

Chebyshev polynomial approximations for some hypergeometric systems

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Abstract

The hypergeometric type differential equations of the second order with polynomial coefficients and their systems are considered. The realization of the LANCZOS Tau Method with minimal residue is proposed for the approximate solution of the second order differential equations with polynomial coefficients. The scheme of Tau method is extended for the systems of hypergeometric type differential equations. A Tau method computational scheme is applied to the approximate solution of a system of differential equations related to the differential equation of hypergeometric type. The case of the discrete systems may be considered also. Various vector perturbations are discussed. Our choice of the perturbation term is a shifted CHEBYSHEV polynomial with a special form of selected transition and normalization. The minimality conditions for the perturbation term are found for one equation. They are sufficiently simple for the verification in a number of important cases. Several approaches for the approximation of kernels of KONTOROVICH–LEBEDEV integral transforms - modified BESSEL functions of the second kind with pure imaginary order and with complex order are elaborated. The codes of the evaluation are constructed and tables of the modified BESSEL functions $K_{1/2+i\beta}(x)$ are published. The advantages of discussed algorithms in accuracy and timing are shown. The effective applications for the solution of some mixed boundary value problems in wedge domains are given.

Key words: LANCZOS *Tau method*, BESSEL *functions*, KONTOROVITCH–LEBEDEV *integral transforms*, CHEBYSHEV *polynomials*, *dual integral equation*

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1 Tau method

The questions of the approximation of the solutions of the linear differential equations with polynomial coefficients by means of polynomials [1] and construction approximations of the KONTOROVITCH–LEBEDEV integral transforms kernels [2] are considered. The Tau method realization with minimal

residue choice for the determination of the polynomial approximations of the solutions of the second order differential equations with polynomial coefficients [3] of the following form

$$(a_0y^2 + a_5y)v''(y) + (a_1y + a_2)v'(y) + a_3v(y) = 0, v(0) = a_4, y \in [0, 1],$$

is supposed. By its using n -th approximation of the solution is sought in the form of the n -th degree polynomial $v_n(y)$, which is the solution of the equation

$$(b_0y^2 + b_5y)v(y) = \int_0^y (b_1x + b_2y + b_3)v(x)dx + b_4y + \tau_{n+2}T_{n+2}^*[(1 - \alpha_{n+2})y + \alpha_{n+2}],$$

where coefficients $a_i, i=0, \dots, 5$, may be expressed by coefficients $b_i, i=0, \dots, 5$, $\alpha_{n+2} = \sin^2(\pi/(4(n+2)))$ - the most left root of the shifted Chebyshev polynomial of the $n+2$ -th degree $T_{n+2}^*(y)$ in the interval $[0, 1]$, τ_{n+2} - undefined coefficient.

The problem about determination of the polynomial $P_n(y) = \sum_{k=0}^n p_k y^k$, which is the least deviated from zero on the interval $[0, 1]$ among all n -th degree polynomials, satisfying the pair of linear correlations on the coefficients $p_0 = 0, \sum_{i=1}^n c_i^{(n)} p_i = 1$ was considered. The following theorem is proved:

Theorem 1. If the sequence of numbers $c_i^{(n)}, i = 1, \dots, n$, is alternating then the polynomial $\tau_n T_n^*[(1 - \alpha_n)y + \alpha_n]$ is the polynomial least deviating from zero in the uniform metric on $[0, 1]$ among all polynomials of degree n , satisfying the indicated pair of linear relations.

On the basis of this theorem it's shown that suggested by us in the Tau method residue, in the number of significant cases, is the minimal in the uniform metric on $[0, 1]$ among all possible polynomial residues permitting the Volterra integral equations solution in the kind of polynomial also.

