Coisotropic deformations of algebraic varieties and integrable systems

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Abstract

Coisotropic deformations of algebraic varieties are defined as those for which an ideal of the deformed variety is a Poisson ideal. It is shown that coisotropic deformations of sets of intersection points of plane quadrics, cubics and space algebraic curves are governed, in particular, by the dKP, WDVV, dVN, d2DTL equations and other integrable hydrodynamical type systems. Particular attention is paid to the study of two- and three-dimensional deformations of elliptic curves. Problem of an appropriate choice of Poisson structure is discussed.

1 Introduction

Algebraic varieties (curves etc.) and their deformations are important ingredients in various branches of mathematics and mathematical physics. Theory of integrable nonlinear differential equations was, probably, the most active area in recent years where these objects and their properties have been intensively studied. Two best known examples of such study are given by the theory of the finite-gap solutions and the theory of the Whitham equations [1]-[8]. The problem of characterization and classification of integrable deformations of algebraic curves has attracted a particular interest. In the papers [7, 8] Krichever formulated a general theory of hierarchies of integrable equations of hydrodynamical type on a Riemann surfaces of arbitrary genus arising in the Whitham averaging method.

An alternative approach for determining and classifying the so-called quasiclassical deformations of algebraic curves has been proposed in [9, 10, 11]. This approach revealed a deep connection between the structure of possible deformations of algebraic curves and their basic algebraic properties like the Galois group [10, 11]. Quite different method of study of the Whitham equations has been discussed recently by Magri [13]. Deformations studied in the papers [9, 10, 11, 13] are all two-dimensional, i.e. parametrized by two variables.

In the present paper we will introduce and study a novel class of deformations of algebraic curves, surfaces and algebraic varieties, the class of coisotropic deformations. A concept of coisotropic deformations of associative algebras has been formulated recently in the papers [14, 15, 16]. Notions of coisotropic submanifold and Poisson ideal are the basic one for this approach. Here we will show that essentially the same idea provides us with a simple and transparent way to define and describe coisotropic deformations of algebraic varieties in affine spaces. Namely, coisotropic deformations of algebraic variety are those for which an ideal of deformed variety is a Poisson ideal, i.e. it is closed with respect to Poisson bracket. We will consider simple examples of algebraic varieties such as sets of intersection points of algebraic curves, algebraic curves and hypersurfaces. It is shown that the coisotropic deformations of these objects are governed by systems of differential equations of hydrodynamical type which in particular cases coincide with well-known integrable systems like the dispersionless KP equation, WDVV equation and dispersionless 2DTL equation.

We will concentrate on the study of the coisotropic deformations on plane and three-dimensional quadrics and cubics. Particular attention will be paid to the study of the three-dimensional coisotropic deformations of elliptic curves. The problem of choice of the Poisson structure is discussed too. It is shown that such a choice is crucial for construction of nontrivial coisotropic deformations.

The paper is organized as follows. General formulation of coisotropic deformations of algebraic varieties is discussed in section 2. Coisotropic deformations of the sets of intersection points of algebraic curves on the plane are considered in section 3. It is shown that two particular classes of deformations are governed by the stationary dKP equation and WDVV equation. Section 4 is devoted to deformations of plane cubics. Deformations of the space curves are discussed in the next section 5. The dKP equation and the dispersionless Veselov-Novikov equation govern coisotropic deformations of the special space curves. Two-dimensional coisotropic deformations of the elliptic curve are studied in section 6. The problem of choice of the Poisson structure is discussed here too. In section 7 we consider three-dimensional deformations of elliptic curves. Deformations of curves and hyperplanes in R^4 described by the Boyer-Finley-d2DTL equation and heavenly equation are presented in section 8.

2 Coisotropic deformations of algebraic varieties

The notion of coisotropic submanifold is a basic ingredient in the formulation and description of coisotropic deformations of associative algebras studied in [14, 15]. Coisotropic submanifold Γ is a submanifold in R^{2n} endowed with the Poisson bracket $\{\ ,\ \}$ such that $\Gamma_{\mathsf{T}} \subset \Gamma$ where Γ_{T} denotes a skew-orthogonal complement of Γ in R^{2n} (see e.g. [17, 18]). Coisotropic submanifold can be defined as the zero locus Γ for the set of functions $f_j(y)$, i.e.

$$f_j(y) = 0, \quad j = 1, ..., m \tag{1}$$

such that

$$\{f_j(y), f_k(y)\}\,|_\Gamma = 0, \quad j, k = 1, ..., m \tag{2}$$

where $y_1, ..., y_n$ are local coordinates in R^{2n}. The definition (1), (2) is equivalent to the condition that all the Hamiltonian fields generated by $f_j(y)$ ate tangent to Γ or to the condition that the ideal $J = \langle f_j \rangle$ generated by the functions $f_j(y)$ is closed $\{J, J\} \subset J$, i.e. it is a Poisson ideal [17, 18]. For associative algebras the functions (1) are of the form [14]

$$f_{jk} = -p_j p_k + \sum_{l=1}^{n} C_{jk}^l(x) p_l, \quad j, k = 1, .., n \tag{3}$$

where $p_j, x_j (j = 1, .., n)$ are canonical Darboux coordinates in R^{2n}. The conditions (2) define the coisotropic deformations of the structure constants $C_{jk}^l(x)$ of a associative algebra in a given basis. It was shown in [15] that this approach is applicable to the other algebraic structures like the Jordan triples. In geometrical terms equations (1), (3) with fixed x represent a set of special quadrics in R^n.

Here we will use the notions of Poisson ideal and coisotropic submanifold in a different setting. It will serve us to define and describe a class of deformations of algebraic varieties. Thus, let us consider an algebraic variety M in R^n defined by the equations

$$f_j(p_1, ..., p_n) = 0, \quad j = 1, .., m \tag{4}$$

where $p_1, ..., p_n$ are local affine coordinates in R^n.

To define deformation of this variety

1) we assume that the coefficients of the polynomials $f_j(p_1, ..., p_n)$ depend on the deformation parameters $x_1, ..., x_n$,

2) we embed the variety M into the space R^{2n} equipped with the Poisson bracket $\{,\}$ and local coordinates $p_1, ..., p_n, x_1, ..., x_n$,

3) then we consider an ideal $J = \langle f_j(p; x) \rangle$ generated by the functions $f_j(p; x)$ and require that this ideal is closed

$$\{J, J\} \subset J \tag{5}$$

or equivalently

$$\{f_j(p;x), f_k(p;x)\} \mid_\Gamma = 0, \quad j,k = 1,..,m \tag{6}$$

where Γ is the locus of common zeros for the functions $f_j(p,x)$, i.e.

$$\Gamma = \{(p,x) \mid f_j(p;x) = 0,\ j = 1,..,m\}\,. \tag{7}$$

In other words we require that the ideal J of the deformed variety M is a Poisson ideal.

Definition 1 *Deformations of the algebraic variety M defined by equations (4) are called coisotropic if the ideal $J = \langle f_j(p;x)\rangle$ of the deformed varieties is a Poisson ideal.*

So, coisotropic deformations of an algebraic variety are those for which coefficients of the functions $f_j(p,x)$ are such that the conditions (6) are satisfied, i.e. the submanifold Γ defined by (7) is a coisotropic submanifold.

The coisotropy conditions (6) impose constraints on the coefficients of the polynomials $f_j(p;x)$. We will refer to the corresponding system of equations for these coefficients as the central system (CS). The concrete form of CS depends on the choice of $f_j(p;x)$ as well as the form of the Poisson bracket $\{,\}$. The choice of the Poisson bracket is a crucial one. For inadequate choice of $\{,\}$ one may have no nontrivial deformation. The consistency of the Poisson structure $\{,\}$ with the polynomials $f_j(p;x)$ is an important point of the approach under consideration.

At $m = n$ the variety M is a set of intersection points of n algebraic curves $f_j(p) = 0$ in R^n and coisotropic deformations of each of these points span Lagrangian submanifolds in R^{2n}. At $m = n-1$ the variety M is a curve in R^n and the conditions (6) define coisotropic deformations of this curve and so on. Clearly, the codimension of the algebraic variety should be greater or equal to two, i.e. $m \geq 2$ in order to be able to define its coisotropic deformation.

