ASYMMETRIC INTEGRABLE QUAD-GRAPH EQUATIONS

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ABSTRACT. Integrable difference equations commonly have more low-order conservation laws than occur for non-integrable difference equations of similar complexity. We use this empirical observation to sift a large class of difference equations, in order to find candidates for integrability. It turns out that all such candidates have an equivalent affine form. These are tested by calculating their algebraic entropy. In this way, we have found several types of integrable equations, two of which seem to be new. One of the new equations occurs as a singular limit of the lattice MKdV equation; the remaining one seems to be isolated from all currently-known discrete integrable systems. We also list all single-tile conservation laws for the integrable equations in the above class.

1. INTRODUCTION

A *quad-graph equation* is a scalar difference equation for $u(k,l)$, where $(k,l) \in \mathbb{Z}^2$, which is of the form

$$\mathcal{F}(k, l, u_{00}, u_{10}, u_{01}, u_{11}) = 0. \tag{1}$$

Here u_{ij} denotes $u(k+i, l+j)$ and we assume that $\mathcal{F}$ depends on all four of these values. Various approaches have been used to discover quad-graph equations that are integrable. Having developed the bilinear formalism for continuous integrable systems, Hirota discretized the bilinear operators for several known integrable systems, obtaining difference equations that had soliton solutions built-in [1–3]. By contrast, Capel *et al.* focused on discretizations of plane wave factors for the singular integral equations that are ubiquitous features of continuous integrable systems [4–6]. Whereas these approaches used discretizations of problems that were known to be integrable, Adler, Bobenko and Suris (ABS) dealt directly with quad-graph equations without reference to continuous systems. They obtained a classification of all integrable quad-graph equations that are consistent on a cube (and thus admit a Lax pair), subject to certain nondegeneracy conditions [7,8]. The idea that consistency on a cube is a sufficient condition for integrability was proposed independently by Nijhoff [9] and Bobenko and Suris [10].

To make further progress, we adopt a different strategy. There is a systematic method for constructing conservation laws of difference equations; this has been used to identify low-order conservation laws of many integrable quad-graph equations [11, 12]. From this work, we observe that integrable difference equations tend to have more low-order conservation laws than non-integrable equations of similar complexity. Although this observation is purely empirical, we use it to sift a large class of quad-graph equations, in order to find equations that admit 'extra' conservation laws. (This approach is dual to that of Levi and Yamilov, who recently obtained some necessary conditions for the existence of higher symmetries – which again indicate integrability – for certain types of quad-graph equations [13]). Having obtained a shortlist of possible candidates for integrability, we test their algebraic entropy.

Zero algebraic entropy is a signature of integrability [14–16]. This occurs for affine linear quad-graph equations when an arbitrary set of initial conditions produces polynomial growth in degree as one moves away from the initial points (see §3 for details). Linear growth in degree implies that the quad-graph equation is linearizable; all known integrable quad-graph equations that are not linearizable exhibit quadratic growth. The calculation of algebraic entropy is a diagnostic test, rather than a constructive method. For instance, Hietarinta discovered a quad-graph equation that is consistent on a cube, but

does not appear in the ABS list [17]. A calculation of algebraic entropy showed that growth in degree for this quad-graph equation is linear; separately, Ramani *et al.* found a clever linearization [18].

In the next section, we determine conditions for the existence of extra conservation laws for a large class of quad-graph equations. Algebraic entropy is calculated in §3, and we find that most of the sifted quad-graph equations exhibit quadratic growth in degree. For completeness, we list the conservation laws in §4, before discussing our results and their consequences in §5.

2. Classification of integrable cases via conservation laws

In this section, we examine the conservation laws of equations of the form

$$u_{11} = \epsilon_1 u_{00} + A(u_{10}) - \epsilon_2 A(u_{01}). \tag{2}$$

Here each ϵ_i is either 1 or -1, and A is a nonlinear complex-valued function that is assumed to be 'differentiable enough' (so that as many derivatives as needed are well-defined, at least locally). Henceforth, we use A_{ij} to denote $A(u_{ij})$. Conservation laws on a single tile satisfy the determining equation

$$F\big(k+1,\, l,\, u_{10},\, \omega\big) - F\big(k,\, l,\, u_{00},\, u_{01}\big) + G\big(k,\, l+1,\, u_{01},\, \omega\big) - G\big(k,\, l,\, u_{00},\, u_{10}\big) = 0, \tag{3}$$

where ω denotes the right-hand side of (2). We solve (3) by deriving a sequence of its differential consequences, each of which eliminates at least one unknown function from the previous equation in the sequence (see [11] for a fuller explanation). This leads to an overdetermined system of functional–differential equations that can be solved completely. Specifically, we apply the commuting differential operators

$$\mathcal{L}_1 = \partial_{10} - \epsilon_1 A'_{10} \partial_{00}, \qquad \mathcal{L}_2 = \partial_{01} + \epsilon_1 \epsilon_2 A'_{01} \partial_{00},$$

(where ∂_{ij} denotes $\partial/\partial u_{ij}$) to obtain

$$\big(\epsilon_2 A'_{10} A'_{01} \partial_{00} + \epsilon_1 A'_{10} \partial_{01}\big)\partial_{00} F\big(k,\, l\, u_{00},\, u_{01}\big) + \big(\epsilon_2 A'_{10} A'_{01} \partial_{00} - \epsilon_1 \epsilon_2 A'_{01}\partial_{10}\big)\partial_{00} G\big(k,\, l,\, u_{00},\, u_{10}\big) = 0. \tag{4}$$