On the example of computing the second kind modified BESSEL function $K_{i\beta}(x)$ this modification's advantages are shown as compared with usual and other tau-method and approximation method's variants. We have the following differential equation with polynomial coefficients for the approximation and computing of the second kind modified Bessel function $K_{i\beta}(x)$:

$$y^2 v''(y) + 2(y+1)v'(y) + \left(\frac{1}{4} + \beta^2\right)v(y) = 0, \quad v(0) = 1,$$

and the Volterra integral equation

$$y^2 v(y) = \int_0^y \left[\left(\frac{9}{4} + \beta^2\right)x - \left(\frac{1}{4} + \beta^2\right)y - 2 \right] v(x) dx + 2y.$$

We obtain the following recurrence formulas for the coefficients of canonical polynomials $Q_m(y) = \sum_{k=0}^m q_{km} y^k$

$$q_{00} = \frac{2}{\frac{1}{4} + \beta^2}, \quad q_{0k} = -\frac{2(k+2)}{k^2 + k + \frac{1}{4} + \beta^2} q_{0k-1}, \quad k = 1, \dots$$

The minimality of the residue suggested by us follows from the Theorem 1 as $q_{0m}/|q_{0m}| = (-1)^m$, $m = 0, 1, \dots$

The advantages of this modification, as compared with usual and other tau-methods [4], is shown.

The new numerical scheme of the Tau method application is proposed for the solution of the second order linear differential equations systems with the second order polynomial coefficients of the following kind:

$$(a_0^{(j)} y^2 + a_1^{(j)} y) v_j''(y) + \sum_{i=1}^k [(a_{3i-1}^{(j)} y - a_{3i}^{(j)}) v_i'(y) + a_{3i+1}^{(j)} v_i(y)] = 0,$$

$$v_j(0) = a_{3k+2}^{(j)}, \quad j = 1, \dots, k, \quad y \in [0, 1],$$

in the unknown vector-function $v(y) = (v_1(y), \dots, v_k(y))$. It is assumed to have only one solution. Integrating twice and carrying an addition in the right part in the kind of the vector-polynomial $P_n(y)$, we derive for the determination of the n -th approximation of the solution $v(y) = (v_1(y), \dots, v_k(y))$ the system of Volterra integral equations with polynomial kernels

$$(b_0^{(j)} y^2 + b_1^{(j)} y) v_j(y) = \int_0^y \left[\sum_{i=1}^k (b_{3i-1}^{(j)} x + b_{3i}^{(j)} y + b_{3i+1}^{(j)}) v_i(x) \right] dx + P_{jn+2}(y),$$

$$j = 1, \dots, k,$$

where coefficients $b_i^{(j)}$ and $a_i^{(j)}$, $i = 0, \dots, 3k+2$ and $j = 1, \dots, k$, are connected in definite way and $P_{jn+2}(y)$, $j = 1, \dots, k$, $-n+2$ -th degree polynomials. The different variables of the vector residue choice and its minimization are analyzed. The recurrent formulas for the canonical vector-polynomials coefficients convenient for the calculations are given.

Consider the system of two second order differential equations ($k = 2$) in more detail. This case is of particular interest for differential equations with complex coefficients.

The scheme of the integral form of the Tau Method described in this paper can be used for deriving polynomial approximations of hypergeometric and confluent hypergeometric functions of the first kind with complex parameters.

The modified KONTOROVITCH - LEBEDEV integral transforms [2] with kernels $\text{Re}K_{1/2+i\beta}(x) = (K_{1/2+i\beta}(x) + K_{1/2-i\beta}(x))/2$ and $\text{Im}K_{1/2+i\beta}(x) = (K_{1/2+i\beta}(x) - K_{1/2-i\beta}(x))/2i$, where $K_s(x)$ is MACDONALD'S function, is of great importance in solving some problems of mathematical physics, in particular mixed boundary value problems for the Helmholtz equation in wedge and cone domains. We find it necessary to compute $\text{Re}K_{1/2+i\beta}(x)$ and $\text{Im}K_{1/2+i\beta}(x)$ to use this transform in practice. These functions also occur in solving some classes of dual integral equations with kernels which contain MacDonald's function of imaginary index $K_{i\beta}(x)$ [2,5]. Therefore, now we consider the second kind modified Bessel function $K_{1/2+i\beta}(x)$ in more detail.