3 Coisotropic deformations on the plane. Deformations of the sets of intersection points and curves

On the plane the variety M is defined by two equations

$$f_1(p,q) = 0 \tag{8}$$

and

$$f_2(p,q) = 0 \tag{9}$$

where the affine coordinates on the plane are denoted by p and q. The coisotropic deformations of the variety M, i.e. the set of intersection points of the curves (8) and (9) is defined by the condition

$$\{f_1(p,q;x,y), f_2(p,q;x,y)\} \mid_\Gamma = 0 \tag{10}$$

where x and y stand for the deformation parameters. The coisotropic submanifold Γ is this case a two-dimensional Lagrangian submanifold.

This construction admits an alternative interpretation. Indeed, let we have the algebraic curve given by equation (8). One can define deformation of this curve in the following way. First, we assume that the coefficients of the polynomial $f_1(p,q)$ depend on deformation parameters x and y. Then we take a function $f_2(p,q;x,y)$ which is polynomial in p and q. Finally, we require that the functions $f_1$and f_2 obey the coisotropy condition (10).

Definition 2 *If the coefficients of the polynomial $f_1(p,q)$ are such that the condition (10) is satisfied then it is said that they define the coisotropic deformation of the curve (8) generated by the function $f_2(p,q;x,y)$.*

Clearly, this definition is a reciprocal one: if the polynomial f_2 generates coisotropic deformation of the curve (8), then at the same time the polynomial f_1 generates coisotropic deformation of the curve (9).

Let us begin with the simplest case of the second order curves (8) and (9). As it is well known any nondegenerate quadric is equivalent to

1. parabola
$$q + p^2 + ap + b = 0 \tag{11}$$

2. or to ellipse
$$ap^2 + bq^2 + cp + dq + f = 0 \tag{12}$$

3. or to hyperbola
$$pq + ap + bq + c = 0. \tag{13}$$

To construct coisotropic deformation we choose the canonical Poisson bracket in R^4, i.e. $\{F, G\} = \frac{\partial F}{\partial p}\frac{\partial G}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial G}{\partial y} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial q}$.

We consider first a parabola and choose f_2 as an arbitrary polynomial $f_2(p, q; x, y)$. The submanifold

$$f_1 = q + p^2 + a(x, y)p + b(x, y) = 0, \tag{14}$$
$$f_2(p, q; x, y) = 0 \tag{15}$$

can be equivalently represented as the zero locus of the functions

$$\begin{aligned} f_1 = q + p^2 + a(x, y)p + b(x, y) = 0, \\ \widetilde{f_2} = \sum_{k=0}^{N} \alpha_k(x, y)p^k = 0 \end{aligned} \tag{16}$$

with certain N and $\alpha_k(x, y)$. In the simplest case N=1, a=0 and $\alpha_1 = 1$ the coisotropy condition (10) gives $\alpha_{0x} = 0, b_y = \alpha_{0y}$, i.e. the deformation is the trivial shift of the parabola: $b = \beta_0(x) + \beta_1(y)$ where $\beta_0(x)$ and $\beta_1(y)$ are arbitrary functions.

In the case N=2 one has $\alpha_2 = 1$ and the CS takes the form

$$\begin{aligned} \alpha_{1y} - 2\alpha_1\alpha_{1x} + (a\alpha_1)_x + 2(\alpha_0 - b)_x = 0, \\ \alpha_{0y} + a\alpha_{0x} - \alpha_1 b_x + 2\alpha_0(a - \alpha_1)_x = 0. \end{aligned} \tag{17}$$

At $\alpha_0 = a = b = 0$ it is the Burgers-Hopf equation $\alpha_{1y} - 2\alpha_1\alpha_{1x} = 0$. Equations (17) describe the coisotropic deformations of the two points of intersection $(p_+, q_+), (p_-, q_-)$ (assuming $\alpha_1^2 \geq 4\alpha_0$)

$$\begin{aligned} p_\pm &= -\frac{\alpha_1(x, y)}{2} \pm \sqrt{\frac{1}{4}\alpha_1^2(x, y) - \alpha_0(x, y)}, \\ q_\pm &= (\alpha_1 - a)p_\pm + \alpha_0 - b \end{aligned} \tag{18}$$

of the curves (16) or equivalently of the parabola

$$q + p^2 + ap + b = 0 \tag{19}$$

and a straight line

$$q + (a - \alpha_1)p + b - \alpha_0 = 0. \tag{20}$$

Now let us consider the cubic polynomial f_2, i.e. a set of intersection points for the curves

$$q + p^2 + ap + b = 0 \tag{21}$$

and

$$p^3 + \alpha_2 p^2 + \alpha_1 p + \alpha_0 = 0. \tag{22}$$

The CS in this case is of the form

$$\begin{aligned} \alpha_{2y} + 2\alpha_{1x} - 3b_x + (a\alpha_2)_x - (\alpha_2^2)_x = 0, \\ \alpha_{1y} + 2\alpha_{0x} + a\alpha_{1x} + 2\alpha_1 a_x - 2\alpha_1\alpha_{2x} - 2\alpha_2 b_x = 0, \\ \alpha_{0y} + a\alpha_{0x} - \alpha_1 b_x + \alpha_0(3a - 2\alpha_2)_x = 0. \end{aligned} \tag{23}$$

This system contains several reductions of interest. There are two distinguished between them. The first is given by $a = \alpha_2 = 0$. The first equation (23) then implies that $2\alpha_1 = 3b$ and the rest two equations take the form

$$\begin{aligned} 3b_y + 4\alpha_{0x} = 0, \\ 4\alpha_{0y} - 3(b^2)_x = 0. \end{aligned} \tag{24}$$

It is the well-known stationary dispersionless Kadomtsev-Petviashvili (KP) or Khokhlov-Zabolotskaya equation (see e.g. [14, 15]). Equations (24) imply the existence of the function F such that $b = 2F_{xx}, \alpha_0 = -\frac{3}{2}F_{xy}$ and the system (24) becomes

$$F_{yy} + 2(F_{xx})^2 = 0. \tag{25}$$

It is the Hirota equation for the stationary dKP equation.

The second reduction is given by the constraints $\alpha_1 = 2b, \alpha_2 = a$. The CS (23) then is converted to

$$\begin{aligned} a_y + b_x = 0, \\ b_y + \alpha_{0x} = 0, \\ \alpha_{0y} + (a\alpha_0 - b^2)_x = 0. \end{aligned} \tag{26}$$

This system of conservation laws implies the existence of the function F such that

$$a = F_{xxx}, \quad b = -F_{xxy}, \quad \alpha_0 = F_{xyy} \tag{27}$$

in terms of which it is reduced to the single equation

$$F_{yyy} + F_{xxx}F_{xyy} - (F_{xxy})^2 = \beta(y) \tag{28}$$

where $\beta(y)$ is an arbitrary function. At $\beta = 0$ it is the celebrated WDVV equation [19, 20, 21]. Note that the system (26) has appeared for the first time in the paper [22].

Thus, the stationary dKP equation and WDVV equation describe coisotropic deformations of the set of intersection points of the curves (21) and (22). The locus of common zeros for the polynomials (21),(22) coincides with zero locus of the parabola

$$q + p^2 + ap + b = 0 \tag{29}$$

and the hyperbola

$$\widetilde{f_2} = pq + (\alpha_2 - a)q + (b - \alpha_1 + a(\alpha_2 - a))p + b(\alpha_2 - a) - \alpha_0 = 0. \tag{30}$$

For the stationary dKP case the equation of hyperbola takes the form

$$pq - \frac{1}{2}bp - \alpha_0 = 0 \tag{31}$$

while at the WDVV case one has

$$pq - bp - \alpha_0 = 0. \tag{32}$$

It would be of interest to clarify the geometrical difference between the dKP and WDVV cases.

