, $\Psi_D^*(x, z)$ near the point P:

$$\Psi_D(x, z) = e^{ikx} \left[1 + \frac{\phi(x)}{k} + O\left(\frac{1}{k^2}\right) \right]$$

$$\Psi_D^*(x, z) = e^{-ikx} \left[1 + \frac{\phi^*(x)}{k} + O\left(\frac{1}{k^2}\right) \right]$$

$$(\Psi_D(x, z), \Psi_D(y, z))_{\kappa_c} = \int_{-\infty}^{+\infty} e^{ik(x-y)} \left[1 + \frac{\phi(x) + \phi^*(y)}{k} \right] dx + \text{regular function} =$$

$$= 2\pi \delta(x - y) + \pi i \, \text{sgn}(x - y)[\phi(x) + \phi^*(y)] + \text{regular function}.$$

For $x = y$ the integrand has no essential singularities and only one first-order pole at P with the residue $\phi(x) + \phi^*(x)$. Therefore $\phi^*(x) = -\phi(x)$, and

$$(\Psi_D(x, z), \Psi_D(y, z))_{\kappa_c} = 2\pi \delta(x - y) + \text{regular function}.$$

Comparing it with (1) we complete the proof.

Now we consider class of functions $\phi(z)$ on the Special Canonical Contour κ_0, such that their **"BA Fourier Transform"** is well defined. We interpret them simply as **"BA Fourier Components"** of function $\tilde{\phi}(x)$ in our **BA Fourier basis** $\Psi_{D^*}(-x, z)$ using the integral:

$$\tilde{\phi}(x) = (\sqrt{2\pi})^{-1} \int_{\kappa_c} \phi(z) \Psi_{D^*}(-x, z) d\mu$$

Statement. For the selected BA function with bounded restriction of $\exp\{ikx\}$ to the Special Canonical Contour κ_0 near $P = \infty$, this integral is well-defined near ∞ if $\phi(k) = o(k^{-1+\epsilon}), \epsilon > 0$.

Proof. The **Inverse BA Fourier Transform** is given by the formula

$$\phi(z) = (\sqrt{2\pi})^{-1} \int \tilde{\phi}(x) \Psi_D(x,z) dx$$

It leads to the same inner product of transformed functions in the x-space treated simply as "Collections of BA Fourier Components" in the previous basis of BA functions $\Psi_D(x,z)$, where x is considered as an "index" numerating the basic vectors:

$$\int \tilde{\phi}_1(x) \tilde{\phi}_2(x) dx = (\phi_1, \phi_2)_{\kappa_c}$$

The space of functions in the x-space obtained by this BA Fourier Transform will be especially discussed below for the important real case, after the proper definition of the space of functions H_{D,κ_0} on corresponding contour in Γ. We describe its construction for the most important hyperelliptic case in the Appendix 2.

For the same BA functions treated as basis in x-space, we obtain a formula

$$(\Psi_D(x,z), \Psi_D(x,w))_x = [\int_x \Psi_D(x,z) \Psi_{D^*}(-x,w) dx] d\mu = 2\pi \delta(z,w) \qquad (2)$$

$= 0, z \neq w$. Here both points $z, w \in \Gamma$ belong to our selected contour κ_c. We assume that these points are nonsingular on this contour. We assume that δ is an one-form in the variable w.

The case of critical contour corresponding to the critical values of the real function p_R should be considered separately. **This formula is meaningful locally only if our BA functions do not contain poles for $x \in R$. It is meaningful globally if our picture is periodic in $x \in R$**, so we have no concentration of poles near $x \to \pm\infty$. We postpone to the next work extension of our results to the quasi-periodic finite-gap case.

Let us discuss, for which classes of functions our BA Fourier Transform is well-defined. It depends on the divisor D and on the geometry of contour κ_c: **Does our BA function contain poles? Does divisor contain infinite point or not? Is our contour critical?**

We postpone the last question.

For the case of BA function with poles we invent following rule: **All integrals above taken along the line $x \in R$, should be taken avoiding pole x_0 in the upper half-plane $x + i\epsilon, \epsilon > 0$.** In order to prove that our inner products written as integral along the x-axis, are well-defined, we prove following:

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Lemma 1 *The expression $\Psi_D(x, z)\Psi_{D^*}(-x, w)$ has residue equal to zero in every pole $x_0 \in R$ as a meromorphic function of the complex variable x in the small strip around the real line.*

The proof follows immediately from Lemma 6 and formula (7) below. We added Appendix 1 to make this proof fully rigorous. In the case of first order poles in the point $x = 0$ both our BA functions have a form like $\alpha/x + O(x)$. Product of such expressions obviously has zero residue.

So the integral defining inner product does not depend on the contour around the pole.

Real Algebraic Curves. Let our data consisting of algebraic curve Γ with selected point $P = \infty$ and local parameter $z = k^{-1}$ near P, be real now. It means precisely that an anti-holomorphic involution is defined

$$\tau : \Gamma \to \Gamma, \tau^2 = 1$$

such that $\tau(P) = P$ and $\tau^*(k) = \bar{k}$.

Our differential dp is such that $\tau^*(dp) = d\bar{p}$. We define p_I such that $\tau^*(p_I) = -p_I$, so the level $\kappa_0 = (p_I = 0)$ is invariant under τ:

$$\tau : \kappa_0 \to \kappa_0$$

and differentials dk, dp are real on κ_0.

Let us point out that our contour κ_0 contains all set of fix-points

$$Fix_\tau \subset \kappa_0$$

where $z \in Fix_\tau$ means $\tau(z) = z$. Following simple geometric statement is useful to clarify relationship between our constructions and some results of the late 1980s (see[5]) about nonsingular real solution to the KPI system with Lax operator $i\partial_x + \partial_y^2 + u(x, y)$:

Lemma 2 *For anti-holomorphic involution τ the fix-point set Fix_τ coincides with canonical contour κ_0 if and only if Fix_τ divides Γ into two parts $\Gamma = \Gamma_+ \bigcup \Gamma_-$.*

Proof of this lemma easily follows from the obvious fact that κ_0 certainly divides Γ but its smaller part never does. We assume that $P \in Fix_\tau$.

We choose divisor D such that $\tau(D) = D^*$ or

$$D + \tau(D) \sim K + 2P$$

where K is divisor of differential forms. So we have $\tau^*(d\mu) = d\bar{\mu}$.

In this case we define a **Hermitian (or sesqui-linear) possibly indefinite Inner Product** for the basic BA functions on the contour κ_0 by the formula

$$< \Psi_D(x,z), \Psi_D((y,z) >_{\kappa_0} = (\Psi_D(x,z), \bar{\Psi}_D(y,\tau(z)))_{\kappa_0} =$$

$$= \int_{\kappa_0} \Psi_D(x,z) \bar{\Psi}_{\tau D}(y,\tau z) d\mu(z)$$

where the integral above is taken with respect to the canonical orientation of the contour κ_0.

We take into account here that $\bar{\Psi}_{\tau D}(y, \tau z)$ is meromorphic in the variable z, has poles in τD and asymptotics $\exp\{-iky\}(1 + O(k^{-1} + ...)$ near P for $y, k \in R$. So for the "real" variables it coincides with our C-linear expression above.

