Expansions on special solutions of the first q-Painlevé equation around the infinity

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Abstract

The first q-Painlevé equation has a unique formal solution around the infinity. This series converges only for $|q| = 1$. If q is a root of unity, this series expresses an algebraic function. In cases that all coefficients are integers, it can be represented by generalized hypergeometric series.

1 Introduction

In this short note, we show a strange phenomenon on an asymptotic series which satisfy the first q-Painlevé equation. It is known that the continuous first Painlevé equation $y'' = 6y^2 + t$ has an asymptotic solution of the form $y \sim \pm\sqrt{-t/6}$ $(t \to -\infty)$, which is called Boutroux's tritronquée solution [1] [2],

We study a q-analogue of Boutorux's solution for the first q-Painlevé equation (q-P_{I}):

$$\bar{f} f^2 \underline{f} = t(1 - f). \tag{1}$$

Here $f = f(t), \bar{f} = f(tq), \underline{f} = f(t/q)$ for $q \in \mathbb{C}^*$. It is known that q-P_{I} reduces to the continuous first Painlevé equation when $q \to 1$.

In Sakai's list [4], the infinite divisor of the initial value of q-P_{I} is of the $A_7^{(1)}$-type. Recently Nishioka showed that q-P_{I} has not any solution which is reduced to the first order q-difference equation [3]. The author does not know any examples of solutions of q-P_{I} and we might show the first example of special solutions of q-P_{I}.

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2 Asymptotic expansions

Assume that q-P_{I} has a formal solution of the form

$$f = \sum_{n=l}^{\infty} a_n(q) t^{-n}$$

for a suitable integer l. It is easy to show that $l = 0$ and $a_0(q) = 1$.

Theorem 1. *There exists a unique formal solution of the form*

$$f = \sum_{n=0}^{\infty} a_n(q) t^{-n} \quad (a_0(q) \neq 0).$$

$$\begin{aligned}
a_0(q) &= 1, \qquad a_1(q) = -1, \\
a_2(q) &= \frac{1}{q} + 2 + q, \\
a_3(q) &= -\frac{1}{q^3} - \frac{2}{q^2} - \frac{5}{q} - 6 - 5q - 2q^2 - q^3, \\
a_4(q) &= \frac{1}{q^6} + \frac{2}{q^5} + \frac{5}{q^4} + \frac{10}{q^3} + \frac{16}{q^2} + \frac{23}{q} + 26 + \\
&\qquad + 23q + 16q^2 + 10q^3 + 5q^4 + 2q^5 + q^6.
\end{aligned}$$

Set $\beta(n) = n(n-1)/2$. Then we have

$$a_n(q) = (-1)^n \sum_{j=-\beta(n)}^{-\beta(n)} c_{n,j} q^j.$$

Here $c_{n,j} \in \mathbb{Z}+$, $c_{n,j} = c_{n,-j}$. Moreover f is a divergent series in case $0 < |q| < 1$ or $1 < |q|$

Proof. It is evident that $a_n(q)$ satisfies a recurrence relation

$$a_n(q) = -F_n(a_1(q), a_2(q), \cdots, a_{n-1}(q), q, 1/q).$$

The coefficients of the polynomial F_n with $(n+1)$-variables are positive and

$$F_n(a_1(q), \cdots, a_{n-1}(q), q, 1/q) = F_n(a_1(q), \cdots, a_{n-1}(q), 1/q, q).$$

Since the leading term on q of F_n is

$$F_n = \left(2 + q^{n-1} + q^{1-n}\right) a_{n-1}(q) + \cdots ,$$

the order of $a_n(q)$ on q is $\beta(n)$. Therefore f diverges for $0 < |q| < 1$ or $1 < |q|$. $\square$