The above construction shows also that the stationary dKP and WDVV equations describe at the same time special classes of coisotropic deformations for the hyperbola (13) generated by parabola (21).

As the last illustrative example in this section we consider coisotropic deformations of a circle

$$f_1 = p^2 + q^2 + u = 0. \tag{33}$$

With the choice

$$f_2 = p^3 - 3pq^2 + ap + bq = 0 \tag{34}$$

the CS takes the form

$$\begin{aligned} (ua)_x + (ub)_y &= 0, \\ 3u_x - a_x + b_y &= 0, \\ 3u_y + a_y + b_x &= 0. \end{aligned}$$

It is the stationary dispersionless Veselov-Novikov (dVN) equation (see [14]). It describes also the coisotropic deformations of the set of intersection points of the circle (33) and the cubic (34).

Considering higher order polynomials f_2, one gets coisotropic deformations described by the stationary higher dKP and dVN equations.

4 Deformations of plane cubics

General form of the plane cubic is ($f_1 \doteq \zeta$)

$$\zeta = p^2 - q^3 - u_4pq - u_3q^2 - u_2p - u_1q - u_0 = 0. \tag{35}$$

Choosing the linear second equation, i.e. the straight line

$$f_2 = \alpha p + \beta q + \gamma = 0 \tag{36}$$

and canonical Poisson bracket, one gets the following CS ($\alpha = 1$)

$$\begin{aligned} &\beta^2 u_{4x} - \beta u_{4y} - 2\beta\beta_y + 3\beta^2\beta_x + 3\gamma_x - u_4\beta_y + u_3\beta_x + \beta u_{3x} - u_{3y} + 2\beta u_4\beta_x = 0, \\ &\beta\gamma u_{4x} - \beta u_4\gamma_x + 3\beta u_2\beta_x - u_{1y} + 2\gamma u_4\beta_x - u_4\gamma_y - u_2\beta_y - \gamma u_{4y} - 2u_3\gamma_x - \beta u_{4y} + 2u_1\beta_x + \\ &+ \beta u_{1x} + 6\beta\gamma\beta_x - 2\gamma\beta_y + \beta^2 u_{2x} - 2\beta\gamma_y = 0, \\ &\beta u_{0x} - u_1\gamma_x - u_2\gamma_y - u_{0y} + 3u_0\beta_x - \gamma u_{2y} - 2\gamma\gamma_y + 3\gamma^2\gamma_y + \beta\gamma u_{2x} - \gamma u_4\gamma_x + 3\gamma u_2\beta_x = 0. \end{aligned} \tag{37}$$

The variety M for the choice (36) consists of at most three points of intersection of the cubic (35) with the straight line (36). Coisotropic deformations of these three points are described by the CS (37) and generate three surfaces in R^4.

In the particular case $u_4 = u_3 = u_2 = 0, \alpha = 1, \gamma = 0$ the CS system takes the form

$$\beta_y - \frac{3}{2}\beta\beta_x = 0, \tag{38}$$

$$u_{1y} - \beta u_{1x} - 2u_1\beta_x = 0, \tag{39}$$

$$u_{0y} - \beta u_{0x} - 3u_0\beta_x = 0. \tag{40}$$

These equations describe deformations of the moduli $u_1$and u_0 of the elliptic curve. The behavior of the moduli is defined completely by the solution of the Burgers-Hopf equation for β. For the discriminant $\Delta = 16(4u_1^3 + 27u_0^2)$ of the elliptic curve (see e.g. [23, 24]) one has the equation

$$\Delta_y - \beta\Delta_x - 6\beta_x\Delta = 0. \tag{41}$$

while deformation of the invariant $j = 12^3\frac{4u_1^3}{4u_1^3+27u_0^2} = 3^3 4^6\frac{u_1^3}{\Delta}$ is defined by the equation

$$j_y - \beta j_x = 0.$$

We see that at the points of the gradient catastrophe for Burgers-Hopf equation where $\beta_x, \beta_y \to \infty$ the moduli u_1, u_0 and the discriminant Δ exhibit gradient catastrophe behavior too.

Stationary solutions of the CS (37) with constant u_4, u_3, u_2, u_1, u_0 are of interest too. It would correspond to Abel's approach to the law of addition on the cubic (see e.g. [24], section 2.14). In the case $u_4 = u_3 = u_2 = 0$ the CS (37) is reduced to the system

$$\begin{aligned} 2\beta\beta_y + 3\beta^2\beta_x + 3\gamma_x &= 0, \\ 2u_1\beta_x + 6\gamma\beta\beta_x + 2(\gamma\beta)_y &= 0, \\ -u_1\gamma_x + 3u_0\beta_x + \gamma\gamma_y + 3\gamma^2\beta_x &= 0. \end{aligned} \tag{42}$$

This overdetermined system, obviously, may have nontrivial solutions only for very special β, γ and constant u_0, u_1.

For general polynomial f_2 the variety M has also the basis

$$\begin{aligned} \zeta &= 0, \\ \widetilde{f_2} &= \alpha(q) + \beta(q)p \end{aligned} \tag{43}$$

where $\alpha(q)$ and $\beta(q)$ are arbitrary polynomials in q.

5 Deformations of space curves

In three and more dimensional spaces a zoology of algebraic varieties is richer and correspondingly their deformations form a much larger collection of different cases.

In the three-dimensional affine space an algebraic curve is defined by two polynomial equations

$$f_1(p_1, p_2, p_3) = 0, \quad f_2(p_1, p_2, p_3) = 0. \tag{44}$$

Coisotropic deformations of this curve are defined by the condition

$$\{f_1(p;x), f_2(p;x)\} \mid_\Gamma = 0. \tag{45}$$

A coisotropic submanifold Γ typically is the four-dimensional submanifold in R^6.

A simple example is provided by the twisted cubic defined by the equations

$$f_1 = p_2 + p_1^2 + u = 0 \tag{46}$$

and

$$f_2 = p_3 + p_1^3 + vp_1 + w = 0 \tag{47}$$

which is one of the first standard examples in all textbooks on algebraic geometry. The CS with the choice of canonical Poisson bracket in R^6 in this case is given by the equations

$$\begin{aligned} u_{x_3} + vu_{x_1}u - w_{x_2} &= 0, \\ v_{x_2} + 2w_{x_1} &= 0, \\ 3u_{x_2} - 2v_{x_1} &= 0. \end{aligned} \tag{48}$$

So, $v = \frac{3}{2}u$ and one has the system

$$\begin{aligned} u_{x_3} + \frac{3}{2}u_{x_1}u - v_{x_2} &= 0, \\ 3u_{x_2} + 4v_{x_1} &= 0 \end{aligned} \tag{49}$$

which is the dKP equation. Thus, the three-dimensional coisotropic deformations of the twisted cubic in R^3 are doverned by the dKP equation. For polynomial solutions of the dKP equation the family of the deformed varieties in R^6 defined by the equations

$$f_1 = p_2 + p_1^2 + u(x_1, x_2, x_3) = 0, \tag{50}$$

$$f_2 = p_3 + p_1^3 + \frac{3}{2}u(x_1, x_2, x_3)p_1 + w(x_1, x_2, x_3) = 0 \tag{51}$$

(i.e. the submanifold Γ) is the algebraic variety too.

We note that on the plane p_2, p_3 the twisted cubic (46),(47) is the cubic curve given by the equation

$$p_3{}^2 + p_2{}^3 + (3\,u - 2\,v)\,p_2{}^2 + 2\,wp_3 + \left(v^2 + 3\,u^2 - 4\,vu\right)p_2 + (u^3 + w^2 - 2\,vu^2 + v^2) = 0.$$

It is obvious, however, that this cubic curve is degenerate for all the values of u and v since it has polynomial parametrization given by (46), (47). Deformations of nondegenerate elliptic curves will be studied in next section.

Choosing

$$f_2 = p_n + p_1^n + \sum_{k=0}^{n-2} v_k p_1^k, \tag{52}$$

one constructs the coisotropic deformations of the n-th order curve in R^3. These deformations are described by the higher dKP equations.