In the x-space we have following inner product of basic BA functions:

$$< \Psi_D(x,z), \Psi_D(x,w) >_x = \int_x [\Psi_D(x,z) \bar{\Psi}_{\tau D}(\bar{x},w) d\mu] dx$$

Let us point out that

$$\bar{\Psi}_{\tau D}(\bar{x}, w) = \Psi_{\tau D}(-x, w)$$

for the real values of the variables k, x. It is meromorphic in x. So the residue in the x-pole is equal to zero for the product $\Psi_D(x,z) \bar{\Psi}_{\tau D}(\bar{x}, \tau(z))$ in the integral because it is the same as in the Lemma 1 above.

We are coming to the following

Lemma 3 *1.The hermitian inner product above on the contour κ_0 is positively defined if and only if $\kappa_0 = Fix_\tau$, and the form $d\mu$ is positive on the contour κ_0. 2. The hermitian inner product in the x-space is well-defined avoiding every pole of Ψ in the upper half-plane in x. It is positive if and only if our BA function $\Psi_D(x,z)$ does not have poles on the real line x.*

The statement 1 makes sense because our form $d\mu$ is real on this contour. We have $\tau(z) = z$ for $z \in Fix_\tau$, and upper part Γ^+ of Γ induces natural orientation of the contour κ_0. It is interesting to compare this result with [5]. The statement 2 is crucial for our work, so we present a full proof in the

Appendix 1 using what we call The Cauchy-Baker-Akhiezer Kernel. This quantity is borrowed from the work [6], but some additional improvements are needed here. Besides that, no full proof was presented in the work [6].

There are following sources for the violation of positivity of the inner product on the contour κ_0:

1. $\kappa_0 \neq Fix_\tau$. We have here $\tau(z) \neq z$ for z outside of fix-point set. Such inner product is always indefinite.

2. $Fix_\tau = \kappa_0$ but the divisor D is chosen such that $d\mu$ has different signs on some components (see chapter 2).

Only poles of Ψ on the real line x are responsible for the non-positivity of the inner product in the x-space. This is central part of our work.

We are going to consider this picture in more details for the important hyperelliptic case in the next chapter.

Chapter 2. The indefinite Inner Product for Hyperelliptic Riemann Surfaces. Schrodinger Operators with singular potential.

Consider now the most important case of nonsingular Hyperelliptic Riemann Surfaces Γ associated with second order periodic operators L: Let Γ is presented in the form

$$w^2 = (u - u_0) \times ...(u - u_{2g}) = R(u)$$

where typical point (except branching points) is written as $\gamma = (z, \pm)$. We take branching point P with $u = \infty$ as our "infinity" with local coordinate $k^{-1} = u^{-1/2} = z$. Every generic divisor $D = \gamma_1 + ... + \gamma_g$ defines a Baker-Akhiezer function $\Psi_D(x, z)$ with standard analytic properties described above. It satisfies to the equation

$$L\Psi = (-\partial_x^2 + U(x))\Psi(x, z) = u(z)\Psi(x, z)$$

Our requirement is that the potential $U(x)$ is periodic $U(x + T) = U(x)$ for real x. From the finite-gap theory we know that necessary and sufficient condition to have real nonsingular potential $U(x)$ (we call it a **Canonical Inverse Spectral Conditions**) consists of two parts:

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1.The Strong Reality Condition for Γ: all branching points u_j are real and distinct. Let $u_0 < u_1 < ... < u_{2g}$.

2.The divisor D is **Proper** i.e. such that $\gamma_k = (\alpha_k, +)$ or $\gamma_k = (\alpha_k, -)$ where $u_{2k-1} \leq \alpha_k \leq u_{2k}, k = 1, 2, ..., g$ (exactly one divisor point is located in every a-cycle).

There are two commuting anti-holomorphic involutions $\tau_{\pm}$ of the Riemann Surface Γ where $\tau_{\pm}(u, +) = (\bar{u}, \pm)$. Let $\tau_+ = \sigma, \tau_- = \tau$. Our contour κ_0 is equal to Fix_τ. It coincides with union of spectral zones. The set Fix_σ coincides with union of spectral gaps:

The union of our a-cycles a_k form the finite part of fix-point set for the anti-involution $\sigma(p) = p$. Their projection on the u-line $u \in R$ coincide with finite "Gaps" $[u_{2k-1}, u_{2k}], k = 1, 2, ..., g$, in the Spectral Theory of operator L in the Hilbert Space $L_2(R)$ of the square-integrable real or complex-valued functions on the real line. So we have $\sigma(D) = D$ and $\tau D = D^*$ where $D + D^* = K + 2P$.

We know however that inverse spectral data lead to the real operators L in other **Non-Canonical Real Cases**:

1.The Riemann Surface Γ is Real. It simply means that the set of branching points $u_j, j = 0.1, ..., u_{2k}$, is invariant under the anti-involution $u \rightarrow \bar{u}$.

2.The Divisor of Poles D should be such that $\sigma(D) = D$ but not necessarily like in the Canonical Case.

If these conditions are satisfied, then the potential $U(x)$ is real. However, this potential is singular . Otherwise, it would be self-adjoint in the positive Hilbert Space which is impossible. So it is singular in all non-canonical real cases. We call our data **Real Semi-Canonical** if Riemann Surface satisfies to the Strong Reality Condition but the divisor D is nor Proper. In particular, our contour κ_0 coincides with fix-point set Fix_τ. The potential $U(x)$ has poles in all real non-canonical cases, and spectrum is real in the semi-canonical case.

Orientation of κ_0 in the Real case is defined by the domain $p_I \geq 0$ and orientation of Γ. For such spectral curves

$$dp = (u - p_1) \times ...(u - p_g)du / \sqrt{(u - u_0) \times ...(u - u_{2g})}, \qquad (3)$$

where all p_k are real and $p_k \in [u_{2k-1}, u_{2k}]$.

Let not all branching points are real: there are complex adjoint pairs between them. In this case we have Fix_τ essentially smaller than the contour

κ_0. So our operator L is singular. It has complex spectrum equal to the projection of the contour κ_0 on the complex u-line.

Such operators are symmetric in the Indefinite Inner Product given by the formulas presented in the Chapter 1.

Using previous results, we are coming to the following

Theorem: *1.Let Riemann Surface and divisor D are real and finite correspondingly. The form*

$$d\mu = (u - \gamma_1) \times ...(u - \gamma_g)du/\sqrt{R(u)}, u = z^{-1/2},$$

is real, nonzero and has a well-defined sign in every component. 2.The set κ_0 is the spectrum of operator L in some space of functions $\tilde{H}_L$ of the variable x depending on the poles of operator L (see description below in the generic case). The Inverse BA Fourier Transform defined above isomorphically maps the space $\tilde{H}_L$ on the space H_{D,κ_0} of functions on the contour κ_0 and back. An Indefinite Inner Product

$$< a, b >_{\kappa_0} = \int_{\kappa_0} a(z)\bar{b}(z)d\mu(z)$$

is defined in the space H_{D,κ_0}. It is isomorphic to the direct sum of ordinary spaces of functions in the components of κ_0, taken with sign provided by the form $d\mu$ and orientation of the contour κ_0; The decay in the infinite component is also specified as above. The linear operator $L = -\partial_x^2 + U(x)$ is symmetric in the space $\tilde{H}_L$, and corresponding "Direct and Inverse BA Fourier Transform" defined in the previous paragraph, is isometric corresponding to these indefinite inner products. 3.In the case where some divisor points are infinite $\gamma_g = \infty$, the form $d\mu$ is holomorphic.

