

## RESEARCH ARTICLE

### *The discrete potential Boussinesq equation and its multisoliton solutions*

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A new discrete potential Boussinesq equation is proposed and its multisoliton solutions are constructed. An ultradiscrete potential Boussinesq equation is also obtained from the discrete potential Boussinesq equation using the ultradiscretization technique. The detail of the multisoliton solutions is discussed by using the reduction technique. The lattice potential Boussinesq equation is also investigated by using the singularity confinement test. The relation between the proposed discrete potential Boussinesq equation and the lattice potential Boussinesq equation is clarified.

**Keywords:** discrete potential Boussinesq equation; multisoliton solutions; bilinear equations

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#### 1. Introduction

In this article, we propose a new discrete potential Boussinesq (BSQ) equation

$$U_{n-1}^{m+2}U_{n-1}^{m-1}(-\delta_1U_n^{m-1} + \delta_2U_n^{m+2}) = U_n^{m+1}U_{n-1}^m(-\delta_1U_{n-1}^{m-1} + \delta_2U_{n-1}^{m+2}), \quad (1)$$

and study the relation to the lattice potential BSQ equation proposed by Nijhoff et al. [1, 2]

$$\begin{aligned} & \frac{p^3 - q^3}{p - q + u_{n+1}^{m+1} - u_{n+2}^m} - \frac{p^3 - q^3}{p - q + u_n^{m+2} - u_{n+1}^{m+1}} \\ & - u_n^{m+1}u_{n+1}^{m+2} + u_{n+1}^m u_{n+2}^{m+1} + u_{n+2}^{m+2}(p - q + u_{n+1}^{m+2} - u_{n+2}^{m+1}) \\ & + u_n^m(p - q + u_n^{m+1} - u_{n+1}^m) \\ & = (2p + q)(u_{n+1}^m + u_{n+1}^{m+2}) - (p + 2q)(u_n^{m+1} + u_{n+2}^{m+1}). \end{aligned} \quad (2)$$

Here  $U = U_n^m$  and  $u = u_n^m$  are the dynamical field variable at the site  $(n, m)$  of a rectangular lattice, and  $\delta_1, \delta_2, p, q$  are the lattice parameters. These are integrable discrete analogues of the potential BSQ equation

$$3w_{tt} + 4c_0w_{xx} - 6w_xw_{xx} - w_{xxxx} = 0, \quad (3)$$

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which leads to the BSQ equation[3, 4]

$$3u_{tt} + 4c_0u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0, \quad (4)$$

by  $u = w_x$ . Note that Eq.(2) is the lowest order members of a hierarchy of lattice equations which is called the lattice Gelfand-Dikii(GD) hierarchy[1]. See also recent works about the lattice potential BSQ equation (2)[5, 6].

In this paper, we propose a new discrete potential BSQ equation and present multisoliton solutions. It is shown that the multisoliton solution for the discrete potential BSQ equations can be constructed from one for the discrete KP (Hirota-Miwa) equation using the reduction technique. Using the discrete potential BSQ equation, we can construct the ultradiscrete potential BSQ equation. We also study the relation between the discrete potential BSQ equation (1) and the lattice potential BSQ equation (2). Bilinear equations of the lattice potential BSQ equation can be derived systematically using the SC test. This reveals the relationship with other discrete potential BSQ equations.

## 2. Multisoliton solutions for the Boussinesq equation

First, we review fundamental results about reductions of the KP equation.

It is well known that the solutions of the KP equation[7]

$$(-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0, \quad (5)$$

can be expressed via a tau function  $\tau(x, y, t)$  as

$$u(x, y, t) = 2 \frac{\partial^2}{\partial x^2} \log \tau(x, y, t), \quad (6)$$

where  $\tau(x, y, t)$  satisfies Hirota's bilinear equation

$$(D_x^4 - 4D_x D_t + 3D_y^2)\tau \cdot \tau = 0. \quad (7)$$

Soliton solutions of Hirota's equation can be written in terms of the Wronskian determinant [8–11]

$$\tau(x, y, t) = \text{Wr}(f_1, \dots, f_N) = \det(f_n^{(n'-1)})_{1 \leq n, n' \leq N}, \quad (8)$$

with  $f_n^{(j)} = \partial^j f_n / \partial x^j$ , and where  $f_1, \dots, f_N$  are a set of linearly independent solutions of the linear system

$$\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial f}{\partial t} = \frac{\partial^3 f}{\partial x^3}. \quad (9)$$