Another simple example corresponds to

$$f_1 = p_1^2 + p_2^2 + u = 0, \tag{53}$$

$$f_2 = p_3 + p_1^3 - 3p_1p_2^2 + ap_1 + bp_2 = 0. \tag{54}$$

This curve M is the intersection of the cylinder defined by the first equation and cubic surface given by the second equation.

The coisotropy condition gives rise to the following CS

$$\begin{aligned} u_{x_3} + (ua)_{x_x} + (ub)_{x_2} &= 0, \\ 3u_{x_1} - a_{x_1} + b_{x_2} &= 0, \\ 3u_{x_2} + a_{x_2} + b_{x_1} &= 0. \end{aligned}$$

It is the dVN equation (see e.g. [15]). For higher order f_2 coisotropic deformations are described by the higher dVN equations.

Finally, let us consider the case

$$\begin{aligned} f_1 &= p_1p_2 + up_1 + v = 0, \\ f_2 &= p_3 + \alpha p_1^2 + \beta p_2^2 + ap_1 + b = 0. \end{aligned}$$

This curve is the intersection of cylindric hyperbola and paraboloid. Its coisotropic deformations is described by the following CS

$$\begin{aligned} u_{x_3} + au_{x_1} - \beta(u^2)_{x_2} + 2\alpha v_{x_1} - b_{x_2} &= 0, \\ v_{x_3} + (av)_{x_1} - 2\beta(uv)_{x_2} &= 0, \\ a_{x_2} - 2\alpha u_{x_1} &= 0, \quad b_{x_1} - 2\beta v_{x_2} = 0. \end{aligned}$$

This hydrodynamical system has distinguished reductions. At $a = 0, \alpha = 0, \beta = \frac{1}{2}$ it is the 2+1-dimensional generalization of the one-layer Benney system proposed in [8, 26]. At $\beta = -\alpha = \frac{1}{2}$ it is the dispersionless Davey-Stewartson system considered in [27].

We note that deformations considered in this section can be treated also as coisotropic deformations of a surface given by equation $f_1(p_1, p_2, p_3) = 0$ generated by a surface defined by equation $f_2(p_1, p_2, p_3) = 0$ (or vice versa).

6 Coisotropic deformations of elliptic curve. Two-dimensional case

Now we will consider coisotropic deformations of nondegenerate cubics. This case is of importance since it provides us with the example of deformations for algebraic curve of nonzero genus.

General cubic is given by equation (35), i.e.

$$\mathcal{E} = {p_3}^2 - {p_2}^3 - u_4 p_2 p_3 - u_3 {p_2}^2 - u_2 p_3 - u_1 p_2 - u_0. \tag{55}$$

where for further convenience we have changed the notation ($p = p_3, q = p_2$). To construct coisotropic deformations of the cubic (55) we choose the canonical Poisson structure in R^6 and functions

$$f_n = p_n - \alpha_n(p_2) - \beta_n(p_2) p_3 = 0 \tag{56}$$

where α and β are polynomials in p_2 with coefficients depending on the deformation variables x_2, x_3, x_n.

It is quite instructive to consider first the particular deformations which correspond to a cyclic variable x_2, i.e. when $u_i = u_i(x_3, x_n)$. In this case p_2 appears in (55), (56) as a parameter $\lambda = p_2$ and the corresponding deformations of the cubic (55) are two-dimensional.

So we consider the cubic

$$\mathcal{E} = p_3^2 - \lambda^3 - u_4 \lambda p_3 - u_3 \lambda^2 - u_2 p_3 - u_1 \lambda - u_0 = 0. \tag{57}$$

We choose the generating function (56) as

$$f_5 = p_5 - v_3 \lambda^2 - v_1 \lambda - v_0 - (v_2 + \lambda) p_3 \tag{58}$$

in order to get a CS allowing deformations for all coefficients u_i of the cubic. The coisotropy condition

$$\{f_5, \mathcal{E}\}|_\Gamma = 0$$

gives the system

$$\begin{aligned}
&\frac{\partial u_4}{\partial x_3} + 2\frac{\partial v_3}{\partial x_3} = 0\\
&\frac{\partial u_3}{\partial x_3} + 2\frac{\partial v_2}{\partial x_3} - u_4\frac{\partial v_3}{\partial x_3} = 0\\
&u_4\frac{\partial v_2}{\partial x_3} - \frac{\partial u_4}{\partial x_5} + \frac{\partial u_4}{\partial x_3}v_2 + \frac{\partial u_2}{\partial x_3} + 2\frac{\partial v_1}{\partial x_3} = 0\\
&-\frac{\partial u_3}{\partial x_5} + \frac{\partial u_1}{\partial x_3} - u_2\frac{\partial v_3}{\partial x_3} + \frac{\partial u_3}{\partial x_3}v_2 - u_4\frac{\partial v_1}{\partial x_3} + 2\frac{\partial v_2}{\partial x_3}u_3 = 0\\
&-\frac{\partial u_2}{\partial x_5} + 2\frac{\partial v_0}{\partial x_3} + \frac{\partial u_2}{\partial x_3}v_2 + u_2\frac{\partial v_2}{\partial x_3} = 0\\
&-\frac{\partial u_1}{\partial x_5} + \frac{\partial u_0}{\partial x_3} + \frac{\partial u_1}{\partial x_3}v_2 - u_4\frac{\partial v_0}{\partial x_3} - u_2\frac{\partial v_1}{\partial x_3} + 2\frac{\partial v_2}{\partial x_3}u_1 = 0\\
&-\frac{\partial u_0}{\partial x_5} + \frac{\partial u_0}{\partial x_3}v_2 - u_2\frac{\partial v_0}{\partial x_3} + 2\frac{\partial v_2}{\partial x_3}u_0 = 0.
\end{aligned} \tag{59}$$

In this and next sections in order to avoid triple indices we write the derivatives in the explicit way. The first two equations imply that

$$v_3 = -\frac{1}{2}u_4 \qquad v_2 = -\frac{1}{2}u_3 + \frac{1}{4}{u_4}^2.$$

and the system (59) becomes the system of five equations for u_0, u_1, u_2, u_3, u_4. The fields v_1 and v_0 can be considered as a couple of gauge fields.

This system admits several reductions. The most interesting one corresponds to the constraint $u_4 = 0, u_2 = 0$. In this case $v_3 = 0,\ 2v_2 = -u_3,\ 2v_1 = -u_2,\ v_0 = 0$ and the above system is converted into the following

$$\begin{aligned}
\frac{\partial u_3}{\partial x_5} &= -\frac{3}{2}\frac{\partial u_3}{\partial x_3}u_3 + \frac{\partial u_1}{\partial x_3},\\
\frac{\partial u_1}{\partial x_5} &= -\frac{1}{2}\frac{\partial u_1}{\partial x_3}u_3 - \frac{\partial u_3}{\partial x_3}u_1 + \frac{\partial u_0}{\partial x_3},\\
\frac{\partial u_0}{\partial x_5} &= -\frac{\partial u_3}{\partial x_3}u_0 - \frac{1}{2}\frac{\partial u_0}{\partial x_3}u_3
\end{aligned} \tag{60}$$

which is the well known 3 component dispersionless KdV equation (see e.g.([25]). Solutions of this system describe 1+1 dimensional coisotropic deformations of the elliptic curve $\mathcal{E}_r = {p_3}^2 - \lambda^3 - u_3\lambda^2 - u_1\lambda - u_0$. generated by the standard symplectic form and the polynomial $f_{5r} = p_5 - \lambda p_3 + \frac{u_3}{2}p_3$.
This evolution coincides with that obtained earlier in a totally different manner in [9]. In order to complete the comparison with the results presented in the paper [9] we will show how coisotropic deformations give rise to corresponding system in terms of Riemann invariants. First we present the elliptic curve in the form $\mathcal{E}_e = {p_3}^2 - (\lambda - e_1)(\lambda - e_2)(\lambda - e_3)$. Then it is a simple check that the coisotropy condition

$$\{\mathcal{E}_e, f_5\}|_\Gamma = 0 \tag{61}$$

is equivalent to the 3 component dKP system in terms of Riemann invariants, i.e

$$\begin{aligned}
\frac{\partial e_1}{\partial x_5} &= \left(\frac{3}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3\right)\frac{\partial e_1}{\partial x_3},\\
\frac{\partial e_2}{\partial x_5} &= \left(\frac{1}{2}e_1 + \frac{3}{2}e_2 + \frac{1}{2}e_3\right)\frac{\partial e_2}{\partial x_3},\\
\frac{\partial e_3}{\partial x_5} &= \left(\frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{3}{2}e_3\right)\frac{\partial e_3}{\partial x_3}.
\end{aligned} \tag{62}$$

