How to detect the integrability of discrete systems

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Abstract

Several integrability tests for discrete equations will be reviewed. All tests considered can be applied directly to a given discrete equation and do not rely on the *a priori* knowledge of the existence of related structures such as Lax pairs. Specifically, singularity confinement, algebraic entropy, Nevanlinna theory, Diophantine integrability and discrete systems over finite fields will be described.

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1 Introduction

We describe in this short review various "integrability detectors" for discrete systems. Here discrete means discrete time: the evolution is governed by recurrence relations, or, by discretising space as well, partial difference equations.

The notion of integrability for continuous systems having a finite number of degrees of freedom goes back to the XIXth century with J. Liouville, and the existence of conserved quantities (in involution) is its basic feature, since it is the ground for the existence of action-angle variables, and the full resolution of the equations of motion.

Remarkably enough, the pioneering work of S. Kowalevskaya and its continuation by P. Painlevé, by an analysis of structural properties, already provided before the end of the XIXth century an integrability detector, that is to say a mean to detect integrability, without having to produce an explicit solution.

Discrete counterparts of essentially all the known properties of continuous integrable systems are now available, but the most recent results have also shown that discrete systems are in a sense more fundamental than the continuous ones. They have a richer structure, and they have become the center of a flourishing activity.

We will present here a cluster of properties which signal the integrability of discrete systems. They are interrelated, even if not equivalent, and they all allow a direct study of the systems.

The review is divided in three parts, each having a different flavour, and reflecting different points of view, but describing the features of what is -by a common consensus- an integrable discrete system. The main concepts discussed are singularity analysis (section 2), growth and complexity (section 3), and analytic and arithmetic approaches (section 4).

2 Singularity confinement

Singularity confinement [1] was the name given to a property of discrete systems integrable by spectral methods, namely that any spontaneously appearing singularity disappears after a few iteration steps. In what follows we shall present the workings of singularity confinement as a discrete integrability detector, some of its applications as well as the associated pitfalls.

2.1 The appearance of singularity confinement

The first occurence of confined singularities was not in studies of integrable systems but rather in the domain of numerical analysis. Indeed, more than 50 years ago, Wynn [2] proposed what he called the ϵ-algorithm which was meant as an accelerator of the convergence of series. It has the form

$$\epsilon_n^{k+1} = \epsilon_{n+1}^{k-1} + \frac{1}{\epsilon_{n+1}^k - \epsilon_n^k} \tag{2.1}$$

where $\epsilon_n^{-1} = 0$ and $\epsilon_n^0 = S_n$, i.e. the series the convergence of which must be accelerated. The numbers ϵ_n^k fill a two-dimensional array where the ϵ's with an odd upper index are auxiliary quantities. One can easily eliminate them leading to Wynn's cross rule:

$$\frac{1}{\epsilon_n^{k+2} - \epsilon_n^k} - \frac{1}{\epsilon_n^k - \epsilon_n^{k-2}} - \frac{1}{\epsilon_{n+2}^k - \epsilon_n^k} + \frac{1}{\epsilon_n^k - \epsilon_{n-2}^k} = 0 \tag{2.2}$$

When implementing the convergence algorithm, in the form (2.1) or (2.2) a division by zero may occur. However it turms out that one may jump over the singularity by using what in numerical analysis are called singular rules and continue the computation. Let us illustrate this by using the cross rule. Solving for ϵ_n^{k+2} we find

$$\epsilon_n^{k+2} = \epsilon_n^k - \frac{(\epsilon_n^k - \epsilon_n^{k-2})(\epsilon_{n+2}^k - \epsilon_n^k)(\epsilon_n^k - \epsilon_{n-2}^k)}{(\epsilon_n^k - \epsilon_n^{k-2})^2 - (\epsilon_n^{k-2} - \epsilon_{n+2}^k)(\epsilon_n^{k-2} - \epsilon_{n-2}^k)} \tag{2.3}$$

Clearly the vanishing of any of the denominators of the three last terms of (2.2) leads simply to $\epsilon_n^{k+2} = \epsilon_n^k$. Thus the singularity does not lead the algorithm to a halt. (On the other hand, from a purely numerical point of view, a division by a number close to zero makes the algorithm unstable due to cancellation errors and particular rules must be introduced. Since this is beyond our scope we shall refer the interested reader to the existing literature).

The notion of confinement made its appearance, in relation with integrability, in the work of Joshi. Indeed, in [3], Joshi observed that integrable systems possess what she called *orbits with pole-like behaviour*. She started by studying possible discretisations of the Riccati equation $z' = \alpha z^2$ focusing on two of them, the logistic mapping $z_{n+1} = az_n(1 - z_n)$ and the homographic one $z_{n+1} = az_n(1 - z_{n+1})$. For the first mapping it is known that it exhibits chaotic behaviour for a whole range of values of the parameter a. (This is true despite the fact that for some special values of a one can find exact solutions of the mapping. For instance, when $a = 2$ we have $z_n = \frac{1}{2}\big(1 - (1 - 2z_0)^{2^n}\big)$. However no contradiction exists. While the solution can be given explicitly in terms of n and the initial point z_0, an inversion of the formula of the solution shows that the initial point z_0 is a multivalued function of the iterates and thus cannot play the role of a conserved quantity. A detailed discussion of solvability and its relation to integrability detectors can be found in the work of some of the present authors [4]). On the other hand the homographic mapping is linearisable: putting $z_n = 1/\zeta_n$ transforms it to a non-homogeneous linear equation $\zeta_{n+1} = 1 + \zeta_n/a$, an equation devoid of any chaotic behaviour. The homographic mapping possesss a family of orbits which reach infinity in a finite number m of steps starting from a finite z_0. For instance when $a = 1$ it suffices to take $z_0 = -1/m$ whereupon we find that z_m diverges but z_{m+1} is finite (in fact equal to 1). Joshi remarks that these orbits are the discrete analogues of the solutions of the Riccati equation which possess movable poles.

The introduction of the notion of singularity confinement [1] and its definitive link with integrability came from the study of the KdV equation:

$$x_j^{i+1} = x_{j+1}^{i-1} + \frac{1}{x_j^i} - \frac{1}{x_{j+1}^i} \tag{2.4}$$

Given the form of (2.4) the question which arises naturally is "what if a singularity appears spontaneously?" How does it evolve under the mapping (2.4)? The result turned out to be the following: a vanishing x at (i, j) leads to divergent x's at both $(i+1, j-1)$ and $(i+1, j)$ and a vanishing x at $(i+2, j-1)$. Then at both sites $(i+3, j-2)$ and $(i+3, j-1)$ a fine cancellation occurs and one obtains finite values: $x_{j-1}^{i+3} = x_j^{i-1} + 1/x_{j-1}^i - 1/x_j^{i+2}$ and a similar one for x_{j-2}^{i+3}. Thus the singularity does not propagate beyond a few lattice points and is confined to a small region. More complicated singularities may exist (x vanishing on more than one point) but it can be shown that they, too, lead to confined singularities.

Moreover the notion of singularity confinement is not limited to lattice systems but exists (and is easier to grasp) also for one- dimensional mappings. The McMillan mapping :

$$x_{n+1} + x_{n-1} = \frac{2\mu x_n}{1 - x_n^2} \tag{2.5}$$

was the acid-test of the method. This mapping is well known for its integrability. In fact, it can be completely integrated in terms of elliptic functions: $x = x_0 \operatorname{cn}(\Omega n, \kappa)$, where $\kappa = x_0 \operatorname{dn}(\Omega)/\operatorname{sn}(\Omega)$ and Ω is related to μ through $\mu = \operatorname{cn}(\Omega)/\operatorname{dn}^2(\Omega)$. A singularity may appear in the recursion (2.5) whenever x passes through the value 1. So let us assume that x_0 is finite and that $x_1 = 1+\epsilon$. (This can be obtained from a perfectly regular x_{-1}). We find then the following values: $x_2 = -\mu/\epsilon - (x_0 + \mu/2) + \mathcal{O}(\epsilon)$, $x_3 = -1 + \epsilon + \mathcal{O}(\epsilon^2)$ and $x_4 = x_0 + \mathcal{O}(\epsilon)$. Thus, not only the singularity is confined at this step but, also, the mapping has recovered the memory of the initial conditions through x_0.

Through a bold step, starting from the remark that for a host of systems, integrable through spectral methods, the spontaneously appearing singularities were confined, the singularity confinement property was elevated to the rank of a discrete integrability criterion.

2.2 Applications of singularity confinement

The most fruitful application of singularity confinement and moreover one that, as we see in what follows, avoids all pitfalls, is the deautonomisation of integrable autonomous mappings. The interest of this approach is that it made possible the derivation of the discrete analogues of Painlevé equations[5]. An illustration is in order at this point.

We start by generalising the McMillan mapping (2.5) to the non- autonomous case

$$x_{n+1} + x_{n-1} = \frac{a + bx_n}{1 - x_n^2} \tag{2.6}$$

where a and b are now functions of the independent variable n. The integrable non-autonomous form of (2.6) will be derived using the singularity confinement property. We assume that for some n we have a regular x_n and $x_{n+1}=\sigma+\epsilon$ where $\sigma=\pm 1$. (In this way we cover the two possibilities of x going through a root of the denominator of the r.h.s.). Iterating further we find:

$$x_{n+2}=-\frac{b_{n+1}+\sigma a_{n+1}}{2\epsilon}+\frac{a_{n+1}-\sigma b_{n+1}}{4}-x_n+\mathcal{O}(\epsilon) \tag{2.7}$$

$$x_{n+3}=-\sigma+\frac{2b_{n+2}-b_{n+1}-\sigma a_{n+1}}{b_{n+1}+\sigma a_{n+1}}\epsilon+\mathcal{O}(\epsilon^2) \tag{2.8}$$

The condition for x_{n+4} to be finite reads:

$$b_{n+1}-2b_{n+2}+b_{n+3}+\sigma(a_{n+1}-a_{n+3})=0 \tag{2.9}$$

which leads to $a_{n+1}=a_{n+3}$ and $b_{n+1}-2b_{n+2}+b_{n+3}=0$. Thus we have $b_n(\equiv z_n)=\alpha n+\beta$ and $a_n=\delta+\gamma(-1)^n$. Ignoring the even-odd dependence we take a as a strict constant. We obtain finally:

$$x_{n+1}+x_{n-1}=\frac{a+z_n x_n}{1-x_n^2} \tag{2.10}$$

This is a form of discrete Painlevé II in agreement with previous results derived through different approaches. (When the even-odd dependence is not neglected, (2.10) is a discrete analogue of Painlevé III, as shown in [6]).

