NON-DECREASING FUNCTIONS, EXCEPTIONAL SETS AND GENERALISED BOREL LEMMAS

R. G. HALBURD AND R. J. KORHONEN

Abstract. The Borel lemma says that any positive non-decreasing continuous function T satisfies $T(r+1/T(r)) \leq 2T(r)$ outside of a possible exceptional set of finite linear measure. This lemma plays an important role in the theory of entire and meromorphic functions, where the increasing function T is either the logarithm of the maximum modulus function, or the Nevanlinna characteristic. As a result, exceptional sets appear throughout Nevanlinna theory, in particular in Nevanlinna's second main theorem. They appear also when using the existence of sufficiently many finite-order meromorphic functions as a criterion to detect difference equations of Painlevé type.

In this paper, we consider generalisations of Borel's lemma. Conversely, we consider ways in which certain inequalities can be modified so as to remove exceptional sets. All results discussed are presented from the point of view of real analysis.

1. Introduction

Non-decreasing functions appear in many contexts in analysis, in particular they appear naturally in the theory of entire and meromorphic functions. Much information about the value distribution of an entire function f is encoded in the asymptotic behaviour of the real-valued non-decreasing maximum modulus function $M_f(r) := \max_{|z|=r} |f(z)|$ as $r \to \infty$. In the case of a meromorphic function, the role of $\log M_f(r)$ is played by the Nevanlinna characteristic $T_f(r)$, which contains information about the distribution of poles of f in $|z| \leq r$, as well as information about how large $|f|$ is on the circle $|z| = r$. The asymptotic behaviour of the non-decreasing function $T_f(r)$ contains information regarding the number of asymptotic directions of f as well as the form of certain types of product representations (the Weierstrass and Hadamard factorizations.)

Recall that for any non-rational meromorphic function f, Picard's Great Theorem says that f takes every value in $\mathbb{C} \cup \{\infty\}$ infinitely many times, with at most two exceptions. The centrepiece of Nevanlinna theory is Nevanlinna's second main theorem [7], which is a vast generalization and quantification of Picard's theorem for meromorphic functions. In 1943, Hermann Weyl referred to the appearance of [7] as *"...one of the few great mathematical events in our century"* [9]. Nevanlinna's second main theorem provides a useful bound on $T_f(r)$ in terms of quantities that are readily interpreted. However, this bound only holds for r outside of some possible exceptional set E of finite linear measure (i.e., $\int_E \mathrm{d}r < \infty$.) The origin of this exceptional set is in an estimate of the logarithmic derivative f'/f, which in turn uses the following lemma due to Borel [1] (see also Hayman [4].) Exceptional sets

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appear also when using Nevanlinna theory to detect difference equations of Painlevé type [3]. Borel's lemma has been generalised by Nevanlinna [8] and Hinkkanen [6] (see also [2, Lemma 3.3.1].)

Lemma 1.1. *(Classical Borel Lemma) Let T be a continuous non-decreasing function on $[r_0, \infty)$ for some r_0 such that $T(r_0) \geq 1$. Then*

$$T\left(r + \frac{1}{T(r)}\right) \leq 2T(r)$$

for all r outside of a possible exceptional set E of (linear) measure at most 2 (i.e., $\int_E \mathrm{d}r \leq 2$.)

Exceptional sets appear throughout Nevanlinna theory. The purpose of the present paper is to explore generalized Borel lemmas and their associated exceptional sets in a purely real setting, independent of (but largely motivated by) Nevanlinna theory. We do so for two reasons. The first is to try to develop a unified approach to many of the results concerning exceptional sets in Nevanlinna theory. To this end we wish to emphasise the common elements of these results, which lie in real rather than complex analysis. The second reason is the authors' belief that these results, which are so important in Nevanlinna theory, should also be of value in other areas of mathematics in which non-decreasing functions naturally arise.

In section 2 we discuss a generalisation of lemma 1.1 and we show that the estimate for the size of the exceptional set (in terms of an arbitrary measure) is the best possible. We then consider a number of applications. In section 3 we consider non-decreasing functions f and g that satisfy inequalities of the form $f(r) \leq g(r)$ outside of some exceptional set E. We show how sufficiently small exceptional sets can be removed by modifying the argument of g so that it is larger than r. As applications we consider functions of finite order and functions of finite type. The order of a positive function T is defined to be

$$\rho(T) := \limsup_{r\to\infty} \frac{\log T(r)}{\log r}.$$

The order is always well-defined but it may be infinite. If $0 < \rho(T) < \infty$, then the type of T is defined to be

$$\tau(T) := \limsup_{r\to\infty} \frac{T(r)}{r^\rho},$$

which again may be infinite. We show that if T is a positive continuous non-decreasing function of finite order ρ and type τ, where $0 < \rho < \infty$ and $0 < \tau < \infty$, then for any $\epsilon > 0$,

$$(\tau - \epsilon)r^\rho \leq T(r) \leq (\tau + \epsilon)r^\rho$$

on a set of infinite linear measure.