Remark 4 *From (3) it follows that the sign of $d\mu$ on real ovals of τ with respect to the orientation of κ_0 coincides with the sign of $dp/d\mu$, or, equivalently, with the sign of the ratio:*

$$(u - p_1) \dots (u - p_g)/(u - \alpha_1) \dots (u - \alpha_g).$$

Remark 5 *As one might see below, we do not describe the exact completion of the spaces $\tilde{H}_L$ and $H_{D,\kappa}$. So our result is incomplete in terms of Modern Functional Analysis.*

Remark 6 *If r point of divisor D are equal to ∞ (i.e. $D = r\infty + (\alpha_1, \pm) + \ldots + (\alpha_{g-r}, \pm)$), we have*

$$d\mu = (u - \alpha_1) \times \ldots (u - \alpha_{g-r}) du / \sqrt{R(u)}$$

The special case $r = g$ all divisor is concentrated in the point ∞. This case was especially considered as a right analog of the ordinary Fourier Transform: It has Remarkable Multiplicative Properties.

Proof. Our Theorem immediately follows from the results of Chapter 1 and Appendix 1.

Description of the Space $\tilde{H}_L$ for generic singular finite-gap real x-periodic operators L:

Let us consider real singular "finite-gap" periodic potentials $U(x)$ with finite number of poles with cyclic order $x_0 = 0 < x_1 < \ldots < x_k < T$ at the circle $[0T]$. In the generic finite-gap case we have

$$U = 2/(x - x_j)^2 + O(x - x_j)$$

near every pole x_j. We define following class of functions $f(x) \in \tilde{H}_{x_0, \ldots, x_k}$ depending on the position of poles only, by the requirement:

$$f(x) = \alpha_j/(x - x_j) + O(x - x_j)$$

where $f(x) - \alpha_j/(x - x_j) = O(x - x_j)$ is C^∞-smooth near the point $x = x_j$. By definition of this class, the operator $L = -\partial^2 + U(x)$ is locally well-defined in it:

$$Lf(x) \in \tilde{H}_{x_0, \ldots, x_k}$$

Describing these spaces, we decompose our space of functions on the line R into the "direct integral" along the Bloch-Floquet multiplier $\varkappa$. All BA (i.e. Bloch-Floquet) eigenfunctions for all points $z \in \kappa_0$ locally belong to this classes for some $\varkappa$ as it follows from the formulas for BA functions obtained in the theory of finite-gap operators.

Question: Are there any L-dependent linear relations between these residues for singular finite-gap periodic operators?

It means that some linear subspace $C^q(L, \varkappa) \subset C^k$ should be chosen in the space of residues. It depends on the Bloch-Floquet multiplier $\varkappa$ and operator L. Its dimension $q = q(L)$ is equal to the number of negative squares

in our inner product for the fixed value of the Bloch Floquet multiplier $\varkappa$–
see the case of higher Lame potentials in the Appendix 2. **Therefore our
indefinite space for every $\varkappa$ consists of all functions $f \in F_{x_0,...,x_k}(\varkappa)$
with first order zero in all poles x_j plus finite-dimensional space
$C^q(L, \varkappa) \subset C^k$:**

$$\tilde{H}_{L,\varkappa} = C^q(L, \varkappa) \bigoplus F_{x_0,...,x_k}(\varkappa)$$

The total space $\tilde{H}_L \subset \tilde{H}_{x_0,...,x_k}$ we realize as a direct integral of the spaces
$\tilde{H}_{L,\varkappa}$ over the circle.

**We proved in many cases that in fact $k = q$, i.e. there are no
relations between these residues. In particular, it means that the
space $\tilde{H}_L$ depends on the poles only (see Appendix 2). According
to our Conjecture this statement is true for all finite-gap periodic
potentials with generic poles. Natural extension of this result for
periodic potentials with isolated poles of the form $n(n+1)/2 + O(x)$
is also described below.**

Take any smooth functions $a(z)$ on the contour κ_0 properly decreasing in
the infinite component for $z \to 0$ or $k \to \infty$. We can see that its BA Fourier
Transform

$$f(x) = \tilde{a}(x) = 1/\sqrt{2\pi} \oint_{\kappa_0} a(z)\Psi_{D^*}(x, z)d\mu$$

belongs to the class described above with some proper decay at $|x| \to \infty$.
We define global behavior and completion of the space in the Appendix
2 using operators on the circle $[0T]$ with periodic (quasi-periodic) bound-
ary conditions at the ends with unimodular Bloch-Floquet multipliers $|\varkappa| =
|\exp\{ip(z)T\}| = 1$ on the contour κ_0 where $p = p_R$.

Examples: The case $g = 1$. There are two different cases here:

1.The Hermit-Lame' Operator. Consider real elliptic curve Γ with genus
$g = 1$ and real branching points u_0, u_1, u_2, ∞. **So this case is real semi-
canonical.** We assume that our divisor $D = \gamma$ coincides with $P = \infty$. The
Baker-Akhiezer Function $\Psi_D(x, z)$ here was found by Hermit simply as some
basis of solutions for all values of spectral parameter $u(z) = z^{-2}$. Operator
L here is the Lame Operator; it has periodic potential $U(x) = 2\wp(x)$ with
poles like $2/x^2$ on the real line x in the points $nT, n \in Z$. Classical XIX
Century people considered this operator on the interval $[0T]$. They obtained
this operator (by the separation of variables in Jacoby coordinates) from the
Laplace-Beltrami operator on the 3-axis ellipsoida. They needed its spec-
trum on the interval $[0T]$ with zero boundary conditions at the ends $0, T$.

It is normally called in the literature "The Dirichlet Spectrum" but we call it The Hermit Spectrum for finite-gap operators. Let us make some useful methodological remarks about the comparison of Hermit (Dirichlet) Spectrum and periodic spectrum of Nonsingular Shifted Periodic Operator $L_{i\omega}$ on the circle $0T$ and with spectrum on the whole line R–**The One-Gap Operator.**

Statement. The Hermit spectrum of Lame operator L is simple. It consists exactly of eigenvalues $\lambda_s, s \in Z^+$, which are the double-degenerate eigenvalues of the shifted nonsingular periodic operator $L_{i\omega}$ on the circle. Here $2i\omega$ is an imaginary period of the function $\wp(x)$.

As we know, the potential $2\wp(x+i\omega)$ plays fundamental role in the Theory of Solitons: this function defines the Traveling Waves ("Soliton Lattices") for the KdV Equation as it was found in XIX Century by Korteweg and De Vries: Consider the corresponding Lax Operator

$$L_{i\omega} = -\partial_x^2 + 2\wp(x + i\omega)$$

The theory of periodic problem for KdV started in 1974 from the "soliton derivation and understanding" of the spectrum of this operator. In particular, its non-degenerate eigenvalues satisfying to the $\pm$-periodic boundary conditions

$$L_{i\omega}\psi(x) = \lambda\psi(x), \psi(x + T) = \pm\psi(x)$$

are exactly the finite branching points u_0, u_1, u_2 of Riemann surface Γ. It has also infinite number of double-degenerate eigenvalues λ_s for $u > u_2$, for the same boundary conditions. It is exactly the set of all Hermit eigenvalues for the nonshifted operator L. They coincide with extremal points of the half-trace $S(u)$ of the monodromy matrix $\hat{T}(u)$:

$$1/2Tr\hat{T} = S(u), \hat{T}(\psi(x,u)) = \psi(x + T, u)$$

along the minimal real period T. We have $-1 \le S(u) \le 1, u \in [u_2, \infty]$, and $S(u) = \pm 1, S'(u) = 0$ for $u = \lambda_s$. In particular, $\lambda_s > u_2$.