Another interesting example may be presented at this point. We start from the integrable mapping

$$x_{n+1}x_{n-1}=\frac{a}{x_n}+\frac{1}{x_n^2} \tag{2.11}$$

and deautonomise it by assuming that a is a function of n. We assume that for some n, x_n is regular and $x_{n+1}=-1/a(n)+\epsilon$. Iterating and taking $\epsilon\to 0$ we find for x_{n+2}, x_{n+3}, x_{n+4} respectively the values 0, ∞ and 0. This is the singularity pattern of this mapping (which, by the way, is exactly the same in the autonomous case). Computing x_{n+5} when ϵ is taken to 0, we find a finite value, $-a(n+1)/a(n+3)^2$. However x_{n+6} is not finite at this limit and unless some constraint is set on a the singularity propagates indefinitely. On the other hand if we take a such that $a(n+2)a(n-2)=a(n)^2$ then x_{n+6} and the subsequent x's are finite. The integration of the constraint on a leads to $a_n=a_{e,o}\lambda^n$, where $a_{e,o}$ indicates an even-odd dependence. However and contrary to the discrete Painlevé II case this dependence is spurious since it can be eliminated through a rescaling of the dependent variable. Thus we have simply $a_n=a_0\lambda^n$. This result is particularly interesting since it leads to a discrete Painlevé of q type.

Singularity confinement's usefulness was not limited to the derivation of discrete Painlevé equations. Many other systems can be treated sucessfully within this approach. Consider for example the system

$$x_{n+1}=x_n+a+\frac{b}{x_n}+\frac{c}{x_{n-1}} \tag{2.12}$$

where a, b and c are functions of n. The singularity analysis of this system is straightforward. A singularity may appear whenever x goes through zero. Thus starting from a finite x_{n-1} we take $x_n = \epsilon$ and iterate further. We find that all subsequent x's diverge when $\epsilon \to 0$. However it at the level of x_{n+2} we choose $c(n+1) = -b(n)$ then x_{n+2} and all the subsequent x's are finite when $\epsilon \to 0$. Thus we expect the mapping to be integrable, and indeed it is. Putting $a(n) = d(n) - d(n-1)$ we can write (2.12) as an exact difference. Absorbing the integration constant into d we have

$$x_{n+1} = d + \frac{b}{x_n} \tag{2.13}$$

Thus (2.12) is the discrete derivative of the homographic mapping. We remark that in this case the singularity analysis does not define completely the functions appearing in (2.12) which remain free up to one constraint between b and c. This existence of free functions is characteristic of mapping the integration of which is obtained through linearisation [7].

2.3 Refining the notion of singularity

The continuous Painlevé transcendents can be viewed as deautonomisations of the elliptic functions. By analogy a systematic derivation of the discrete Painlevé equations may proceed through the deautonomisation of the QRT [8] mapping. The latter is an integrable family of mappings, the solution of which is expressed in terms of elliptic functions. The "symmetric" form of the QRT mapping is the following

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1} f_2(x_n)}{f_2(x_n) - x_{n-1} f_3(x_n)} \tag{2.14}$$

where f_i are specific quartic polynomial, involving in all 5 free parameters. The derivation of the discrete Painlevé equations proceeds through the deautonomisation of these parameters. However the application of the singularity confinement approach in this case necessitates a more precise definition of what we mean by "singularity". Clearly an infinite value for x_i, $i = n, n \pm 1$, does not play any particular role. In fact, relation (2.14) is 'bi-homographic' and thus infinity can be taken to any finite value by a simple homographic transformation of variables. However (2.14) may pose a subtler problem. It may turn out that for a certain n the mapping (apparently) loses one degree of freedom. This occurs when x_{n+1} is defined independenly of x_{n-1} and this happens whenever: $f_1(x_n)f_3(x_n) - f_2(x_n)^2 = 0$. Thus we consider that a singularity appears whenever x_{n+1} is independent of x_{n-1} the singularity being associated to the loss of a degree of freedom. It is then natural to ask how this singularity can be confined, i.e. how the mapping can recover the lost degree of freedom. For rational mappings of the kind we are considering, this can be realized if some of the mapping's variables assume an indeterminate form, for instance 0/0. In that case new free parameters can be introduced and the mapping recovers its full dimensionality.

The situation we just described is perhaps better illustrated in the case of linearisable mappings of the form

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1} f_2(x_n)}{f_4(x_n) - x_{n-1} f_3(x_n)} \tag{2.15}$$

where f_i are linear in x_n: $f_i = a_i x_n + b_i$. Here the mapping loses one degree of freedom whenever:

$$f_1(x_n) f_3(x_n) - f_2(x_n) f_4(x_n) = 0 \tag{2.16}$$

Once x_n is obtained from (2.16) one can compute x_{n+1} simply as $x_{n+1} = f_1(x_n)/f_4(x_n) = f_2(x_n)/f_3(x_n)$, unless x_{n-1} was such that both the numerator and the denominator of the fraction defining x_{n+1} vanished, that is

$$x_{n-1} = f_1(x_n)/f_2(x_n) = f_4(x_n)/f_3(x_n) \tag{2.17}$$

Thus one sees two ways in which the singularity confinement can be preserved: either relation (2.17) is satisfied or it is not, in which case x_{n+1} is determined and is independent of x_{n-1}. In the latter case one degree of freedom will be definitely lost, as x_{n+2} will be determined in terms of x_n only, unless both the numerator and the denominator of the fraction that define it vanish, that is $x_n = f_1(x_{n+1})/f_2(x_{n+1}) = f_4(x_{n+1})/f_3(x_{n+1})$. In the case where (2.15) is satisfied, on the other hand, it would appear that a degree of freedom suddenly appears at step $n+1$. The only way out is to demand that x_n was determined by x_{n-1} only, independent of x_{n-2}, which means that one already had at the previous step: $x_n = f_1(x_{n-1})/f_4(x_{n-1}) = f_2(x_{n-1})/f_3(x_{n-1})$.

The ideas presented above can be easily generalized to an N-component rational mapping:

$$x'_i = f_i(x_1, x_2, \ldots, x_N) \quad i = 1, 2, \ldots, N \tag{2.18}$$

Normally for such an N-component mapping, N free parameters, introduced by the initial conditions, must be present at every step. Now, it may happen that at some iteration one (or more) degress of freedom are lost. The condition for this to occur is that the Jacobian of $(x'_1, x'_2, \ldots, x'_N)$ with respect to $(x_1, x_2, \ldots, x_N)$ vanishes. This signals the appearance of a singularity which can only be confined if at some subsequent step an indeterminate form appears allowing the lost parameter to be re-introduced.

Some remarks are in order before concluding this section. The whole idea of confinement is that a singularity appearing spontaneously (due to the choice of initial conditions) must disappear after a few iteration steps. A non-confined singularity is one which appears at some iteration and does not disappear for all subsequent ones. However there may exist situations where a singularity exists for *all* iterations. We consider that such a singularity is the analogue of what we call a fixed singularity in the continuous case. The existence of singularities of this type are not a counterindication for integrability. The situation may become even more complicated if at some iteration, in the middle of such a "fixed" singularity, the singularity disappears only to reappear after a few iteration steps. We call this situation anti-confinement and we consider that again it should not hinder integrability. Thus an unconfined singularity is one extending only either to plus or to minus infinity in the iteration index. All other situations are considered as situations of confinement (including the case of anticonfinement and that of a fixed singularity).

2.4 Singularity confinement as the discrete Painlevé property

If one wonders what the discrete analogue of the Painlevé property is, singularity confinement appears to be an excellent candidate. Both are based on the local study of singularities: the special structure these singularities possess when the system is integrable. In the continuous case the Painlevé property is based on the requirement that the solutions of a given equation be devoid of multivaluedness- inducing singularities (and thus one can, in the sense of Poincaré, integrate the equation). In the discrete case the singularity confinement property is based on the requirement that the singularities not lead to intederminate points, i.e. points where the iterates of the mapping is not well defined. Thus, in Kruskal's sense, the mapping has a meaning as a dynamical system. Obviously an integrability detector is related to a specific type of integrability, of which there exist several kinds, the term integrability being conveniently rather vague. The Painlevé property is characteristic of systems the integration of which proceeds through spectral methods. The same holds true for singularity confinement in the discrete setting. Thus on the basis of these analogies it is quite reasonable to posit that singularity confinement is the discrete analogue of the Painlevé property. (We should make clear here that this is not a rigorous statement and, in the light of what follows, one that should be assorted to a caveat).

While for continuous systems the Painlevé property is almost tautologically identified to integrability, the situation is not as favourable for singularity confinement. Already in her work on orbits with pole-like behaviour [3] Joshi remarked that there existed systems which, while apparently nonintegrable do possess such orbits. The example she presented is the mapping $z_{n+1} = z_n^2/(z_n^2 - a^2)$ with $a \neq 1$, which possesses orbits with pole-like behaviour, namely those including the sequence: $\pm a, \infty, 1, \ldots$. This particular example may be dismissed on the basis of the observation that the mapping is not well defined in both evolution directions. If one tries to evolve towards diminishing n's the preimages of the initial point proliferate, i.e. their number grows exponentially. As was commented in [9], (and in a more general setting, related to correspondences, in [10]) this is a feature deemed incompatible with integrability. Still the problem persists. There exist well-defined mappings which do possess the singularity confinement property but are *not* integrable. We shall illustrate this point with the mapping studied in [11]:

$$\frac{x_{n+1}}{x_{n-1}} = x_n - \frac{1}{x_n} \tag{2.19}$$

Its singularity structure can be easily obtained. We find two singularity patterns $\{\pm 1, 0, \infty, \mp 1\}$ and the singularity is confined. The nonintegrable character of this mapping was studied in [11] where it was shown that there exists a deep link between its dynamics and that of the Fibonacci recurrence. This is not the only example of nonintegrable confining mapping. Whole families of such mappings do exist. Thus it appears that the analogy of singularity confinement with the Painlevé property breaks down at this point. Singularity confinement is not sufficient for integrability. We shall not go into detailed explanations here. It suffices to say that for discrete systems to be integrable, a proper local singularity structure is not enough. The growth properties of the solutions at infinity enter into play. The best way to qualify this, as shall be explained later in this review, is through the Nevanlinna approach [12]. To put it in a nutshell, for a discrete system to be integrable the requirement is that the Nevanlinna order of the solution be finite (which guarantees not too fast a growth) and moreover that its singularities be confined. A practical way to study the growth properties of the solution of a given mapping is through the algebraic entropy method [13, 14], as shall be explained later in this article.

The parallel between singularity confinement and the Painlevé property is deeper than what hinted at till now: both turn out not to be necessary for a specific kind of integrability, namely linearisability. Indeed, there exist equations which are integrable through linearisation and which do *not* possess the Painlevé property [15]. This is true both in the continuous and the discrete case. We shall illustrate this through two examples. We start from the linear equation

$$\frac{tx'' + (at - 1/2)x' + btx}{x'' + ax' + bx} = K \tag{2.20}$$

and take its derivative so as to eliminate K, obtaining a third order equation. Next we show that the same third order equation can be obtained if we start from the nonlinear equation

$$x''x' + 2ax'^2 + 3bx'x + (2ab - b')x^2 = M \tag{2.21}$$

and take its derivative so as to eliminate M. Here a and b are not free. We have $b = a^2 - a'/2$ and a satisfying the equation $a''' = 6a''a + 7a'^2 - 16a'a^2 + 4a^4$ which is equation XII in the Chazy classification. So, equation (2.21) is integrable by linearisation through equation (2.20). It is straightforward to show that (2.21) violates the Painlevé property. Solving it for x'', we find terms proportional to x^2/x' (and $1/x'$) which were shown to be incompatible with the Painlevé property. A *caveat* is in order at this point. While there exist large classes of linearisable equations without the Painlevé property, there does also exist a large class of linearisable equations which do satisfy the Painlevé criterion. The best known example of equations belonging to this class are the Riccati equation and its higher order analogues.