For example, ordinary  $N$ -soliton solutions are obtained by taking  $f_n = e^{\theta_{2n-1}} + e^{\theta_{2n}}$  for  $n = 1, \dots, N$ , where

$$\theta_m(x, y, t) = -k_m x + k_m^2 y - k_m^3 t + \theta_{m;0}, \quad (10)$$

for  $m = 1, \dots, 2N$ , where the  $4N$  parameters  $k_1 < \dots < k_{2N}$  and  $\theta_{1;0}, \dots, \theta_{2N;0}$  are

real constants. For  $N = 1$ , one obtains the single-soliton solution of KP equation:

$$u_{i,j}(x, y, t) = \frac{1}{2}(k_j - k_i)^2 \operatorname{sech}^2 \left[ \frac{1}{2}(\theta_i - \theta_j) \right], \quad (11)$$

where  $i = 1$  and  $j = 2$ . The most general form of the  $N$ -soliton solution is given by

$$\tau(x, y, t) = \det(A\Theta K) = \sum_{1 \leq m_1 < \dots < m_N \leq M} V_{m_1, \dots, m_N} A_{m_1, \dots, m_N} \exp \theta_{m_1, \dots, m_N}, \quad (12)$$

where  $A = (a_{n,m})$  is the  $N \times M$  coefficient matrix,  $\Theta = \operatorname{diag}(e^{\theta_1}, \dots, e^{\theta_M})$ , and the  $M \times N$  matrix  $K$  is given by  $K = (k_m^{n-1})$  [12–17].  $V_{m_1, \dots, m_N}$  is the Vandermonde determinant  $V_{m_1, \dots, m_N} = \prod_{1 \leq j < j' \leq N} (k_{m_{j'}} - k_{m_j})$ , and  $A_{m_1, \dots, m_N}$  is the  $N \times N$ -minor whose  $n$ -th column is respectively given by the  $m_n$ -th column of the coefficient matrix for  $n = 1, \dots, N$ . The only time dependence in the tau function comes from the exponential phases  $\theta_{m_1, \dots, m_N}$ . Also, for all  $G \in \operatorname{GL}(N, \mathbb{R})$ , the coefficient matrices  $A$  and  $A' = GA$  produce the same solution of the KP equation. Thus without loss of generality one can consider  $A$  to be in row-reduced echelon form (RREF). One can also multiply each column of  $A$  by an arbitrary positive constant which can be absorbed in the definition of  $\theta_{1;0}, \dots, \theta_{M;0}$ .

Real nonsingular (positive) solutions of the KP equation are obtained if  $k_1 < \dots < k_M$  and all minors of  $A$  are nonnegative. Under these assumptions and some fairly general irreducibility conditions on the coefficient matrix, Eq.(29) produces  $(N_-, N_+)$ -soliton solutions of the KP equation with  $N_- = M - N$  and  $N_+ = N$ , as in the simpler case of fully resonant solutions. Asymptotic line solitons are given by Eq.(11) with the indices  $i$  and  $j$  labeling the phases  $\theta_i$  and  $\theta_j$  being swapped in the transition between two dominant phase combinations along the line  $\theta_i = \theta_j$ . Asymptotic solitons can thus be uniquely characterized by an index pair  $[i, j]$  with  $1 \leq i < j \leq M$ . Recently, line soliton solutions of the KP II equation were classified using this formulation [12–17]. Elastic 2-soliton solutions are classified into three classes: ordinary (O-type), asymmetric (P-type) and resonant (T-type). The coefficient matrices corresponding to these classes have the following RREFs:

$$A_O = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & b \end{pmatrix}, \quad A_P = \begin{pmatrix} 1 & 0 & 0 & -b \\ 0 & 1 & a & 0 \end{pmatrix}, \quad A_T = \begin{pmatrix} 1 & 0 & -c & -d \\ 0 & 1 & a & b \end{pmatrix}, \quad (13)$$

where  $a, b, c, d > 0$  are free parameters with  $ad - bc > 0$ . These three types of solutions cover disjoint sectors of the 2-soliton parameter space of amplitudes and directions. Moreover, their interaction properties are also different. This difference is obvious in the case of T-type solutions, but also applies to O-type and P-type solutions, since P-type solutions only exist for unequal amplitude, and the interaction phase shift has the opposite sign for O-type and P-type solutions. Inelastic 2-soliton solutions fall into four categories.

It is known that the KP equation reduces to the KdV equation

$$-4u_t + 6uu_x + u_{xxx} = 0 \quad (14)$$

by the constraint  $\partial u / \partial y = 0$  [13, 18–20]. In the bilinear form, the bilinear KP equation (7) is reduced to the bilinear KdV equation

$$(D_x^4 - 4D_x D_t) \tau \cdot \tau = 0, \quad (15)$$

by the constraint of omitting terms including  $D_y$ , which is the so-called 2-reduction. As mentioned in , this constraint implies that all the solitons of the KdV equation are parallel to the  $y$ -axis. Thus we must have a condition  $k_j = -k_i$  for each  $[i, j]$ -soliton. From the ordering  $k_1 < k_2 < \dots < k_{2N}$ , we must assume

$$k_1 < k_2 < \dots < k_N < 0, \quad k_{N+j} = -k_{N-j+1} \text{ for } j = 1, \dots, N.$$

This allow only P-type soliton solution in which  $A$ -matrix is

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 & a_{1,2N} \\ 0 & 1 & \dots & 0 & 0 & \dots & a_{2,2N-1} & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_{N,N} & \dots & 0 & 0 \end{pmatrix}.$$