Dividing by $\epsilon_2 A'_{10} A'_{01}$, then differentiating with respect to u_{01} yields the partial differential equation

$$\partial_{01}\Big(\partial_{00} + \frac{\epsilon_1 \epsilon_2}{A'_{01}}\partial_{01}\Big)\partial_{00} F\big(k,\, l,\, u_{00},\, u_{01}\big) = 0, \tag{5}$$

whose general solution is

$$F\big(k,\, l,\, u_{00},\, u_{01}\big) = f_1\big(k,\, l,\, \epsilon_1 u_{00} - \epsilon_2 A_{01}\big) + f_2\big(k,\, l,\, u_{00}\big) + f_3\big(k,\, l,\, u_{01}\big). \tag{6}$$

Without loss of generality, set $f_3 = 0$ (absorbing the resulting trivial conservation law into f_2 and G). Then (4) amounts to

$$\Big(\partial_{00} - \frac{\epsilon_1}{A'_{10}}\partial_{10}\Big)\partial_{00} G\big(k,\, l,\, u_{00},\, u_{10}\big) = -\partial_{00}^2 f_2\big(k,\, l,\, u_{00}\big),$$

whose general solution is

$$G\big(k,\, l,\, u_{00},\, u_{10}\big) = g_1\big(k,\, l,\, \epsilon_1 u_{00} + A_{10}\big) + g_2\big(k,\, l,\, u_{10}\big) - f_2\big(k,\, l,\, u_{00}\big). \tag{7}$$

At this stage, it is convenient to substitute (6) and (7) into the determining equation (3), using the difference equation (2) to eliminate u_{00}. This puts the determining equation in the form

$$f_1\big(k+1,\, l,\, \epsilon_1 u_{10} - \epsilon_2 A_{11}\big) - f_1\big(k,\, l,\, u_{11} - A_{10}\big) + f_2\big(k+1,\, l,\, u_{10}\big) - f_2\big(k,\, l+1,\, u_{01}\big)$$

$$+ g_1\big(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}\big) - g_1\big(k,\, l,\, u_{11} + \epsilon_2 A_{01}\big) + g_2\big(k,\, l+1,\, u_{11}\big) - g_2\big(k,\, l,\, u_{10}\big) = 0. \tag{8}$$

Applying $\partial_{01}\partial_{11}$ to (8), we obtain

$$\epsilon_1 A'_{11} g''_1\big(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}\big) = \epsilon_2 A'_{01} g''_1\big(k,\, l,\, u_{11} + \epsilon_2 A_{01}\big), \tag{9}$$

where g_1'' is the second derivative of g with respect to its third argument. This condition holds trivially if g_1 is linear in the third argument, which leads to two 'universal' conservation laws for which F and G are each linear in A_{ij}. These are

$$F_1\big(k,\, l,\, u_{00},\, u_{01}\big) = \big(\sqrt{\epsilon_1\epsilon_2}\big)^{k+l-1}\epsilon_2^l\big(\epsilon_2 u_{00} - \sqrt{\epsilon_1\epsilon_2}A_{01}\big),$$
$$G_1\big(k,\, l,\, u_{00},\, u_{10}\big) = \big(\sqrt{\epsilon_1\epsilon_2}\big)^{k+l}\epsilon_2^l\big(\epsilon_2 u_{10} + A_{00}\big), \tag{10}$$

and

$$F_2\big(k,\, l,\, u_{00},\, u_{01}\big) = \big(-\sqrt{\epsilon_1\epsilon_2}\big)^{k+l-1}\epsilon_2^l\big(\epsilon_2 u_{00} + \sqrt{\epsilon_1\epsilon_2}A_{01}\big),$$
$$G_2\big(k,\, l,\, u_{00},\, u_{10}\big) = \big(-\sqrt{\epsilon_1\epsilon_2}\big)^{k+l}\epsilon_2^l\big(\epsilon_2 u_{10} + A_{00}\big), \tag{11}$$

In order to find all functions A for which there are additional conservation laws on a tile, we now restrict attention to the case $g_1'' \neq 0$.[1] Dividing (9) by A_{01}', then applying the operator $\partial_{01} - \epsilon_2 A_{01}'\partial_{11}$ and rearranging the result, we obtain

$$\frac{g_1'''\big(k,\, l+1, \epsilon_1 u_{01} + A_{11}\big)}{g_1''\big(k,\, l+1, \epsilon_1 u_{01} + A_{11}\big)} = \frac{A_{11}'A_{01}'' + \epsilon_2\big(A_{01}'\big)^2 A_{11}''}{A_{01}'A_{11}'\big(\epsilon_1 - \epsilon_2 A_{01}'A_{11}'\big)}.$$

It is convenient to write $A_{ij}' = B(A_{ij}) \equiv B_{ij}$, so that $A_{ij}'' = B_{ij}B_{ij}'$ (which is nonzero, as A_{ij} is a nonlinear function of u_{ij}) and $A_{ij}''' = \big(B_{ij}\big)^2 B_{ij}'' + B_{ij}\big(B_{ij}'\big)^2$. Then

$$\frac{g_1'''\big(k,\, l+1, \epsilon_1 u_{01} + A_{11}\big)}{g_1''\big(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}\big)} = \frac{B_{01}' + \epsilon_2 B_{01}B_{11}'}{\epsilon_1 - \epsilon_2 B_{01}B_{11}}. \tag{12}$$