Next we turn to a discrete example. We examine the mapping [11]:

$$x_{n+1} = x_n\Big(\frac{x_n}{x_{n-1}} + a_n\Big) \tag{2.22}$$

When x_n passes through the value zero (which may well happen for some nonzero initial conditions) it is clear from the form of the mapping that all the subsequent values of x will be zero and the memory of the initial conditions is forever lost. Thus this singularity is nonconfined. On the other hand, (2.22) is linearisable in a straightforward way since it can be written as:

$$y_n = y_{n-1} + a_n \tag{2.23}$$

$$x_{n+1} = x_n y_n \tag{2.24}$$

where one has to solve two linear equations in cascade. Of course the remark concerning the existence of linearisable systems with the Painlevé property has its analogue here for linearisable mappings with confined singularities: all mappings of the "projective" family fall into this class.

A natural question here is what is the usefulness of non-sufficient integrability criterion. First, it is clear that *if the slow growth at infinity of the solutions is guaranteed*, as is the case for some integrable autonomous mappings, the singularity confinement criterion is quite adequate for its deautonomisation. Moreover it does present some advantage over the algebraic entropy approach since one does study one singularity at a time (which leads to more manageable calculations) and not their combined effect, as in the algebraic entropy applications. Linearisable discrete systems are a class of their own, but this is also true in the continuous case. When dealing with a linearisable system the singularity confinement approach may lead to a unnecessarily constrained system (or, as in the case of (2.22), perhaps miss it altogether). We can illustrate this point through a specific example. We start with the mapping [16]:

$$\frac{1}{x_n + x_{n+1}} + \frac{1}{x_n + x_{n-1}} = \frac{k}{x_n} \tag{2.25}$$

A singularity appears whenever the value of x becomes 0. Let us assume that for some n we have a regular x_{n-2} and $x_{n-1} = 0$. Iterating the mapping we find that all subsequent x's are zero unless some constraint holds. The interesting result is that the singularity can be confined at *any* step. Thus if we require that the confinement is attained at the level of x_{n+m} we find that k must be of the form $k = m/(m+1)$. All these mappings are integrable, but not just them. As a matter of fact (2.25) is integrable, through linearisation, for k an arbitrary function of n.

While linearisability does not require confined singularities the solutions must still have finite Nevanlinna order. In fact, as will be explained later in this review, the detailed study of the growth properties of the solutions of some rational mapping does furnish indications as to its linearisability. On the other hand, for continuous systems no general linearisability criterion is known to date.

2.5 Further applications of singularity confinement

While singularity confinement was discovered in a purely discrete context its extension to differential-difference systems [17] does not pose fundamental problems. The discrete part of the system is considered as a recursion allowing one to compute a given term from the knowledge of the preceding ones. The idea is to look for the possible singularities and their propagation under this recursion. We shall illustrate such an application with the classical example of the Toda system. We start from

$$\ddot{x}_n = e^{x_{n+1}-x_n} - e^{x_n-x_{n-1}} \tag{2.26}$$

and transform it into a purely algebraic form through the transformation: $a_n = e^{x_{n+1}-x_n}$, $b_n = \dot{x}_n$, leading to:

$$\dot{a}_n = a_n(b_{n+1} - b_n) \tag{2.27}$$

$$\dot{b}_n = a_n - a_{n-1} \tag{2.28}$$

We look for the spontaneous appearance of a singularity for some n (where the particle number is interpreted as the number of steps in the recursion). This means that we do not study the solutions that are allowed to be singular for every n but only those that *become* singular at some n. In this context relation (2.27)–(2.28) is to be interpreted as a recursion:

$$a_n = a_{n-1} + \dot{b}_n \tag{2.29}$$

$$b_{n+1} = b_n + \frac{\dot{a}_n}{a_n} \tag{2.30}$$

We start by assuming that both b_n and a_n are non-divergent and that the singularity appears in step $n+1$. In fact, due to the presence of the logarithmic derivative in (2.30), a pole may appear in b_{n+1} if a_n vanishes at some time t_0. Let us start with the simplest case of a single zero i.e. $a_n = \alpha\tau$ where $\tau = t - t_0$ and $\alpha = \alpha(t)$ with $\alpha(t_0) \neq 0$. Substituting in (2.30) we find: $b_{n+1} = 1/\tau + \ldots$, $a_{n+1} = -1/\tau^2 + \ldots$ Iterating further we obtain: $b_{n+2} = -1/\tau + \ldots$, $a_{n+2} = A\tau + \ldots$ where A is a quantity depending on α and b_n. Iterating further we obtain a finite result for b_{n+3}. Thus the singularity that appeared at b_{n+1} due to the simple root in a_n is confined after two steps. The vanishing-a_n behaviour just examined and which induces the divergence of b_{n+1} is not the only one. One can imagine higher order zeros of the type $a_n = \alpha\tau^k$. Depending on the value of k, more and more intermediate steps will be necessary for the confinement of the singularity: in principle, the confinement of singularity $a_n \propto \tau^k$ would necessitate $k+1$ steps. However the simplest singular behaviour is also the most generic one and, for systems comprising parameters to be determined, its study yields the most important integrability constraints for the system.

The analysis just presented should be interpreted as follows. First, the singularities that appear do have the Painlevé property (absence of branching). Second, they do not propagate *ad infinitum* under the recursion (2.29)–(2.30) but are confined to a few iteration steps. The first is the usual, Painlevé property and the second is the singularity confinement. Both are thus required for the integrability of differential-difference systems. Extention of such "hybrid" methods to other systems like, differential-delay or integrodifferential ones have also been explored. We are not going to discuss them but proceed to a different application of the notion of singularity confinement (although, as we shall explain, the term "singularity" is not quite appropriate in that context).

In [18] Joshi and Lafortune have transposed the notion of confinement to the ultradiscrete case and proposed an analogue to the singularity confinement property. In the ultradiscrete systems the nonlinearity is mediated by terms involving the max operator. Typically one is in presence of terms like $\max(X_n, 0)$. When, depending on the initial conditions, the value of X_n crosses zero, the result of the $\max(X_n, 0)$ operation becomes non-analytic: when X is slightly smaller than 0 the result is zero, while for $X > 0$ the result is X, and the derivative at 0 does not exist. It is this non-analyticity that plays the role of the singularity. Typically if we put $X = \epsilon$, a term $\mu = \max(\epsilon, 0)$ propagates with the iterations of the mapping and perpetuates the non-analyticity *unless* by some coincidence it disappears. This disappearance is the equivalent of the singularity confinement for ultradiscrete systems. In order to give an illustrative example based on the ultradiscrete Painlevé I equation

$$X_{n+1} + X_{n-1} = A + \max(0, X_n) - 2X_n \tag{2.31}$$

We shall examine the behaviour of a singularity appearing at, say, $n = 1$ where $X_1 = \epsilon$, while X_0 is regular and look at the propagation of this singularity both forwards and backwards. The presence of $\mu \equiv \max(\epsilon, 0)$ indicates that the value of X is "singular". Below we present only the results corresponding to $A > 0$, those corresponding to $A < 0$ leading to similar conclusions. First we examine the case $X_0 > A$. We find $\dots$, $X_{-3} = A - \epsilon$, $X_{-2} = X_0 - A + 2\epsilon$, $X_{-1} = -X_0 + A - \epsilon$, $X_0 = X_0$, $X_1 = \epsilon$, $X_2 = A - X_0 - 2\epsilon + \mu$, $X_3 = 2X_0 - A + 3\epsilon - 2\mu$, $X_4 = A - X_0 - \epsilon + \mu$, $X_5 = -\epsilon$, $X_6 = X_0 + 2\epsilon$, $\dots$. Here the solution is regular until X_1 then singular, confined, between X_2 and X_4 and regular from X_5 on. Next we consider $X_0 < 0$ and $|X_0| < A$ and obtain the following sequence: $\dots$, $X_{-13} = X_{-7} - 2X_{-5}$, $X_{-12} = X_{-6} - 2X_{-5}$, $X_{-11} = X_{-5}$, $X_{-10} = X_{-7} - X_{-5}$, $X_{-9} = X_{-6} - X_{-5}$, $X_{-8} = X_{-5}$, $X_{-7} = A + \epsilon$, $X_{-6} = -X_0 - 2\epsilon + \mu$, $X_{-5} = X_0 + \epsilon - \mu$, $X_{-4} = A - X_0 - \epsilon + \mu$, $X_{-3} = -\epsilon$, $X_{-2} = X_0 + \epsilon$, $X_{-1} = A - 2X_0 - \epsilon$, X_0, $X_1 = \epsilon$, $X_2 = A - X_0 - 2\epsilon + \mu$, $X_3 = X_0 + \epsilon - \mu$, $X_4 = -X_0 + \mu$, $X_5 = A - \epsilon$, $X_6 = X_3$, $X_7 = X_4 - X_3$,

$X_8 = X_5 + X_3$, $X_9 = X_3$, $X_{10} = X_4 - 2X_3$, $X_{11} = X_5 + 2X_3$, $\ldots$. One can see a regular zone between X_{-3} and X_1 and a singular pattern from X_2 on as well as until X_{-4}, as can be seen from the persistence of the singular valued X_{-5} and X_3. This is an anti-confined case, in the sense that a (small) regular region exist surrounded by singular values extending all the way to infinity in both directions. As we explained already such a behaviour is deemed compatible with integrability. The cases $0 < X_0 < A$ and $X_0 < -A$ lead to similar, anti-confined, patterns. Thus in all cases we have either a confined singularity (a central singular zone with regular behaviour outside) or an anti-confined singularity (a central regular zone with singular behaviour outside). Both behaviours are deemed compatible with integrability. The two points which we consider important in this analysis are that a) one must study all possible sectors of initial conditions and/or parameters and b) one must consider the possibility of anti-confined solutions.