Let us consider multisoliton solutions of the BSQ equation. It is also known that the KP equation is reduced to the BSQ equation without  $u_{xx}$

$$3\delta_0 u_{t't'} - 3(u^2)_{xx} - u_{xxxx} = 0, \quad (16)$$

by the constraint  $\partial u / \partial t = 0$  [18–20]. Here we introduced a new independent variable  $t'$  such that  $y = \sqrt{-\delta_0} t'$ ,  $\delta_0 = \pm 1$ . In the bilinear form, the bilinear KP equation (7) is reduced to

$$(D_x^4 - 3\delta_0 D_{t'}^2) \tau \cdot \tau = 0, \quad (17)$$

by the constraint of omitting terms including  $D_t$ , which is the so-called 3-reduction. This condition implies that all the solitons of Eq.(16) are parallel to the  $t$ -axis. Thus we must have a condition  $k_j^2 + k_j k_i + k_i^2 = 0$ , i.e.  $k_j = \omega k_i$  or  $k_j = \omega^2 k_i$  ( $\omega = -1/2 + i\sqrt{3}/2$ ,  $\omega^2 = -1/2 - i\sqrt{3}/2$ ,  $\omega^3 = 1$ ), for  $[i, j]$ -soliton. In the BSQ equation,  $k_1, \dots, k_{2N}$  take complex values in general. Since  $k_1, \dots, k_{2N}$  are complex values, we cannot consider the ordering of  $k_1, \dots, k_{2N}$  which was assumed when we consider KP soliton solutions. However, from the constraint we have a restriction to the  $A$ -matrix such that each row has only 2 or 3 nonzero elements and each column has only one element. For 2-soliton solutions, 2 types of 2-soliton interactions are possible, i.e. 2 elastic soliton interactions (O-type and P-type), other 2-soliton interactions are impossible because some columns in  $A$ -matrix have 2 elements. However, there is no distinction between O-type and P-type solitons because we don't have the ordering of  $k_1, \dots, k_{2N}$ . Thus 2-soliton interaction of the BSQ equation is actually only one type. To get real solutions, we must remove imaginary numbers by the following ways.

The case of 2 elements:

(i) Suppose that we have 2 elements in the  $i$ -th row of the  $A$ -matrix. Let the corresponding wave numbers of these elements be  $k_{j_1}, k_{j_2}$ ; (ii) Let  $k_{j_2} = \omega k_{j_1}$  (or  $k_{j_2} = \omega^2 k_{j_1}$ ); (iii) Using the gauge invariance of  $\tau$ -function, each element in the Wronskian determinant can be  $f_i \sim 1 + a_{i,j_2} \exp((\omega - 1)k_{j_1}x + \sqrt{-\delta_0}(\omega^2 - 1)k_{j_1}^2 t' + \theta_{j_2,0})$  (or  $f_i \sim 1 + a_{i,j_2} \exp((\omega^2 - 1)k_{j_1}x + \sqrt{-\delta_0}(\omega - 1)k_{j_1}^2 t' + \theta_{j_2,0})$ ); (iv) Reparametrize  $k_{j_1}$  and  $k_{j_2}$  by  $\kappa_i = (\omega - 1)k_{j_1}$  and  $\Omega_i = \sqrt{-\delta_0}(\omega^2 - 1)k_{j_1}^2$  (or  $\kappa_i = (\omega^2 - 1)k_{j_1}$  and  $\Omega_i = \sqrt{-\delta_0}(\omega - 1)k_{j_1}^2$ ). Then we have  $f_i \sim 1 + a_{i,j_2} \exp(\kappa_i x + \Omega_i t' + \theta_{j_2,0})$  with linear dispersion relations  $\kappa_i^4 - 3\delta_0 \Omega_i^2 = 0$ ; (vi) So a set of  $f_i \sim 1 + a_{i,j_2} \exp(\kappa_i x + \Omega_i t' + \theta_{j_2,0})$  with linear dispersion relations  $\kappa_i^4 - 3\delta_0 \Omega_i^2 = 0$  gives the real and regular multisoliton solutions.

The case of 3 elements:

(i) Suppose that we have 3 elements in the  $i$ -th row of the  $A$ -matrix. Let the corresponding wave numbers of these elements be  $k_{j_1}, k_{j_2}, k_{j_3}$ ; (ii) Let  $k_{j_2} = \omega k_{j_1}$  and  $k_{j_3} = \omega^2 k_{j_1}$ ; (iii) Using the gauge invariance of  $\tau$ -function, each element in the Wronskian determinant can be  $f_i \sim 1 + a_{i,j_2} \exp((\omega - 1)k_{j_1}x + (\omega^2 - 1)k_{j_1}^2 y + \theta_{j_2,0}) + a_{i,j_3} \exp((\omega^2 - 1)k_{j_1}x + (\omega - 1)k_{j_1}^2 y + \theta_{j_3,0})$ ; (iv) Reparametrize parameters  $k_{j_1}, k_{j_2}$  and  $k_{j_3}$  by  $\kappa_{i,2} = k_{j_2} - k_{j_1}, \kappa_{i,3} = k_{j_3} - k_{j_1}, \Omega_{i,2} = \sqrt{-\delta_0}(k_{j_2}^2 - k_{j_1}^2)$  and  $\Omega_{i,3} = \sqrt{-\delta_0}(k_{j_3}^2 - k_{j_1}^2)$ . Then we have  $f_i \sim 1 + a_{i,j_2} \exp(\kappa_{i,2}x + \Omega_{i,2}t' + \theta_{j_2,0}) + a_{i,j_3} \exp(\kappa_{i,3}x + \Omega_{i,3}t' + \theta_{j_3,0})$  with linear dispersion relations  $\kappa_{i,j}^4 - 3\delta_0\Omega_{i,j}^2 = 0$  for  $j = 1, 2$ ; For example, consider  $N = 1$ . This gives Y-shape soliton resonance interaction.