We also consider other measures of growth and other ways of describing the size of exceptional sets. Let $\log^{\circ 1} x := \log x$ and for $n \geq 2$ define the iterated logarithm by $\log^{\circ n} x := \log^{\circ(n-1)} x$. For $n \geq 1$, the n-order of a (sufficiently large) function T is defined to be

$$\rho_n(T) := \limsup_{r\to\infty} \frac{\log^{\circ n} T(r)}{\log r}.$$

The case $n = 1$ gives the usual order $\rho_1 = \rho$. The case $n = 2$ is usually referred to as the *hyperorder* of T.

2. A generalised Borel lemma

We begin by presenting a generalisation of lemma 1.1 (and of [2, Lemma 3.3.1]). In the following, $F^{\circ k}$ means F composed with itself k times.

Lemma 2.1. *(The generalised Borel lemma) Let T and μ be positive continuous functions of r for $r \in [r_0, \infty)$ for some r_0. Suppose further that T is non-decreasing and that μ is differentiable and increasing. Let ψ and F be positive and continuous on $[T(r_0), \infty)$. Suppose that on $[T(r_0), \infty)$, ψ is non-increasing, F is non-decreasing and $\lim_{k\to\infty} F^{\circ k}(T(r_0)) = \infty$. Let $s(r) = \mu^{-1}(\mu(r) + \psi(T(r)))$ and define*

$$E := \{r \geq r_0 \, : \, T(s(r)) \geq F(T(r))\}. \tag{1}$$

Then

$$\int_{r \in E \cap [r_0, r)} \mathrm{d}\mu(r) \leq \sum_{n=1}^{\nu_r} \psi\left(F^{\circ\{n-1\}}\left(T(r_0)\right)\right), \tag{2}$$

where ν_r is the largest integer such that

$$F^{\circ\{\nu_r - 1\}}(T(r_0)) \leq T(r). \tag{3}$$

This lemma is presented in a very general form but we will soon specialise to some important cases. The classical Borel lemma corresponds to the case $F(x) = 2x$, $\mu(r) = r$ and $\psi(x) = 1/x$. The set E in lemma 2.1 corresponds to the exceptional set in lemma 1.1. Other choices of ψ that give stronger estimates with a larger exceptional set include $\psi(x) = 1/x^\epsilon$ and

$$\psi(x) = 1/((\log x)(\log\log x)(\log\log\log x)\cdots(\log^{\circ\{n-1\}} x)(\log^{\circ n} x)^{1+\epsilon}),$$

where $\epsilon > 0$.

Proof:
If E is empty there is nothing to prove, so we suppose that E is non-empty. We define two (possibly finite) sequences (r_n) and (s_n) by induction. Let $r_1 = \min(E \cap [r_0, \infty))$. Assuming that we have defined r_n for some integer n, we define $s_n = s(r_n)$. If $E \cap [s_n, \infty) \neq \emptyset$, then we let $r_{n+1} = \min(E \cap [s_n, \infty))$.

Next we show that if the sequence (r_n) has infinitely many terms, then $\lim_{n\to\infty} r_n = \infty$. Suppose that this is not the case. Then since $r_{n+1} \geq s_n \geq r_n$, it follows that (r_n) has a finite limit $r_\infty < \infty$. Then for all n,

$$\mu(r_{n+1}) - \mu(r_n) \geq \mu(s_n) - \mu(r_n) \geq \psi(T(r_n)) \geq \psi(T(r_\infty)).$$

Since $\psi(T(r_\infty)) > 0$ and independent of n, it follows that $\lim_{n\to\infty} \mu(r_n) = \infty$. But the continuity of μ implies that $\lim_{n\to\infty} \mu(r_n) = \mu(r_\infty) < \infty$. So we have shown that either r_n is defined for only finitely many n or $\lim_{n\to\infty} r_n = \infty$. It follows that

$$E \cap [r_0, r) \subseteq \bigcup_{n=1}^{N} [r_n, s_n],$$

where N is the largest integer such that $r_N \leq r$. Therefore

$$\int_{r \in E \cap [r_0, r)} \mathrm{d}\mu(r) \leq \sum_{n=1}^{N} \int_{r_n}^{s_n} \mu'(r)\,\mathrm{d}r \leq \sum_{n=1}^{N} \psi(T(r_n)). \tag{4}$$

