DIFFERENCE PICARD THEOREM FOR MEROMORPHIC FUNCTIONS OF SEVERAL VARIABLES

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Abstract. It is shown that if $n \in \mathbb{N}$, $c \in \mathbb{C}^n$, and three distinct values of a meromorphic function $f : \mathbb{C}^n \to \mathbb{P}^1$ of hyper-order $\varsigma(f)$ strictly less than $2/3$ have forward invariant pre-images with respect to a translation $\tau : \mathbb{C}^n \to \mathbb{C}^n$, $\tau(z) = z+c$, then f is a periodic function with period c. This result can be seen as a generalization of M. Green's Picard-type theorem in the special case where $\varsigma(f) < 2/3$, since the empty pre-images of the usual Picard exceptional values are by definition always forward invariant. In addition, difference analogues of the lemma on the logarithmic derivative and of the second main theorem of Nevanlinna theory for meromorphic functions $\mathbb{C}^n \to \mathbb{P}^1$ are given, and their applications to partial difference equations are discussed.

1. Introduction

The purpose of this paper is to find difference analogues of the lemma on the logarithmic derivative and of the second main theorem of Nevanlinna theory for meromorphic functions, where the operation of partial differentiation in the ramification term has been replaced by the genuine shift operator $\Delta_c f := f(z_1 + c_1, \ldots, z_n + c_n) - f(z_1, \ldots, z_n)$, $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$, operating on a meromorphic function $f : \mathbb{C}^n \to \mathbb{P}^1$ of hyper-order strictly less than $2/3$. Hyper-order is defined by

$$\varsigma(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}, \tag{1.1}$$

where $T(r, f)$ is the Nevanlinna characteristic function of f (see Section 3 below for a short review of Nevanlinna theory of several variables). These results will have two main applications. First, we will obtain a difference analogue of Picard's theorem in several variables, which says that if $n \in \mathbb{N}$, $c \in \mathbb{C}^n$, and three distinct values of a meromorphic function $f : \mathbb{C}^n \to \mathbb{P}^1$ such that $\varsigma(f) < 2/3$ have forward invariant pre-images with respect to a translation $\tau : \mathbb{C}^n \to \mathbb{C}^n$, $\tau(z) = z + c$, then f is a periodic function with period c. In the special case of $\varsigma(f) < 2/3$ this result can be seen as a generalization of M. Green's Picard-type theorem, since the (empty) pre-images of the usual Picard exceptional values are special cases of forward invariant pre-images. The second application can be described as a Malmquist type theorem for partial difference equations. We will show that the existence of one meromorphic

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solution $w : \mathbb{C}^n \to \mathbb{P}^1$ such that $\varsigma(w) < 2/3$ is enough to reduce a large class of partial difference equations into a difference Riccati equation.

The remainder of the paper is organized as follows. The difference analogue of Picard's theorem (Theorem 2.1 below) is stated in Section 2. Section 3 contains difference analogues of the lemma on the logarithmic derivative and of the second main theorem (Theorems 3.1 and 3.3 below). Applications of these results to partial difference equations are discussed in Section 4. The difference analogue of the lemma on the logarithmic derivative in several variables is proved in Sections 5 and 6, while Section 7 contains the proof of the difference version of the second main theorem. Finally, the difference analogue of Picard's theorem in several variables is proved in Section 8.

2. A difference analogue of Picard's theorem

Picard's theorem states that any non-constant entire function $f(z)$ assumes all values in the complex plane with at most one possible exception [25]. Fatou [9, 10] has constructed an example of a biholomorphic mapping $f : \mathbb{C}^2 \to \mathbb{C}^2$ such that the set of Picard exceptional values $\mathbb{C}^2 \setminus f(\mathbb{C}^2)$ contains a non-empty open set. At first sight this example appears to imply severe difficulties in generalizing Picard's theorem to meromorphic functions of several variables. However, it turns out that there is a natural generalization which can be found by rephrasing Picard's theorem in terms of projective spaces. Green [12] showed that any holomorphic mapping from $\mathbb{C}^n$ into the projective space $\mathbb{P}^m$ that misses $2m + 1$ hyperplanes in general position is a constant, thus improving an earlier Picard-type theorem by Wu [32]. Moreover, extensions of Nevanlinna's second main theorem to several variables can be regarded as deep generalizations of Picard's theorem, see, for instance, [3, 13, 31, 33, 5].

We will show that forward invariance with respect to a translation of the pre-image of a target value is, in the sense of Picard exceptionality, as restrictive for non-periodic meromorphic functions $\mathbb{C}^n \to \mathbb{P}^1$ such that $\varsigma(f) < 2/3$, as omitting the target value completely. We say that the pre-image of $a \in \mathbb{P}^1$ is under f is forward invariant with respect to the translation τ if $\tau(f^{-1}(\{a\})) \subset f^{-1}(\{a\})$ where $\tau(f^{-1}(\{a\}))$ and $f^{-1}(\{a\})$ are considered to be multisets in which each point is repeated according to its multiplicity. By this definition the (empty and thus forward invariant) pre-images of the usual Picard exceptional values become special cases of forward invariant pre-images. The following theorem is a difference analogue of Picard's theorem for meromorphic functions in several variables.

Theorem 2.1. *Let $f : \mathbb{C}^n \to \mathbb{P}^1$ be a meromorphic function such that $\varsigma(f) < 2/3$, and let $\tau(z) = z + c$, where $\tau : \mathbb{C}^n \to \mathbb{C}^n$ and $c \in \mathbb{C}^n$. If three distinct values of f have forward invariant pre-images with respect to τ, then f is a periodic function with period c.*

Theorem 2.1 is proved in Section 8 below. A simple example from [19] shows that the condition on growth of f cannot be removed, at least not completely. By taking $g(z) = \exp(\exp(z))$, the pre-image of each of the m^{th} roots of unity is forward invariant with respect to the translation $\tau(z) = z + \log(m + 1)$. Since clearly $g(z) \not\equiv g(z + \log(m + 1))$, it follows that a slightly weaker growth condition in Theorem 2.1 would allow a non-periodic meromorphic function with *arbitrarily many* values having forward invariant pre-images.