Applying the operator

$$\partial_{01} - \frac{\epsilon_1}{A_{11}'}\,\partial_{11} = B_{01}\frac{\partial}{\partial A_{01}} - \epsilon_1\frac{\partial}{\partial A_{11}}$$

to (12) gives (after simplification)

$$\big(1 - \epsilon_1\epsilon_2 B_{01}B_{11}\big)\big(B_{01}'' - \epsilon_1\epsilon_2 B_{11}''\big) - B_{01}\big(B_{11}'\big)^2 + \epsilon_1\epsilon_2 B_{11}\big(B_{01}'\big)^2 = 0. \tag{13}$$

This is the classifying equation that yields all functions A for which there exist conservation laws other than (10) and (11). As (13) stands, the functions B_{01} and B_{11} are thoroughly entwined, but this can be resolved by one further differentiation, which yields the necessary condition

$$\big(B_{ij}'''/B_{ij}'\big)' = 0.$$

A simple calculation shows that $\big(B_{ij}'\big)^2$ is a nonzero quadratic function of B_{ij}; substituting this into (13) and solving the resulting conditions gives

$$\big(B_{ij}'\big)^2 = c_1^2\big(B_{ij}^2 + 1\big) + (1 + \epsilon_1\epsilon_2)c_2 B_{ij}, \qquad c_1, c_2 \in \mathbb{C}. \tag{14}$$

This splits into four cases, as follows.

Case I: $c_1 = 0$.

In this case, we require $\epsilon_2 = \epsilon_1$ and $c_2 \neq 0$, in order that B_{ij}' is nonzero. Then

$$A_{ij}' = B_{ij} = \frac{c_2}{2}\big(A_{ij} + c_3\big)^2,$$

so

$$A_{ij} = \frac{c}{\big(c_4 - u_{ij}\big)} - c_3, \qquad \text{where} \quad c = 2/c_2 \neq 0. \tag{15}$$

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[1]If $g_1'' = 0$ but $f_1'' \neq 0$, similar calculations lead to precisely the same classifying equation (14), so nothing is lost by this assumption.

Then the solution of (12), after absorbing the linear terms into f_2 and g_2, is

$$g_1\big(k,\,l+1,\,\epsilon_1 u_{01}+A_{11}\big)=c_5\ln\big(\epsilon_1 c_4-c_3-(\epsilon_1 u_{01}+A_{11})\big),\qquad c_5\neq 0. \tag{16}$$

This satisfies (9) if $\epsilon_1=\epsilon_2=1$, but if $\epsilon_1=\epsilon_2=-1$ then (9) gives the further constraint $c_3=c_4$. So this case leads to two possible equations, namely

$$u_{11}=u_{00}-c\Big(\frac{1}{u_{10}}-\frac{1}{u_{01}}\Big), \tag{17}$$

and

$$u_{11}=-u_{00}-c\Big(\frac{1}{u_{10}}+\frac{1}{u_{01}}\Big), \tag{18}$$

where the constant c_4 has been absorbed into u_{ij}. The transformation $u_{00}\mapsto(-1)^k u_{00}$ maps (18) into (17), which is the lattice KdV equation (simplified slightly from the form stated in [19]). By solving (8) for the remaining unknown functions, we obtain five conservation laws for (17), which are listed later (after all integrable quad-graph equations of the form (2) have been identified). The corresponding conservation laws for (18) follow from the above transformation.

Case II: $c_1\neq 0$, $\epsilon_2=\epsilon_1$, $c_2^2=c_1^4$.

In this case set $c_2=\epsilon_3 c_1^2$, where $\epsilon_3=\pm 1$. Then the general solution of (14) leads to the result

$$e^{c_1 A_{ij}+c_3}=\frac{\epsilon_3}{1-z_{ij}^{\epsilon_3}}, \tag{19}$$

where the notation $z_{ij}=e^{c_1 u_{ij}+c_4}$ is used henceforth. Then (12) amounts to

$$\frac{g_1'''\big(k,\,l+1,\,\epsilon_1 u_{01}+A_{11}\big)}{g_1''\big(k,\,l+1,\,\epsilon_1 u_{01}+A_{11}\big)}=\frac{c_1\big[\epsilon_1+z_{01}^{\epsilon_3}e^{c_1 A_{11}+c_3}\big]}{\epsilon_3-z_{01}^{\epsilon_3}e^{c_1 A_{11}+c_3}}. \tag{20}$$

If $\epsilon_3=\epsilon_1$, the general solution of (20) is

$$g_1''\big(k,\,l+1,\,\epsilon_1 u_{01}+A_{11}\big)=\frac{a(k,l+1)\exp\big\{c_1(\epsilon_1 u_{01}+A_{11})+c_3+\epsilon_1 c_4\big\}}{\big[1-\epsilon_1\exp\big\{c_1(\epsilon_1 u_{01}+A_{11})+c_3+\epsilon_1 c_4\big\}\big]^2},$$

where $a(k,l)$ is an arbitrary nonzero function. So when $\epsilon_3=\epsilon_1=1$, the condition (9) gives only $a(k,l)=\alpha(k)$, whereas when $\epsilon_3=\epsilon_1=-1$ it also gives the constraint $e^{2(c_3-c_4)}=1$. Writing (2) in terms of z_{ij}, we obtain

$$\frac{z_{11}}{z_{00}}=\frac{z_{01}-1}{z_{10}-1}\,,\qquad\text{(when }\epsilon_3=\epsilon_1=1), \tag{21}$$

and

$$z_{00}z_{11}=\frac{z_{10}z_{01}}{(z_{10}-1)(z_{01}-1)}\,,\qquad\text{(when }\epsilon_3=\epsilon_1=-1). \tag{22}$$