In perfect analogy to the discrete case there exist nonintegrable systems with confined singularities and integrable systems with unconfined singularities [19]. In section (2.4) we presented an example of a nonintegable mapping which did pass the confinement test. Up to a minor change of sign, which does not modify the singularity structure, we can rewrite it as

$$x_{n+1} = x_{n-1}\left(x_n + \frac{1}{x_n}\right) \tag{2.32}$$

Its ultradiscretisation is straightforward. We find

$$X_{n+1} = X_{n-1} + |X_n| \tag{2.33}$$

using the absolute value of X instead of its equivalent $\max(X, 0) + \max(-X, 0)$. We shall examine the behaviour of a singularity appearing at, say, $n = 1$ where $X_1 = \epsilon$, while X_0 is regular. We distinguish two different sectors $X_0 < 0$ and $X_0 > 0$. In the first case ($X_0 < 0$) we find the sequence: $\ldots$, $X_{-3} = 3X_0$, $X_{-2} = 2X_0 - \epsilon$, $X_{-1} = X_0 + \epsilon$, X_0, $X_1 = \epsilon$, $X_2 = X_0 - \epsilon + 2\mu$, $X_3 = -X_0 + 2\epsilon - 2\mu$, $X_4 = \epsilon$, $X_5 = -X_0 + \epsilon$, $\ldots$. We can see readily that the singularity, indicated by the presence of μ, is confined (to X_2 and X_3 only). Turning to the case $X_0 > 0$ we find the sequence: $\ldots$, $X_{-4} = -X_0 + 2\mu + \epsilon$, $X_{-3} = -X_0 + 2\mu$, $X_{-2} = \epsilon$, $X_{-1} = -X_0 + \epsilon$, X_0, $X_1 = \epsilon$, $X_2 = X_0 + 2\mu - \epsilon$, $X_3 = -X_0 + 2\mu$, $X_4 = 2X_0 + 4\mu - \epsilon$, $\ldots$. In this case we are in presence of an anti-confined solution: a regular part around $n = 0$ is surrounded by unconfined singularities both for large positive and large negative n's. Thus the ultradiscrete mapping (2.33) has confined singularities despite its nonintegrable character (the latter being inferred in [19] from the growth properties of its iterates).

The converse situation, of a mapping which while integrable does not possess confined singularities, does also exist. As expected an example is to be sought among linearisable systems. In [11] we discovered the "multiplicative" linearisable mapping

$$\frac{x_{n+1}}{x_{n-1}} = a\frac{x_n + a}{x_n + 1} \tag{2.34}$$

Without loss of generality the mapping can be ultradiscretised to

$$X_{n+1} = X_{n-1} + A + \max(X_n, A) - \max(X_n, 0) \tag{2.35}$$

with $A > 0$. The complete description of the solution would require examining several sectors exist but in order to show that there exist unconfined singularities it suffices to exhibit such a situation in one sector. It turns out that the case where X_0 has a large negative value is one leading to unconfined singularities. We find: $\ldots$, $X_{-4} = -X_0 - 4A$, $X_{-3} = -4A + \epsilon$, $X_{-2} = X_0 - 2A$, $X_{-1} = -2A + \epsilon$, X_0, $X_1 = \epsilon$, $X_2 = X_0 + 2A - \mu$, $X_3 = 2A + \epsilon$, $X_4 = X_0 + 3A - \mu$, $X_5 = 4A + \epsilon$, $X_6 = X_0 + 4A - \mu$, $X_7 = 6A + \epsilon$, $\ldots$. We remark readily that while for negative indices the solution is regular, a singularity, mediated by μ, appears for positive n's and is never confined.

Clearly a better understanding of ultradiscrete integrability would necessitate some input from a Nevanlinna-like theory for ultradiscrete systems.

3 Algebraic entropy

3.1 Introduction

Wondering about the integrability of a recurrence relation, or more generally a map, naturally invites to analyse its iterates. It is unfortunately usually impossible to calculate explicitly these iterates by hand or even with any state-of-the-art formal calculus software, simply because the expressions one should manipulate are rational fractions of increasing degree of the various initial conditions. The complexity and size of the calculation make it impossible to conduct.

It was nevertheless seen early enough, that "integrable" maps are not as complex as generic ones. This was done primarily experimentally, by an accumulation of examples, and later by the elaboration of the concept of algebraic entropy which we will review here (see [20, 21, 22, 23, 13]).

The basic idea, given a rational map defined on n-dimensional space, is to examine the growth of the degree of its iterates, and extract a canonical quantity, which is an index of complexity of the map. This will be the algebraic entropy (or its avatar the dynamical degree). We will restrict ourselves to *birational* maps, that is to say maps of which the inverse is also rational.

The first step is to properly define the degree we consider, ensure that the entropy is well defined, and is independent of the coordinate system, so that it is canonical (section 3.2). This leads us to use complex projective space as a space of initial conditions. If one is interested in recurrences, the dimension of the space to consider is just the order of the recurrence.

The next step is to explain what influences the value of the entropy. This makes the link with the singularity analysis: actually, the singularity structure entirely governs the value of the entropy (section 3.3).

Section 3.4 describes how to calculate the algebraic entropy in practice. The first method is to list sufficiently many terms of the sequence of degrees of the iterates, and guess the full sequence from its first few terms. This heuristic approach has proved extremely efficient. The second method is to examine the singularity structure to find the exact value of the entropy. The latter yields proofs, but cannot always be used.

At this point one can evaluate the interest of the entropy as an integrability detector: integrable systems have a vanishing entropy, non integrable one have a non vanishing entropy.

Section 3.5.2 describes the natural extension of the notion to non-autonomous maps, and to the so-called lattice maps, which are to maps what partial difference equations are to difference equations. There again the vanishing of the entropy can be used as an integrability detector.

We present conjectures on the value of the entropy, and partial proofs. The main conjecture is that the algebraic entropy of any map over projective space of any dimension is the logarithm of an algebraic integer. This puts a limit on the set of values the entropy can assume, and there is a further conjecture on these values: in a given dimension there is a minimum for this value. In other words, there exists an *entropy gap*, i.e. one cannot approach arbitrarily close to integrability.

3.2 Definition

Suppose we are given a rational evolution map φ acting in an n-dimensional space. We first write it in a canonical way, using projective space, and the $(n+1)$ homogeneous coordinates $\{x_0, x_1, \dots, x_n\}$ for n-dimensional projective space, as a polynomial transformation in the homogeneous coordinates.

$$x_i \longrightarrow \phi_i(x_0, x_1, \dots, x_n), \quad i = 0 \dots n$$

If one factors out any common polynomial factors, the degree is well defined, in a given system of coordinates, although it is not invariant by changes of coordinates.

Definition: Let d_k be the degree of the k-th iterate of φ. Define the entropy[1] as

$$\epsilon = \lim_{k\to\infty} \frac{1}{k} \, \mathrm{Log}(d_k). \tag{3.1}$$

Proposition: The entropy is always defined and is invariant under changes of coordinates: it is a birational invariant associated to the transformation. If $\epsilon = 0$, and the growth is polynomial, of the form $d_k \simeq \alpha \, k^\nu$ then ν and α are canonically defined (birationally invariant).

This is a direct consequence of the elementary property that any pair of birational maps φ and ψ,

$$d_{\psi\cdot\varphi} \le d_\psi \, d_\varphi, \tag{3.2}$$

To give a flavour of the calculations, let us anticipate on section (3.4), and examine three simple examples in the two-dimensional plane.

[1] One may also define the dynamical degree as the exponential of the algebraic entropy.

Example 1: the Henon map. It is a map in two dimension given in non homogeneous coordinates (u, v), as

$$u \longrightarrow 1 + v - \alpha u^2 \tag{3.3}$$
$$v \longrightarrow \beta u \tag{3.4}$$

Going to homogeneous coordinates $[x, y, z]$ means replacing u by y/x and v by z/x. This leads to the map

$$\varphi_H : \quad [x, y, z] \longrightarrow [x^2, x^2 + xz - \alpha y^2, \beta xy] \tag{3.5}$$

One gets as a series of degrees of the iterates, for generic values of the parameters:

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, ??? \tag{3.6}$$

Notice that the degree of the map (3.5) is 2. The sequence d_k is not only bounded by 2^k, but it saturates this bound. The growth of the sequence of degrees is maximised. In fact, the map being polynomial in non-homogeneous coordinates, there cannot be any drop of the degree, and the entropy is just $\log(2)$.

This is not always the case as the following two examples show.

Example 2: exponential, but not maximal growth

Consider the map

$$\varphi_- : \quad [x, y, z] \longrightarrow [yz + 2\,xz - 2\,xy, yz - xy, yz + xz - 2\,xy] \tag{3.7}$$

We get

$$1, 2, 4, 7, 12, 20, 33, 54, 88, 143, 232, 376, 609, 986, 1596, 2583, ??? \tag{3.8}$$

After the third iterate, the bound $d_k \leq d_1^k = 2^k$ is not saturated anymore. However growth of still exponential, but with rate $(1 + \sqrt{5})/2$, as we will see later. This means that when one evaluates the iterates, some common factors appear in the homogeneous coordinates, and this leads to a drop of the degree. The entropy is non vanishing but is lower than $\log(2)$.

Example 3: very low growth

Consider the map:

$$\varphi_+ : \quad [x, y, z] \longrightarrow [yz + 4\,xz + 4\,xy, yz - 2\,xz + xy, yz + xz - 2\,xy] \tag{3.9}$$

Notice that this map is very similar to the previous one. We get

$$1, 2, 4, 7, 12, 18, 25, 34, 44, 55, 68, ??? \tag{3.10}$$

This is not only below the maximal growth, which would again be 2^k, since the map is quadratic, but is not even exponential.

Comparing to the previous example, it appears that the first four terms of the sequence of degrees are identical, but additional drops appear at the level of the fifth iterate. The drop of the degree is such that the growth is polynomial (quadratic), *the entropy vanishes* and ν as defined above is 2 . This map is indeed algebraically integrable: it possesses an algebraic invariant, defining a linear pencil of elliptic curves, to which the orbits are confined.

3.3 The importance of being singular

One may wonder about the origin of the drop of the degree. It is actually geometrically very simple, and comes from the singularity structure.

We need to recall what is singularity for a map φ of projective space. A point $[x_0, x_1, \ldots, x_n]$ is singular if all the homogeneous coordinates of the image by φ vanish. The set of these points is thus given by $n+1$ homogeneous equations. This set has codimension at least 2: it will be points in CP_2, complex curves and points in CP_3, and so on. One important point is that, as soon as the map is non-linear, and this is the case we will be interested in, *there always are singular points.* The vanishing of all homogeneous coordinates means that there is no image point in CP_n. The mere vanishing of a few, but not all coordinates means that the image "goes to infinity", but this is harmless for us, contrary to what happens in affine space. This is what projective space has been invented for: to cope with points at infinity, which are not to be forgotten when one consider algebraic varieties and rational maps. Moreover, using complex projective space simplifies a lot the counting of intersection points, by Bezout theorem.

The maps we consider are almost invertible. They are diffeomorphisms on a Zariski open set, i.e. they are invertible everywhere except on an algebraic variety, which we may find as follows: suppose the map φ and its inverse $\psi = \varphi^{-1}$ are written with homogeneous coordinates. The composed maps $\varphi \cdot \psi$ and respectively $\psi \cdot \varphi$ are then just multiplication of all coordinates by some polynomial κ_φ and resp. κ_ψ

$$\psi \cdot \varphi(m) = \kappa_\varphi(m).id(m) \quad \text{and} \quad \varphi \cdot \psi(m) = \kappa_\psi(m).id(m) \tag{3.11}$$

The map φ is clearly not invertible on the image of the variety of equation $\kappa_\varphi(m) = 0$.