3. Second main theorem

One of the key components in Nevanlinna's original proof of the second main theorem is a technical result referred to as the lemma on the logarithmic derivative. This lemma has also been used as an important tool in the study of value distribution of meromorphic solutions of differential equations in the complex plane [20, 23, 14]. The original proof of the second main theorem in several variables was based on a differential geometric method due to Ahlfors and F. Nevanlinna, see, e.g., [31], instead of Nevanlinna's method based on the lemma on the logarithmic derivative. The first generalization of the lemma on the logarithmic derivative to several complex variables was given by Vitter [30], who used the method of non-negative curvature developed by Carlson, Cowen, Griffiths and King [3, 13, 8]. Biancofiore and Stoll used an alternative method based on a technique they call "fiber integration" to prove their version of the lemma on the logarithmic derivative in several complex variables [2]. Further improvements and generalizations of the lemma on the logarithmic derivative has been given, for instance, by Cherry [4] and Ye [33, 34].

The purpose of this section is to present difference analogues of the lemma on the logarithmic derivative and of the second main theorem in several complex variables. Before stating these two key results of this paper, we will briefly recall some of the standard notation of Nevanlinna theory in $\mathbb{C}^n$ [27, 22, 28] (see also, for instance, [30, 2, 34]).

Let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, and let $r > 0$. Introducing the differential operators $d := \partial + \overline{\partial}$ and $d^c := (\partial - \overline{\partial})/4\pi i$, we define $\omega_n(z) := dd^c \log |z|^2$ and $\sigma_n(z) := d^c \log |z|^2 \wedge \omega_n^{n-1}(z)$ where $z \in \mathbb{C}^n \setminus \{0\}$ and $|z|^2 := |z_1|^2 + \cdots + |z_n|^2$. Then $\sigma_n(z)$ defines a positive measure with total measure one on the boundary $\partial B_n(r) := \{z \in \mathbb{C}^n : |z| = r\}$ of the ball $B_n(r) := \{z \in \mathbb{C}^n : |z| < r\}$. In addition, by defining $\upsilon_n(z) := dd^c|z|^2$ and $\rho_n(z) := \upsilon_n^n(z)$ for all $z \in \mathbb{C}^n$, it follows that $\rho_n(z)$ is the Lebesgue measure on $\mathbb{C}^n$ normalized such that the ball $B_n(r)$ has measure r^{2n}.

Let f be a meromorphic function in $\mathbb{C}^n$ in the sense that f can be written as a quotient of two relatively prime holomorphic functions. We will write $f = (f_0, f_1)$ where $f_0 \not\equiv 0$, and regard f as a meromorphic map $f : \mathbb{C}^n \to \mathbb{P}^1$ such that $f^{-1}(\infty) \neq \mathbb{C}^n$. The standard definition of Nevanlinna characteristic function of f is given by

$$T_f(r, s) := \int_s^r \frac{A_f(t)}{t}\, dt$$

where $0 < s < r$ and

$$A_f(t) = \frac{1}{t^{2n-2}} \int\limits_{B_n(t)} f^*\omega \wedge \upsilon_n^{n-1} = \int\limits_{B_n(t)} f^*\omega \wedge \omega_n^{n-1} + A_f(0)$$

is a measure of the spherical area covered by the image of $B_n(t)$ under f. Here the pullback $f^*\omega$ satisfies

$$f^*\omega = dd^c \log(|f_0|^2 + |f_1|^2)$$

for all z outside of the set of indeterminacy $I_f := \{z \in \mathbb{C}^n : f_0(z) = f_1(z) = 0\}$ of f.

A *divisor* on $\mathbb{C}^n$ is an integer valued function which is locally the difference between the zero-multiplicity functions of two holomorphic functions, in our case f_0 and f_1. Let $a \in \mathbb{P}^1$ such that $f^{-1}(a) \neq \mathbb{C}^n$. Then the a-divisor ν_f^a of $f = (f_0, f_1)$ is the divisor associated to the holomorphic functions $f_1 - af_0$ and f_0. By denoting

$S(r) := \overline{B}_n(r) \cap \operatorname{supp} \nu_f^a$, where $\overline{B}_n(r) = \{z \in \mathbb{C}^n : |z| \leq r\}$ and $\operatorname{supp} \nu_f^a$ denotes the closure of the set $\{z \in \mathbb{C}^n : \nu_f^a(z) \neq 0\}$, we may define the *counting function* of ν_f^a as

$$n_f(r,a) := r^{2-2n} \int\limits_{S(r)} \nu_f^a \upsilon_n^{n-1}$$

for all $n \geq 1$ and for all $r > 0$.

There are slightly different ways to continue the formulation of Nevanlinna theory from here. Stoll [28] defines the (integrated) *counting function* of ν_f^a as

$$N_f(r,s,a) := \int_s^r \frac{n_f(t,a)}{t}\, dt$$

for all $0 < s < r$, and the *compensation function* as

$$m_f(r,a) := \int\limits_{\partial B_n(r)} \log \frac{1}{\|f,a\|}\, \sigma_n(z),$$

where $||f,a||$ denotes the chordal distance from f to $a \in \mathbb{P}^1$. Then the first main theorem of Nevanlinna theory becomes

$$T_f(r,s) = N_f(r,s,a) + m_f(r,a) - m_f(s,a)$$

where $0 < s < r$.

We choose a slightly different approach (see e.g. [34]) by denoting $N(r,f) := N_f(r,0,\infty)$ and $N(r,1/(f-a)) := N_f(r,0,a)$, where $a \neq \infty$ and we have assumed that $f(0) \neq a, \infty$. Then by the Jensen formula,

$$N\left(r, \frac{1}{f}\right) - N(r,f) = \int\limits_{\partial B_n(r)} \log|f(z)|\sigma_n(z) - \log|f(0)| \tag{3.1}$$

for all $r > 0$, provided that $f(0) \neq 0, \infty$. By defining the *proximity function* of f as

$$m(r,f) := \int\limits_{\partial B_n(r)} \log^+|f(z)|\sigma_n(z),$$

and if $a \neq \infty$,

$$m\left(r, \frac{1}{f-a}\right) := \int\limits_{\partial B_n(r)} \log^+ \frac{1}{|f(z)-a|}\sigma_n(z),$$

the Jensen formula (3.1) becomes

$$T(r,f) = m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right) - \log\frac{1}{|f(0)-a|} \tag{3.2}$$

where $T(r,f) = m(r,f) + N(r,f)$ and f is a meromorphic function on $\mathbb{C}^n$ satisfying $f(0) \neq a, \infty$. The order of growth of f is defined by

$$\rho(f) := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

The following theorem is a difference analogue of the lemma on the logarithmic derivative in several complex variables. It generalizes the one dimensional result [16, Theorem 2.1] by Halburd and the author. Recall the definition of hyper-order from (1.1).