Equation (21) is equivalent under a point transformation to an equation that was recently discovered by Levi and Yamilov (see (26) and (31) in [13]); they found higher symmetries, a Lax pair and two conservation laws. When $\epsilon_3=-\epsilon_1$, equation (20) yields

$$g_1''(k,\,l+1,\,\epsilon_1 u_{01}+A_{11})=a(k,l+1)\exp\big\{-c_1(\epsilon_1 u_{01}+A_{11})-c_3-\epsilon_1 c_4\big\}$$

Equation (9) produces the constraint $a(k,l)=\alpha(k)$ when $\epsilon_3=-\epsilon_1=-1$, but when $\epsilon_3=-\epsilon_1=1$ it gives $a(k,l)=0$. So we obtain only one further equation, namely

$$\frac{z_{11}}{z_{00}}=\frac{z_{10}(z_{01}-1)}{z_{01}(z_{10}-1)},\qquad\text{(when }\epsilon_3=-\epsilon_1=-1). \tag{23}$$

Note that (21), (22) and (23) are affine linear in each z_{ij}.

Case III: $c_1 \neq 0$, $\epsilon_2 = \epsilon_1$, $c_2^2 \neq c_1^4$.

In this case
$$B_{ij} = \bar{c}_2 \sinh\left(c_1 A_{ij} + c_3\right) - \tilde{c}_2$$
where $\tilde{c}_2 = c_2/c_1^2$ and $\bar{c}_2^2 = 1 - \tilde{c}_2^2 \neq 0$. Then the general solution of $A'_{ij} = B_{ij}$ is
$$e^{c_1 A_{ij} + c_3} = \frac{1 + \tilde{c}_2 + (1 - \tilde{c}_2)z_{ij}}{\bar{c}_2(1 - z_{ij})},$$
and therefore (12) amounts to
$$\frac{g_1'''\left(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}\right)}{g_1''\left(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}\right)} = \frac{c_1\left[1 + \epsilon_1\tilde{c}_2 + \epsilon_1\bar{c}_2 \exp\left\{c_1(\epsilon_1 u_{01} + A_{11}) + c_3 + \epsilon_1 c_4\right\}\right]}{\left[1 + \epsilon_1\tilde{c}_2 - \epsilon_1\bar{c}_2 \exp\left\{c_1(\epsilon_1 u_{01} + A_{11}) + c_3 + \epsilon_1 c_4\right\}\right]}.$$
Hence
$$g_1''(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}) = \frac{a(k, l+1) \exp\left\{c_1(\epsilon_1 u_{01} + A_{11}) + c_3 + \epsilon_1 c_4\right\}}{\left[1 + \epsilon_1\tilde{c}_2 - \epsilon_1\bar{c}_2 \exp\left\{c_1(\epsilon_1 u_{01} + A_{11}) + c_3 + \epsilon_1 c_4\right\}\right]^2},$$
and so (9) produces the constraint $a(k, l) = \alpha(k)$ when $\epsilon_1 = 1$. The resulting difference equation is
$$\frac{z_{11}}{z_{00}} = \frac{(z_{10} + c)(z_{01} - 1)}{(z_{01} + c)(z_{10} - 1)}, \qquad c \notin \{-1, 0\}, \tag{24}$$
where $c = (1 + \tilde{c}_2)/(1 - \tilde{c}_2)$. This is equivalent under a point transformation to the lattice MKdV equation[2] (see [20–22]); in particular, a Lax pair for this equation is given in [22].

When $\epsilon_1 = -1$, the condition (9) yields $a(k, l) = \alpha(k)c^l$, together with $e^{2(c_3 - c_4)} = 1$. This leads to the difference equation
$$z_{00}z_{11} = \frac{(z_{10} + c)(z_{01} + c)}{c(z_{10} - 1)(z_{01} - 1)}, \qquad c \notin \{-1, 0\}. \tag{25}$$

Again, we have found equations that are affine linear in z_{ij} by seeking all functions A that satisfy (9) with nonlinear g_1. However, the remaining determining equation (8) provides an extra constraint on (25), namely that $c = 1$. (This is the only instance where (9) is insufficient to determine the equations that admit more than two conservation laws on a tile.) The point transformation
$$z_{ij} \mapsto (-1)^{k+i} z_{ij}^{(-1)^{k+i}} \tag{26}$$
maps (25) with $c = 1$ to (24), also with $c = 1$. Note that when $c = 0$, (24) reduces to (23). Furthermore, (21) is the limit of (24) as $c \to \infty$ with z_{ij} fixed. So (21) and (23) are each singular limits of the lattice MKdV equation.

Case IV: $c_1 \neq 0$, $\epsilon_2 = -\epsilon_1$.