Proof. The Hermit problem is real, so we look for the real eigenfunctions $\psi(x, \lambda_s) = \psi_s(x)$ equal to zero at the ends $0, T$. The space of nonsingular solutions to the equation $L\psi = \lambda_s\psi$ is one-dimensional, so this function ψ should be also an eigenvector of the monodromy matrix $\hat{T}$ with real eigenvalue. So it is either the double-degenerate point indicated in our statement or it belongs to the gap of the spectrum. In the last case we point out that

BA functions in the gaps always have poles at the points nT for the Lame operator. Our statement is proved.

For the points $u = \lambda_s$ monodromy matrix $\hat{T}$ is equal to ± 1. One of corresponding two eigenfunctions is nonsingular and equal to zero in the points nT defining the eigenfunction of the Hermit Spectrum. So it is completely determined by the Riemann Surface Γ.

We return now to spectral theory of the singular Lame operator L on the whole line $x \in R$ and canonical real contour $z \in \kappa_0$.

The Space of functions H_{D,κ_0} is a direct sum of 2 spaces

$$H = H_0 \bigoplus H_\infty$$

Here H_0 consists of functions on the compact circle $c_1 \subset \Gamma$ located over the spectral zone $[u_0, u_1]$ (the finite zone of spectrum). The second subspace H_∞ consists of functions on $R \subset \Gamma$ located over the infinite zone of spectrum $[u_2, \infty]$ and homeomorphic to $R = S^1 \backslash \infty$. They have specific asymptotic at infinity indicated above in the chapter 1.

Statement: The Indefinite Inner Product in the space $H_{\infty,\kappa_0} = H_0 \bigoplus H_\infty$ is positive at H_∞ and negative at H_0.

Proof. We have in this case $d\mu = dz/\sqrt{(u - u_0)(u - u_1)(u - u_2)}$, and orientation of the contour $\kappa_0 = c_1 \bigcup c_\infty$ is such that $d\mu_{c_1} < 0, d\mu|_{c_\infty} > 0$. This statement is proved.

For comparison good to consider the "selfadjoint" case such that $\gamma' = (\alpha', \pm)$ where $\alpha' \in [u_1, u_2]$ is located in the finite gap. In this case we have

$$d\mu' = (u - \alpha')du/\sqrt{(u - u_0)(u - u_1)(u - u_2)}$$

So we have $d\mu' = (u - \alpha')d\mu$. Taking into account that the function p_I is the same in both cases, we see that the factor $(u - \alpha')$ has opposite signs in the gaps c_1 and c_∞. Therefore in this case the Inner Product is positive (as we knew before).

2. The complex branching points. Let $g = 1$ but $u_0 = \bar{u}_1 \in C, u_3 \in R$ for the Riemann Surface Γ. **This case is real but not semicanonical.** We take divisor point at infinity $D = \infty$. Corresponding potential is also real function $2\wp(x)$ with pole $2/x^2$ and real period T but corresponding lattice is romb-like (the complex period is not orthogonal to the real one, but their lengths are equal to each other.) Our contour κ_0 is connected and critical. It consists of two circles crossing each other transversally in two real points

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$u^*, \pm$ on the infinite component over $[u_2, \infty]$. So the spectrum of operator $L = -\partial_x^2 + 2\wp(x)$ on the whole line discussed in this work, contains non-real part where anti-involution τ is not identity $Fix_\tau \neq \kappa_0$. The Hermit Spectrum is also well-defined (zero boundary conditions at the poles $0, T$). It is also equal to the set of real points $S(u) = \pm 1$, where $S = 1/2 Tr\hat{T}$ is the trace of monodromy matrix (which is real for real x). All these points are located at the infinite real zone $[u_3, \infty]$.

The Standard Fourier Transform we have for the case of genus zero: The Riemann Surface Γ has 2 branching points $u_0 = 0, u_1 = \infty$. The spectral zone in Γ has one component c_∞ only located over $[0, \infty]$. It is isomorphic to $R = S^1 \backslash \infty$. The measure $d\mu$ coincides with standard measure. So our Hilbert Space is exactly $H = H_\infty = L_2(R)$, and inner product is positive. For genus more than zero we can have positive inner product only for smooth potentials where divisor points are located in the finite gaps (one gap–one point).

Return now to the generic singular finite-gap potentials. **In all cases where our divisor contains infinite point or any point located at the infinite gap, we have indefinite inner product**. We can always move this point by some time shift to infinity.

In all cases where our divisor contains two (or more) points located in the same finite gap, we have indefinite inner product.

We can easily describe the sign corresponding to the cycle $c_j \subset \Gamma$ located over the zone $[u_{2j-1}, u_{2j}]$, i.e. how it enters the Indefinite Space:

Take divisor points $\gamma_s = (\alpha_s, \pm)$, where $\alpha_s \in [u_{2q_s-1}, u_{2q_s}], s = 1, ..., r$, and $\gamma_{r+k} = \infty$ for all $k > r$. As we know, it simply coincides with sign of the expression $dp/d\mu$ where:

$$d\mu = (u - \alpha_1) \times ...(u - \alpha_r) dz / \sqrt{(u - u_0) \times ...(u - u_{2g})}$$

on the cycle c_k, taking into account the orientation of the contour κ_0 provided by the function p_I as it was explained in the Chapter 1. **For example, for $r = 0$ (the case of BA Fourier Transfom with important Multiplicative Properties based on the Hermit-Lame' potentials $n(n+1)\wp(x) = U(x)$), the signs corresponding to c_k, are alternating.** Here we have

$$d\mu = du / \sqrt{R(u)}$$

We shall discuss this case in the Appendix 2.

Appendix 1. The Cauchy-Baker-Akhiezer Kernel

Following [6], let us define the Cauchy-Baker-Akhiezer Kernel $\omega(x, z, w)$, $x \in \mathbb{C}$, $z \in \Gamma \backslash P$, $w \in \Gamma \backslash P$ by the following analytic properties:

1. For a fixed x the kernel $\omega(x, z, w)$ is a meromorphic function in z and a meromorphic 1-form in w.

2. For fixed x, w the kernel $\omega(x, z, w)$ has exactly $g + 1$ simple poles in z at the points $\gamma_1, \ldots, \gamma_g, w$.