Theorem 3.1. *Let f be a non-constant meromorphic function in $\mathbb{C}^n$ such that $f(0) \neq 0, \infty$, let $c \in \mathbb{C}^n$, and let $\varepsilon > 0$. If $\varsigma(f) = \varsigma < 2/3$, then*

$$\int_{\partial B_n(r)} \log^+ \left| \frac{f(z+c)}{f(z)} \right| \sigma_n(z) = o\left(\frac{T(r,f)}{r^{1-\frac{3}{2}\varsigma-\varepsilon}} \right) \tag{3.3}$$

for all $r > 0$ outside of a possible exceptional set $E \subset [1, \infty)$ of finite logarithmic measure $\int_E 1/dt < \infty$.

The proof of Theorem 3.1 can be found in Sections 5 and 6 below. Recall that we have adopted the notation $\Delta_c f := f(z+c) - f(z)$ for $c \in \mathbb{C}^n$ and $f : \mathbb{C}^n \to \mathbb{P}^1$. The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let a and c be constants in $\mathbb{C}^n$, let f be a non-constant meromorphic function in $\mathbb{C}^n$ such that $f(0) \neq a, \infty$, and let $\varepsilon > 0$. If $\varsigma(f) = \varsigma < 2/3$, then*

$$m\left(r, \frac{\Delta_c f}{f-a}\right) = o\left(\frac{T(r,f)}{r^{1-\frac{3}{2}\varsigma-\varepsilon}} \right)$$

for all $r > 0$ outside of a possible exceptional set $E \subset [1, \infty)$ of finite logarithmic measure.

Corollary 3.2 can be applied to prove a difference analogue of the second main theorem of Nevanlinna theory for meromorphic functions $f : \mathbb{C}^n \to \mathbb{P}^1$, which extends [16, Theorem 2.4] to meromorphic functions of several variables.

Theorem 3.3. *Let $c \in \mathbb{C}^n$, let $\varepsilon > 0$, and let f be a meromorphic function in $\mathbb{C}^n$ such that $\Delta_c f \not\equiv 0$. Let $q \geq 2$, and let $a_1, \ldots, a_q \in \mathbb{P}^1$ be distinct finite constants such that $f(0) \neq a_j, \infty$ for all $j = 1, \ldots, q$. If $\varsigma(f) = \varsigma < 2/3$, then*

$$m(r,f) + \sum_{j=1}^{q} m\left(r, \frac{1}{f-a_j}\right) \leq 2T(r,f) - N_\Delta(r,f) + o\left(\frac{T(r,f)}{r^{1-\frac{3}{2}\varsigma-\varepsilon}} \right),$$

where

$$N_\Delta(r,f) = 2N(r,f) - N(r, \Delta_c f) + N\left(r, \frac{1}{\Delta_c f}\right),$$

and r lies outside of a possible exceptional set $E \subset [1, \infty)$ of finite logarithmic measure.

The proof of Theorem 3.3 can be found from Section 7.

4. Applications to partial difference equations

Ablowitz, Halburd and Herbst [1] have suggested that the existence of sufficiently many finite-order meromorphic solutions could be used as a detector of Painlevé type difference equations. Halburd and the author used one-dimensional difference analogues [15, 16] of some of the main results of Nevanlinna theory to prove that the existence of at least one finite-order meromorphic solution, which is not simultaneously a solution of a first-order difference Riccati equation, is enough reduce a large class of difference equations into a list of equations consisting exactly of known discrete equations of Painlevé type [17, 18].

The purpose of this section is to extend some of the methods used in [1] to partial differences, and apply these generalized results to single out the difference Riccati equation out of a large class of first-order partial difference equations. We start by stating the main result of this section.

Let $\mathcal{S}(f) = \{g : \mathbb{C}^n \to \mathbb{P}^1 \text{ meromorphic } : T(r,g) = o(T(r,f))\}$ where $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. A meromorphic solution $w : \mathbb{C}^n \to \mathbb{P}^1$ of a difference equation is called *admissible* if all coefficients of the equation are in $\mathcal{S}(f)$ (see [23, p. 192]).

Theorem 4.1. *Let $c \in \mathbb{C}^n$. If the difference equation*

$$w(z+c) = R(z, w(z)), \tag{4.1}$$

where $R(z,u)$ is rational in u having meromorphic coefficients in $\mathbb{C}^n$, has an admissible meromorphic solution $w : \mathbb{C}^n \to \mathbb{P}^1$ such that $\varsigma(w) < 2/3$, then $\deg_w(R) = 1$.

The first result needed in the proof of Theorem 4.1 is due to Valiron [29] and Mohon'ko [24].

Theorem 4.2 ([24, 29])**.** *Let $R(z,u)$ be a rational function of u whose coefficients are meromorphic functions $h(z)$ in $\mathbb{C}^n$ satisfying $T(r,h) = O(\phi(r))$, where ϕ is a fixed positive increasing function on $[0,\infty)$. Then for every meromorphic function $f : \mathbb{C}^n \to \mathbb{P}^1$ we have*

$$T(r, R(z, f(z))) = \deg_f T(r,f) + O(\phi(r)).$$

According to an identity due to Valiron [29] and Mohon'ko [24] (see also, e.g., [11, p. 31] and [23, p. 29])

$$\deg_f(R)T(r,f) = T(r, R(z, f(z))) + O(\phi(r)), \tag{4.2}$$

whenever f is a non-constant meromorphic function in the complex plane. As was observed in [11, Appendix B., p. 453], by following the proof of (4.2) in [24] (see also [23]) it can be seen that the identity (4.2) holds for any non-decreasing characteristic function $T(r,f)$ which satisfies the basic Nevanlinna inequalities, the first main theorem, and the property $T(r,f^2) = 2T(r,f)$. Therefore, in particular, the assertion of Theorem 4.2 follows.