Now

$$T(r_n) \geq T(s_{n-1}) \geq F(T(r_{n-1})) \geq F \circ F(T(r_{n-2}))$$

$$\geq \cdots \geq F^{\circ\{n-1\}}(T(r_1)) \geq F^{\circ\{n-1\}}(T(r_0)). \tag{5}$$

In particular,

$$T(r) \geq T(r_N) \geq F^{\circ\{N-1\}}(T(r_0)).$$

Hence $N \leq \nu_r$. The proposition is proved on substituting the inequality (5) into (4). □

Lemma 2.1 shows that if

$$\sum_{n=1}^{\infty} \psi\left(F^{\circ\{n-1\}}\left(T(r_0)\right)\right) = L < \infty, \tag{6}$$

then $T(s(r)) < F(T(r))$ for all r outside a possible exceptional set E of μ-measure no greater than L. The following example shows that this is optimal.

Example 1. *Let μ and ψ be positive continuous functions of r for $r \in [r_0, \infty)$ for some $r_0 \leq 1$, and let F be non-decreasing and continuous on $[1, \infty)$ such that $F(1) \geq 1$. Moreover, let $\varepsilon > 0$ and let $(r_n)_{n=1}^{\infty}$ be a sequence of points such that $\mu(r_j) - \mu(r_{j-1}) \geq \psi(F^{\circ\{j-1\}}(1)) + \varepsilon/2^j$ for all $j \in \mathbb{N}$, and $r_j \to \infty$ as $j \to \infty$. Define T as follows:*

$$T(x) = \begin{cases} F^{\circ\{j-1\}}(1), & x \in [r_{j-1}, \mu^{-1}\left(\mu(r_j) - \frac{\varepsilon}{2^j}\right)] \\ \frac{2^j}{\varepsilon}(F^{\circ j}(1) - F^{\circ\{j-1\}}(1))(x - r_j) + F^{\circ j}(1), & x \in [\mu^{-1}\left(\mu(r_j) - \frac{\varepsilon}{2^j}\right), r_j], \end{cases}$$

where $j \in \mathbb{N}$. If $s(r) = \mu^{-1}(\mu(r) + \psi(T(r)))$ for $r \in [r_0, \infty)$, then it follows by the definition of T that the set E of points r such that

$$T(s(r)) \geq F(T(r))$$

contains all $r \in [\mu^{-1}(\mu(r_j) - \psi(F^{\circ\{j-1\}}(1))), \mu^{-1}(\mu(r_j) - \varepsilon/2^j)]$, where $j \in \mathbb{N}$. Since $T(r_0) = 1$, we have that the μ-measure of E is at least

$$\begin{aligned} \int_{r \in E \cap [r_0, \infty)} \mathrm{d}\mu(r) &\geq \sum_{j=1}^{\infty} \mu\left(r_j\right) - \frac{\varepsilon}{2^j} - \left(\mu(r_j) - \psi(F^{\circ\{j-1\}}(1))\right) \\ &= \sum_{j=1}^{\infty} \psi\left(F^{\circ\{j-1\}}\left(T(r_0)\right)\right) - \varepsilon. \end{aligned}$$

Therefore the constant L in (6) cannot be replaced by $L - \epsilon$ for any $\epsilon > 0$.

The most common applications of lemma 2.1 involve the choice $F(x) = Cx$ for some constant $C > 1$. In this case equation (3) gives

$$\nu_r = 1 + \left[\log_C \frac{T(r)}{T(r_0)}\right],$$

where $[\lambda]$ denotes the largest integer not exceeding λ. The inequality (2) then becomes

$$\int_{r \in E \cap [r_0, r)} \mathrm{d}\mu(r) \leq \sum_{n=1}^{\nu_r} \psi\left(C^{n-1} T(r_0)\right) \leq \psi(T(r_0)) + \int_0^{\nu_r - 1} \psi(C^x T(r_0)) \mathrm{d}x$$

$$\leq T(r_0) + \frac{1}{\log C} \int_{T(r_0)}^{T(r)} \psi(u) \frac{\mathrm{d}u}{u}. \tag{7}$$

The next theorem follows immediately.