3. For fixed x, z the kernel $\omega(x, z, w)$ has simple zeroes in w at the points $\gamma_1, \ldots, \gamma_g$, and a simple pole with residue 1 at the point z. In local coordinates we have

$$\omega(x, z, w) = \frac{dw}{w - z} + \text{regular terms} \quad \text{as} \quad w \to z. \qquad (4)$$

4. For fixed x, w the kernel $\omega(x, z, w)$ has an essential singularity in the variable z at the point $P = \infty$:

$$\omega(x, z, w) = e^{ik(z)x} \left(O\left(\frac{1}{k(z)} \right) \right). \qquad (5)$$

5. For fixed x, z the kernel $\omega(x, z, w)$ has an essential singularity in the variable w at the point $P = \infty$:

$$\omega(x, z, w) = e^{-ik(w)x} \left(O\left(\frac{1}{k(w)} \right) \right) dk(w). \qquad (6)$$

For generic spectral data the kernel $\omega(x, z, w)$ exists and is unique, by the Riemann-Roch Theorem. The proof is analogous to the proof of existence and uniquiness for the Baker-Akhiezer function. Following the idea of the work [6], we prove one of the most important properties of this Kernel:

Fundamental Lemma. Following Formula is Valid:

$$\partial_x \omega(x, z, w) = -i\Psi(x, z)\Psi^*(x, w)d\mu(w). \qquad (7)$$

Proof. For a fixed w the right-hand side of (7) has the following analytic properties:

1. $\partial_x \omega(x, z, w)$ is meromorphic in the variable z on $\Gamma \backslash P$ and has exactly g simple poles at the points $\gamma_1, \ldots, \gamma_g$.

2. $\partial_x \omega(x, z, w) = O(1)e^{ik(z)x}$ as $z \to P$.

Therefore for a fixed w the expression $\partial_x \omega(x, z, w)$ is proportional to $\Psi(x, z)$. Similarly for a fixed z the expression $\partial_x \omega(x, z, w)$ is proportional to $\Psi^*(x, w)d\mu(w)$. Therefore $\partial_x \omega(x, z, w) = c\Psi(x, z)\Psi^*(x, w)d\mu(w)$. Assuming z and w are both close to P, we obtain $c = -i$. Our Fundamental Lemma is proved.

Remark 7 *Let $x = 0$. Then the kernel $\omega(0, z, w)$ coincides with the standard meromorphic analog of Cauchy kernel on the closed Riemann surfaces (see [8], [9]).*

Remark 8 *It is natural to define Cauchy-Baker-Akhiezer kernel $\omega(\vec{t}, z, w)$ depending on all KP times $\vec{t} = (t_1, t_2, t_3 \ldots)$, $x = t_1$, $y = t_2$, $t = t_3$ (see [6]). Essential singularities for the kernel $\omega(\vec{t}, z, w)$ have the following form:*

$$\omega(\vec{t}, z, w) = e^{i \sum_{j=1}^{\infty} t_j k^j(z)} \left(O\left(\frac{1}{k(z)}\right) \right), \quad z \to P, \tag{8}$$

$$\omega(\vec{t}, z, w) = e^{-i \sum_{j=1}^{\infty} t_j k^j(w)} \left(O\left(\frac{1}{k(w)}\right) \right) dk(w), \quad w \to P. \tag{9}$$

Here we assume that only finite number of variables t_n are different from 0.

To stress the dependence of $\omega(\vec{t}, z, w)$ on the divisor $D = \gamma_1 + \ldots + \gamma_g$, we shall write $\omega_D(\vec{t}, z, w)$ if necessary.

Lemma 4 *Denote by $D(\vec{t})$ the divisor of zeros for the function $\Psi_D(\vec{t}, z)$. Then for any $\vec{t'}$ we have following transformation law:*

$$\omega_D(\vec{t}, z, w) = \frac{\Psi_D(\vec{t'}, z)}{\Psi_D(\vec{t'}, w)} \omega_{D(\vec{t'})}(\vec{t} - \vec{t'}, z, w) \tag{10}$$

The next special case plays leading role in our investigation because it is the most natural source for the singular operators:

Lemma 5 *Assume, that exactly one point of the divisor D lies at the infinite points $P = \infty$: $D = \gamma_1 + \gamma_2 + \ldots + \gamma_{g-1} + P$. Then for generic $\gamma_1, \ldots, \gamma_{g-1}$ one can write an especially simple formula for the kernel $\omega(\vec{t}, z, w)$:*

$$\omega(\vec{t}, z, w) = \frac{\theta[\sum_j \vec{U}_j t_j + \vec{A}(z) - \vec{A}(w) - \vec{A}(\gamma_1) - \ldots - \vec{A}(\gamma_{g-1}) - \vec{K})]}{\theta[\sum_j \vec{U}_j t_j - \vec{A}(\gamma_1) - \ldots - \vec{A}(\gamma_{g-1}) - \vec{K}]} \times$$

$$\times \frac{C \cdot d\mu(w)}{\theta[\vec{A}(z) - \vec{A}(w) - \vec{A}(\gamma_1) - \ldots - \vec{A}(\gamma_{g-1}) - \vec{K}]} \cdot \exp\left[i \sum_j t_j \int_w^z \Omega_j\right] \quad (11)$$

Here Ω_j are meromorphic differentials with an unique pole at the point P,

$$\Omega_j = d(k^j) + regular \; tems$$

and zero a-periods, U_j denotes the normalized vector of b-periods for Ω_j:

$$U_j^k = \frac{1}{2\pi} \oint_{b_k} \Omega_j,$$

$\vec{A}(\gamma)$ denotes the Abel transform with the starting point P, $\vec{K}$ is the vector of Riemann constants, $d\mu$ is the holomorphic differential with the zeroes $\gamma_1, \ldots, \gamma_{g-1}$. Let ν be a local coordinate near P such, that $d\mu = d\nu(1 + o(1))$. Then the normalization constant C is defined by:

$$C = \partial_\nu\Big|_{\nu=0} \theta[-\vec{A}(v) - \vec{A}(\gamma_1) - \ldots - \vec{A}(\gamma_{g-1}) - \vec{K}]. \quad (12)$$

Using standard arguments, one can easily check, that the expression (11) is single-valued in Γ, and for generic $\vec{t}$ it has the proper poles in z and w. Let $z \to w$. Then

$$\omega(\vec{t}, z, w) \sim \frac{C \cdot d\mu(w)}{\theta[\vec{A}(z) - \vec{A}(w) - \vec{A}(\gamma_1) - \ldots - \vec{A}(\gamma_{g-1}) - \vec{K}]} \quad (13)$$

For z different from $\gamma_1, \ldots, \gamma_g, \gamma_1^*, \ldots, \gamma_g^*$ denominator of (13) has a first-order zero at $w = z$. If $z = \gamma_j$, $j = 1, \ldots, g$, then the denominator vanishes identically. If $z = \gamma_j^*$, $j = 1, \ldots, g$, then denominator has a second-order pole. Therefore the zeroes of the numerator coincides with the zeroes of the denominator's differential, and the residue of (13) at $w = z$ is regular in

Γ. Therefore this residue is constant. Normalization (12) means, that the residue is equal to 1 at $z = \infty$. It completes the proof.