Chiang and Feng [7] have shown that if f is a finite-order meromorphic function in the complex plane and $\eta \in \mathbb{C}$, then

$$T(r, f(z+\eta)) = T(r,f) + O(r^{\rho-1+\varepsilon}), \quad r \to \infty, \tag{4.3}$$

where $\rho = \rho(f)$ is the order of f and $\varepsilon > 0$. A similar estimate

$$T(r, f(z+\eta)) = T(r,f) + o(T(r,f)), \tag{4.4}$$

where $r \to \infty$ outside of an exceptional set of finite logarithmic measure, follows by combining [16, Theorem 2.1] with [17, Lemma 2.1]. The following theorem is a generalization of the asymptotic relations (4.3) and (4.4) to several variables.

Theorem 4.3. *Let $f : \mathbb{C}^n \to \mathbb{P}^1$ be a meromorphic function, let $c \in \mathbb{C}^n$ and let $\varepsilon > 0$. If $\varsigma(f) = \varsigma < 2/3$, then*

$$T(r, f(z+c)) = T(r,f) + o\left(\frac{T(r,f)}{r^{1-\frac{3}{2}\varsigma-\varepsilon}}\right) \tag{4.5}$$

where $r \to \infty$ outside of an exceptional set of finite logarithmic measure.

Proof. First we observe that $N(r, f(z+c)) \leq N(r+|c|, f)$ by the definition of the counting function. Therefore, by defining

$$\lambda_2 := \limsup_{r\to\infty} \frac{\log\log N(r,f)}{\log r}$$

and applying [19, Lemma 8.3], it follows that

$$N(r, f(z+c)) \leq N(r+|c|, f) = N(r,f) + o\left(\frac{N(r,f)}{r^{1-\lambda_2-\varepsilon}}\right), \tag{4.6}$$

where r tends to infinity outside of an exceptional set of finite logarithmic measure. Second, by Theorem 3.1 we have

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\frac{3}{2}\varsigma-\varepsilon}}\right) \tag{4.7}$$

where r lies again outside of an exceptional set of finite logarithmic measure. The upper bound in the asymptotic relation (4.5) follows by combining (4.6) and (4.7) with the inequality

$$T(r, f(z+c)) \leq N(r+|c|, f) + m(r,f) + m\left(r, \frac{f(z+c)}{f(z)}\right),$$

and using the facts $\lambda_2 \leq \varsigma$ and $N(r,f) \leq T(r,f)$. The lower bound follows similarly by combining

$$T(r, f(z)) \leq N(r+|c|, f(z+c)) + m(r, f(z+c)) + m\left(r, \frac{f(z)}{f(z+c)}\right),$$

with (4.6) and (4.7), applied with the function $f(z+c)$ and the shift $-c$. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Suppose that (4.1) has a meromorphic solution $w : \mathbb{C}^n \to \mathbb{P}^1$ such that $\varsigma(w) < 2/3$. By applying Theorems 4.2 and 4.3 respectively to the right and left sides of (4.1), it follows that

$$T(r,w) = \deg_w(R)T(r,w) + o(T(r,w))$$

as $r \to \infty$ outside of an exceptional set of finite logarithmic measure. Therefore, $\deg_w(R) = 1$. □

5. Estimates on integrated difference quotients in $\mathbb{C}$ and $\mathbb{C}^n$

In this section we lay the foundations for the proof of Theorem 3.1 by obtaining growth estimates for integrated difference quotients of a meromorphic function f in $\mathbb{C}$ and in $\mathbb{C}^n$. We start with the one-dimensional case.

Lemma 5.1. *Let $f(z)$ be a meromorphic function in $\mathbb{C}$ such that $f(0) \neq 0, \infty$, and let $c \in \mathbb{C}$ and $\delta \in (0,1)$. Then for all $r > 0$ and $s > r + |c|$,*

$$\int_{\partial B_1(r)} \log^+ \left|\frac{f(z+c)}{f(z)}\right| \sigma_1(z) \leq \frac{8\pi|c|^\delta}{\delta(1-\delta)r^\delta}\left(n(s,f) + n\left(s, \frac{1}{f}\right)\right)$$
$$+ \frac{4\pi|c|}{(1-\delta)(s-r-|c|)} \cdot \left(\frac{s}{s-r}\right)^{1-\delta}\left(m(s,f) + m\left(s,\frac{1}{f}\right)\right).$$

Similar estimates to Lemma 5.1 have been obtained before in [15, Lemma 2.3], [7, Theorem 2.4] and [19, Lemma 8.2] by using similar methods to here. The improved factor in front of the function $m(s,f)+m(s,1/f)$ in Lemma 5.1 enables us to get the inequality (5.10) below in the proof of Theorem 3.1, instead of a weaker estimate which would follow by using, for instance, [19, Lemma 8.2]. The reason why this is important is the fact that the estimate (5.10) is ultimately the cause for the slightly unsatisfactory growth condition $\varsigma(f) < 2/3$ in Theorem 3.1. By applying [19, Lemma 8.2] instead of Lemma 5.1 we would be lead to the condition $\varsigma(f) < 2/5$. This also means that if one is interested in extending Theorem 3.1 to meromorpfic functions of hyper-order less than one, say, then inequality (5.10) is a good place to start looking for potential improvements.

Proof of Lemma 5.1. The Poisson-Jensen formula [20, Theorem 1.1] implies

$$\begin{aligned}\log\left|\frac{f(z+c)}{f(z)}\right| &= \int_0^{2\pi} \log|f(se^{i\theta})|\mathrm{Re}\left(\frac{se^{i\theta}+z+c}{se^{i\theta}-z-c}-\frac{se^{i\theta}+z}{se^{i\theta}-z}\right)\frac{d\theta}{2\pi}\\ &\quad+\sum_{|a_n|<s}\log\left|\frac{s(z+c-a_n)}{s^2-\bar{a}_n(z+c)}\cdot\frac{s^2-\bar{a}_n z}{s(z-a_n)}\right|\\ &\quad-\sum_{|b_m|<s}\log\left|\frac{s(z+c-b_m)}{s^2-\bar{b}_m(z+c)}\cdot\frac{s^2-\bar{b}_m z}{s(z-b_m)}\right|,\end{aligned}\tag{5.1}$$

where $|z|=r$, $s>r+|c|$, and $\{a_j\}$ and $\{b_m\}$ are the sequences of zeros and poles of f, respectively. By denoting $\{q_k\}:=\{a_j\}\cup\{b_m\}$ and integrating (5.1) over the set $\{\xi\in[0,2\pi) : \left|\frac{f(re^{i\xi}+c)}{f(re^{i\xi})}\right|\geq 1\}$, it follows that