The KP equation is reduced to the BSQ equation

$$3\delta_0 u_{t't'} + 4c_0 u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0, \tag{18}$$

by the constraint  $\frac{\partial}{\partial t} = c_0 \frac{\partial}{\partial x}$ . Here we introduced a new independent variable  $t'$  such that  $y = \sqrt{-\delta_0}t'$ . In the bilinear form, the bilinear KP equation (7) is reduced to

$$(D_x^4 - 4c_0 D_x^2 - 3\delta_0 D_{t'}^2)\tau \cdot \tau = 0. \tag{19}$$

by the constraint of replacing  $D_t$  by  $c_0 D_x$ , which is the so-called 3-pseudo reduction[20]. To realize this constraint in the multisoliton solutions, we can add the constraint

$$k_j^3 - k_i^3 = c_0(k_j - k_i), \tag{20}$$

i.e.

$$k_j = \frac{1}{2} \left( -k_i \pm \sqrt{4c_0 - 3k_i^2} \right), \tag{21}$$

to the soliton solutions of the KP equation. Note that  $k_j$  can be real if  $4c_0 > 3k_i^2$ . So we can assume the ordering of  $k_1 < \dots < k_{2N}$ . From the constraint we have a restriction to the  $A$ -matrix such that each row has only 2 or 3 nonzero elements and each column has only one element. For 2-soliton solutions, 2 types of 2-soliton interactions are possible, i.e. 2 elastic soliton interactions (O-type and P-type), other 2-soliton interactions are impossible because some columns in  $A$ -matrix have 2 elements.

Let us consider explicit real and regular multisoliton solutions. For the case of  $\delta_0 = -1$ , (i) Suppose that we have 2 elements in the  $i$ -th row of the  $A$ -matrix. Let the corresponding wave numbers of these elements be  $k_{j_1}, k_{j_2}$ ; (ii) Let  $k_{j_2} = \frac{1}{2} \left( -k_{j_1} \pm \sqrt{4c_0 - 3k_{j_1}^2} \right)$ ; (iii) Each element in the Wronskian determinant can be  $f_i \sim a_{i,j_1} \exp(k_{j_1}x + k_{j_1}^2 t' + \theta_{j_1,0}) + a_{i,j_2} \exp(k_{j_2}x + k_{j_2}^2 t' + \theta_{j_2,0})$ . This gives multisoliton solutions for the BSQ equation.

If we have 3 elements in the  $i$ -th row of the  $A$ -matrix, each element in the Wronskian determinant can be  $f_i \sim a_{i,j_1} \exp(k_{j_1}x + k_{j_1}^2 t' + \theta_{j_1,0}) + a_{i,j_2} \exp(k_{j_2}x + k_{j_2}^2 t' + \theta_{j_2,0}) + a_{i,j_3} \exp(k_{j_3}x + k_{j_3}^2 t' + \theta_{j_3,0})$ . For  $N = 1$ , this gives Y-shape resonant soliton solution.

For the case of  $\delta_0 = 1$ , (i) Suppose that we have 2 elements in the  $i$ -th row of the  $A$ -matrix. Let the corresponding wave numbers of these elements be  $k_{j_1}, k_{j_2}$ ; (ii) Let  $k_{j_2} = \frac{1}{2} \left( -k_{j_1} \pm \sqrt{4c_0 - 3k_{j_1}^2} \right)$ ; (iii) Using the gauge invariance of  $\tau$ -function, each

element in the Wronskian determinant can be  $f_i \sim 1 + a_{i,j_2} \exp((k_{j_2} - k_{j_1})x + i(k_{j_2}^2 - k_{j_1}^2)t' + \theta_{j_2,0})$ ; (iv) Reparametrize  $k_{j_1}$  and  $k_{j_2}$  by  $\kappa_i = k_{j_2} - k_{j_1}$  and  $\Omega_i = i(k_{j_2}^2 - k_{j_1}^2)$ . Then we have  $f_i \sim 1 + a_{i,j_2} \exp(\kappa_i x + \Omega_i t' + \theta_{j_2,0})$  with linear dispersion relations  $\kappa_i^4 - 4c_0 \kappa_i^2 - 3\delta_0 \Omega_i^2 = 0$ .