From the algebro-geometrical characterization viewpoint three of five parameters $u_i, i = 0, 1, 2, 3, 4$ for an elliptic curve are redundant (see e.g.[23]). Only two special combinations of u_i (moduli g_2 and g_3) are essential and the canonical form of elliptic curve is

$$\mathcal{E}_c = {\pi_3}^2 - ({\pi_2}^3 + g_2\pi_2 + g_3) = 0. \tag{63}$$

Moduli g_2 and g_3 are given by the formulae [23]

$$g_2 = u_1 - \frac{1}{3}{u_3}^2 \qquad g_3 = u_0 + \frac{2}{27}{u_3}^3 - \frac{1}{3}u_1u_3. \tag{64}$$

where for sake of simplicity we choose $u_4 = u_2 = 0$. General cubic (55) is converted into the canonical form by admissible transformation ($u_4 = u_2 = 0$) [23]

$$p_2 = \pi_2 - \frac{1}{3}u_3 \quad p_3 = \pi_3. \tag{65}$$

Direct calculation gives the following equations for moduli

$$\begin{aligned}
\frac{\partial g_2}{\partial x_5} &= \frac{\partial g_3}{\partial x_3} - \frac{5}{6}\frac{\partial g_2}{\partial x_3}u_3 - \frac{2}{3}\frac{\partial u_3}{\partial x_3}g_2,\\
\frac{\partial g_3}{\partial x_5} &= -\frac{5}{6}\frac{\partial g_3}{\partial x_3}u_3 - \frac{1}{3}\frac{\partial g_2}{\partial x_3}g_2 - \frac{\partial g_3}{\partial x_3}u_3,\\
\frac{\partial u_3}{\partial x_5} &= \frac{\partial g_2}{\partial x_3} - \frac{5}{6}\frac{\partial u_3}{\partial x_3}u_3.
\end{aligned} \tag{66}$$

This system contains not only moduli but also the function u_3. One can, in principal, express u_3 in terms of g_2 and g_3 solving the third of above equations (Burgers-Hopf equation with the source) or equivalently solving the Hamilton-Jacobi equation

$$\phi_{x_5} + \frac{5}{12}(\phi_{x_3})^2 - g_2 = 0 \tag{67}$$

where $\phi_{x_3} = u_3$.

Thus, the system (66) governs the coisotropic deformations of moduli of elliptic curve parametrized by two variables x_3 and x_5. Equations (66) allows us also to find deformations of the discriminant δ and invariant J of elliptic curve. The corresponding equations are rather complicated and we do not present them here. For illustration we consider the following simple solution of the system (66) (C is an arbitrary constant)

$$\begin{aligned} u_3 &= C, \\ u_1 &= x_5 - \frac{C^2}{4}, \\ u_0 &= x_3 - \frac{C}{2}x_5 + \frac{C^3}{6} \end{aligned} \tag{68}$$

which provides us with simple, linear deformation of the elliptic curve. For this solution the discriminant is equal to

$$\Delta(x_3, x_5) = 16(4\,{x_5}^3 + \frac{47}{4}\,{x_5}^2C^2 - \frac{21}{2}\,x_5C^4 + \frac{49}{24}\,C^6 + 27\,{x_3}^2 - 45\,x_3x_5C + \frac{35}{2}\,x_3C^3). \tag{69}$$

In the particular case $C = 0$ we have $\Delta(0, 0) = 0$ and so at $x_3 = x_5 = 0$ the curve is singular. Deformation (variation of x_3 and x_5) however generically desingularize it. Conversely, if $C \neq 0$ then $\Delta(0, 0) \neq 0$, but the deformation produces Δ which may be different from zero not everywhere. This deformation changes J and, therefore, changes the elliptic curve. We note also that for this solution the family of deformed cubics is the algebraic variety too.

Thus, it is very natural to look for coisotropic deformations of moduli starting directly from the canonical form (63) of elliptic curve. One immediately finds that the canonical Poisson structure considered before is not appropriate in this case since the coisotropy condition with such a Poisson bracket gives rise to only trivial deformations. Thus, one should search for an adequate Poisson structure.

One way to find it is to study transformation of the canonical Poisson structure under the transformations of Darboux coordinates p_i, x_i to new coordinates π_i, τ_i given by the formula (65) and by $\pi_5 = p_5$, $\tau_i = x_i,\ i = 2, 3, 5$. Direct calculation gives the following transformed Poisson structure

$$\begin{aligned} \{\tau_i, \tau_j\}_{\tau\pi} &= 0, \qquad i, j = 2, 3, 5 \\ \{\tau_i, \pi_j\}_{\tau\pi} &= \delta_{ij}, \qquad i, j = 2, 3, 5 \\ \{\pi_i, \pi_j\}_{\tau\pi} &= \frac{1}{3}\frac{\partial u_3}{\partial x_3}\delta_{i3}\delta_{j2} + \frac{1}{3}\frac{\partial u_3}{\partial x_5}\delta_{i5}\delta_{j2}, \qquad i, j = 2, 3, 5. \end{aligned} \tag{70}$$

It is not difficult to check that the coisotropy condition

$$\{\mathcal{E}_c, J^{(5)}\}_{\tau\pi}|_\Gamma = 0 \tag{71}$$

with the Poisson bracket (70) and $J^{(5)}|_{p_2=\pi_2-\frac{1}{3}u_3} = \pi_5 - \pi_2\pi_3 + \frac{u_3}{6}\pi_3$ gives the system (66).

One can perform similar transformations in other cases. The problem of consistency of an algebraic curve and corresponding Poisson structure which allow to construct nontrivial coisotropic deformations will be discussed elsewhere.

Finally, we note that the results of this section can be extended to hyperelliptic curves. Indeed, of one takes a hyperelliptic curve

$${p_{2n+1}}^2 = {p_2}^{2n+1} + \sum_{i=0}^{2n} v_i(x_{2n+1}, x_{2n+3}){p_2}^i, \tag{72}$$

choose the polynomial functions of the form $f_m = p_{2n+3} - (\sum_{i=0}^{m} u_i(x_{2n+1}, x_{2n+3}){p_2}^i)p_{2n+1}$ then the coisotropy condition with the canonical Poisson bracket reproduces hydrodynamical type systems derived by a different method in the paper [9].