This is similar to Case III; the solution of (14) is
$$B_{ij} = \sinh\left(c_1 A_{ij} + c_3\right).$$
Therefore
$$e^{c_1 A_{ij} + c_3} = \frac{1 + z_{ij}}{1 - z_{ij}}, \tag{27}$$

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[2]We thank Frank Nijhoff and Kenichi Maruno for alerting us to this.

and so

$$\frac{g_1'''(k,\, l+1,\, \epsilon_1 u_{01} + A_{11})}{g_1''(k,\, l+1,\, \epsilon_1 u_{01} + A_{11})} = \frac{c_1 \big[1 - \epsilon_1 \exp \big\{ c_1(\epsilon_1 u_{01} + A_{11}) + c_3 + \epsilon_1 c_4 \big\} \big]}{\big[1 + \epsilon_1 \exp \big\{ c_1(\epsilon_1 u_{01} + A_{11}) + c_3 + \epsilon_1 c_4 \big\} \big]}.$$

Hence

$$g_1''(k,\, l+1,\, \epsilon_1 u_{01} + A_{11}) = \frac{a(k, l+1) \exp \big\{ c_1(\epsilon_1 u_{01} + A_{11}) + c_3 + \epsilon_1 c_4 \big\}}{\big[1 + \epsilon_1 \exp \big\{ c_1(\epsilon_1 u_{01} + A_{11}) + c_3 + \epsilon_1 c_4 \big\} \big]^2}.$$

When $\epsilon_1 = 1$, (9) gives the constraints $a(k, l) = \alpha(k)$ and $e^{2c_3} = -1$, and the resulting difference equation is

$$\frac{z_{11}}{z_{00}} = -\frac{(z_{10} + 1)(z_{01} + 1)}{(z_{10} - 1)(z_{01} - 1)}. \tag{28}$$

When $\epsilon_1 = -1$, we obtain similarly $a(k, l) = \alpha(k)$, $e^{2c_4} = -1$, which leads to

$$z_{00} z_{11} = -\frac{(z_{10} + 1)(z_{01} - 1)}{(z_{10} - 1)(z_{01} + 1)}. \tag{29}$$

Once again, the process has produced affine linear equations. It turns out that (28) can be mapped to (29) by the point transformation (26).

3. Algebraic entropy

To test the integrability of the previous lattice maps, we evaluate their algebraic entropy [23–25]. The system has an infinite dimensional space of initial conditions. We choose initial conditions on a diagonal regular staircase, which is shown in Figure 1.

$$\Delta = \big\{ u_{nm} : \ n + m \in \{0, 1\} \big\}. \tag{30}$$

This defines a forward evolution towards the upper right corner of the lattice, and a backward evolution towards the lower left corner.
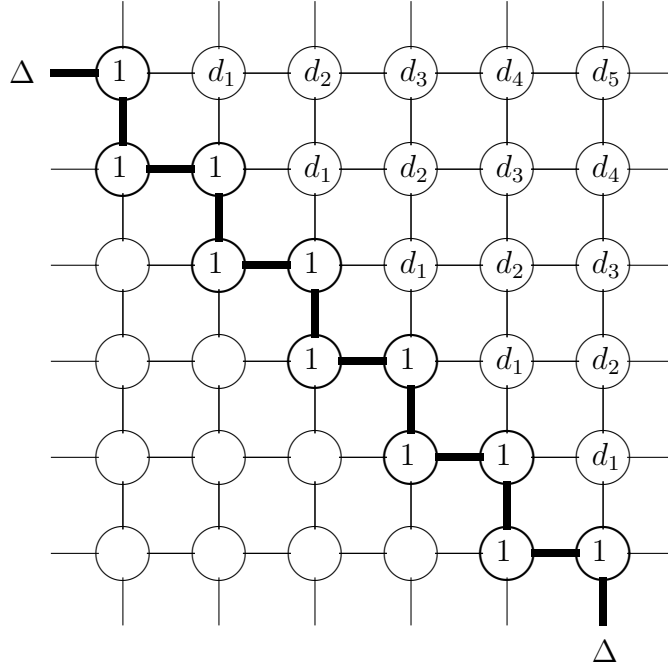

Figure 1: The distribution of degrees over the lattice.

The method is to let the system evolve, calculating u_{nm} away from the diagonal by using (recursively) the defining relation on an elementary tile of the lattice. Each u_{nm} is a rational polynomial in terms of the initial conditions; the degree of the denominator is evaluated. The space of initial conditions is infinite dimensional but, for any quad-graph equation, we need to specify only $2k + 1$ initial conditions to evaluate k iterates. This gives a sequence of degrees $\{d_n\}$, as shown in Figure 1. The growth of that sequence gives the entropy

$$\epsilon = \lim_{n \to \infty} \frac{1}{n} \ln(d_n). \tag{31}$$

Vanishing of the entropy is the hallmark of integrability [14–16].

Although we are able to calculate only a limited number of terms of the sequence, it is possible to infer the exact value of the entropy. The reason is the existence of a finite recurrence with integer coefficients that is satisfied by the sequence of degrees. The most efficient way to find this recurrence is to fit the sequence with a Padé approximant. The existence of the recurrence on the degrees ensures that the generating function for the sequence of degrees is a rational fraction.