Next, we consider the case of complex parameters. Suppose that a parameter  $k_i$  corresponds to the pivot of  $A$ -matrix. If  $4c_0 < 3k_i^2$ , then we have a complex parameter

$$k_j = \frac{1}{2} \left( -k_i \pm i\sqrt{3k_i^2 - 4c_0} \right),$$

among  $k_1, \dots, k_{2N}$ . Since some of  $k_1, \dots, k_{2N}$  are complex values, we cannot consider the ordering of  $k_1, \dots, k_{2N}$  which was assumed when we consider KP soliton solutions. However, from the constraint we have a restriction to the  $A$ -matrix such that each row has only 2 or 3 nonzero elements and each column has only one element. For 2-soliton solutions, 2 types of 2-soliton interactions are possible, i.e. 2 elastic soliton interactions (O-type and P-type), other 2-soliton interactions are impossible because some columns in  $A$ -matrix have 2 elements. However, there is no distinction between O-type and P-type solitons because we don't have the ordering of  $k_1, \dots, k_{2N}$ . Thus 2-soliton interaction of the BSQ equation is actually only one type again. To get real solutions, we must remove imaginary numbers by the following ways.

The case of 2 elements:

(i) Suppose that we have 2 elements in the  $i$ -th row of the  $A$ -matrix. Let the corresponding wave numbers of these elements be  $k_{j_1}, k_{j_2}$ ; (ii) Let  $k_{j_2} = \frac{1}{2} \left( -k_{j_1} \pm i\sqrt{3k_{j_1}^2 - 4c_0} \right)$ ; (iii) Using the gauge invariance of  $\tau$ -function, each element in the Wronskian determinant can be  $f_i \sim 1 + a_{i,j_2} \exp((k_{j_2} - k_{j_1})x + (k_{j_2}^2 - k_{j_1}^2)y + \theta_{j_2,0})$ ; (iv) Reparametrize  $k_{j_1}$  and  $k_{j_2}$  by  $\kappa_i = k_{j_2} - k_{j_1}$  and  $\Omega_i = \sqrt{-\delta_0}(k_{j_2}^2 - k_{j_1}^2)$ . Then we have  $f_i \sim 1 + a_{i,j_2} \exp(\kappa_i x + \Omega_i t' + \theta_{j_2,0})$  with linear dispersion relations  $\kappa_i^4 - 4c_0 \kappa_i^2 - 3\delta_0 \Omega_i^2 = 0$ ; (vi) So a set of  $f_i \sim 1 + a_{i,j_2} \exp(\kappa_i x + \Omega_i t' + \theta_{j_2,0})$  with linear dispersion relations  $\kappa_i^4 - 4c_0 \kappa_i^2 - 3\Omega_i^2 = 0$  gives the real and regular multisoliton solutions.

The case of 3 elements:

(i) Suppose that we have 3 elements in the  $i$ -th row of the  $A$ -matrix. Let the corresponding wave numbers of these elements be  $k_{j_1}, k_{j_2}, k_{j_3}$ ; (ii) Let  $k_{j_2} = \frac{1}{2} \left( -k_{j_1} + i\sqrt{3k_{j_1}^2 - 4c_0} \right)$  and  $k_{j_3} = \frac{1}{2} \left( -k_{j_1} - i\sqrt{3k_{j_1}^2 - 4c_0} \right)$ ; (iii) Using the gauge invariance of  $\tau$ -function, each element in the Wronskian determinant can be  $f_i \sim 1 + a_{i,j_2} \exp((k_{j_2} - k_{j_1})x + \sqrt{-\delta_0}(k_{j_2}^2 - k_{j_1}^2)t' + \theta_{j_2,0}) + a_{i,j_3} \exp((k_{j_3} - k_{j_1})x + \sqrt{-\delta_0}(k_{j_3}^2 - k_{j_1}^2)t' + \theta_{j_3,0})$ ; (iv) Reparametrize  $k_{j_1}, k_{j_2}$  and  $k_{j_3}$  by  $\kappa_{i,2} = k_{j_2} - k_{j_1}$ ,  $\kappa_{i,3} = k_{j_3} - k_{j_1}$ ,  $\Omega_{i,2} = \sqrt{-\delta_0}(k_{j_2}^2 - k_{j_1}^2)$  and  $\Omega_{i,3} = \sqrt{-\delta_0}(k_{j_3}^2 - k_{j_1}^2)$ . Then we have  $f_i \sim 1 + a_{i,j_2} \exp(\kappa_{i,2} x + \Omega_{i,2} t' + \theta_{j_2,0}) + a_{i,j_3} \exp(\kappa_{i,3} x + \Omega_{i,3} t' + \theta_{j_3,0})$  with linear dispersion relations  $\kappa_{i,j}^4 - 4c_0 \kappa_{i,j}^2 - 3\delta_0 \Omega_{i,j}^2 = 0$  for  $j = 1, 2$ ; For example, consider  $N = 1$ . This gives Y-shape soliton resonance interaction.