$$m\left(r,\frac{f(z+c)}{f(z)}\right)\leq S_1(r)+S_2(r),\tag{5.2}$$

where

$$S_1(r)=\int_0^{2\pi}\int_0^{2\pi}\left|\log|f(se^{i\theta})|\mathrm{Re}\left(\frac{2cse^{i\theta}}{(se^{i\theta}-re^{i\varphi}-c)(se^{i\theta}-re^{i\varphi})}\right)\right|\frac{d\theta}{2\pi}\frac{d\varphi}{2\pi}\tag{5.3}$$

and

$$\begin{aligned}S_2(r)&=\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1+\frac{c}{re^{i\varphi}-q_k}\right|\frac{d\varphi}{2\pi}+\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1-\frac{c}{re^{i\varphi}+c-q_k}\right|\frac{d\varphi}{2\pi}\\ &\quad+\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1+\frac{c}{re^{i\varphi}-\frac{s^2}{\bar{q}_k}}\right|\frac{d\varphi}{2\pi}+\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1-\frac{c}{re^{i\varphi}+c-\frac{s^2}{\bar{q}_k}}\right|\frac{d\varphi}{2\pi}.\end{aligned}$$

By Fubini's theorem the order of integration in (5.3) may be changed, which results in

$$\begin{aligned}S_1(r)&=\int_0^{2\pi}\left|\log|f(se^{i\theta})|\right|\int_0^{2\pi}\left|\mathrm{Re}\left(\frac{2cse^{i\theta}}{(se^{i\theta}-re^{i\varphi}-c)(se^{i\theta}-re^{i\varphi})}\right)\right|\frac{d\varphi}{2\pi}\frac{d\theta}{2\pi}\\ &\leq\frac{2|c|s}{(s-r-|c|)(s-r)^{1-\delta}}\int_0^{2\pi}\left|\log|f(se^{i\theta})|\right|\int_0^{2\pi}\frac{1}{|se^{i\theta}-re^{i\varphi}|^{\delta}}\frac{d\varphi}{2\pi}\frac{d\theta}{2\pi}.\end{aligned}\tag{5.4}$$

By the change of variables $\varphi' = \theta - \varphi$, we have

$$\begin{aligned}\int_0^{2\pi} \frac{1}{|se^{i\theta} - re^{i\varphi}|^\delta} \frac{d\varphi}{2\pi} &= \int_0^{2\pi} \frac{1}{|se^{i(\theta-\varphi)} - r|^\delta} \frac{d\varphi}{2\pi} = -\int_\theta^{\theta-2\pi} \frac{1}{|se^{i\varphi'} - r|^\delta} \frac{d\varphi'}{2\pi} \\ &= \int_0^{2\pi} \frac{1}{|se^{i\varphi'} - r|^\delta} \frac{d\varphi'}{2\pi} \leq \frac{2\pi}{s^\delta(1-\delta)}\end{aligned}$$

(see, e.g., [11, p. 89] for the last inequality). Hence (5.4) becomes

$$S_1(r) \leq \frac{4\pi|c|}{(1-\delta)(s-r-|c|)} \cdot \left(\frac{s}{s-r}\right)^{1-\delta} \left(m(s,f) + m\left(s, \frac{1}{f}\right)\right). \tag{5.5}$$

Moreover, since

$$\begin{aligned}\int_0^{2\pi} \log^+ \left|1 + \frac{c}{re^{i\varphi} - d}\right| \frac{\varphi}{2\pi} &\leq \frac{1}{\delta} \int_0^{2\pi} \log^+ \left|1 + \frac{c}{re^{i\varphi} - d}\right|^\delta \frac{\varphi}{2\pi} \\ &\leq \frac{1}{\delta} \int_0^{2\pi} \left|\frac{c}{re^{i\varphi} - d}\right|^\delta \frac{\varphi}{2\pi} \leq \frac{2\pi|c|^\delta}{\delta(1-\delta)r^\delta}\end{aligned}$$

for any $d \in \mathbb{C}$, it follows that

$$S_2(r) \leq \frac{8\pi|c|^\delta}{\delta(1-\delta)r^\delta} \left(n(s,f) + n\left(s, \frac{1}{f}\right)\right). \tag{5.6}$$

The assertion follows by combining the inequalities (5.2), (5.5) and (5.6). □

We will now extend Lemma 5.1 to several complex variables. The basic idea is to combine a method, which Biancofiore and Stoll refer to as "fiber integration" [2] (see also [34]) with Lemma 5.1. For the sake of brevity we adopt the notation

$$\begin{aligned} m_f(r,\infty,0) &:= m(r,f) + m\left(r, \frac{1}{f}\right), \\ n_f(r,\infty,0) &:= n_f(r,\infty) + n_f(r,0). \end{aligned}$$

Lemma 5.2. *Let f be a non-constant meromorphic function in $\mathbb{C}^n$ such that $f(0) \neq 0, \infty$, let $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$, let $0 < \delta < 1$, and denote $\tilde{c}_j := (0, \ldots, 0, c_j, 0, \ldots, 0)$. Then*

$$\begin{aligned}\int_{\partial B_n(r)} \log^+ \left|\frac{f(z+\tilde{c}_j)}{f(z)}\right| \sigma_n(z) &\leq \frac{8\pi|c_j|^\delta C}{\delta(1-\delta)} \left(\frac{R}{r}\right)^{2n-2} \frac{n_f(R,\infty,0)}{r^\delta} \\ &+ \frac{4\pi|c_j|}{1-\delta} \left(\frac{R}{r}\right)^{2n-2} \left(\frac{R}{R-(r+|c_j|)}\right) \left(\frac{R}{R-r}\right)^{1-\delta} \frac{m_f(R,\infty,0)}{\sqrt{R^2-r^2}}\end{aligned}$$

for all $R > r + |c_j| > |c_j|$.