Theorem 2.2. *Let T and μ be positive continuous functions of r for $r \in [r_0, \infty)$ for some r_0. Suppose further that T is non-decreasing and that μ is differentiable and increasing. Let ψ be a positive, continuous and nonincreasing function on $[T(r_0), \infty)$. Let $s(r) = \mu^{-1}(\mu(r) + \psi(T(r)))$ and let $C > 1$. If*

$$\int_k^\infty \psi(u) \frac{\mathrm{d}u}{u} < \infty$$

for some k then

$$T(s(r)) \leq CT(r)$$

outside of a possible exceptional set of finite μ-measure.

The upper and lower logarithmic densities of a subset $E \subset \mathbb{R}$ are given by

$$\limsup_{r\to\infty} \frac{1}{\log r} \int_{E\cap[r_0,r)} \frac{\mathrm{d}r}{r} \text{ and } \liminf_{r\to\infty} \frac{1}{\log r} \int_{E\cap[r_0,r)} \frac{\mathrm{d}r}{r}$$

respectively.

Corollary 2.3. *Let T be a positive continuous nondecreasing function on $[r_0, \infty)$ for some r_0 such that $T(r_0) \geq \exp^{\circ\{n-1\}}(0)$. Let $A > 0$ and $C > 1$ be constants and let n be a positive integer. Define*

$$\begin{aligned} \sigma(u) &:= \exp\left(Au\frac{\mathrm{d}}{\mathrm{d}u}\log^{\circ n} u\right) \\ &= \begin{cases} \exp(A), & n = 1; \\ \exp\left(A\left((\log u)(\log\log u)\cdots(\log^{\circ\{n-1\}} u)\right)^{-1}\right), & n \geq 2. \end{cases} \end{aligned}$$

Let

$$E := \{r \geq r_0 \,:\, T\left(r\,\sigma\left(T(r)\right)\right) \geq CT(r)\}. \tag{8}$$

Then

(1) *If T has finite n-order*

$$\limsup_{r\to\infty} \frac{\log^{\circ n} T(r)}{\log r} = \rho,$$

then the upper logarithmic density of E is at most $\dfrac{A\rho}{\log C}$.

(2) *If T has finite lower n-order*

$$\liminf_{r\to\infty} \frac{\log^{\circ n} T(r)}{\log r} = \lambda,$$

then the lower logarithmic density of E is at most $\dfrac{A\lambda}{\log C}$.

The $n = 1$ case of corollary 2.3 is essentially the same as lemma 4 in Hayman [5]. Hayman's result is expressed in terms of the derivative of a meromorphic function but his proof shows that the result is real analytic in nature.

Proof: Apply equation (7) with $\mu(r) = \log r$ and $\psi(u) = Au\frac{\mathrm{d}}{\mathrm{d}u}\log^{\circ n} u$. □

Corollary 2.4. *Let T be a non-decreasing function and let $\alpha > 0$ and $C > 1$ be constants. If*

$$T(r+\alpha) \geq CT(r),$$

on a set of infinite logarithmic measure, then the hyperorder of T is at least one, i.e.

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} \geq 1.$$

Proof:
Suppose to the contrary that the hyperorder of T is less than one. Then for sufficiently small $\epsilon > 0$ and sufficiently large r,

$$(\log T(r))^{1+\epsilon} \leq r.$$

Let

$$E := \{r \,:\, T(r+\alpha) \geq CT(r)\}.$$

Then

$$E \subseteq \widetilde{E} := \{r \,:\, T\left(r\left[1 + \frac{\alpha}{(\log T(r))^{1+\epsilon}}\right]\right) \geq CT(r)\}.$$

So, by applying theorem 2.2 with $\mu(r) = \log r$ and

$$\psi(u) = \log\left(1 + \frac{\alpha}{(\log u)^{1+\epsilon}}\right),$$

it follows that

$$\int_{E\cap[1,\infty)} \frac{\mathrm{d}r}{r} \leq \int_{\widetilde{E}\cap[1,\infty)} \frac{\mathrm{d}r}{r} < \infty,$$

which is a contradiction. □

Another interesting choice for F in lemma 2.1 is $F(x) = x^C$, for some constant $C > 1$. In this case, if $T(r_0) > \mathrm{e}$,

$$\nu_r = 1 + \left[\frac{\log\log T(r) - \log\log T(r_0)}{\log C}\right].$$

In this case

$$\int_{E\cap[r_0,r)} \mu'(r)\,\mathrm{d}r \leq \psi(T(r_0)) + \frac{1}{\log C}\int_{T(r_0)}^{T(r)} \psi(u)\frac{\mathrm{d}u}{u\,\log u}.$$

So, for example, we get the following analogue of Hayman's result. If T has hyperorder ρ then

$$T(\alpha r) \leq T(r)^C,$$

outside a set of logarithmic density at most $\rho\log\alpha/\log C$. Similarly, the results that we have described in this paper for $F(x) = Cx$ are easily extended to the obvious analogues for $F(x) = x^C$.