The following table gives the sequences of degrees and the corresponding entropy for the various quad-graph equations in §2.

eq.#	Sequence	$\{d_n\}$	ϵ
(17)	$1, 3, 7, 13, 21, 31, 43, 57, \ldots$	$1 + n + n^2$	0
(21)	$1, 2, 4, 7, 11, 16, 22, 29, \ldots$	$1 + (n^2 + n)/2$	0
(22)	$1, 3, 6, 10, 14, 18, 22, 26, \ldots$	$4n - 2, \ (n \geq 2)$	0
(23)	$1, 3, 6, 11, 18, 27, 38, 51, \ldots$	$n^2 + 2, (n \geq 1)$	0
(24)	$1, 3, 7, 13, 21, 31, 43, 57, \ldots$	$1 + n + n^2$	0
(25) $c \neq 1$	$1, 3, 7, 17, 41, 99, 239, 577, \ldots$	$((1 + \sqrt{2})^{n+1} + (1 - \sqrt{2})^{n+1})/2$	$\ln(1 + \sqrt{2})$
(25) $c = 1$	$1, 3, 7, 13, 21, 31, 43, 57, \ldots$	$1 + n + n^2$	0
(29)	$1, 3, 7, 13, 21, 31, 43, 57, \ldots$	$1 + n + n^2$	0

Equation (22) gives linear growth of the degree, which indicates that this equation is linearizable. This result is confirmed by the existence of an infinite family of conservation laws on a single tile (see §4). For generic values of the parameter c, equation (25) is not integrable; neither does it have more than two conservation laws on the tile.

All other cases exhibit quadratic growth of the degree, and therefore are claimed to be integrable, but not linearizable. This raises the question of whether any of the new integrable quad-graph equations can be mapped to any known equation. This will be discussed in §5.

4. The conservation laws

Although the lattice KdV equation (17) and the lattice MKdV equation (24) are not new, their conservation laws (on a single tile) have not previously been listed. Throughout this section, the universal

conservation laws (10) and (11) span the first two conservation laws in each list (up to the addition of trivial conservation laws). So the 'extra' conservation laws are (F_i, G_i), where $i \geq 3$; here F_i denotes $F_i(k, l, z_{00}, z_{01})$ and G_i denotes $G_i(k, l, z_{00}, z_{10})$. Lattice KdV has the following conservation laws.

$$F_1 = u_{00} + c/u_{01}, \qquad\qquad\qquad G_1 = u_{10} - c/u_{10};$$

$$F_2 = (-1)^{k+l}\big(u_{00} + c/u_{01}\big), \qquad\qquad G_2 = (-1)^{k+l+1}\big(u_{10} + c/u_{10}\big);$$

$$F_3 = \ln\big(u_{00} + c/u_{01}\big), \qquad\qquad\qquad G_3 = \ln(u_{10});$$

$$F_4 = \ln(u_{00}), \qquad\qquad\qquad\qquad G_4 = \ln\big(u_{10} - c/u_{00}\big);$$

$$F_5 = -k\ln(u_{00}) + l\ln\big(u_{00} + c/u_{01}\big), \qquad G_5 = -k\ln\big(u_{10} - c/u_{00}\big) + (l-1)\ln(u_{10}).$$

For lattice MKdV, the single-tile conservation laws are:

$$F_1 = \ln\left(\frac{z_{00}(z_{01} - 1)}{z_{01} + c}\right), \qquad\qquad G_1 = \ln\left(\frac{z_{10}(z_{10} + c)}{z_{10} - 1}\right);$$

$$F_2 = (-1)^{k+l}\ln\left(\frac{z_{01} + c}{z_{00}(z_{01} - 1)}\right), \qquad G_2 = (-1)^{k+l}\ln\left(\frac{z_{10}(z_{10} - 1)}{z_{10} + c}\right);$$

$$F_3 = \ln\left(\frac{z_{00}z_{01} - z_{00} - z_{01} - c}{z_{01} + c}\right), \qquad G_3 = \ln(z_{10} + c);$$

$$F_4 = \ln\left(\frac{z_{00} + c}{z_{00}}\right), \qquad\qquad G_4 = \ln\left(\frac{z_{00}z_{10} + cz_{00} + cz_{10} - c}{z_{10}(z_{00} + c)}\right);$$

$$F_5 = k\ln\left(\frac{z_{00}(z_{01} + c)}{(z_{01} - 1)(z_{00} + c)^2}\right) + l\ln\left(\frac{(z_{00}z_{01} - z_{00} - z_{01} - c)^2}{z_{00}(z_{01} + c)(z_{01} - 1)}\right),$$

$$G_5 = k\ln\left(\frac{z_{10}(z_{10} - 1)(z_{00} + c)^2}{(z_{10} + c)(z_{00}z_{10} + cz_{00} + cz_{10} - c)^2}\right) + l\ln\left(\frac{(z_{10} + c)(z_{10} - 1)}{z_{10}}\right) + \ln\left(\frac{z_{10}}{(z_{10} + c)^2}\right).$$

We now list the conservation laws corresponding to the affine linear quad-graph equations that were derived in Cases II and IV. For brevity, we restrict attention to equations (21), (22), (23) and (29); all of our other new equations are related to one of these by the point transformation (26).