We consider the case $|q| = 1$. Since $c_{n,j} = c_{n,-j}$, $a_n(q)$ is real for $|q| = 1$. If we set $q = e^{i\theta}$,

$$\begin{aligned}
a_1(q) &= -1\\
a_2(q) &= 2(1 + \cos\theta)\\
a_3(q) &= 2(3 + 5\cos\theta + 2\cos 2\theta + \cos 3\theta)\\
a_4(q) &= 4\cos^2\theta/2(5 + 16\cos\theta + 4\cos 2\theta + 8\cos 3\theta + 2\cos 5\theta)
\end{aligned}$$

and

$$|a_n(q)| \leqq (-1)^n a_n(1).$$

We show that $\{a_n(1)\}$ is nothing but a generalized hypergeometric series. We set

$$\begin{aligned}
G_1(t) &:= {}_4F_3\left(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}; \frac{2}{3}, \frac{3}{3}, \frac{4}{3}; -\frac{256}{27t}\right),\\
G_2(t) &:= {}_4F_3\left(\frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \frac{5}{4}; \frac{3}{3}, \frac{4}{3}, \frac{5}{3}; -\frac{256}{27t}\right),\\
G_3(t) &:= {}_4F_3\left(\frac{3}{4}, \frac{4}{4}, \frac{5}{4}, \frac{6}{4}; \frac{4}{3}, \frac{5}{3}, \frac{6}{3}; -\frac{256}{27t}\right).
\end{aligned}$$

Theorem 2. *When $|q| = 1$, f converges at least for $|t| > 256/27$. In case $q^N = 1$, f is algebraic. Iff $q = e^{k\pi i/3}$ $(k = 0, 1, 2, 3, 4, 5)$ or $q = \pm i$, $a_n(q)$ are integers for any $n = 1, 2, 3, 4, \ldots$.*

In case $q = 1$, $f^4 = t(1 - f)$ and $f = G_1(t)$.

In case $q = -1$, $f^2(2 - f)^2 = t(1 - f)$ and $f = 1 - G_2(t^2)/t$.

In case $q = e^{2\pi i/3}$, $f^4 = t(1 - f)(1 - 3f + 3f^2)$ and

$$f = G_1(t^3) - \frac{1}{t}G_2(t^3) + \frac{1}{t^2}G_3(t^3).$$

In case $q = i$, f satisfies a complicated algebraic relation and

$$f = 1 - \frac{2}{t} + \left(\frac{1}{t} + \frac{2}{t^2}\right) G_1(t^4) - \frac{2}{t^3}G_2(t^4) + \frac{2}{t^4}G_3(t^4).$$

We can prove the theorem by direct calculation. It is open to represent $\{a_n(e^{\pi i/3})\}$ as a hypergeometric series, since it may satisfy a more higer order equation.

3 Conclusion

We can consider the similar series for other q-Painlevé equations. For the continuous Painlevé equations except the sixth, generic asymptotic series around the infinity are always divergent. When the series are convergent, the solutions should be rational or confluent hypergeometric.

For q-Painlevé equations, asymptotic series might be also divergent for $|q| \neq 1$ except rational solutions and hypergeometric-type solutions. But the case $|q| = 1$ is interesting in the study of q-Painlevé equations.

Acknowledgements

This work was enlightened at the Isaac Newton Institute for Mathematical Sciences in Cambridge during the programme Discrete Integrable Systems. This work was supported by JSPS Grant-in-Aid for Scientific Research (C) 21540217.

References

[1] Boutroux, P.; Recherches sur les transcendantes de M. Painlevé et l'étude asymptotique des équations différentielles du second ordre. *Ann. Sci. École Norm. Sup. (3)* Sér. 3, **30** (1913), 255–375; **31** (1914), 99–159.

[2] Joshi, N. and Kitaev, A; On Boutroux's tritronquée solutions of the first Painlevé equation, *Stud. Appl. Math.* **107** (2001) 253–291.

[3] Nishioka, S.; "On Solutions of q-Painlevé Equation of Type $A_7^{(1)}$, *Funkcial. Ekvac.* **52** (2009), 41–51.

[4] Sakai, H.; Rational surfaces associated with affine root systems and geometry of the Painlevé equations. *Comm. Math. Phys.* **220** (2001), 165–229.