3. Removing exceptional sets

Lemma 3.1. *Let μ be a positive increasing differentiable function of r for all r greater than some r_0 and let f and g be non-decreasing functions for all $r > r_0$. Furthermore, suppose that $f(r) \le g(r)$ for all $r \in (r_0, \infty) \setminus E$, where the exceptional set $E \subset (r_0, \infty)$ satisfies*

$$\int_{r \in E} \mathrm{d}\mu = \int_E \mu'(r)\, \mathrm{d}r < \infty. \tag{9}$$

Then given $\epsilon > 0$, there is an $\hat{r} \ge r_0$ such that for all $r > \hat{r}$, $f(r) \le g(s(r))$, where $s(r) = \mu^{-1}(\mu(r) + \epsilon)$.

Proof:
Suppose that there is an infinite sequence $(r_n)_{n=1}^{\infty} \subset (r_0, \infty)$ such that $r_{n+1} \ge s_n := s(r_n)$ and $(r_n, s_n) \subset E$, for all $n \in \mathbb{N}$. Then

$$\int_E \mu'(r)\, \mathrm{d}r \ge \sum_{m=1}^{\infty} \int_{r_n}^{s_n} \mu'(r)\, \mathrm{d}r = \sum_{n=1}^{\infty} \epsilon = \infty,$$

which contradicts the finite measure condition (9). Therefore, there must be a number $\hat{r} \ge r_0$ such that for any $r > \hat{r}$, there exists $t \in (r, s(r)) \setminus E$. Since f and g are non-decreasing, it follows that

$$f(r) \le f(t) \le g(t) \le g(s(r)). \qquad \square$$

Theorem 3.2. *Let f and g be positive non-decreasing functions of r for all r greater than some r_0. Let μ be a positive differentiable increasing function, fix $\epsilon > 0$ and set $s(r) = \mu^{-1}(\mu(r) + \epsilon)$ for all $r > r_0$. Suppose that*

$$\limsup_{r \to \infty} \frac{g(s(r))}{g(r)} = 1$$

and that

$$\limsup_{r \to \infty} \frac{f(r)}{g(r)} = \lambda,$$

for some nonzero finite λ. Then for any $\delta > 0$,

$$\left| \frac{f(r)}{g(r)} - \lambda \right| < \delta$$

on a set F of infinite μ-measure (i.e., such that $\int_{r \in F} d\mu(r) = \infty$).

Proof:
It follows from the definition of lim sup that there is an $r_1 \ge r_0$ such that $f(r) \le (\lambda + \delta) g(r)$ for all $r > r_1$. Now suppose that $f(r) \le (\lambda - \delta) g(r)$ outside of a set of finite μ-measure. From lemma 3.1 with $\tilde{f} = f$ and $\tilde{g} = (\lambda - \delta) g$, we see that $f(r) \le (\lambda - \delta) g(s(r))$ for all sufficiently large r. Hence

$$\limsup_{r \to \infty} \frac{f(r)}{g(r)} \le (\lambda - \delta) \limsup_{r \to \infty} \frac{g(s(r))}{g(r)} = \lambda - \delta < \lambda,$$

which contradicts the definition of λ. So $f(r) \le (\lambda - \delta) g(r)$ outside of a set of infinite μ-measure. $\square$

Corollary 3.3. *Let T be a positive non-decreasing function of order ρ, where $0 < \rho < \infty$. Then for any $\epsilon > 0$,*

$$r^{\rho-\epsilon} \leq T(r) \leq r^{\rho+\epsilon}$$

on a set of infinite logarithmic measure.

Proof:
Apply theorem 3.2 using $f(r) = \log T(r)$, $g(r) = \log r$, $\mu(r) = \log r$, $\delta = \epsilon$ and $\lambda = \rho$. □

Corollary 3.4. *Let T be a positive non-decreasing function of finite order ρ and type τ, where $0 < \rho < \infty$ and $0 < \tau < \infty$. Then for any $\epsilon > 0$,*

$$(\tau - \epsilon)r^{\rho} \leq T(r) \leq (\tau + \epsilon)r^{\rho}$$

on a set of infinite linear measure.

Proof:
Apply theorem 3.2 using $f(r) = T(r)$, $g(r) = r^{\rho}$, $\mu(r) = r$, $\delta = \epsilon$ and $\lambda = \tau$. □

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