### 3. Discrete analogues of the potential Boussinesq equation

#### 3.1. The discrete potential Boussinesq equation

Here, we present the main theorem in this article.

**Theorem 3.1:** *The difference-difference equation*

$$U_{n-1}^{m+2}U_{n-1}^{m-1}(-\delta_1U_n^{m-1} + \delta_2U_n^{m+2}) = U_n^{m+1}U_{n-1}^m(-\delta_1U_{n-1}^{m-1} + \delta_2U_{n-1}^{m+2}), \quad (22)$$

where  $\delta_1 = a_2(a_1 - a_3)$  and  $\delta_2 = a_3(a_1 - a_2)$  and  $a_1, a_2, a_3$  are arbitrary real constants, is an integrable discrete analogue of the potential BSQ equation (3).

Moreover, the discrete potential BSQ equation has multisoliton solutions

$$U_n^m = \frac{\tau_{n+1}^m}{\tau_n^m},$$

$$\tau_n^m = \det(\mathcal{A}\Theta P), \quad (23)$$

where  $\mathcal{A} = (\alpha_{n,m})$  is the  $N \times 2N$  coefficient matrix,  $\Theta = \text{diag}(e^{\theta_1}, \dots, e^{\theta_{2N}})$ ,  $e^{\theta_j} = p_j^s(1-p_ja_1)^{-n}(1-p_ja_2)^{-m}$ , and the  $M \times N$  matrix  $P$  is given by  $P = (p_m^{n-1})$ . where the  $2N$  parameters  $p_1, \dots, p_{2N}$  are real constants. The  $\mathcal{A}$ -matrix has a restriction such that each row has only 2 or 3 nonzero elements and each column has only one element. In the case having 2 elements  $(i, j_1)$  and  $(i, j_2)$  in the  $i$ -th row of the  $\mathcal{A}$ -matrix,  $p_{j_2}$  must satisfy a reduction condition

$$p_{j_2} = \frac{1}{a_2} + \frac{(1 - a_3p_{j_1})}{2a_3} \pm \frac{\sqrt{a_2(1 - a_3p_{j_1})(a_2 - 4a_3 + 3a_2a_3p_{j_1})}}{2a_2a_3}. \quad (24)$$

**Proof:** The Hirota-Miwa (discrete KP) equation is written as

$$\begin{aligned} & a_1(a_2 - a_3)\tau(n_1 + 1, n_2, n_3)\tau(n_1, n_2 + 1, n_3 + 1) \\ & + a_2(a_3 - a_1)\tau(n_1, n_2 + 1, n_3)\tau(n_1 + 1, n_2, n_3 + 1) \\ & + a_3(a_1 - a_2)\tau(n_1, n_2, n_3 + 1)\tau(n_1 + 1, n_2 + 1, n_3) = 0, \end{aligned} \quad (25)$$

where  $\tau$  depends on three discrete independent variables  $n_1, n_2$  and  $n_3$ , and  $a_1, a_2$  and  $a_3$  are the difference intervals for  $n_1, n_2$  and  $n_3$ , respectively [21].

The Casorati determinant solution for the Hirota-Miwa equation (25) is as follows[22]:

$$\tau(n_1, n_2, n_3) = \det(\psi_i(n_1, n_2, n_3; s + j - 1))_{1 \leq i, j \leq N}, \quad (26)$$

where  $\psi_1, \dots, \psi_N$  are a set of linearly independent solutions of the linear system

$$\Delta_{n_j}\psi_i(n_1, n_2, n_3; s) = \psi_i(n_1, n_2, n_3; s + 1), \quad (j = 1, 2, 3).$$

Here  $\Delta_{n_j}$  are the backward difference operators:

$$\Delta_{n_j}f(n_j) \equiv \frac{f(n_j) - f(n_j - 1)}{a_j}, \quad (j = 1, 2, 3). \quad (27)$$

For example, ordinary  $N$ -soliton solutions are obtained by taking

$$\begin{aligned} \psi_i(n_1, n_2, n_3; s) &= \alpha_{2i-1}p_{2i-1}^s(1 - p_{2i-1}a_1)^{-n_1}(1 - p_{2i-1}a_2)^{-n_2}(1 - p_{2i-1}a_3)^{-n_3} \\ &+ \alpha_{2i}p_{2i}^s(1 - p_{2i}a_1)^{-n_1}(1 - p_{2i}a_2)^{-n_2}(1 - p_{2i}a_3)^{-n_3}, \end{aligned} \quad (28)$$

for  $n = 1, \dots, N$  where the  $4N$  parameters  $p_1 < \dots < p_{2N}$  and  $\alpha_1, \dots, \alpha_{2N}$  are positive real constants. The most general form of the  $N$ -soliton solution is given

by

$$\tau(x, y, t) = \det(\mathcal{A}\Theta P) = \sum_{1 \leq m_1 < \dots < m_N \leq M} V_{m_1, \dots, m_N} \mathcal{A}_{m_1, \dots, m_N} \exp \theta_{m_1, \dots, m_N}, \quad (29)$$