What may happen is that further action of φ on these points leads to images in the singular set of φ. This means that $\kappa_\varphi(m)$ (or a piece of it if it is decomposable) *has to factorize from all the components.* This is the origin of the drop of the degree! This is the link between singularity (in the projective sense) and the degree sequence.

We may illustrate this with the example given by eqn (3.7), when looking at the iterates of φ on a generic point $[x, y, z]$. Here $\kappa_\varphi = xyz$, and the factor dropping at the third iterate is x. The situation is a little more intricate for the further iterates, but this is the essence of the phenomenon.

At his point it is possible to understand what singularity confinement is: if for all components of the variety $\kappa_\varphi(m) = 0$ one encounters singular points of φ

in such a way that some finite order iterate of φ -once the common factors are trimmed- define non ambiguously a proper image in CP_n, we have "singularity confinement" (see section 2 and references therein, as well as section 3.4.2).

3.4 How to calculate the entropy

3.4.1 Heuristic method

As was said above, it is hopeless to attempt a full calculation of the iterates of a map, the size of the expressions to manipulate being too large in practice.

On may circumvent this difficulty by looking at the successive images of a generic projective line, i.e. a degree 1 curve, with some running parameter (say t), and numerical coefficients. The effect at the level of the calculation is to handle only univariate polynomials (in t), drastically simplifying the computation. This is the most efficient way to produce the first terms of the sequence of degrees, and it has been successfully used for map on spaces of large dimension.

This also has a simple geometrical image: one is counting the intersection of the images with a fixed generic hyperplane. This is exactly in the spirit of the notion of complexity proposed by Arnold for diffeomorphisms [24].

At this point it is necessary to recall that the image of a variety V by a (bi)-rational map possibly contains two parts: the "proper image" and some additional pieces, the ensemble forming the "total image". The image to consider in relation to Arnold's definition is of course the proper image. The total image actually contains, in addition to the proper image, the blow-ups of possible singular points located on V.

Once one has listed the beginning of the sequence of degrees, it is necessary to evaluate its growth. The heuristic way is to complete the list in a reasonable way.

One first try is to calculate the discrete derivatives of the sequence

$$d'_n = d_{n+1} - d_n, \qquad d''_n = d'_{n+1} - d'_n, \qquad \dots \tag{3.12}$$

and look for relations between the successive derivatives.

Looking again at the three examples given above, we see that for the Hénon map (eq. 3.6)

$$d'_n = d_n, \qquad d_n = 2^n, \tag{3.13}$$

For sequence (3.8) we have

$$\{d'_n\} = 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ... \tag{3.14}$$

where we recognise (the beginning of) a Fibonacci sequence, which has exponential rate of growth.

For sequence (3.10), we have

$$\{d'_n\} = 1, 2, 3, 5, 6, 7, 9, 10, 11, 13, ... \tag{3.15}$$

$$\{d''_n\} = 1, 1, 2, 1, 1, 2, 1, 1, 2, ... \tag{3.16}$$

The second derivative is periodic, which indicates quadratic growth of the degree.

Another possibility is to write down the generating function of the sequence of degrees

$$g(s) = \sum_{k=0}^{\infty} d_k \, s^k \tag{3.17}$$

and try to fit it with a Pade approximant.

For Hénon (eq. 3.6)

$$g_{\text{Hénon}} = \frac{1}{1-2\,s}. \tag{3.18}$$

Remarkably this method works in many cases and the generating function we find is a rational fraction with integer coefficients!

For the two other examples of section (3.2) we have respectively for φ_- (eq. 3.8) and φ_+ (eq. 3.10).

$$g_{\varphi_-} = \frac{1}{(1-s)\,(1-s-s^2)} \tag{3.19}$$

$$g_{\varphi_+} = \frac{1+s^2+2\,s^4}{(s^2+s+1)\,(1-s)^3} \tag{3.20}$$

The growth of the sequence of degrees is given by the location of the pole of g which has the smallest modulus. As soon as there is a zero of the denominator of g inside the unit circle, the entropy does not vanish.

If, as is the case for g_{φ_+}, all poles have modulus 1, the multiplicity m of the root 1 ($m = 3$ for the case) gives the type of polynomial growth. It is of the degree $\nu = m - 1$ (quadratic for $\nu = 3$).

One could accumulate many example of maps, and believe that the sequence of degrees can always be fitted by a rational generating function, but the situation is more complicated [25].

If the generating function g is a rational fraction with integer coefficients and constant coefficient of the denominator equal to 1, then two propositions are true:

- The sequence of degrees verifies a finite recurrence relation with integer coefficients
- The entropy is the logarithm of an algebraic integer

The first part of this proposition is now proven for maps of CP_2 (see next paragraph and [26, 27, 28]), and there are counterexamples in higher dimensions [25]. The second part is still a conjecture for (bi)rational map of projective space of arbitrary dimension [13].

3.4.2 Use of the singularity structure

This is the approach taken in [29, 26, 30].

It is known that given a birational map $\varphi : X \to Y$, it is possible to remove the singularities and the non invertibility by blowing up a number of sub-varieties of X and Y respectively.

$$\begin{array}{ccccc} & & \tilde{\varphi} & & \\ & \tilde{X} & \longrightarrow & \tilde{Y} & \\ \pi_X & \downarrow & & \downarrow & \pi_Y \\ & X & \longrightarrow & Y & \\ & & \varphi & & \end{array}$$

The problem is that we want to iterate the map, i.e. we need to have $\tilde{X} = \tilde{Y}$. This leads us in general to an infinite sequence of successive blow-ups. Fortunately it is sometimes possible to realize this regularization with a finite number of blow-ups. This is precisely what has been called "discrete singularity confinement" (see section 2 and reference therein). We have in this case

$$\begin{array}{ccccc} & & \tilde{\varphi} & & \\ & \tilde{X} & \longrightarrow & \tilde{X} & \\ \Pi & \downarrow & & \downarrow & \Pi \\ & P & \longrightarrow & P & \\ & & \varphi & & \end{array}$$

The projection Π is a product of a finite number of blow-ups. The lifted map $\tilde{\varphi}$ is a smooth map on a rational variety $\tilde{X}$.

The two-dimensional case is particularly interesting, because we can use intersection theory of curves drawn on two-dimensional varieties. The Picard group of P_2 has one generator. Since the singular varieties always have codimension at least 2, we have to blow-up only points, and each blow-up adds one generator to the Picard group, with self-intersection -1 (see for example [31, 32])

There exists a (non-positive) scalar product on the Picard group $Pic(\tilde{X})$ of $\tilde{X}$. The map $\tilde{\varphi}$ induces an *isometry* $\Phi*$ on $Pic(\tilde{X})$. It is possible to represent the isometry $\Phi*$ with a matrix $\mu(\varphi)$. It is then possible to read the number of intersections of the images of a generic line under φ from the powers of $\mu(\varphi)$. This proves the existence of a finite recurrence relation on the successive degrees: it is just the characteristic polynomial of $\mu(\varphi)$, and since the coefficient of the leading term is 1, the entropy is the logarithm of an algebraic integer.

Moreover, since the metric is Lorentzian, we know that all (except at most two) of the eigenvalues lie on the unit circle. Consequently the entropy is the logarithm of a Salem number (see section (3.6.2). It was also shown that, as soon as there are enough rational invariants of the map, the entropy vanishes [33, 34]. In the two-dimensional case, this leads to $\nu = 1$ or $\nu = 2$ only. For higher dimensions, the polynomial growth may have $\nu > 2$ [35].

Remark: We also have examples in higher dimensions, where is is possible to prove that the sequence of degrees verifies a finite recurrence relation with integer coefficients [13, 36], even if it is not a general property of the sequence of degrees.

3.5 Beyond maps

3.5.1 Non autonomous maps

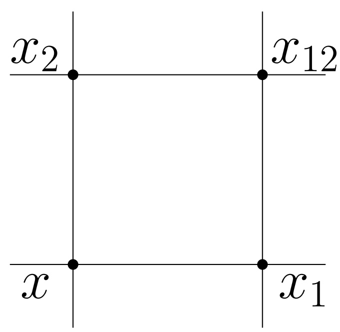
The notion of entropy may be applied to sequences of maps. This is particularly important for non autonomous iterations: suppose one has a family of maps $f(A)$ depending on set of parameters $A = [\alpha_1, \ldots, \alpha_2, \alpha_r]$. We may consider a sequence $\{f_n\}$ of such maps with parameters $A_n = [\alpha_{1,n}, \alpha_{2,n}, \ldots, \alpha_{r,n}]$. Rather than iterating a fixed map f, construct the sequence $F_0 = f_0, F_1 = f_1 \cdot F_0, \ldots, F_n = f_n \cdot F_{n-1}$. It is clear that the definition of entropy is possible from the sequence of degrees of the F_n's.

Here again the vanishing of the entropy detects integrability. The prototype of such integrable non-autonomous maps is the set of discrete Painlevé equations (see section 2). For all of these the entropy vanishes, and $\nu = 2$ [29, 37, 38].

3.5.2 Lattice maps

The notion of entropy can be extended [39, 40, 41] to the so called lattice equations, which are to maps what partial differential equations are to ordinary differential equations. For these systems, the characterisation of integrability also started around 1990 (see [42, 43, 44, 45, 46, 47, 48, 49]).

For simplicity we will consider the two dimensional case, where the elementary plaquette of the lattice is a square, but this applies straightforwardly to higher dimensions and other lattices.



The unknown variable $y[m,n]$ is defined at all vertices of the lattice, and the equation is a constraint of the form

$$f(x, x_1, x_2, x_{12}) = 0 \tag{3.21}$$

with $x = y[m,n]$, $x_1 = y[m+1,n]$, $x_2 = y[m,n+1]$, $x_{12} = y[m+1,n+1]$ for all vertices $[m,n]$.

In the simplest case, the above relation is multilinear, so as to determine any of the values at a corner of the plaquette rationally in terms of the three others. It defines a 1+1 dimensional evolution.

The space of initial data is infinite dimensional. Initial date are given on a line which must allow the determination of the values at all points of the lattice. The simplest possible choice is to take a regular diagonal staircase going diagonally.

We iterate the relation by calculating the values on diagonals moving away from the initial staircase, and define a sequence[2] of degrees d_k.

The most straightforward definition is the same as the one of maps:

$$\epsilon = \lim_{k\to\infty} \frac{1}{k} \log(d_k). \tag{3.22}$$

Proposition: The limit ϵ defined above exists, is independent of the coordinate system, and if $\epsilon = 0$ and the growth is polynomial, the degree of that polynomial is canonically defined.

The reason is the same as for maps (sub-additivity of the $\log(d_k)$). The difference comes from the infinite dimensionality of the space of initial data. However, due to the structure of the recurrence relation (causality) we need only $2q+1$ initial values if we want to calculate q steps !

We may evaluate explicitly the degrees with the same trick as for maps: initial data are given a fractional linear value in terms of some unknown t, all with the same denominator.

When the entropy vanishes, the growth of the degree is polynomial, and the degree of that polynomial is a secondary characterisation of the complexity.