We have a system of two second order differential equations [6,7]

$$\begin{aligned} y^2 v_1'' + 2(y+1)v_1' + \beta^2 v_1 + \beta v_2 &= 0, \\ y^2 v_2'' + 2(y+1)v_2' - \beta v_1 + \beta^2 v_2 &= 0, \\ v_1(0) = 1, v_2(0) &= 0, \end{aligned}$$

or the system of VOLTERRA integral equations

$$y^2 v_1(y) = \int_0^y ((2 + \beta^2)x - (2 + \beta^2 y))v_1(x)dx + \beta \int_0^y (x - y)v_2(x)dx + 2y,$$

$$y^2 v_2(y) = \beta \int_0^y (y - x)v_1(x)dx + \int_0^y ((2 + \beta^2)x - (2 + \beta^2 y))v_2(x)dx,$$

$$K_{1/2+i\beta}(x) = (\pi/(2x))^{1/2} e^{-x} (v_1(1/x) + i v_2(1/x)), x \geq 1.$$

By means of computations is shown that the choice of the residue in the form $P_{jn+2}(y) = \tau_{jn+2} T_{n+2}[(1 - \alpha_{n+2})y + \alpha_{n+2}]$, $j = 1, 2$, is optimal as compared with other known variants in this case too.

2 Mixed boundary value problems

The definition of two pairs of direct and inverse modified KONTOROVITCH-LEBEDEV integral transforms [2] are cited

$$F_+(\tau) = \int_0^\infty f(x) \text{Re}K_{1/2+i\tau}(x) dx, 0 \leq \tau \leq \infty,$$

$$f(x) = (4/\pi^2) \int_0^\infty ch(\pi\tau) F_+(\tau) \text{Re}K_{1/2+i\tau}(x) d\tau, 0 < x < \infty,$$

and

$$F_-(\tau) = \int_0^\infty f(x) \operatorname{Im} K_{1/2+i\tau}(x) dx, 0 \leq \tau \leq \infty,$$

$$f(x) = (4/\pi^2) \int_0^\infty \operatorname{ch}(\pi\tau) F_-(\tau) \operatorname{Im} K_{1/2+i\tau}(x) d\tau, 0 < x < \infty.$$

The sufficient conditions of the existence of these transforms and the validity of the inversion formulas are given.

It's shown that the inversion formulas of the modified KONTOROVITCH–LEBEDEV integral transforms can be deduced from the inversion formulas of the "usual" KONTOROVITCH–LEBEDEV transforms and the corresponding theorem is proven.

For the case of nonnegative finite functions with restricted variation the conditions of present theorem are reduced to one condition, which is necessary and sufficient then.

The verification of the solution of singular integral equations of the form

$$\phi(x) = f(x) + \lambda \int_0^\infty (K_1(x+y)_-^+ K_0(x+y)) \phi(y) dy, 0 < x < \infty,$$

where $f(x)$ - given function, λ - parameter, complying with the condition $\lambda < 1/\pi$, is given by means of the modified KONTOROVITCH–LEBEDEV transforms [8]. The proof of the PARSEVAL equalities for these transforms is conducted [9].

The problem of the evaluation of the modified KONTOROVITCH–LEBEDEV transforms is greatly simplified by means of their decomposition into the form of compositions of more simple integral transforms, in particular FOURIER and LAPLACE transforms. The expression of the modified KONTOROVITCH–LEBEDEV integral transforms over the general MEYER integral transforms of special index and argument is given.