where  $\mathcal{A} = (\alpha_{n,m})$  is the  $N \times M$  coefficient matrix,  $\Theta = \text{diag}(e^{\theta_1}, \dots, e^{\theta_M})$ ,  $e^{\theta_j} = p_j^s (1 - p_j a_1)^{-n} (1 - p_j a_2)^{-m}$ , and the  $M \times N$  matrix  $P$  is given by  $P = (p_m^{n-1})$ .  $V_{m_1, \dots, m_N}$  is the Vandermonde determinant  $V_{m_1, \dots, m_N} = \prod_{1 \leq j < j' \leq N} (p_{m_{j'}} - p_{m_j})$ , and  $\mathcal{A}_{m_1, \dots, m_N}$  is the  $N \times N$ -minor whose  $n$ -th column is respectively given by the  $m_n$ -th column of the coefficient matrix for  $n = 1, \dots, N$ . For all  $G \in \text{GL}(N, \mathbb{R})$ , the coefficient matrices  $\mathcal{A}$  and  $\mathcal{A}' = G \mathcal{A}$  produce the same solution of the KP equation. Thus without loss of generality one can consider  $\mathcal{A}$  to be in RREF.

Let us consider a reduction of the Hirota-Miwa equation. We assume that there exists a nonzero constant  $\Phi$  such that for arbitrary  $n_1, n_2$  and  $n_3$

$$\tau(n_1, n_2, n_3) = \Phi \tau(n_1, n_2 - 2, n_3 - 1), \quad (30)$$

which is a reduction condition (3-reduction). Applying the reduction condition, we can omit the dependency of  $n_3$  and obtain the bilinear form

$$\begin{aligned} a_1(a_2 - a_3)\tau(n_1 + 1, n_2)\tau(n_1, n_2 - 1) + a_2(a_3 - a_1)\tau(n_1, n_2 + 1)\tau(n_1 + 1, n_2 - 2) \\ + a_3(a_1 - a_2)\tau(n_1, n_2 - 2)\tau(n_1 + 1, n_2 + 1) = 0. \end{aligned} \quad (31)$$

After the change of variables  $n_1 \rightarrow n$ ,  $n_2 \rightarrow m$ ,  $\tau(n_1, n_2) \rightarrow \tau_n^m$ , we obtain

$$a_1(a_2 - a_3)\tau_{n+1}^{m+1}\tau_n^m + a_2(a_3 - a_1)\tau_n^{m+2}\tau_{n+1}^{m-1} + a_3(a_1 - a_2)\tau_n^{m-1}\tau_{n+1}^{m+2} = 0, \quad (32)$$

which is the bilinear form of the discrete potential BSQ equation.

Now we impose a constraint on the parameters of the solution so that the reduction condition is satisfied. For simplicity, we consider the case in which  $\psi_1, \dots, \psi_N$  have 2 terms. Then we observe

$$\begin{aligned} &\psi_i(n_1, n_2 + 2, n_3 + 1; s) \\ &= p_{j_1}^s (1 - p_{j_1} a_1)^{-n_1} (1 - p_{j_1} a_2)^{-n_2 - 2} (1 - p_{j_1} a_3)^{-n_3 - 1} \\ &\quad + p_{j_2}^s (1 - p_{j_2} a_1)^{-n_1} (1 - p_{j_2} a_2)^{-n_2 - 2} (1 - p_{j_2} a_3)^{-n_3 - 1} \\ &= p_{j_1}^s (1 - p_{j_1} a_1)^{-n_1} (1 - p_{j_1} a_2)^{-n_2 - 2} (1 - p_{j_1} a_3)^{-n_3 - 1} \\ &\quad \times \left[ 1 + C_i \left( \frac{p_{j_2}}{p_{j_1}} \right)^s \left( \frac{1 - p_{j_2} a_1}{1 - p_{j_1} a_1} \right)^{-n_1} \left( \frac{1 - p_{j_2} a_2}{1 - p_{j_1} a_2} \right)^{-n_2} \left( \frac{1 - p_{j_2} a_3}{1 - p_{j_1} a_3} \right)^{-n_3} \right], \end{aligned} \quad (33)$$

where

$$C_i = \left( \frac{1 - p_{j_2} a_2}{1 - p_{j_1} a_2} \right)^{-2} \left( \frac{1 - p_{j_2} a_3}{1 - p_{j_1} a_3} \right)^{-1}.$$

If we apply the reduction condition

$$(1 - p_{j_1} a_2)^2 (1 - p_{j_1} a_3) = (1 - p_{j_2} a_2)^2 (1 - p_{j_2} a_3), \quad (34)$$



i.e.

$$p_{j_2} = \frac{1}{a_2} + \frac{(1 - a_3 p_{j_1})}{2a_3} \pm \frac{\sqrt{a_2(1 - a_3 p_{j_1})(a_2 - 4a_3 + 3a_2 a_3 p_{j_1})}}{2a_2 a_3}, \quad (35)$$

we obtain

$$\psi_i(n_1, n_2, n_3; s) = (1 - p_{j_2} a_2)^{-2} (1 - p_{j_2} a_3)^{-1} \psi_i(n_1, n_2 - 2, n_3 - 1; s), \quad (36)$$