Equation (21)

$$F_1 = \ln\big(z_{00}(z_{01} - 1)\big), \qquad\qquad G_1 = \ln\left(\frac{z_{10}}{z_{10} - 1}\right);$$

$$F_2 = (-1)^{k+l}\ln\big(z_{00}(z_{01} - 1)\big), \qquad G_2 = (-1)^{k+l+1}\ln\big(z_{10}(z_{10} - 1)\big);$$

$$F_3 = z_{00}(1 - z_{01}), \qquad\qquad\qquad G_3 = z_{10};$$

$$F_4 = \ln(z_{00}), \qquad\qquad\qquad\qquad G_4 = \ln\left(\frac{z_{10}}{z_{00} + z_{10} - 1}\right);$$

$$F_5 = k\ln\left(\frac{z_{00}}{z_{01} - 1}\right) + l\ln\big(z_{00}(z_{01} - 1)\big), \qquad G_5 = k\ln\left(\frac{z_{10}(z_{10} - 1)}{(z_{00} + z_{10} - 1)^2}\right) + l\ln\left(\frac{z_{10} - 1}{z_{10}}\right) + \ln(z_{10}).$$

Levi and Yamilov [13] recently derived an alternative form of (21) and listed two of its conservation laws, which are equivalent to (F_1, G_1) and (F_4, G_4).

Equation (22)

This equation has an infinite set of conservation laws, which depend upon two arbitrary functions α, β:

$$F_\alpha = \alpha(l+1)\ln\left(\frac{z_{00}z_{01} - z_{00} - z_{01}}{z_{00}(z_{01}-1)}\right) + \alpha(l)\ln\left(\frac{z_{00}z_{01} - z_{00} - z_{01}}{z_{01}}\right), \qquad G_\alpha = \alpha(l)\ln(1-z_{10});$$

$$F_\beta = \beta(k)\ln(1-z_{01}), \qquad G_\beta = \beta(k+1)\ln\left(\frac{z_{00}z_{10} - z_{00} - z_{10}}{z_{00}(z_{10}-1)}\right) + \beta(k)\ln\left(\frac{z_{00}z_{10} - z_{00} - z_{10}}{z_{10}}\right).$$

This is an indicator that, unlike the other new quad-graph equations, (22) is linearizable.

Equation (23)

$$F_1 = \ln\left(\frac{z_{00}(z_{01}-1)}{z_{01}}\right), \qquad\qquad G_1 = \ln\left(\frac{z_{10}^2}{z_{10}-1}\right);$$

$$F_2 = (-1)^{k+l}\ln\left(\frac{z_{01}}{z_{00}(z_{01}-1)}\right), \qquad\qquad G_2 = (-1)^{k+l}\ln(z_{10}-1);$$

$$F_3 = \ln\left(\frac{z_{00}z_{01} - z_{00} - z_{01}}{z_{01}}\right), \qquad\qquad G_3 = \ln(z_{10});$$

$$F_4 = \frac{1}{z_{00}}, \qquad\qquad G_4 = \frac{z_{00}-1}{z_{00}z_{10}};$$

$$F_5 = k\ln\left(\frac{z_{01}}{z_{00}(z_{01}-1)}\right) + l\ln\left(\frac{(z_{00}z_{01} - z_{00} - z_{01})^2}{z_{00}z_{01}(z_{01}-1)}\right), \quad G_5 = k\ln\left(\frac{z_{10}-1}{z_{10}^2}\right) + l\ln(z_{10}-1) - \ln(z_{10}).$$

Note: Most of these conservation laws can be obtained by substituting $c = 0$ into the conservation laws for the lattice MKdV equation. However, this does not apply to (F_4, G_4); one must first divide the corresponding MKdV conservation law by c before taking the limit as $c \to 0$. More generally, at any singular limit of a class of difference equations, some conservation laws may become trivial. Furthermore, the limiting equation may have additional conservation laws. Thus it is safest to calculate all conservation laws of the limiting equation from scratch.

Equation (29)

$$F_1 = \cos\left(\tfrac{(k+l)\pi}{2}\right)\ln\left(\frac{iz_{00}(z_{01}+1)}{z_{01}-1}\right), \qquad G_1 = \cos\left(\tfrac{(k+l)\pi}{2}\right)\ln\left(\frac{z_{10}-1}{z_{10}+1}\right) - \sin\left(\tfrac{(k+l)\pi}{2}\right)\ln(iz_{10});$$

$$F_2 = \sin\left(\tfrac{(k+l)\pi}{2}\right)\ln\left(\frac{iz_{00}(z_{01}+1)}{z_{01}-1}\right), \qquad G_2 = \sin\left(\tfrac{(k+l)\pi}{2}\right)\ln\left(\frac{z_{10}-1}{z_{10}+1}\right) + \cos\left(\tfrac{(k+l)\pi}{2}\right)\ln(iz_{10});$$

$$F_3 = \ln\left(\frac{(z_{00}+1)^2(z_{01}-1)}{z_{00}(z_{01}+1)}\right), \qquad G_3 = \ln\left(\frac{(z_{00}z_{10} - z_{00} + z_{10} + 1)^2(z_{10}+1)}{z_{10}(z_{00}+1)^2(z_{10}-1)}\right) + i\pi l/2;$$

$$F_4 = \ln\left(\frac{(z_{00}z_{01} + z_{00} - z_{01} + 1)^2}{z_{00}(z_{01}+1)(z_{01}-1)}\right), \qquad G_4 = \ln\left(\frac{(z_{10}+1)(z_{10}-1)}{z_{10}}\right) + i\pi l/2;$$