It follows from (11) that the Cauchy-Baker-Akhiezer Kernel $\omega(\vec{t}, z, w)$ is meromorphic in all t_j. Combining (11) with (10) we obtain following:

Lemma 6 *For any divisor D such, that $\Psi_D(x, z)$ is defined for generic x, the kernel $\omega_D(x, z, w)$ is meromorphic in x.*

Remark 9 *Assume, that operators, associated with the curve Γ are strictly periodic in x with period T. Then following formula is true:*

$$\omega(x + T, z, w) = \omega(x, z, w) \cdot e^{i[p(z)-p(w)]x}. \tag{14}$$

Let us derive the orthogonality relation for BA functions treated as basis in the space of functions in x-space (2). Their inner products already were discussed in the work [7]. We have

$$\int_{-nT}^{nT} \Psi(x, z)\Psi^*(x, w)d\mu dx = i\omega(x, z, w)\Big|_{-nT}^{nT} =$$

$$= i\omega(0, z, w)\left[e^{i[p(z)-p(w)]nT} - e^{-i[p(z)-p(w)]nT}\right] =$$

$$= [p(w) - p(z)]\omega(0, z, w)\int_{-nT}^{nT} e^{i[p(z)-p(w)]x}dx.$$

Therefore

$$\lim_{n\to\infty}\int_{-nT}^{nT} \Psi(x, z)\Psi^*(x, w)d\mu dx = 2\pi[p(w) - p(z)]\omega(0, z, w)\delta(p(z) - p(w)),$$

and

$$\lim_{n\to\infty}\int_{-nT}^{nT} \Psi(x, z)\Psi^*(x, w)d\mu dx = 0 \text{ for } z \neq w, \quad z, w \in \kappa_c.$$

Let $w \to z$. Substituting (4) and taking into account, that the orientation on the canonical contour κ_c is defined by dp, we obtain our final result:

Following Orthogonality Relations for BA functions as a basis in the x-space, are true:

$$\left(\int_{-\infty}^{\infty} \Psi(x, z)\Psi^*(x, w)dx\right)d\mu =$$

$$= 2\pi(p(w) - p(z)) \left[\frac{dp(w)}{p(w) - p(z)} + \text{regular terms} \right] \delta(p(z) - p(w)) = 2\pi\delta(z, w)$$

Appendix 2. The Hyperelliptic Case. The Periodic Boundary Conditions.

The Cauchy-Baker-Akhiezer Kernel.

In the hyperelliptic case there exists a simple explicit formula for the Cauchy-Baker-Akhiezer Kernel:

$$\omega(x, z, w) = i \frac{\Psi(x, z)\Psi_x^*(x, w) - \Psi_x(x, z)\Psi^*(x, w)}{w - z} d\mu(w)$$

It is easy to check, that all analytic properties are fulfilled. Moreover,

$$\partial_x \omega(x, z, w) = [\Psi(x, z)\Psi_{xx}^*(x, w) - \Psi_{xx}(x, z)\Psi^*(x, w)] \frac{i}{w - z} d\mu =$$

$$= [(-z - U(x))\Psi(x, z)\Psi^*(x, w) - (-w - U(w))\Psi(x, z)\Psi^*(x, w)] \frac{i}{w - z} d\mu =$$

$$= -i\Psi(x, z)\Psi^*(x, w)d\mu.$$

The periodic boundary conditions.

We assume that our finite-gap operators are periodic with the period T. In addition to the spectral problem in the whole line one can consider the periodic boundary problem with an fixed unitary multiplier:

$$\Psi(x + T, z) = \varkappa\Psi(x, z), \quad |\varkappa| = 1. \tag{15}$$

For regular potentials this problem is self-adjoint and has only discrete spectrum. Let us enumerate the points z_j in Γ, $j = 1, 2, \ldots, \infty$ such, that

$$e^{iTp(z_j)} = \varkappa.$$

All these points lie in the canonical contour κ_0. Each finite oval contains only finite number of points z_j.

Lemma 7 *The scalar product for the basic eigenfunctions is given by:*

$$\int_0^T \Psi(x, z_j)\Psi^*(x, z_k)dx = \delta_{jk}\frac{dp(z_j)}{d\mu(z_j)}. \tag{16}$$

As above, we deform the integration contour in the x-plane to avoid singularities.

Let us point out that in the real self-adjoint case with positive inner product this formula was known–see the work [4], formula 30. Our lemma extends this result to the case of singular potentials and indefinite inner product.

   Proof. For $j \neq k$

$$\int_0^T \Psi(x, z_j)\Psi^*(x, z_k)dx = \frac{i\omega(x, z_j, z_k)}{d\mu(z_k)}\Big|_0^T = \frac{i\omega(0, z_j, z_k)}{d\mu(z_k)}\left[e^{iT[p(z_j)-p(z_k)]} - 1\right] = 0.$$

Let $k = j$.

$$\int_0^T \Psi(x, z_j)\Psi^*(x, z_j)dx = \lim_{w \to z_j}\frac{i\omega(x, z_j, w)}{d\mu(w)}\Big|_0^T =$$

$$= \lim_{w \to z_j}\left[\frac{idp(w)}{d\mu(w)[p(w) - p(z_j)]} + \text{regular terms}\right]\left[e^{iT[p(z_j)-p(w)]} - 1\right] = \frac{dp(z_j)}{d\mu(z_j)}.$$

Assume now, that the spectral data Γ, D satisfy the reality constraints. Taking into account that for real case $\Psi^*(x, z) = \bar{\Psi}(x, \tau z)$, we obtain following:

   Theorem. *Let us define the scalar product by*

$$(\Psi(x, z_j), \Psi(x, z_k))_x = \int_0^T \Psi(x, z_j)\bar{\Psi}(\bar{x}, z_k).dx \tag{17}$$

For singular potentials this scalar product is indefinite. The dimension of negative subspace is finite and coincides with the number of points z_j such, that $dp(z_j)/d\mu(z_j) < 0$. Therefore we have a **Pontryagin-Sobolev** *space of functions where our singular finite-gap operator is symmetric.*

   Example: The Higher Lame Potentials. Let us consider the operators $L = -\partial^2 + n(n+1)\wp(x)$ for the semi-canonical case where elliptic function is defined by the Riemann Surface $y^2 = (u - u_0)(u - u_1)(u - u_2)$, $u_j \in R$. For $n = 1$ this case was discussed in Chapter 2. For all $n \in Z^+$ these potential are singular finite gap. They correspond to some algebraic curve Γ_n with

genus $g = n$ and poles in the points $mT, m \in Z$. Here T is a real period of the function $\wp(x)$. The operator $L_{n,i\omega} = -\partial^2 + n(n+1)\wp(x+i\omega)$ is smooth finite-gap for the imaginary period $2i\omega$. It corresponds to the same algebraic curve Γ_n.

The Hermit Spectrum here also coincides with the set of real points $S_n(u) = \pm 1, S'_n(u) = 0$ as for $n = 1$. Here $S_n = 1/2 Tr \hat{T}_n$ is the trace on monodromy matrix for the operators L_n or $L_{n,i\omega}$ on the real line. For the case L_n our divisor is equal to $D = n\infty$. For the operator $L_{n,i\omega}$ with even nonsingular periodic potential, our divisor is located in the upper boundaries of the finite gaps of spectrum. They are the branching points $u^*_{n,2k}, k = 1, 2, ..., 2n$ for the algebraic curve Γ_n. All Hermit Spectrum here is located in the infinite spectral zone $[u^*_{n,2n}, \infty]$ as for $n = 1$ (see chapter 2).