Proof. Let $r > 0$, and let h be a function on $\partial B_n(r)$ such that $h\sigma_n$ is integrable over $\partial B_n(r)$. Then, according to [2, Lemma 3.1],

$$\int_{\partial B_n(r)} h(z)\sigma_n(z) = \frac{1}{r^{2n-2}} \int_{\overline{B}_{n-1}(r)} \int_{\partial B_1(p_r(w))} h(w,\zeta)\sigma_1(\zeta)\rho_{n-1}(w), \tag{5.7}$$

where $p_r(w) = \sqrt{r^2 - |w|^2}$. Write $f_{[w]}(z) = f(w,z)$ for $w \in \mathbb{C}^{n-1}$. By applying (5.7) with $h(z) = \log^+ |f(z+\tilde{c}_j)/f(z)|$, we obtain

(5.8)
$$\int_{\partial B_n(r)} \log^+ \left| \frac{f(z+\tilde{c}_j)}{f(z)} \right| \sigma_n(z) = \frac{1}{r^{2n-2}} \int_{\overline{B}_{n-1}(r)} \int_{\partial B_1(p_r(w))} \log^+ \left| \frac{f_{[w]}(\zeta + c_j)}{f_{[w]}(\zeta)} \right| \sigma_1(\zeta)\rho_{n-1}(w).$$

Since $p_R(w) > p_r(w) + |c_j|$ whenever $R > r + |c_j|$, Lemma 5.1, applied with (5.8), implies that

(5.9)
$$\begin{aligned} &\int_{\partial B_n(r)} \log^+ \left| \frac{f(z+\tilde{c}_j)}{f(z)} \right| \sigma_n(z) \\ &\leq \frac{1}{r^{2n-2}} \int_{\overline{B}_{n-1}(r)} \left(\frac{4\pi |c_j|}{(1-\delta)(p_R(w) - p_r(w) - |c|)} \cdot \left(\frac{p_R(w)}{p_R(w) - p_r(w)} \right)^{1-\delta} \right. \\ &\qquad \left. \times\, m_{f_{[w]}}(p_R(w), \infty, 0) \right) \rho_{n-1}(w) \\ &\quad + \frac{1}{r^{2n-2}} \int_{\overline{B}_{n-1}(r)} \frac{8\pi |c_j|^\delta}{\delta(1-\delta)p_r(w)^\delta} n_{f_{[w]}}(p_R(w), \infty, 0)\rho_{n-1}(w) \\ &=: I_m + I_n \end{aligned}$$

for all $R > r + |c_j|$.

We will now proceed to estimate terms I_m and I_n separately, starting with I_m. Since $p_r(w) + |c_j| \leq p_{r+|c_j|}(w)$ for all $r > 0$, and since $p_R(w) \geq \sqrt{R^2 - r^2}$ and $p_r(w) \leq r \cdot p_R(w)/R$ for all $w \in \overline{B}_{n-1}(r)$, it follows that

$$\frac{1}{p_R(w) - p_r(w) - |c_j|} \leq \frac{1}{p_R(w)\left(1 - \frac{p_{r+|c_j|}(w)}{p_R(w)}\right)} \leq \frac{R}{(R - (r+|c_j|))\sqrt{R^2 - r^2}}$$

and

$$\left(\frac{p_R(w)}{p_R(w) - p_r(w)} \right)^{1-\delta} \leq \left(\frac{R}{R-r} \right)^{1-\delta}.$$

Therefore

$$I_m \leq \left(\frac{R}{R - (r+|c_j|)} \right) \frac{4\pi |c_j| r^{2-2n}}{(1-\delta)\sqrt{R^2 - r^2}} \left(\frac{R}{R-r} \right)^{1-\delta} \int_{\overline{B}_{n-1}(R)} m_{f_{[w]}}(p_R(w), \infty, 0)\rho_{n-1}(w).$$

Since

$$\frac{1}{R^{2n-2}} \int_{\overline{B}_{n-1}(R)} m_{f_{[w]}}(p_R(w), \infty, 0)\rho_{n-1}(w) = m_f(R, \infty, 0)$$

by equation (5.7), we finally have

$$I_m \leq \frac{4\pi |c_j|}{1-\delta} \left(\frac{R}{r} \right)^{2n-2} \left(\frac{R}{R - (r+|c_j|)} \right) \left(\frac{R}{R-r} \right)^{1-\delta} \frac{m_f(R, \infty, 0)}{\sqrt{R^2 - r^2}} \tag{5.10}$$

for all $R > r + |c_j|$.

Consider now the term I_n. We may assume, without loss of generality, that $\delta > 1/4$. Then, denoting the integer part of a real number x by $[x]$, it follows that $q(\delta) := [1/(1-\sqrt{\delta})] \geq 2$, and so Hölder's inequality yields

(5.11)
$$\int\limits_{\overline{B}_{n-1}(r)} \frac{n_{f_{[w]}}(p_R(w),\infty,0)}{p_r(w)^{\delta}} \rho_{n-1}(w) \leq \left(\int\limits_{\overline{B}_{n-1}(r)} n_{f_{[w]}}^{q(\delta)}(p_R(w),\infty,0)\rho_{n-1}(w) \right)^{\frac{1}{q(\delta)}} \times \left(\int\limits_{\overline{B}_{n-1}(r)} p_r(w)^{-\frac{\delta q(\delta)}{q(\delta)-1}} \rho_{n-1}(w) \right)^{\frac{q(\delta)-1}{q(\delta)}}.$$

Since $0 < \frac{\delta q(\delta)}{q(\delta)-1} < 1$, it follows that

$$\int\limits_{\overline{B}_{n-1}(r)} p_r(w)^{-\frac{\delta q(\delta)}{q(\delta)-1}} \rho_{n-1}(w) \leq C r^{2n-2-\frac{\delta q(\delta)}{q(\delta)-1}} \tag{5.12}$$

where

$$C = \int\limits_{\overline{B}_{n-1}(1)} \frac{1}{(1-\xi^2)^{\frac{\delta q(\delta)}{2(q(\delta)-1)}}} \rho_{n-1}(\xi).$$

On the other hand, by [26, Hilfssatz 7] applied with a weighted counting function $\widetilde{n}$ such that $\widetilde{n}(r) = n_f^{q(\delta)}(R,\infty,0)$, it follows that

$$\begin{aligned} n_f^{q(\delta)}(R,\infty,0) &= \widetilde{n}(R) \\ &\geq \frac{1}{R^{2n-2}} \int\limits_{\overline{B}_{n-1}(R)} \widetilde{n}_{f_{[w]}}(p_R(w))\rho_{n-1}(w) \\ &\geq \frac{1}{R^{2n-2}} \int\limits_{\overline{B}_{n-1}(r)} n_{f_{[w]}}^{q(\delta)}(p_R(w),\infty,0)\rho_{n-1}(w). \end{aligned} \tag{5.13}$$