The entropy for lattice maps is again a very good integrability detector[39, 40, 41, 50], and enjoys similar properties, especially for what concerns the values it can assume (e.g. being the log of an algebraic integer).

3.6 What values for the algebraic entropy?

3.6.1 Vanishing entropy

Of first importance is the value 0 for ϵ, as it signals the fall-off of complexity which is one of the signatures of integrability [14, 51, 28]. Moreover when $\epsilon = 0$ the secondary invariant ν classifies different polynomial growth rates.

If $\nu = 1$, the system is believed to be linearisable [52, 53].

The $\nu = 2$ case is particularly interesting: appears in two dimensions as soon as there is an algebraic invariant and is related to automorphisms of rational elliptic surfaces.

Higher values of ν appear in dimensions larger than 2 [35].

It is probable that all values of ν can be attained at the price of increasing the dimension, but explicit realisations remain to be found.

[2]We may actually define a number of different entropies for lattice equations (see [40]))

3.6.2 Non vanishing entropy

The set of non-vanishing values taken by the entropy ϵ is also of interest, because these values are not arbitrary. For two dimensional maps for example, ϵ is known to be the logarithm of a Salem number (see section 3.4.2). This is a quite strong constraint: as such numbers are believed to be bounded below, this in turn implies that there is a minimum for the entropy of two-dimensional maps.

It is natural to wonder what values ϵ can assume in general.

Definition: Let $\mu(n)$ be the infimum of $\exp(\epsilon)$ over birational maps of P_n, ϵ being the algebraic entropy.

Proposition

- This infimum exists
- $\mu(n+1) \leq \mu(n)$
- $\mu(2) \leq s_{\text{Lehmer}}$
- $\mu(k) < s_{\text{Lehmer}}, \quad \forall k \geq 3$
- $\lim_{n\to\infty} \mu(n) = 0$

Here s_{Lehmer} denote the root of

$$P_L = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1 \tag{3.23}$$

of modulus larger than 1. This number is approximately 1.17628. This is conjectured (Lehmer's conjecture) to be the smallest possible Salem number.

Proof

The first part of the proposition is trivial. The second part is elementary since it is always possible, given a map over projective space of dimension n to add a dimension which is just a spectator. The resulting map has the same entropy as the original one.

To prove the third part of the proposition, it is sufficient to produce a map with entropy exactly $\log(s_{\text{Lehmer}})$, see [27, 28]. Actually, if Lehmer's conjecture is true we have the equality $\mu(2) = s_{\text{Lehmer}}$.

To prove the fourth part of the proposition, it sufficient to produce a map with entropy lower than $\log(s_{\text{Lehmer}})$. Take for example the monomial map (in dimension $n = 3$). given by

$$[x, y, z, t] \longrightarrow [xz, yt, yz, xy] \tag{3.24}$$

The characteristic polynomial of the matrix associated to the map is $s^3 - s^2 + 1$ and yields $\exp(\epsilon) \simeq 1.1509639$ which is strictly lower than $\mu(2)$. Notice that the algebraic integer giving the entropy is complex.

To prove the last part of the proposition, monomial maps are again useful. Their entropy is given by the matrix of the exponents appearing in the map and is easy to evaluate. Consider the map of P_n

$$\Lambda : [x_0, x_1, \ldots, x_n] \longrightarrow [x_1 x_0,\ x_2, x_n,\ x_3 x_n, \ldots,\ x_0 x_n,\ x_n^2] \tag{3.25}$$

which is essentially a permutation with a quadratic perturbation.

The entropy may be calculated from the characteristic polynomial of the matrix associated to Λ. It is the maximal root of the polynomial

$$Q_n = s^n - s^{n-1} - 1 \tag{3.26}$$

We find a sequence of entropies

$$\epsilon_n \simeq \frac{\log(n)}{n} \tag{3.27}$$

This ends the proof, and leads to the conjecture:

Conjecture: *One may conjecture that the sequence* $\log(\mu(n))$ *is a monotonically decreasing sequence of strictly positive numbers, going to zero as* $n \to \infty$.

In other words: in a given dimension one cannot get arbitrarily close to integrability, as measured with the algebraic entropy. The price to pay is to increase the dimension (i.e. the order of the equation). This also means than any arbitrarily small non integrable perturbation of an integrable systems *takes it to a finite distance form integrability*, if this distance is measured with algebraic entropy.

4 Analytic and arithmetic approaches

4.1 Introduction

In this section we will explore an analogue of the Painlevé property for difference equations that is complex analytic in nature. The Painlevé property for differential equations concerns the singularity structure of solutions in the complex domain. In order to extend these ideas to the discrete world we consider difference equations, such as

$$F(z; y(z-1), y(z), y(z+1)) = 0, \tag{4.1}$$

for some function F. Equation (4.1) is a functional relation that needs to be satisfied for all $z \in \mathbb{C}$. This is in contrast to the discrete setting in which one considers a solution of the corresponding discrete equation,

$$F(n; y_{n-1}, y_n, y_{n+1}) = 0, \tag{4.2}$$

to be a sequence (y_n), rather than a function y on $\mathbb{C}$. Note that in this section, the fundamental objects studied are solutions of equations of the form (4.1). We do not consider starting from a sequence (y_n) of equation (4.2) and extending it to a solution $y(z)$ of equation (4.1), which is a highly non-unique process.

The general solutions of difference equations contain arbitrary periodic functions. For example, the general solution of

$$y(z+1) - 5y(z) + 6y(z+1) = 0$$

is

$$y(z) = 2^z \pi_1(z) + 3^z \pi_2(z),$$

where π_1 and π_2 are arbitrary period 1 functions. Clearly the general solution can have arbitrarily bad singularities if we choose the periodic functions appropriately. On the other hand, large classes of difference equations are known to admit many meromorphic solutions. In particular, if R is a nonconstant rational function, then the first-order difference equation

$$w(z+1) = R(w(z))$$

is known to admit the family of solutions

$$w(z) = W(z + \pi(z)),$$

where W is a particular nonconstant meromorphic function and π is an arbitrary period one function [54, 55, 56, 57]. This holds for integrable as well as nonintegrable equations.

In [52], Ablowitz, Halburd and Herbst suggested that a difference analogue of the Painlevé property is that the equations should admit sufficiently many finite-order meromorphic solutions. Nevanlinna theory provides a sophisticated theory of the value distribution of meromorphic functions. Those meromorphic functions of finite order are particularly well behaved and provide a natural class of "nice" meromorphic functions.

We will outline how Nevanlinna theory can be used to obtain strong necessary conditions for a difference equation to have admissible meromorphic solutions of finite order. Roughly speaking, a meromorphic solution of a particular difference equation is called admissible if it more complicated than all of the coefficients of the difference equation. The strong necessary conditions mentioned above include precise forms for the coefficients of the equation.

The remarkable formal similarity between Nevanlinna theory and Diophantine approximation has been noted by several authors including Osgood and Vojta. This similarity suggests that discrete equations with solutions in a number field k should be considered to be of Painlevé type if the logarithmic height grows polynomially rather than exponentially. In the case $k = \mathbb{Q}$, the logarthmic height of the non-zero rational number $x = a/b$, with a and b coprime, is $h(x) = \log \max\{|a|, |b|\}$. Preliminary classification results are described. We will conclude with the work of Roberts and Vivaldi on discrete equations over finite fields.

4.2 Basic terminology

Let f be a meromorphic function. The main idea of Nevanlinna theory is to encode important information about f in the asymptotic behaviour of certain real-valued functions of a positive variable r. These functions represent averages of certain functions of f over the disc $|z| \leq r$ or over the circle $|z| = r$. For more information on Nevanlinna theory, see [58].

The *proximity function* is

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta,$$

where $\log^+ x := \max(\log x, 0)$. The *enumerative function* is

$$N(r, f) := \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

where $n(r, f)$ is the number of poles of f (counting multiplicities) in $|z| \leq r$. The *Nevanlinna characteristic function*

$$T(r, f) = m(r, f) + N(r, f)$$

measures "the affinity" of f for infinity. For $a \in \mathbf{C}$,

$$T\left(r, \frac{1}{f - a}\right) = m\left(r, \frac{1}{f - a}\right) + N\left(r, \frac{1}{f - a}\right)$$

measures the "affinity" of f for the value a.

Nevanlinna's First Main Theorem says that for any meromorphic function f and any $a \in \mathbb{C}$,

$$T\left(r, \frac{1}{f - a}\right) = T(r, f) + O(1), \qquad r \to \infty.$$

This says that asymptotically (as $r \to \infty$), the affinity of f for the value ∞ is the same as its affinity for the value a.

Several important classes of meromorphic functions are characterised by the rate of growth of $T(r, f)$. For example, f is a constant if and only if $T(r, f)$ is bounded and f is rational if and only if $T(r, f) = O(\log r)$. The order of a meromorphic function is defined to be

$$\rho(f) := \limsup_{r \longrightarrow \infty} \frac{\log T(r, f)}{\log r}.$$

The order is well-defined for any meromorphic function but it may be infinite. Finite-order meromorphic functions play a special role in the theory.

The Nevanlinna characteristic is a natural measure of the complexity of a meromorphic function. We will denote by $S(r, f)$, any positive function of $r \geq r_0$, for some r_0, such that

$$S(r, f) = o(T(r, f)),$$

for all $r \in [r_0, \infty) \backslash E$, where E is a set of finite logarithmic measure (i.e. $\int_E \mathrm{d}r < \infty$). The appearance of so called exceptional sets, such as E, is ubiquitous in Nevanlinna theory as they arise naturally in a number of important technical lemmas. Furthermore, we denote by $\mathcal{S}(f)$ the set of all meromorphic functions g such that $T(r, g) = S(r, f)$.

Let $R(z, f)$ be a rational function of y with coefficients a_μ that are meromorphic functions of z. Let $d = \deg_f R(z, f)$ be the degree of R as a rational function of f. Then an extremely useful identity originally due to Valiron [59] and generalized by Mohon'ko [60], says that if the characteristic function of the coefficients are small compared to the characteristic of f, more precisely, if $a_\mu \in \mathcal{S}(f)$, then

$$T(r, R(z, f)) = dT(r, f) + S(r, f). \tag{4.3}$$

An important estimate for meromorphic solutions of difference equations due to Yanagihara [56] is that, given $\epsilon > 0$,

$$T(r, f(z \pm 1)) \leq (1 + \epsilon)T(r + 1, f(z)) + O(1), \tag{4.4}$$

for sufficiently large r.

4.3 Finite-order solutions

Consider the first-order difference equation

$$y(z + 1) = R(z, y(z)), \tag{4.5}$$

where R is as described above. A meromorphic solution y of equation (4.5) is called *admissible* if the coefficient functions $a_\mu(z)$ in R satisfy $a_\mu \in \mathcal{S}(y)$. Note that if $R(z, y) \equiv R(y)$ has constant coefficients, then a meromorphic solution is admissible precisely if it is non-constant. Similarly, if the coefficients a_μ of R are all rational functions, then a meromorphic solution is admissible precisely if it is non-rational.