Let us construct the space H_L for the Lame operators $L = L_n$. Elementary local analysis with Laurent series near the pole $x = 0$ leads to the following result: **All eigenfunctions of the operator L_n have following singularities**:

$$n = 2k - 1 : \psi(x) = \alpha_1/x^n + \alpha_2/x^{n-2} + ... + \alpha_k/x + \phi(x)$$

$$n = 2k : \psi(x) = \alpha_1/x^n + \alpha_2/x^{n-2} + ... + \alpha_k/x^2 + \phi(x)$$

where

$$\phi(0) = ... = \phi^{(2r)}(0) = ... = \phi^{(2k-2)}(0) = 0$$

for $n = 2k - 1$, and

$$\phi'(0) = ...\phi^{2r}(0) = ... = \phi^{2k-2}(0)$$

for $n = 2k$[4] Therefore all their linear combinations also satisfy to these linear relations. We are coming to the following result:

Consider the spaces of C^∞-functions $f(x) \in F_{n,\varkappa}, f(x+T) = \varkappa f(x)$, and

$$f(x) - f(-x) = O(x^n), n = 2k$$

$$f(x) + f(-x) = O(x^n), n = 2k - 1$$

We introduce in these spaces the standard positive inner product. Consider the spaces of function $F_{n,\varkappa} \bigoplus C^k(\varkappa)$ where the space $C^k(\varkappa)$ over C is generated by the singular BA eigenfunctions with negative inner squares defined

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[4]Local analytic properties of wave functions for singular potential with such singularities in case $n = 1$ were discussed in [11].

above, with Bloch-Floquet multiplier $\varkappa$. We have $k = [(n+1)/2]$ so all this extension exactly corresponds to the set of "residues" $\alpha_1, ..., \alpha_k$. They have singularities with Laurent part described here. The direct integral of these spaces over the circle $\varkappa \in S^1$ is our space $\tilde{H}_{L_n}$ with indefinite metric.

Construct KdV solution with initial value $U(x,0) = n(n+1)\wp(x)$ where

$$U_t = 6UU_x - U_{xxx}$$

It has a form like

$$U(x,t) = \sum_{j=1}^{j=n(n+1)/2} 2\wp(x - x_j(t))$$

such that $x_j(0) = 0$. We assume that there exists and unique solution such that $x_j \neq x_p$ for small t.

Statement. We have asymptotically $x_j \sim a_j t^{1/3}$ for $t \to +0$ and $a_j \neq a_p$. Here a_j in general are complex numbers. This set is invariant under transformation $a_j \to \eta a_j$ where $\eta^3 = 1$. Every orbit of the group Z_3 generated by η, enter this set only ones. This group acts freely if $n(n+1)/2$ is divisible by 3. There is exactly one zero orbit $a_{n(n+1)/2} = 0$ if $n(n+1)/2 = 1 (mod 3)$. Other orbits are free and distinct, i.e. $a_j^3 \neq a_k^3$ if they belong to different orbits.

In order to prove that, we consider rational solutions $U_0 = \sum_j 2/(x - x_j(t))^2$ such that $x_j(0) = 0$. For the poles we have an equation

$$\partial_t x_j(t) = -12 \sum_{p \neq j} 1/(x_j - x_p)^2$$

Changing time, we remove inessential factor -12 keeping the same notation for time. The solution U_0 approximates U near the pole $t \to +0, x \to 0$. Our functions $x_j = a_j t^{1/3}$ exactly satisfy it. So we obtain algebraic equation for the set a_j^3. Our statement easily follows from this algebraic equation.

Conjecture. There exists exactly $k = [(n+1)/2]$ "real" orbits (i.e. $a_j^3 \in R$).

Let us point out that rational solutions to the KdV hierarchy with initial function $U_0(x,0) = n(n+1)/x^2$ were investigated in the work [13]. Some general formulas were obtained for this solutions. In particular, they proved that at least one pole is real (because corresponding Lax Operator is not self-adjoint).

According to our work, this number is exactly equal to the number of negative squares in the indefinite inner product for every value of $\varkappa$. It is obvious that the number of real orbits is $k' \geq k = [(n+1)/2]$ but opposite inequality should be proved. Our space $\tilde{H}_{L_t}$ for the operators $L_t = -\partial_x^2 + U(x,t)$ can be obtained by the extension of the space of C^∞-functions with zeroes in the poles $x_j(t)$. So the dimension of additional part is exactly equal to the number of poles if and only if there are no relations between residues. The dimension of negative part for L_t is stable under KdV deformation (i.e. remains constant).

We proved this conjecture numerically for $n < 9$.

For $n = 1$ we have $a_1 = 0$.

For $n = 2$ we have $a_j = (1, \eta, \eta^2)$.

For $n = 3$ we have $a_j = (a, a\eta, a\eta^2, b, b\eta, b\eta^2)$. Here we have

$$w = a^3/b^3 = 1/2(-7 + \sqrt{45})$$

and both orbits are real

$$a^3 = 1 + 9w(w + 2)/(w - 1)^2, b^3 = 1 + 9w^{-1}(w^{-1} + 2)/(w^{-1} - 1)^2$$

As a Corollary, we obtain an important result: **There are no relations between residues of the real poles for the generic finite-gap real singular potential in this specific case. So the space $\tilde{H}_{L_t}$ of functions in the variable x for $L = -\partial_x^2 + U(x,t)$ depends on the poles only.**

For Higher Lame Potentials we have **"The Intermediate Cases"**

$$D = r\infty + \gamma_1 + ... + \gamma_{n-r}$$

where $n = g$ and all points $\gamma_j = (\lambda_j^*, \pm)$ are located in some finite gaps (no more than one point in one finite gap). The Hermit Spectrum contains the same points as for higher Lame potential: they were described through the half-trace $S(u)$ of monodromy matrix $S = 1/2Tr\hat{T}_n$ as $S = \pm 1, S' = 0$. However, here the Hermit Spectrum contains also finite number of points λ_j^* coming from the divisor points in finite gaps. These additional spectral points λ_j^* can be obtained using some carefully chosen chain of ,,Backlund-Darboux Transformations" starting from the smooth potential $L_{n,i\omega}$: sometimes they simply lead to the special shifts along the Jacoby Variety in the theory of finite-gap potentials.

Generalizations: The Intermediate and One Pole finite-gap singular potentials.

The Intermediate Potentials. Consider any nonsingular hyperelliptic curve Γ given as $y^2 = (u - u_0)...(u - u_{2n})$ with real $u_0 < ... < u_{2n}$. We chose divisor $D = r\infty + \gamma_1 + ... + \gamma_{n-r}$ where all finite points $\gamma_j = (\lambda_j^*, \pm)$ belong to some finite gaps $a_k = [u_{2k-1}u_{2k}], k \geq 1$, (no more than one point in the gap). Let us assume that this potential is x-periodic for real x, with minimal period T. Corresponding potential $U_{(x)}$ has some number of poles on the circle $0T$, of the form $r(r+1)/x^2 + O(x)$ each. We can prove that the Hermit spectrum in this case consists of all points $S(u) = \pm 1, S' = 0$, plus all points λ_j^* in the finite gaps. Minimal set of Hermit eigenvalues we have for the case $r = n$. Maximal set we have for $r = 0$, i.e. for the nonsingular finite-gap potentials.

Other potentials of that family can be obtained from the nonsingular one with $r = 0$ by the special sequence of Backlund-Darboux Transformations (Γ is fixed).