Finally, by (5.11), (5.12) and (5.13), we have

$$I_n \leq \frac{8\pi|c_j|^{\delta} C}{\delta(1-\delta)} \left(\frac{R}{r}\right)^{2n-2} \frac{n_f(R,\infty,0)}{r^{\delta}}. \tag{5.14}$$

The assertion of the lemma follows by combining the estimates (5.9), (5.10) and (5.14). □

6. Proof of Theorem 3.1

Since

$$n_f(r,\infty,0) \leq \frac{R}{R-r}\left(N(R,f) + N\left(R,\frac{1}{f}\right)\right)$$

for all $R > r$, it follows by the first main theorem (3.2) and Lemma 5.2 that there exists a positive constant K_1, depending only on c_j and δ, such that

$$\int\limits_{\partial B_n(r)} \log^+ \left|\frac{f(z+\tilde{c}_j)}{f(z)}\right| \sigma_n(z) \leq K_1 K_2(r,R)\left(T(R,f) + \log\frac{1}{|f(0)|}\right) \tag{6.1}$$

for all $R > r + |c_j| > |c_j|$, where

$$K_2(r,R) = \left(\frac{R}{r}\right)^{2n-2}\left(\frac{R}{R-(r+|c_j|)}\right)\left(\frac{1}{\sqrt{R^2-r^2}}\left(\frac{R}{R-r}\right)^{1-\delta}+\frac{1}{r^\delta}\right). \tag{6.2}$$

Let $\xi(x)$ and $\phi(r)$ be positive, nondecreasing, continuous functions defined for $e \leq x < \infty$ and $r_0 \leq r < \infty$, respectively, where r_0 is such that $T(r+|c|,f) \geq e$ for all $r \geq r_0$. Then by Hinkkanen's Borel type growth lemma [21, Lemma 3] (see also [6, Lemma 3.3.1])

$$T\left(r+|c|+\frac{\phi(r+|c|)}{\xi(T(r+|c|,f))},f\right) \leq 2T(r+|c|,f)$$

for all r outside of a set E satisfying

$$\int_{E\cap[r_0,s]}\frac{dr}{\phi(r)} \leq \frac{1}{\xi(e)}+\frac{1}{\log 2}\int_e^{T(s+|c|,f)}\frac{dx}{x\xi(x)}$$

where $s < \infty$. Therefore, by choosing $\phi(r) = r$ and $\xi(x) = (\log x)^{1+\tilde{\varepsilon}}$ with $\tilde{\varepsilon} > 0$, and defining

$$R = r+|c_j|+\frac{r+|c_j|}{(\log T(r+|c_j|,f))^{1+\tilde{\varepsilon}}}, \tag{6.3}$$

we have

$$T(R,f) = T\left(r+|c_j|+\frac{\phi(r+|c_j|)}{\xi(T(r+|c_j|,f))},f\right) \leq 2T(r+|c_j|,f) \tag{6.4}$$

for all r outside of a set E of finite logarithmic measure. By substituting (6.3) and (6.4) into (6.1), we obtain

$$\int_{\partial B_n(r)} \log^+\left|\frac{f(z+\tilde{c}_j)}{f(z)}\right|\sigma_n(z) = o\left(\frac{T(r+|c_j|,f)(\log T(r+|c_j|,f))^{(1+\tilde{\varepsilon})(\frac{5}{2}-\delta)}}{r^\delta}\right) \tag{6.5}$$

where r runs to infinity outside of an exceptional set of finite logarithmic measure. (From now on $E \subset [1,+\infty)$ denotes a set, which is not necessarily the same at each occurrence, but which always has finite logarithmic measure.)

Since the hyper-order of f is $\varsigma(f) = \varsigma$, we have $\log T(r+|c_j|,f) \leq r^{\varsigma+\tilde{\varepsilon}}$ for all r sufficiently large. On the other hand, by [17, Lemma 2.1] (see also [19, Lemma 8.3]), we have $T(r+|c_j|,f) = T(r,f)+o(T(r,f))$ for all r outside of an exceptional set E of finite logarithmic measure. Therefore, (6.5) yields

$$\int_{\partial B_n(r)} \log^+\left|\frac{f(z+\tilde{c}_j)}{f(z)}\right|\sigma_n(z) = o\left(\frac{T(r,f)}{r^{\delta(1+\varsigma)-\frac{5+\varepsilon}{2}\varsigma}}\right), \tag{6.6}$$

where $\varepsilon > 0$ is arbitrary small (and depends only on $\tilde{\varepsilon}$), and $r \notin E$.

In the general case any $c \in \mathbb{C}^n$ can be written as $c = \sum_{j=0}^{n} \tilde{c}_j$ where $\tilde{c}_0 := 0$. Therefore, by (6.6), we have

$$
\begin{aligned}
\int_{\partial B_n(r)} \log^+ \left| \frac{f(z+c)}{f(z)} \right| \sigma_n(z) &= \int_{\partial B_n(r)} \log^+ \prod_{k=1}^{n} \left| \frac{f(z+\sum_{j=0}^{k} \tilde{c}_j)}{f(z+\sum_{j=0}^{k-1} \tilde{c}_j)} \right| \sigma_n(z) \\
&\leq \sum_{k=1}^{n} \int_{\partial B_n(r)} \log^+ \left| \frac{f(z+\sum_{j=0}^{k} \tilde{c}_j)}{f(z+\sum_{j=0}^{k-1} \tilde{c}_j)} \right| \sigma_n(z) \qquad (6.7) \\
&= \sum_{k=1}^{n} o\left(\frac{T(r, f(z+\sum_{j=0}^{k-1} \tilde{c}_j))}{r^{\delta(1+\varsigma)-\frac{5+\varepsilon}{2}\varsigma}} \right)
\end{aligned}
$$

for all $r \notin E$. On the other hand, by [17, Lemma 2.1] (see also [19, Lemma 8.3]) it follows that for any $s > 0$ which does not depend on r we have $N(r+s, f) = N(r, f) + o(N(r, f))$, where $r \notin E$. Hence, by (6.6), we have

$$
\begin{aligned}
T(r, f(z+\tilde{c}_j)) &= m(r, f(z+\tilde{c}_j)) + N(r, f(z+\tilde{c}_j)) \\
&\leq m\left(r, \frac{f(z+\tilde{c}_j)}{f(z)}\right) + m(r, f) + N(r+|c_j|, f) \qquad (6.8) \\
&= T(r, f) + o(T(r, f))
\end{aligned}
$$

for all $r \notin E$. Since $c = \sum_{j=0}^{n} \tilde{c}_j$, it follows by repeated application of (6.8) that

$$
T(r, f(z+c)) = T(r, f) + o(T(r, f)) \qquad (6.9)
$$

where $r \notin E$. Relation (3.3) follows by combining (6.7) and (6.9), and by substituting $\delta = 1 - \varepsilon/(2+2\varsigma)$. □