7 Three-dimensional deformations of elliptic curves and Poisson structures

We will consider now fully three-dimensional deformations of the general cubic (55). We take the canonical Poisson bracket in R^6, choose the generating function as $f_4 = p_4 - {p_2}^2 - u_2 p_3 - u_1 p_2 - u_0$ and denote deformation variables by x_2, x_3, x_4. The coisotropy condition takes the form

$$\{p_4 - P_4(p_2, p_3), \mathcal{E}\}|_\Gamma = c_7\, {p_2}^2 p_3 + c_6\, {p_2}^3 + c_5\, p_2 p_3 + c_4\, {p_2}^2 + c_3\, p_3 + c_2\, p_2 + c_0 = 0 \tag{73}$$

where

$$\begin{aligned}
c_7 =& 2\,\frac{\partial u_4}{\partial x_2} - 3\,\frac{\partial v_2}{\partial x_2}\\
c_6 =& 2\,\frac{\partial u_3}{\partial x_2} - 3\,\frac{\partial v_1}{\partial x_2} - u_4\frac{\partial v_2}{\partial x_2} + 2\,\frac{\partial v_2}{\partial x_3}\\
c_5 =& v_1\frac{\partial u_4}{\partial x_2} - u_4\frac{\partial v_1}{\partial x_2} - 2\,u_3\frac{\partial v_2}{\partial x_2} + v_2\frac{\partial u_4}{\partial x_3} + u_4\frac{\partial v_2}{\partial x_3} - {u_4}^2\frac{\partial v_2}{\partial x_2} - \frac{\partial u_4}{\partial x_4} + 2\,\frac{\partial u_2}{\partial x_2} + 2\,\frac{\partial v_1}{\partial x_3}\\
c_4 =& -u_4u_3\frac{\partial v_2}{\partial x_2} + v_1\frac{\partial u_3}{\partial x_2} - \frac{\partial u_3}{\partial x_4} + 2\,\frac{\partial u_1}{\partial x_2} - 3\,\frac{\partial v_0}{\partial x_2} - 2\,u_3\frac{\partial v_1}{\partial x_2} + v_2\frac{\partial u_3}{\partial x_3} - u_4\frac{\partial v_1}{\partial x_3} + 2\,u_3\frac{\partial v_2}{\partial x_3}\\
c_3 =& -u_4u_2\frac{\partial v_2}{\partial x_2} - \frac{\partial u_2}{\partial x_4} + 2\,\frac{\partial v_0}{\partial x_3} + v_1\frac{\partial u_2}{\partial x_2} - u_4\frac{\partial v_0}{\partial x_2} - u_1\frac{\partial v_2}{\partial x_2} + v_2\frac{\partial u_2}{\partial x_3} + u_2\frac{\partial v_2}{\partial x_3}\\
c_2 =& -u_4u_1\frac{\partial v_2}{\partial x_2} - \frac{\partial u_1}{\partial x_4} + 2\,\frac{\partial u_0}{\partial x_2} + v_1\frac{\partial u_1}{\partial x_2} - 2\,u_3\frac{\partial v_0}{\partial x_2} - u_1\frac{\partial v_1}{\partial x_2} + v_2\frac{\partial u_1}{\partial x_3} - u_4\frac{\partial v_0}{\partial x_3} - u_2\frac{\partial v_1}{\partial x_3} + 2\,u_1\frac{\partial v_2}{\partial x_3}\\
c_0 =& v_1\frac{\partial u_0}{\partial x_2} - u_1\frac{\partial v_0}{\partial x_2} + v_2\frac{\partial u_0}{\partial x_3} - u_2\frac{\partial v_0}{\partial x_3} - \frac{\partial u_0}{\partial x_4} - u_4u_0\frac{\partial v_2}{\partial x_2} + 2\,u_0\frac{\partial v_2}{\partial x_3}.
\end{aligned} \tag{74}$$

Thus, the coisotropy condition (73) is equivalent to the equations $c_i = 0$. The conditions $c_7 = 0$ and $c_6 = 0$ imply that

$$v_2 = \frac{2}{3}u_4 \quad , \qquad v_1 = \frac{2}{3}u_3 - \frac{2}{9}{u_4}^2 + \frac{4}{3}\frac{\partial}{\partial x_2}^{-1}\frac{\partial u_4}{\partial x_3}. \tag{75}$$

The rest of these conditions give rise to the system

$$\begin{aligned}
\frac{\partial u_4}{\partial x_4} =& -\frac{2}{3}\frac{\partial}{\partial x_2}(u_2u_3) - \frac{5}{9}{u_4}^2\frac{\partial u_4}{\partial x_2} + \frac{4}{9}u_4\frac{\partial u_4}{\partial x_3} + 2\frac{\partial u_2}{\partial x_2} + \frac{4}{3}\frac{\partial u_3}{\partial x_3} + \frac{4}{9}\frac{\partial u_4}{\partial x_2}\frac{\partial}{\partial x_2}^{-1}\frac{\partial u_4}{\partial x_3} + \frac{8}{9}\frac{\partial}{\partial x_2}^{-1}\frac{\partial^2 u_4}{\partial {x_3}^2}\\
\frac{\partial u_3}{\partial x_4} =& -\frac{2}{3}u_4u_3\frac{\partial u_4}{\partial x_2} + v_1\frac{\partial u_3}{\partial x_2} + 2\,\frac{\partial u_1}{\partial x_2} - 3\,\frac{\partial v_0}{\partial x_2} - 2\,u_3\frac{\partial v_1}{\partial x_2} + \frac{2}{3}u_4\frac{\partial u_3}{\partial x_3} - u_4\frac{\partial v_1}{\partial x_3} + \frac{4}{3}\,u_3\frac{\partial u_4}{\partial x_3}\\
\frac{\partial u_2}{\partial x_4} =& -\frac{2}{3}u_4u_2\frac{\partial u_4}{\partial x_2} + 2\,\frac{\partial v_0}{\partial x_3} + v_1\frac{\partial u_2}{\partial x_2} - u_4\frac{\partial v_0}{\partial x_2} - \frac{2}{3}u_1\frac{\partial u_4}{\partial x_2} + \frac{2}{3}u_4\frac{\partial u_2}{\partial x_3} + \frac{2}{3}u_2\frac{\partial u_4}{\partial x_3}\\
\frac{\partial u_1}{\partial x_4} =& -\frac{2}{3}u_4u_1\frac{\partial u_4}{\partial x_2} + 2\,\frac{\partial u_0}{\partial x_2} + v_1\frac{\partial u_1}{\partial x_2} - 2\,u_3\frac{\partial v_0}{\partial x_2} - u_1\frac{\partial v_1}{\partial x_2} + \frac{2}{3}u_4\frac{\partial u_1}{\partial x_3} - u_4\frac{\partial v_0}{\partial x_3} - u_2\frac{\partial v_1}{\partial x_3} + \frac{4}{3}\,u_1\frac{\partial u_4}{\partial x_3}\\
\frac{\partial u_0}{\partial x_4} =& v_1\frac{\partial u_0}{\partial x_2} - u_1\frac{\partial v_0}{\partial x_2} + \frac{2}{3}u_4\frac{\partial u_0}{\partial x_3} - u_2\frac{\partial v_0}{\partial x_3} - \frac{2}{3}u_4u_0\frac{\partial u_4}{\partial x_2} + \frac{4}{3}\,u_0\frac{\partial u_4}{\partial x_3}
\end{aligned} \tag{76}$$

where again $v_1 = \frac{2}{3}u_3 - \frac{2}{9}{u_4}^2 + \frac{4}{3}\frac{\partial}{\partial x_2}^{-1}\frac{\partial u_4}{\partial x_3}$.
This system governs three-dimensional coisotropic deformations of the elliptic curve (55). From the viewpoint of deformations of space curves it describes deformations of a space curve which is the intersection of a cylindrical surface generated by the cubic (55) and a space quadric defined by the equation $f_4 = p_4 - {p_2}^2 - u_2p_3 - u_1p_2 - u_0 = 0$.

We note that this system contains an arbitrary field v_0 . It is associated with the gauge freedom for the system. In the gauge

$$v_0 = \frac{1}{27}u_4^2u_3 - \frac{1}{6}u_3^2 - \frac{2}{3}u_1 - \frac{4}{9}\frac{\partial}{\partial x_2}^{-1}\frac{\partial}{\partial x_3}\left(u_2 - \frac{1}{27}u_4^3 + \frac{1}{6}u_3u_4\right) \tag{77}$$

this system coincides with the dispersionless limit of the first hidden KP system considered in [12]. So, it is natural to refer to the system (76) as the genus 1 KP system.