Let y be an admissible solution of equation (4.5). Taking the Nevanlinna characteristic of both sides of equation (4.5) and using the estimates (4.3) and (4.4) yields

$$(1 + \epsilon)T(r + 1, y) \geq dT(r, y) + S(r, y). \tag{4.6}$$

Ignoring the $S(r, y)$ term in the inequality (4.6) implies that $T(r, y)$ grows exponentially if $d > 1$. Even though, by definition, the $S(r, y)$ term has a possible small exceptional set associated with it, this conclusion remains valid (see [61]). So the only difference equation of the form (4.5) with a finite-order admissible meromorphic solution is the difference Riccati equation

$$y(z + 1) = \frac{a_1(z)y(z) + a_2(z)}{a_3(z)y(z) + a_4(z)}, \tag{4.7}$$

where $a_j \in \mathcal{S}$ for $j \in 1, \ldots, 4$. The case in which R is rational in z and y was first done by Yanagihara [56]. It is well known that equation (4.7) can be linearised.

This is reminiscent of the fact that the only differential equation of the form $y' = R(z, y)$ with the Painlevé property is the (differential) Riccati equation.

It is clear that some sort of admissibility condition is required in the above, so long as we only consider a single solution. Otherwise, we could construct equations of essentially any form as follows. Let y be any finite-order meromorphic function and define $f(z) = y(z+1) - y(z)^7$. Then by construction, there is a finite-order meromorphic solution of the equation

$$y(z+1) = y(z)^7 + f(z). \tag{4.8}$$

This is a very special solution of equation (4.8) and it is not admissible because, intuitively, the complexity of f is approximately the same as the complexity of y.

There are several elementary inequalities involving the Nevanlinna characteristics of combinations of meromorphic functions f, g, h, including

$$T(r, fg) \quad \leq \quad T(r, f) + T(r, g), \tag{4.9}$$

$$T(r, f + g), \quad \leq \quad T(r, f) + T(r, g) + \log 2. \tag{4.10}$$

Also, Grammaticos, Tamizhmani, Ramani and Tamizhmani [12] showed that

$$T(r, fg + gh + hf) \leq T(r, f) + T(r, g) + T(r, g) + \log 3. \tag{4.11}$$

For higher order difference equations, these identities are useful in obtaining restrictions on the degrees of functions that appear in equations with finite-order admissible solutions. For example, if either of the equations

$$y(z+1) + y(z-1) = R(z, y(z)) \quad \text{or} \quad y(z+1)y(z-1) = R(z, y(z))$$

has a finite-order admissible meromorphic solution, then $d := \deg_y R(z, y) \leq 2$. The proof is essentially the same as the first-order case only we use the identities (4.9–4.10) [52]. Similarly Ramani, Grammaticos, Tamizhmani and Tamizhmani [62] considered the equation

$$(y(z+1) + y(z))(y(z) + y(z-1)) = \frac{P(z, y(z)}{Q(z, y(z)},$$

where $P(z, y)$ and $Q(z, y)$ are polynomials in y with no common factors. They showed that $\deg_y P \leq 4$ and $\deg_y Q \leq 2$. These sorts of results are useful as a first step in classifying equations, however, more subtle arguments are required to obtain precise information on the forms of the coefficient functions.

The most complete classification result based on the existence of finite-order meromorphic solutions obtained to date is the following [61].

Theorem 4.1 *If the equation*

$$y(z+1) + y(z-1) = R(z, y(z)), \tag{4.12}$$

where $R(z,y)$ is rational in y and meromorphic in z, has an admissible meromorphic solution of finite order, then either y satisfies a difference Riccati equation

$$y(z+1) = \frac{p(z+1)\,y(z) + q(z)}{y(z) + p(z)}, \tag{4.13}$$

where $p, q \in \mathcal{S}(y)$, or equation (4.12) can be transformed by a linear change in y to one of the following equations:

$$\begin{aligned}
y(z+1) + y(z) + y(z-1) &= \frac{\pi_1 z + \pi_2}{y(z)} + \kappa_1 && (4.14)\\
y(z+1) - y(z) + y(z-1) &= \frac{\pi_1 z + \pi_2}{y(z)} + (-1)^z \kappa_1 && (4.15)\\
y(z+1) + y(z-1) &= \frac{\pi_1 z + \kappa_1}{y(z)} + \frac{\pi_2}{y(z)^2} && (4.16)\\
y(z+1) + y(z-1) &= \frac{\pi_1 z + \pi_3}{y(z)} + \pi_2 && (4.17)\\
y(z+1) + y(z-1) &= \frac{(\pi_1 z + \kappa_1) y(z) + \pi_2}{(-1)^{-z} - y(z)^2} && (4.18)\\
y(z+1) + y(z-1) &= \frac{(\pi_1 z + \kappa_1) y(z) + \pi_2}{1 - y(z)^2} && (4.19)\\
y(z+1)y(z) + y(z)y(z-1) &= p && (4.20)\\
y(z+1) + y(z-1) &= p\,y(z) + q && (4.21)
\end{aligned}$$

where $\pi_k, \kappa_k \in \mathcal{S}(y)$ are arbitrary finite-order periodic functions with period k.

The following partial classification result was obtained for the product case in [63].

Theorem 4.2 *Let y be an admissible finite-order meromorphic solution of the equation*

$$y(z+1)y(z-1) = \frac{c_2(y(z) - c_+)(y(z) - c_-)}{(y(z) - a_+)(y(z) - a_-)} =: R(z, y(z)), \tag{4.22}$$

where the coefficients are meromorphic functions, $c_2 \not\equiv 0$ and $\deg_y(R) = 2$. If the order of the poles of y is bounded, then either y satisfies a difference Riccati equation

$$y(z+1) = \frac{p y(z) + q}{y(z) + s} \tag{4.23}$$

where $p, q, s \in \mathcal{S}(y)$, or equation (4.22) can be transformed by a bilinear change in y to one of the equations

$$\begin{aligned}
y(z+1)y(z-1) &= \frac{\underline{\gamma} y(z)^2 + \delta \lambda^z y(z) + \gamma \mu \lambda^{2z}}{(y(z) - 1)(y(z) - \gamma)} && (4.24)\\
y(z+1)y(z-1) &= \frac{y(z)^2 + \delta e^{i\pi z/2} \lambda^z y(z) + \mu \lambda^{2z}}{y(z)^2 - 1} && (4.25)
\end{aligned}$$

where $\lambda \in \mathbb{C}$, and $\delta, \mu, \gamma \in \mathcal{S}(y)$ are arbitrary finite-order periodic functions such that δ and γ have period 2 and μ has period 1.

The proofs of Theorems 4.1 and 4.2 have to main parts. The first uses Nevanlinna theory to show that admissible solutions have many poles and that the behaviour of $T(r, y)$ is dominated by $N(r, y)$. The main new tool here is the following difference analogue of the lemma on the logarithmic derivative.

Theorem 4.3 ([64, 65]) *Let $f(z)$ be a meromorphic function of finite order and let $c \in \mathbb{C}$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f).$$

This theorem is the main source of the possible exceptional sets that have already been mentioned. A similar result to Theorem 4.3 was obtained independently by Chiang and Feng [66]. Their result does not contain an exceptional set, however, it does not give a direct comparison between $m\left(r, \frac{f(z+c)}{f(z)}\right)$ and $T(r, f)$. We need such a comparison for our purposes.

The second main part of the proofs of Theorems 4.1 and 4.2 is a counting argument, showing that for every point in a disc of radius r at which y takes one of the special values for which the right side of the equation becomes infinite, then one can find sufficiently many poles of y (counting multiplicities) in the disc of radius $r + 1$, unless the equation is one of the special cases listed in the theorem. This part of the argument has many similarities with the standard singularity (non-)confinement calculations (see [61]).

Recently it has been shown in [67] that the conclusions of Theorems 4.1, 4.2 and 4.3 hold if we replace the assumption that y is of finite order by the weaker assumption that

$$\limsup_{r \to \infty} \frac{\log^+ \log^+ T(r, y)}{\log r} < 1.$$

This is equivalent to saying that y has hyper-order strictly less than one. This assumption perhaps looks closer in spirit to the zero algebraic entropy condition.

4.4 Vojta's dictionary and Diophantine integrability

Mathematicians since the late nineteenth century have commented on the strong formal similarity between number theory and complex function theory. Kronecker [68] points out that it is useful to think of the numbers in a number field k as functions of the prime ideals of the ring of integers $\mathcal{O}_k$. Just over a century later, Osgood [69] and Vojta [70] independently drew attention to the remarkable formal similarity between Nevanlinna theory and Diophantine approximation. Vojta went so far as to produce a "dictionary" relating, or translating, key ideas, definitions and theorems between the two fields.

Vojta's dictionary is just a heuristic but it is a very powerful source of precise conjectures. For example, a meromorphic function f in Nevanlinna theory corresponds to an infinite subset of a number field k in Diophantine approximation. The Nevanlinna characteristic $T(r, f)$ corresponds to the logarithmic height $h(x) = \log H(x)$ of some $x \in k$. If $k = \mathbb{Q}$, then the height of any nonzero $x \in \mathbb{Q}$ is $H(x) = \max\{|a|, |b|\}$, where a and b are coprime integers such that $x = a/b$. A weak version of Nevanlina's second main theorem corresponds to the well known approximation theorem of Roth. A Diophantine analogue of the lemma on the logarithmic derivative and a new analogue of Nevanlinna's second main theorem have recently appeared in [71].

Let us apply Vojta's dictionary to the idea that a difference equation of "Painlevé type" in Nevanlinna theory is one with many finite-order meromorphic solutions. A related infinite list of numbers in a number field can obviously be generated by moving back to the corresponding discrete equation with coefficients and initial conditions in the number field. Demanding that the meromorphic solution be of finite-order corresponds to demanding that the logarithmic height of the solution of the discrete equation does not grow exponentially. Such discrete equations are referred to in [72] as being Diophantine integrable.

Slow height growth over the rationals has been used by a several authors as a detector of integrability and as a tool for numerically calculating entropies. Abarenkova, Anglès d'Auriac, Boukraa, Hassani, and Maillard [73] used slow height to determine values of a parameter for which a family of maps is integrable. Height growth and integrability have also been explored in [74, 75, 76, 77].