7. Proof of Theorem 3.3

The first main theorem yields

$$
\begin{aligned}
\sum_{k=1}^{p} m\left(r, \frac{1}{f-a_k}\right) &= \sum_{k=1}^{p} T\left(r, \frac{1}{f-a_k}\right) - \sum_{k=1}^{p} N\left(r, \frac{1}{f-a_k}\right) \qquad (7.1) \\
&= pT(r, f) - N\left(r, \frac{1}{P(f)}\right) + O(1),
\end{aligned}
$$

where

$$
P(f) = \prod_{k=1}^{p} (f - a_k).
$$

By partial fraction decomposition there exist constants $\alpha_k \in \mathbb{C}$ such that

$$
\frac{1}{P(f)} = \sum_{k=1}^{p} \frac{\alpha_k}{f-a_k},
$$

and so, since we have assumed that f is finite at the origin and $f(0) \neq a_j$ for $j = 1, \ldots, q$, Corollary 3.2 yields

$$
m\left(r, \frac{\Delta_c f}{P(f)}\right) \leq \sum_{k=1}^{p} m\left(r, \frac{\Delta_c f}{f-a_k}\right) + O(1) = o\left(\frac{T(r, f)}{r^{\delta}}\right)
$$

for all r outside of an exceptional set of finite logarithmic measure. Therefore,

$$m\left(r,\frac{1}{P(f)}\right)=m\left(r,\frac{\Delta_c f}{P(f)}\frac{1}{\Delta_c f}\right)\leq m\left(r,\frac{1}{\Delta_c f}\right)+o\left(\frac{T(r,f)}{r^\delta}\right) \tag{7.2}$$

outside of an exceptional set. By applying Theorem 4.2, it follows that $pT(r,f)=T(r,P(f))+O(1)$, and so by using the first main theorem and (7.2), Eq. (7.1) becomes

$$\begin{aligned}\sum_{k=1}^{p} m\left(r,\frac{1}{f-a_k}\right) &= m\left(r,\frac{1}{P(f)}\right)+o\left(\frac{T(r,f)}{r^\delta}\right)\\ &\leq m\left(r,\frac{1}{\Delta_c f}\right)+o\left(\frac{T(r,f)}{r^\delta}\right)\\ &= T(r,\Delta_c f)-N\left(r,\frac{1}{\Delta_c f}\right)+o\left(\frac{T(r,f)}{r^\delta}\right),\end{aligned}$$

where r runs to infinity outside of an exceptional set of finite logarithmic measure. Therefore,

$$\begin{aligned}m(r,f)+\sum_{k=1}^{p} m\left(r,\frac{1}{f-a_k}\right) \leq\ & T(r,f)+N(r,\Delta_c f)+m(r,\Delta_c f)\\ &-N\left(r,\frac{1}{\Delta_c f}\right)-N(r,f)+o\left(\frac{T(r,f)}{r^\delta}\right)\end{aligned}$$

outside the exceptional set. Since

$$m(r,\Delta_c f)=m\left(r,f\frac{\Delta_c f}{f}\right)\leq m(r,f)+m\left(r,\frac{\Delta_c f}{f}\right)=m(r,f)+o\left(\frac{T(r,f)}{r^\delta}\right)$$

by Corollary 3.2, it follows that

$$\begin{aligned}m(r,f)+\sum_{k=1}^{p} m\left(r,\frac{1}{f-a_k}\right) \leq\ & 2T(r,f)+N(r,\Delta_c f)-N\left(r,\frac{1}{\Delta_c f}\right)\\ &-2N(r,f)+o\left(\frac{T(r,f)}{r^\delta}\right)\end{aligned}$$

for all r outside of an exceptional set of finite logarithmic measure. □

8. Proof of Theorem 2.1

By composing f with an appropriate Möbius transformation, if necessary, it may be assumed that $a_j\in\mathbb{C}$ and $f(0)\neq a_j$ for $j=1,2,3$. Consider the composition of f with the function $\tau(z)=z+c$. Since, by Theorem 3.1,

$$m(r,f\circ\tau)=m(r,f)+o(T(r,f)),$$

and by Theorem 4.3,

$$T(r,f\circ\tau)=T(r,f)+o(T(r,f))$$

for all r outside of an exceptional set of finite logarithmic measure, it follows that

$$\begin{aligned} N(r,\Delta_c f) &\leq N(r, f\circ\tau) + N(r,f) \\ &= T(r, f\circ\tau) + T(r,f) - m(r,f\circ\tau) - m(r,f) \\ &= 2T(r,f) - 2m(r,f) + o(T(r,f)) \\ &= 2N(r,f) + o(T(r,f)) \end{aligned}$$

outside of an exceptional set. Therefore, by Theorem 3.3 it follows that either

$$T(r,f) \leq \sum_{k=1}^{3} N\left(r, \frac{1}{f-a_k}\right) - N\left(r, \frac{1}{f\circ\tau - f}\right) + o(T(r,f)) \tag{8.1}$$

for all r outside of a small exceptional set, or $f\circ\tau \equiv f$. Since by the assumption $\tau(f^{-1}(\{a_j\})) \subset f^{-1}(\{a_j\})$ for $j = 1, 2, 3$, it follows that

$$\sum_{k=1}^{3} N\left(r, \frac{1}{f-a_k}\right) \leq N\left(r, \frac{1}{f\circ\tau - f}\right)$$

and thus (8.1) leads to a contradiction. Therefore $f \equiv f\circ\tau$. □

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