$$F_5 = k\ln\left(\frac{z_{00}(z_{01}+1)}{(z_{00}+1)^2(z_{01}-1)}\right) + l\ln\left(\frac{(z_{00}z_{01} + z_{00} - z_{01} + 1)^2}{z_{00}(z_{01}+1)(z_{01}-1)}\right) + i\pi k(k-1)/4 + i\pi kl/2,$$

$$G_5 = k\ln\left(\frac{z_{10}(z_{00}+1)^2(z_{10}-1)}{(z_{10}+1)(z_{00}z_{10} - z_{00} + z_{10} + 1)^2}\right) + l\ln\left(\frac{(z_{10}+1)(z_{10}-1)}{z_{10}}\right) + \ln\left(\frac{z_{10}}{(z_{10}+1)^2}\right).$$

5. Comments

It is remarkable that, although the original Ansatz (2) contained an arbitrary function A, each of the equations that we have found by sifting can be written in an affine form, after a simple change of variable. This has actually made the entropy calculation possible, because it gives rational evolution.

It is natural to ask at this point how the equations we get compare to the known affine linear quad-graph equations. We have already seen several cases where an equation that we have derived turns out to be equivalent under a point transformation to a known equation. Therefore it is important to characterize this equivalence, which can be done using the appoach introduced in [8] Any affine linear quad-graph equation can be written in polynomial form:

$$Q(v_1, v_2, v_3, v_4) = 0,$$

where v_i, $i = 1 \ldots 4$, are the values (of u_{ij} or z_{ij} as appropriate) at the four corners. For any choice of a pair of indices $1 \leq i < j \leq 4$, define h_{ij} by

$$Q(v_1, v_2, v_3, v_4) \longrightarrow h_{ij}(v_k, v_l) = \partial_{v_i} Q \cdot \partial_{v_j} Q - Q \cdot \partial_{v_i} \partial_{v_j} Q, \qquad i \neq j \neq k \neq l \qquad (32)$$

It is then possible to associate to each of the four corners a polynomial

$$r_k(v_k) = (\partial_{v_l} h_{ij})^2 - 2 h_{ij} (\partial_{v_l}^2 h_{ij}). \qquad (33)$$

These polynomials play a central role in the classification of [8], because (after a Möbius tranformation, if necessary), they can take one of six canonical forms, according to their root distribution.

For example, the lattice MKdV equation (24) yields

$$
\begin{aligned}
h_{z_{00}z_{01}} &= (1+c)(z_{10}-1)(z_{10}+c)z_{11}; \\
h_{z_{00}z_{10}} &= -(1+c)(z_{01}-1)(z_{01}+c)z_{11}; \\
h_{z_{00}z_{11}} &= -(z_{01}-1)(z_{01}+c)(z_{10}-1)(z_{10}+c); \\
h_{z_{01}z_{10}} &= -(1+c)^2 z_{00} z_{11}; \\
h_{z_{01}z_{11}} &= -(1+c)(z_{10}-1)(z_{10}+c)z_{00}; \\
h_{z_{10}z_{11}} &= (1+c)(z_{01}-1)(z_{01}+c)z_{00}.
\end{aligned}
$$

All of the functions h_{ij} are products of linear factors; this is the case for every equation in our classification. In other words, all of these equations are 'degenerate' in the sense used in [8]. Moreover

$$
\begin{aligned}
r_{00} &= (1+c)^4 z_{00}^2; & (34) \\
r_{11} &= (1+c)^4 z_{11}^2; & (35) \\
r_{10} &= (1+c)^2 (1-z_{10})^2 (z_{10}+c)^2; & (36) \\
r_{01} &= (1+c)^2 (1-z_{01})^2 (z_{01}+c)^2. & (37)
\end{aligned}
$$

These are in the canonical forms, but are not in any of the cases that were classified in Theorem 2 of [8]. Hence none of the equations that we have studied are equivalent to any equation in the ABS classification.

In summary, it is feasible to look for new integrable difference equations by searching for equations that admit 'extra' conservation laws. The class that we have studied has been particularly fruitful, although only two of the equations (up to equivalence) seem to be unknown. A useful by-product is that one obtains a list of conservation laws, most of which are new (even for the known equations). The calculation of algebraic entropy is a clear indicator of integrability or linearizability.

One of the new equations is a singular limit of the lattice MKdV equation. In fact, each of the limits $c = 0$ and $c \to \infty$ in (24) trivialise this equation in the form (2.49) of [22] (where they amount to taking $p = 0$ and $q = 0$ respectively). This may explain why they were not discovered sooner. Our other new equation (29) has maximal asymmetry within the form of Ansatz (2), because $\epsilon_1 = -\epsilon_2$. Although the affine form of the equation has real coefficients, its conservation laws are complex.

Two particularly important questions remain: do the new equations have a Lax pair description, and are they 3D-consistent? If we wanted to check directly the consistency around the cube, we should first choose an Ansatz for the form of the relations we want to use on the six faces of a cube. This leads to ask what are the possible deformations of our models. These could be Möbius transformations or deformations that do not lie within the assumed Ansatz (2). The analysis of the singularity pattern might be a way to tackle this problem. However, one should be prepared to accept deformed equations that are not affine; this is beyond the scope of the current paper.

Acknowledgements

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