Numerically testing for slow height growth is perhaps the quickest and simplest of the standard tests, provided the equation has fixed parameters in a number field (e.g., $\mathbb{Q}$). As an example we consider the so-called q-$\mathrm{P_{VI}}$ equation, which is the system

$$\begin{aligned} \frac{f_n f_{n+1}}{cd} &= \frac{g_{n+1} - \alpha q^{n+1}}{g_{n+1} - \gamma}\,\frac{g_{n+1} - \beta q^{n+1}}{g_{n+1} - \delta}, \\ \frac{g_n g_{n+1}}{\gamma\delta} &= \frac{f_n - a q^n}{f_n - c}\,\frac{f_n - b q^n}{f_n - d}, \end{aligned} \tag{4.26}$$

subject to the constraint $q = \alpha\beta\gamma\delta/abcd$. The system (4.26) was discovered by Jimbo and Sakai as the compatibility condition for an isomonodromy problem [78] and is an integrable discretization of the sixth Painlevé equation ($\mathrm{P_{VI}}$). Figure 1 is a plot of $\log\log\max\{H(f_n), H(g_n)\}$ against $\log n$ for iterates of equation (4.26) with the initial conditions $f_0 = 2/3$, $g_0 = 3/4$ and the choice of parameters $(\alpha, \beta, \gamma, \delta, a, b, c, d) = (15/7, 4/3, 1/2, 1, 8/7, 5/7, 2, 1/7)$. The two graphs represent two different choices for q, namely, $q = 1/2(= \alpha\beta\gamma\delta/abcd$, i.e., the integrable case corresponding to the asymptotically linear graph) and $q = 2$. From the graph we see that the logarithmic heights $h(f_n)$ and $h(g_n)$ appear to grow polynomially in the integrable case and exponentially in the non-integrable case.

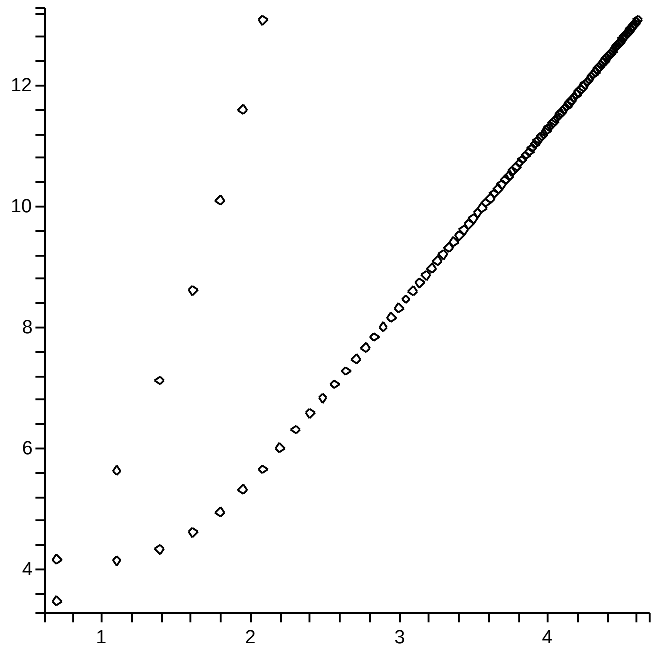


Figure 1: Plot of $\log\log\max\{H(f_n), H(g_n)\}$ vs $\log n$ for equation (4.26)

4.5 Height growth and the discrete Painlevé equations

In this section we describe in outline some methods for proving a number of estimates about height growth. Many of the techniques used to obtain necessary condtions for a difference equation to have a finite-order meromorphic solution also appear to have Diophantine analogues. For example, the identities (4.9–4.11) correspond to

$$\begin{aligned} h(xy) &\leq h(x) + h(y); \\ h(x+y) &\leq h(x) + h(y) + \log 2; \\ h(yz + zx + xy) &\leq h(x) + h(y) + h(z) + \log 3. \end{aligned}$$

There is also a simple analogue of the Valiron-Mohon'ko estimate (see [72]). In order to obtain more precise information on the coefficient functions appearing in equations with solutions having slow height growth, we consider an analogue of singularity confinement.

The logarithmic height on a number field can be expressed as a certain sum over the places (equivalence classes of absolute values) of the number field. In the simplest case $k = \mathbb{Q}$, the only nontrivial absolute values are the the p-adic absolute values $|\cdot|_p$ and usual absolute value, $|\cdot|_\infty$, which is often referred to as the absolute value associated with the "prime at infinity". The p-adic value of the nonzero rational number x is $|x|_p = p^{-r}$, where r is defined to be the unique integer such that $x = p^r a/b$, where p divides neither a nor b. In terms of these absolute values we have for any nonzero $x \in \mathbb{Q}$,

$$h(x) = \sum_{p \leq \infty} \log^+ |x|_p,$$

where the sum is over all primes (finite and infinite). An analogue of singularity confinement can be described in terms of absolute values.

Let us consider the equation

$$y_{n+1} + y_{n-1} = \frac{\alpha_n + \beta_n y_n}{y_n^2}, \tag{4.27}$$

where $\alpha_n \neq 0$ and β_n are rational functions of n with coefficients in $\mathbb{Q}$. If $\alpha_n = A$ and $\beta_n = Bn + C$, where A, B and C are constants, then equation (4.27) is a known discrete Painlevé equation with a continuum limit to the first Painlevé equation. If α_n and β_n are both constants then equation (4.27) can be solved in terms of the Weierstrass elliptic function.

For the rest of this section, we assume that at least one of the functions $\alpha_{n+2} - \alpha_n$ or $\beta_{n+2} - 2\beta_{n+1} + \beta_n$ is not identically zero. With respect to a particular absolute value on $\mathbb{Q}$ we can define a scale, ϵ_n, against which we can compare the size of the iterate y_n. Since $\alpha_{n+2} - \alpha_n$ is a rational function of n, then either it is identically zero or there is an integer n_0 such that $\alpha_{n+2} - \alpha_n \neq 0$ for all $n > n_0$. The same is true for the rational function $\beta_{n+2} - 2\beta_{n+1} + \beta_n$.

Let $\kappa_p = 1$ for all primes $p < \infty$ and $\kappa_\infty = 3$. For sufficiently small $\delta > 0$, we define $\epsilon_n > 0$ as follows.

1. If $|\alpha_{n+2} - \alpha_n| \neq 0$ then

$$\begin{aligned} \epsilon_n^{-\delta} &= \kappa_p \max\{1, |\alpha_n|^{-1}, |\alpha_n|, |\beta_n|, |\alpha_{n+1}|, |\alpha_{n-1}|, |\beta_{n-1}|, |\beta_{n+1}|, \\ &\quad 2|\alpha_{n-2}|^{-1}, |\alpha_{n-2}|, |\beta_{n-2}|, |\alpha_n - \alpha_{n-2}|, |\alpha_n - \alpha_{n-2}|^{-1}\} \end{aligned} \tag{4.28}$$

2. If $|\alpha_{n+2} - \alpha_n| \equiv 0$ and $|\beta_{n+2} - 2\beta_{n+1} + \beta_n| \neq 0$ then

$$\begin{aligned} \epsilon_n^{-\delta} &= \kappa_p \max\{1, |\alpha_n|, |\alpha_n|^{-1}, |\beta_n|, |\alpha_{n+1}|, |\alpha_{n-1}|, |\beta_{n+1}|, \\ &\quad |\beta_n - 2\beta_{n-1} + \beta_{n-2}|^{-1}\}. \end{aligned} \tag{4.29}$$

The following estimates (see [79]) represent a version of the singularity (non-)confinement calculation interpreted in terms of a particular absolute value on $\mathbb{Q}$.

Theorem 4.4 *Let $(y_n)_{n=k-1}^{k+3} \subseteq \mathbb{Q}/\{0\}$ with $k - 1 \geq r_0$ satisfy equation (4.27) where $\alpha_n \not\equiv 0$, $\alpha_n, \beta_n \in \mathbb{Q}(n)$ and at least one of the functions $\alpha_{n+2} - \alpha_n$ or $\beta_{n+2} - 2\beta_{n+1} + \beta_n$ is not identically zero. If $|y_{k-1}| \leq |y_k|^{-1/2}$ and, for sufficiently small $\delta > 0$, $|y_k| < \epsilon_k$ then*

1. $y_{k+1} = \frac{\alpha_k}{y_k^2} + \frac{\beta_k}{y_k} + A_k$, *where* $|A_k| \leq |y_k|^{-1/2}$.

2. $y_{k+2} = -y_k + \frac{\beta_{k+1}}{\alpha_k} y_k^2 + B_k$, *where* $|B_k| \leq |y_k|^{3-4\delta}$

3. $y_{k+3} = \frac{\alpha_{k+2}-\alpha_k}{y_{k+2}^2} + \frac{\beta_{k+2}-2\frac{\alpha_{k+2}}{\alpha_k}\beta_{k+1}+\beta_k}{y_{k+2}} + C_k$,
where $|C_k| \leq \max\left\{|\frac{\alpha_{k+2}-\alpha_k}{\alpha_k}||y_{k+2}|^{1-\delta}, |y_{k+2}|^{-1/2}\right\}$ *for non-Archimedean absolute values and* $|C_k| \leq 2|\frac{\alpha_{k+2}-\alpha_k}{\alpha_k}||y_{k+2}|^{1-\delta}+3|y_k|^{-1/2}$ *for Archimedean absolute values.*

In [79], this theorem is used to prove that if equation (4.27) has an admissible solution with polynomial height growth, then $\alpha_n = A$ and $\beta_n = Bn + C$, where A, B and C are constants, corresponding to the discrete Painlevé equation mentioned above. This supports the idea that slow height growth leads to discrete integrable equations.

4.6 Finite fields

So far we have considered discrete equations over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}$. We conclude this review with a brief description of the work of Roberts and Vivaldi and Jogia [80, 81] on rational symplectic maps over finite fields and the use of statistics concerning the lengths of orbits as a detector of integrability. The main theoretical idea underlying this analysis is the celebrated Hasse-Weil bound, which provides a sharp estimate for the number of points on a given algebraic curve over a finite field.

For any prime number p, let $\mathbb{F}_p$ denote the field of integers modulo p. Let $C(\mathbb{F}_p)$ be an irreducible algebraic curve over $\mathbb{F}_p$ of genus g. Then the number $\sharp C(\mathbb{F}_p)$ of points on this curve is constrained by the Hasse-Weil bound

$$p + 1 - 2g\sqrt{p} \leq \sharp C(\mathbb{F}_p) \leq p + 1 + 2g\sqrt{p}.$$

Now consider a rational map $f : \mathbb{F}_p^2 \to \mathbb{F}_p^2$. Since the phase space is finite, all orbits of f are periodic. If f has a polynomial first integral, $I(x, y)$, then all points on a particular orbit must lie on a curve C of the form $I(x, y) = \alpha$, for some $\alpha \in \mathbb{F}_p$. Hence the Hasse-Weil bound provides a necessary condition for the existence of such a first integral that is irreducible and of genus g.

For any rational map on $\mathbb{R}^2$ or $\mathbb{C}^2$ of infinite order possessing a rational integral, the genus of each level set is 0 or 1 [82]. Generically the genus is 1 unless the curve possesses singularities. In [80, 81], the authors considered rational maps with coefficients in $\mathbb{Q}$. They considered reductions of these maps modulo various primes p and looked at the lengths of orbits (either the maximum orbit length, or the average orbit length for each p). By exploring different values of parameters, they found that the orbit lengths compared to the Hasse-Weil (upper) bound clearly indicates the values of the parameters for which the map is integrable. When the integrable case corresponds to a non-integer rational value of the parameter, a sieve method can be used to deduce the value by exploring a number of choices of the prime p.

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