The dual integral equations with MACDONALD'S function of the imaginary order $K_{i\tau}(x)$ in the kernel of the following form were introduced by LEBEDEV and SKALSKAYA [2]

$$\int_0^\infty M(\tau) \tau \tanh(\alpha\tau) K_{i\tau}(kr) d\tau = rg(r), 0 < r < a,$$

$$\int_0^\infty M(\tau) K_{i\tau}(kr) d\tau = f(r), r > a,$$

where $g(r)$ and $f(r)$ - given functions. They showed [2] that solutions of this equations may be determined in the form of single quadratures from solutions

of second kind FREDHOLM integral equations with symmetric kernel containing MACDONALD'S function of the complex order $K_{1/2+i\tau}(x)$.

$$M(\tau) = \frac{2\text{sqr}t(2) \sinh(\pi\tau)}{\pi\text{sqr}t(\pi) \sinh(\alpha\tau)} \int_0^\infty \psi(t) \text{Re}K_{1/2+i\tau}(kt) dt,$$

$$\psi(t) = h(t) - \int_0^\infty K(s, t) \psi(s) ds, a \leq t < \infty,$$

where $\text{Re}K_{1/2+i\tau}(z)$ - real part of MACDONALD'S function of complex order $1/2 + i\tau$. In the case $g(r) = 0$

$$h(t) = -\frac{\text{sqr}t(k) \exp(kt)}{\pi} \frac{d}{dt} \int_0^\infty \frac{\exp(-kr) f(r)}{\text{sqr}t(r-t)} dr$$

$$K(s, t) = \frac{4}{\pi} \int_0^\infty \frac{\sinh[(\pi - \alpha)\tau]}{\sinh(\alpha\tau)} \text{Re}K_{1/2+i\tau}(ks) \text{Re}K_{1/2+i\tau}(kt) d\tau.$$

The numerical solution is conducted. The economical methods of the evaluation of kernels of the integral equations based on GAUSS quadrature formulas on LAGERRE polynomial's knots are proposed. The procedures of the preliminary transformation of integrals and extraction of the singularity in the integrand are used for the increase of accuracy and speed of algorithms. The cases of dual integral equations admitting complete analytical solution are considered. Observed examples demonstrate the efficiency of this approach in the numerical solution of the mixed boundary value problems of elasticity and combustion in the wedge domains.

The application of the integral KONTOROVITCH-LEBEDEV transforms and dual integral equations to the solution of the mixed boundary value problems are considered. The diffusion and elastic problems reduced to the solution of the proper mixed boundary value problem for the HELMHOLTZ equation

$$\Delta u - k^2 u = 0, \quad \frac{\partial u}{\partial \eta} \Big|_{\varphi=\pm\alpha, 0 < r < a}(r) = g(r), \quad u \Big|_{\varphi=\pm\alpha, r > a}(r) = f(r),$$

$$u \Big|_{r \rightarrow 0} - \text{restricted}, \quad u \Big|_{r \rightarrow \infty} - \text{restricted}.$$

The solution of the problem as derived by LEBEDEV is determined by the next way in the form of the integral KONTOROVITCH-LEBEDEV transform

$$u(r, \varphi) = \int_0^\infty M(\tau) \frac{\cosh \varphi\tau}{\cosh \alpha\tau} K_{i\tau}(kr) d\tau,$$

where $M(\tau)$ is the solution of dual integral equation.

It is shown that the above-mentioned problems solution for the HELMHOLTZ equation are present in the form of single quadrature from HELMHOLTZ

equation are present in the form of single quadrature from FREDHOLM integral equation type. The dimension of the problem is lowered on unit by this, what is the essential advantage of this method. The examples permitting the complete analytical solution of the problem are given.

The numerical solution of the mixed boundary value problems and received dual integral equations is carried out. It consists of numerical solution of the second kind FREDHOLM integral equation with symmetric kernels and the followed taking of quadratures from their solution. The estimation of error is given. The control calculations results give the precision for the solution in 6-7 digits after comma. The considered examples demonstrate the efficiency of the dual integral method in the solution of the mixed boundary value problems for the HELMHOLTZ equation in the wedge domains.

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