Now we will extend the observation on the transformation of the Poisson structure done in the previous section to the three-dimensional case and will analyze relation between different Poisson structures which produce the same integrable systems. We consider the general cubic (55). As before admissible transformations are those which do not change the genus. Let us denote new coordinates by $\xi_{2,3,4}, \pi_{2,3,4}$. The only admissible transformations are the graded transformations $\pi_2 = p_2 + \alpha(x_2, x_3, x_4)$ and $\pi_3 = p_3 + \beta_1(x_2, x_3, x_4)p_2 + \beta_0(x_2, x_3, x_4)$. In order to preserve the gradation we perform similar change of the coordinate $\pi_4 = p_4 + \gamma_3(x_2, x_3, x_4){p_2}^2 + \gamma_2(x_2, x_3, x_4)p_3 + \gamma_1(x_2, x_3, x_4)p_2 + \gamma_0(x_2, x_3, x_4)$. Finally, the polynomiality in the π_is variables of the family of Poisson tensors is preserved only if an adequate change of variables $\xi_i = \xi_i(x_2, x_3, x_4)$ for $i = 2, 3, 4$ is performed.
The inverse to such transformation is of the form

$$\begin{aligned} x_i &= x_i(\xi_2, \xi_3, \xi_4) \qquad i = 2, 3, 4 \\ p_2 &= \pi_2 - \alpha \\ p_3 &= \pi_3 - \beta_1\pi_2 + (\alpha\beta_1 - \beta_0) \\ p_4 &= \pi_4 - \gamma_3{\pi_2}^2 - \gamma_2\pi_3 + (2\alpha\gamma_3 + \gamma_2\beta_1 - \gamma_1)\,\pi_2 + \left(-\alpha^2\gamma_3 - \alpha\beta_1\gamma_2 + \beta_0\gamma_2 + \alpha\gamma_1 - \gamma_0\right) \end{aligned} \tag{78}$$

where $\alpha = \alpha(x = x(\xi))$ and analogously for the other coefficients. Under this transformation the elliptic curve is converted to

$$\begin{aligned} \mathcal{E}_g =& {\pi_3}^2 - {\pi_2}^3 - (u_4 + 2\,\beta_1)\pi_2\pi_3 - (-u_4\beta_1 - {\beta_1}^2 - 3\,\alpha + u_3){\pi_2}^2 - (-u_4\alpha + u_2 - 2\,\beta_1\alpha + 2\,\beta_0)\pi_3 \\ &- (-2\,\beta_1\beta_0 - u_4\beta_0 + 2\,u_4\beta_1\alpha + 2\,{\beta_1}^2\alpha + 3\,\alpha^2 + u_1 - 2\,u_3\alpha - u_2\beta_1)\pi_2 \\ &- (u_2\beta_1\alpha + u_0 + 2\,\beta_1\alpha\,\beta_0 - {\beta_1}^2\alpha^2 - {\beta_0}^2 + u_3\alpha^2 - u_2\beta_0 - \alpha^3 - u_4\beta_1\alpha^2 + u_4\alpha\,\beta_0 - u_1\alpha) \end{aligned} \tag{79}$$

and the deformation function becomes

$$\begin{aligned} {J^{(4)}}_g =& \pi^4 - (\gamma_3 + 1){\pi_2}^2 - (\gamma_2 + v_2)\pi_3 - (-2\,\gamma_3\alpha + \gamma_1 - \gamma_2\beta_1 - 2\,\alpha - v_2\beta_1 + v_1)\pi_2 \\ &- (\gamma_3\alpha^2 + \gamma_2\beta_1\alpha - \gamma_2\beta_0 - \gamma_1\alpha + \gamma_0 + \alpha^2 + v_2\beta_1\alpha - v_2\beta_0 - v_1\alpha + v_0). \end{aligned}$$

Then the canonical Poisson structure is transformed into the following

$$
\begin{aligned}
\{\xi_i,\xi_j\} =& 0 \qquad i,j=2,3,4\\
\{\xi_i,\pi_2\} =& \frac{\partial\xi_i}{\partial x_2}\Big|_{x=x(\xi)} \qquad i=2,3,4\\
\{\xi_i,\pi_3\} =& \frac{\partial\xi_i}{\partial x_3}\Big|_{x=x(\xi)}+\beta_1\frac{\partial\xi_i}{\partial x_2}\Big|_{x=x(\xi)} \qquad i=2,3,4\\
\{\xi_i,\pi_4\} =& 2\gamma_3\frac{\partial\xi_i}{\partial x_2}\Big|_{x=x(\xi)}\pi_2+\frac{\partial\xi_i}{\partial x_4}\Big|_{x=x(\xi)}+\gamma_2\frac{\partial\xi_i}{\partial x_3}\Big|_{x=x(\xi)}+(\gamma_1-2\gamma_3\alpha)\frac{\partial\xi_i}{\partial x_2}\Big|_{x=x(\xi)}\\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i=2,3,4\\
\{\pi_2,\pi_3\} =& -\frac{\partial\beta_1}{\partial x_2}\Big|_{x=x(\xi)}\pi_2+\left(\frac{\partial\alpha}{\partial x_3}+\frac{\partial}{\partial x_2}(\alpha\beta_1-\beta_0)\right)\Big|_{x=x(\xi)}\\
\{\pi_2,\pi_4\} =& \frac{\partial\gamma_3}{\partial x_2}\Big|_{x=x(\xi)}{\pi_2}^2-\frac{\partial\gamma_2}{\partial x_2}\Big|_{x=x(\xi)}\pi_3+\left(\frac{\partial}{\partial x_2}(2\alpha\gamma_3-\gamma_1)+\beta_1\frac{\partial\gamma_2}{\partial x_2}\right)\Big|_{x=x(\xi)}\pi_2\\
&+\left(\frac{\partial}{\partial x_2}(\alpha\gamma_1-\gamma_0-\alpha^2\gamma_3)+(\alpha\beta_1-\beta_0)\frac{\partial\gamma_2}{\partial x_2}+\gamma_2\frac{\partial\alpha}{\partial x_3}+\frac{\partial\alpha}{\partial x_4}\right)\Big|_{x=x(\xi)}\\
\{\pi_3,\pi_4\} =& \left(2\gamma_3\frac{\partial\beta_1}{\partial x_2}-\beta_1\frac{\partial\gamma_3}{\partial x_2}-\frac{\partial\gamma_3}{\partial x_3}\right)\Big|_{x=x(\xi)}{\pi_2}^2+\left(-\frac{\partial\gamma_2}{\partial x_3}-\beta_1\frac{\partial\gamma_2}{\partial x_2}\right)\Big|_{x=x(\xi)}\pi_3\\
&+\left(2\alpha\frac{\partial\gamma_3}{\partial x_3}+2\alpha\beta_1\frac{\partial\gamma_3}{\partial x_2}-4\alpha\gamma_3\frac{\partial\beta_1}{\partial x_2}+\beta_1\frac{\partial\gamma_2}{\partial x_3}+{\beta_1}^2\frac{\partial\gamma_2}{\partial x_2}+2\gamma_3\frac{\partial\beta_0}{\partial x_2}-\frac{\partial\gamma_1}{\partial x_3}\right.\\
&\left.-\beta_1\frac{\partial\gamma_1}{\partial x_2}+\frac{\partial\beta_1}{\partial x_4}+\gamma_2\frac{\partial\beta_1}{\partial x_3}+\gamma_1\frac{\partial\beta_1}{\partial x_2}\right)\Big|_{x=x(\xi)}\pi_2+\left(-\alpha^2\frac{\partial\gamma_3}{\partial x_3}-\alpha^2\beta_1\frac{\partial\gamma_3}{\partial x_2}\right.\\
&+2\alpha^2\gamma_3\frac{\partial\beta_1}{\partial x_2}-\alpha\beta_1\frac{\partial\gamma_2}{\partial x_3}+\beta_0\frac{\partial\gamma_2}{\partial x_3}-\alpha{\beta_1}^2\frac{\partial\gamma_2}{\partial x_2}+\beta_0\beta_1\frac{\partial\gamma_2}{\partial x_2}-2\alpha\gamma_3\frac{\partial\beta_0}{\partial x_2}+\alpha\frac{\partial\gamma_1}{\partial x_3}\\
&\left.+\alpha\beta_1\frac{\partial\gamma_1}{\partial x_2}-\alpha\frac{\partial\beta_1}{\partial x_4}-\alpha\gamma_2\frac{\partial\beta_1}{\partial x_3}-\alpha\gamma_1\frac{\partial\beta_1}{\partial x_2}-\frac{\partial\gamma_0}{\partial x_3}+\frac{\partial\beta_0}{\partial x_4}+\gamma_2\frac{\partial\beta_0}{\partial x_3}+\gamma_1\frac{\partial\beta_0}{\partial x_2}-\beta_1\frac{\partial\gamma_0}{\partial x_2}\right)\Big|_{x=x(\xi)}.
\end{aligned}
\tag{80}
$$

With the choice $\alpha=\frac{1}{3}u_3+\frac{1}{12}(u_4)^2, \beta_0=-\frac{1}{2}u_2, \beta_1=-\frac{1}{2}u_4$ equation (79) takes the canonical form and the Poisson structure (80) becomes an appropriate one for constructing three-dimensional deformations of moduli g_2 and g_3 of the elliptic curves. The corresponding equations are too complicated to be presented here.