No problem to describe here (for all $r \leq n$) the number of eigenvalues with negative inner square for periodic spectrum: Put first the sign $+$ on the infinite spectral zone and move to the next one from the left. Put sign $-$ on the next spectral zone if there is no divisor points between them. Otherwise put the same sign on the next zone. After that continue to move left with the same rule. So we have exactly r changes of sign. It is equal to the number of finite gaps empty from the divisor points. Additional negative periodic singular eigenfunction is coming from the points $S(u) = 1, S' = 0$ inside of finite spectral zones with the negative sign. We can think about that point as "degenerate gap" of the length zero with additional divisor point in it, so it does not change sign. So the maximal number of negative signs we obtain for the case $r = 1$ where the empty gap is exactly the maximal finite gap $[u_{2n-1}u_{2n}]$. However, the detailed description of indefinite functional space in the x variable depends here on the number of poles which is hard to predict.

Construct the Infinite Gap limit with fixed r equal to the number of empty finite gaps. This class of potentials might be especially interesting. We can introduce the infinite-gap limit of them: We keep the number r with corresponding empty finite gaps, real period T and the form and number of poles unchanged for $n \to \infty$, opening more and more small gaps (branching points) with new divisor points inside of infinite spectral zone. We open them near the points $S(u) = \pm 1$ such that their length tends to zero rapidly enough. Probably, it is possible to construct such singular infinite-gap periodic potentials simply by the sequences of specially chosen Backlund-Darboux transformations from the smooth periodic potentials. We

need to prove that all such "infinite-gap" intermediate potentials can be approximated by the singular finite-gap potentials with the same period and fixed r.

The One Pole Potentials. Consider the special case $D = n\infty$ periodic in x with minimal period T. Let corresponding operator (i.e. its algebraic curve Γ and period T) is small enough perturbation of the Lame potential $n(n+1)\wp(x)$. In this case the potential $U(x)$ has only one real pole on the circle $0T$ with singularity $n(n+1)/x^2 + O(x^2)$. All results formulated above for Lame operators are valid here (some of the them require evenness of the perturbed potential).

More general one-pole potentials can be constructed: Let $U(x)$ be any smooth periodic finite-gap potential with minimal period T. Following [12], one can apply a special Backlund (i.e. Darboux) transformations (the Crum transformations) to $U(x)$. The resulting potential $U_1(x)$ is also periodic with the same period. It has the same spectral curve Γ and has exactly one real pole on the circle $0T$ at the point 0 with singularity $2/x^2 + O(x)$. The Dirichlet spectrum (Hermit spectrum in our paper) of $U_1(x)$ is obtained from the Dirichlet-Hermit spectrum of $U(x) = U_0$ by removing the ground state. One can iterate this procedure $U_j \to U_{j+1}$, removing the ground state at each step and increasing the leading coefficient of the pole at the point 0 from $j(j+1)/x^2 + O(x)$ to $(j+1)(j+2)/x^2 + O(x)$.

Statement. Denote by $U_r(x)$ the potential, obtained from the smooth finite-gap potential $U(x) = U_0$ by r iterations of the Crum transformation over the ground state functions. Then $U_r(x)$ is periodic with period T and has exactly one pole of type $r(r+1)/x^2 + O(x)$ at the point $x = 0$. The spectral curve of $U_r(x)$ coincides with the spectral curve for $U(x) = U_0$, and the Hermit-Dirichlet spectrum of $U_r(x)$ coincides with the Hermit-Dirichlet spectrum of $U(x)$ with r lowest states removed.

This procedure can also be applied to smooth periodic infinite-gap potentials with minimal period T.

Examples: Degenerate Cases. Consider now the degenerate singular one-gap potentials corresponding to the degenerations of spectral elliptic curves associated with Weierstrass elliptic functions and Lame operators:

a. Trigonometric (hyperbolic) singular degeneration:

The real period tends to infinity, and imaginary period remains finite

$$2\wp(x) + const \to 2c^2/\sinh^2(cx) = U_c(x)$$

Here elliptic functions correspond to the algebraic curve $\Gamma_t, t \to 0$. Realizing

it as a spectral curve of our operators, we obtain branching points $u_0^t < u_1^t < u_2^t < \infty$ such that $u_j^t \in R$. In process of degeneration $t \to 0$ we have

$$u_0, u_1 \to -c^2, u_2 \to 0$$

The divisor point γ coincides with ∞ for all t. Easy to see that the spectrum of operator $L_c = -\partial_x^2 + U_c(x)$ in the corresponding space $\tilde{H}_{L_c}$ on the whole line R occupies all set $\lambda \geq 0$ plus one negative discrete eigenvalue. We have singular eigenfunction

$$\psi_c = 1/\sinh(cx)$$

for $\lambda_0 = -c^2$. Our spectrum consists of the union of the half-line $\lambda \geq 0$ and one-dimensional space for $\lambda_0 = -c^2$. We have exactly one negative simple eigenvalue $-c^2 = \lambda_0 < 0$. It has negative inner square in the Pontryagin-Sobolev space constructed above. All other eigenvalues are positive. The number of negative squares on the whole line is equal to one.

b. Rational degeneration $c \to 0$: Here $U \to 2/x^2$. Both periods became infinite, and $u_j^t \to 0, j = 0, 1, 2$. The spectrum of operator $L = -\partial_x^2 + 2/x^2$ on the whole line became $\lambda \geq 0$. For $\lambda = 0$ we have singular eigenfunction $\psi_0 = 1/x$ (the discrete eigenvalue on the bottom of the continuous spectrum) whose inner square is negative in our inner product.

c. Easy to consider also trigonometric degeneration $U = 2c^2/\sin^2(cx)$ with antiperiodic eigenstate corresponding to the endpoint of limiting "finite spectral zone" $0 \leq \lambda \geq c^2$,

$$\psi_0 = 1/\sin(cx), L_c\psi_0 = c^2\psi_0$$

Infinite spectral zone also starts in the same point $\lambda \geq c^2$. Here real period remains finite and imaginary period tends to zero. This case corresponds to the degeneration of real elliptic curves corresponding to the zero length of finite gap $u_1 = u_2 = c^2$ and $u_0 = 0$. The divisor point is equal to ∞. We have spectrum $\lambda \geq u_0$. For every Bloch-Floquet multiplier $\varkappa \in S^1$ we have exactly one singular eigenfunction with eigenvalue $0 \leq \lambda_0(\varkappa) \leq c^2$ whose inner square is negative in our indefinite metric. The groundstate $\phi_0 = \cos(cx)/\sin(cx)$ for $\lambda = 0$ corresponds to the periodic case $\varkappa = 1$. This spectrum can be obtained from the constant potential $U = 0$ by the Crum-Darboux Transformation discussed above (here $c = 1$):

$$L \to \tilde{L}, L = -(\partial_x + a)(\partial_x - a) = -\partial_x^2 + U, L\eta = \lambda'\eta$$

$$\tilde{L} = -(\partial_x - a)(\partial_x + a) = -\partial_x^2 + \tilde{U}, a = (\log \eta)_x$$
$$[Lf = \mu f, \tilde{f} = (\partial_x - a)f] \rightarrow [\tilde{L}\tilde{f} = \mu \tilde{f}]$$
$$\tilde{U} = U - 2a_x = U - 2(\log \eta)_{xx}$$

Taking here $U = 0, \lambda' = 1, \eta = \sin(x)$, we obtain

$$\tilde{U} = 2/\sin^2(x), [\mu = 0, f = 1] \rightarrow \tilde{f} = \phi_0 = \cos(x)/\sin(x)$$

and

$$[\mu = 1, f = \cos(x)] \rightarrow [\tilde{f} = (\partial_x - a)\cos(x) = -1/\sin(x)]$$

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