The observations made in this and previous section naturally lead to introduction of a notion of equivalent Poisson structures in the framework of the theory of coisotropic deformations. This remark is due to Jean-Claude Thomas and Volodya Rubtsov.

8 Coisotropic deformations of curves and surfaces in R^4

Deformations of curves in R^4 defined by three equations can be studied analogously to the three-dimensional case. For the curves given by equations

$$f_1 = p_1p_2+ap_1+bp_2+c=0, \tag{81}$$

$$f_2 = p_3+\sum_{k=1}^{n}\alpha_k p_1^k=0, \quad f_3=p_4+\sum_{k=1}^{m}\beta_k p_2^k=0 \tag{82}$$

with arbitrary n and m the coisotropic deformations are governed by the CSs which coincide with equations of the universal Whitham hierarchy on Riemann sphere with two punctures (see [14]).

An interesting particular case corresponds to

$$f_1 = p_1p_2 - 1 = 0, \tag{83}$$
$$f_2 = p_3 + ap_1 - a = 0, \quad f_3 = p_4 + bp_2 - b = 0 \tag{84}$$

for which the curve is the intersection of the cylindrical hyperbola and two hyperplanes. In order to get nontrivial coisotropic deformations in this case one has to choose the Poisson bracket in the form (see [15])

$$\{f, g\} = \sum_{k=1}^{4} \gamma_k \left(\frac{\partial f}{\partial p_k}\frac{\partial g}{\partial x_k} - \frac{\partial f}{\partial x_k}\frac{\partial g}{\partial p_k} \right) \tag{85}$$

where $\gamma_1 = p_1, \gamma_2 = -p_2, \gamma_3 = \gamma_4 = 1$. Then the coisotropy conditions

$$\{f_j, f_k\} \mid_\Gamma = 0, \quad j, k = 1, 2, 3 \tag{86}$$

gives rise to the following CS

$$a_{x_1} + a_{x_2} = 0, \quad b_{x_1} + b_{x_2} = 0, \tag{87}$$
$$a_{x_4} + ab_{x_1} = 0, \quad b_{x_3} - ba_{x_2} = 0. \tag{88}$$

This system implies that the variable $\Theta = \log(ab)$ obeys the equation

$$\Theta_{x_3x_4} + (\exp\Theta)_{x_1x_1} = 0 \tag{89}$$

which is well-known Boyer-Finley or dispersionless two-dimensional Toda lattice (2DTL) equation. Choosing f_2 and f_3 as polynomials of any order in p_1 and p_2, respectively, one gets coisotropic deformations of curves in R^4 described by higher d2DTL equations.

Coisotropic deformations can be constructed also for algebraic varieties of other types in R^4. For instance, let us consider a pencil of hyperplanes in R^4 defined by the equations

$$f_1 = p_3 + (a - \lambda)p_1 + bp_2 = 0, \tag{90}$$
$$f_2 = p_4 + cp_1 + (d - \lambda)p_2 = 0 \tag{91}$$

where λ is a parameter. The coisotropy condition $\{f_1, f_2\} \mid_\Gamma = 0$ for all values of λ with the canonical Poisson bracket in R^8 gives rise to

$$a = \Phi_{x_1}, \quad b = \widetilde{\Phi}_{x_1}, \quad c = \Phi_{x_2}, \quad d = \widetilde{\Phi}_{x_2} \tag{92}$$

and the equations

$$\begin{aligned} \Phi_{x_1x_4} - \Phi_{x_2x_3} + \Phi_{x_2}\Phi_{x_1x_1} - \Phi_{x_1}\Phi_{x_1x_2} + \widetilde{\Phi}_{x_2}\Phi_{x_1x_2} - \widetilde{\Phi}_{x_1}\Phi_{x_2x_2} &= 0, \\ \widetilde{\Phi}_{x_1x_4} - \widetilde{\Phi}_{x_2x_3} + \widetilde{\Phi}_{x_2}\widetilde{\Phi}_{x_1x_2} - \widetilde{\Phi}_{x_1}\widetilde{\Phi}_{x_2x_2} + \Phi_{x_2}\widetilde{\Phi}_{x_1x_1} - \Phi_{x_1}\widetilde{\Phi}_{x_1x_2} &= 0. \end{aligned} \tag{93}$$

This CS describes the coisotropic deformations of the pencil of the hyperplanes. The system (93) admits the constraint $\Phi = \Theta_{x_2}, \widetilde{\Phi} = -\Theta_{x_1}$ under which it is reduced to the single equation

$$\Theta_{x_1x_4} - \Theta_{x_2x_3} + \Theta_{x_1x_1}\Theta_{x_2x_2} - (\Theta_{x_1x_2})^2 = \alpha(x_1, x_3, x_4) \tag{94}$$

where $\alpha(x_1, x_3, x_4)$ is an arbitrary function. At $\alpha = 0$ it is the celebrated heavenly equation [28].

This last example has a natural extension to the spaces of any dimension. Indeed, let us consider a rational pencil of hyperplanes in R^n defined by the equations

$$f_1 = \sum_{k=1}^{n} a_k(\lambda)p_k = 0, \quad f_2 = \sum_{k=1}^{n} b_k(\lambda)p_k = 0 \tag{95}$$

where $a_k(\lambda)$ and $b_k(\lambda)$ are certain rational functions of the parameter λ. The coisotropy condition $\{f_1, f_2\} \mid_\Gamma = 0$ with the canonical Poisson bracket in R^{2n} for all values of λ is equivalent to the system of differential equations for the coefficients of the rational functions $a_k(\lambda)$, $b_k(\lambda)$. This system coincides with that considered in [29] in connection with the commutativity condition of multidimensional vector fields. This coincidence is not accidental. It is a well-known fact that the expression for the commutator of vector fields is in one-to one correspondence with the expression of the Poisson bracket of functions linear in momenta p_j ($p_j \longleftrightarrow \frac{\partial}{\partial x_j}$).

9 Conclusion

Coisotropic deformations studied in this paper form a special class of deformations of algebraic varieties. CSs describing such deformations represent themselves the differential constraints on coefficients of the functions f_j, i.e. on the coordinates of parameter space for algebraic variety (for this notion see e.g. [30]). Solutions of CSs generate particular subvarieties of dimension n (surfaces, hypersurfaces etc) in the parameter space. Examples of CSs considered in the paper are the integrable hydrodynamical type systems. They have number of remarkable properties (infinite sets of integrals, symmetries etc). These properties are inherited by the deformed algebraic varieties and they could be of algebro-geometrical relevance.

For general solutions of CSs each member (for fixed $x_i, i = 1, ..., n$) of the family of deformed varieties is an algebraic variety in R^n, but the totality of them (i.e. submanifold Γ) is not. Polynomial solutions of CSs are then of particular interest. For them the whole family of deformed algebraic varieties is an algebraic variety too (in R^{2n}). Thus, for such solutions of CSs families of coisotropically deformed algebraic varieties belong to the class of families of algebraic varieties which "vary algebraically with parameters". This class is "one that is fundamental in much of algebraic geometry" [30] (p.41).

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