Finally we have a reduction condition

$$\tau(n_1, n_2, n_3) = \prod_{k=1}^N (1 - p_k a_2)^{-2} (1 - p_k a_3)^{-1} \tau(n_1, n_2 - 2, n_3 - 1). \quad (37)$$

In a similar way discussed in section 2, we can consider the general multisoliton solutions of the discrete potential BSQ equation.  $\square$

The reality condition is

$$p_{j_1} < \frac{1}{a_3}, \quad p_{j_1} > \frac{-a_2 + 4a_3}{3a_2 a_3},$$

for  $a_2 < a_3$ , and

$$p_{j_1} > \frac{1}{a_3}, \quad p_{j_1} < \frac{-a_2 + 4a_3}{3a_2 a_3},$$

for  $a_2 > a_3$ . With parameters satisfying these conditions, we can construct real and regular multisoliton solutions using the formula in Theorem 3.1.

Otherwise, we must use the technique of reparametrization which was used in section 2. The procedure is as follows. (i) Suppose that we have 2 elements in the  $i$ -th row of the  $A$ -matrix. Let the corresponding complex wave numbers of these elements be  $p_{j_1}$  and  $p_{j_2}$ ; (iii) Using the gauge invariance of  $\tau$ -function, each element in the Wronskian determinant can be  $f_i \sim 1 + \alpha_{i,j_2} K_i^{-n} \Omega_i^{-m}$  where  $K_i = (1 - a_1 p_{j_2}) / (1 - a_1 p_{j_1})$ ,  $\Omega_i = (1 - a_2 p_{j_2}) / (1 - a_2 p_{j_1})$ ; (iv) Choose new parameters  $K_i$  and  $\Omega_i$  to be real numbers.

It is easy to take the ultradiscrete limit in Eq.(22).

**Theorem 3.2:** *The ultradiscrete potential BSQ equation is*

$$V_{n-1}^{m+2} + V_{n-1}^{m-1} + L_n^{m+2} = V_n^{m+1} + V_{n-1}^m + L_{n-1}^{m+2}, \quad (38)$$

$$V_n^{m+2} = \max(L_n^{m+2} + c_1 + c_2, V_n^{m-1} + c_2). \quad (39)$$

**Proof:** Use the standard procedure of ultradiscretization. Equation (22) is rewritten in the form of

$$U_{n-1}^{m+2} U_{n-1}^{m-1} I_n^{m+2} = U_n^{m+1} U_{n-1}^m I_{n-1}^{m+2}, \quad (40)$$

$$I_n^{m+2} = -\delta_1 U_n^{m-1} + \delta_2 U_n^{m+2}. \quad (41)$$

Introduce new variables  $U_n^m = \exp(V_n^m / \epsilon)$ ,  $I_n^m = \exp(L_n^m / \epsilon)$ ,  $\delta_1 = \exp(c_1 / \epsilon)$ ,  $1 / \delta_2 = \exp(c_2 / \epsilon)$ . Then take the limit  $\epsilon \rightarrow 0^+$  using the formula

$\lim_{\epsilon \rightarrow 0^+} \epsilon \ln(\exp(A/\epsilon) + \exp(B/\epsilon)) = \max(A, B)$  for  $A, B \in \mathbb{R}$ [23].  $\square$

**Remark:**

Date et al. proposed another discrete potential BSQ equation[24]

$$\begin{aligned} v_{n-1}^m v_n^{m-1} (a_1(a_2 - a_3)v_{n+1}^m + a_2(a_3 - a_1)v_n^{m+1}) \\ = v_{n-1}^{m-1} v_{n+1}^{m+1} (a_1(a_2 - a_3)v_n^{m-1} + a_2(a_3 - a_1)v_{n-1}^m). \end{aligned} \quad (42)$$

By the transformation  $v_n^m = \tau_{n+1}^{m+1}/\tau_n^m$ , we obtain a bilinear equation

$$a_1(a_2 - a_3)\tau_{n+1}^m \tau_{n-1}^m + a_2(a_3 - a_1)\tau_n^{m+1} \tau_n^{m-1} + a_3(a_1 - a_2)\tau_{n-1}^{m-1} \tau_{n+1}^{m+1} = 0. \quad (43)$$

This bilinear equation is obtained by adding the reduction condition

$$\tau(n_1, n_2, n_3) = \Phi \tau(n_1 - 1, n_2 - 1, n_3 - 1), \quad (44)$$

which gives yet another 3-reduction.

For this discrete potential BSQ equation, we can also make an ultradiscrete analogue of the potential BSQ equation

$$X_{n-1}^m + X_n^{m-1} + Y_n^{m+1} = X_{n-1}^{m-1} + X_{n+1}^{m+1} + Y_{n-1}^m, \quad (45)$$

$$Y_n^m = \max(X_{n+1}^{m-1} + c_1, X_n^m + c_2), \quad (46)$$

taking the ultradiscrete limit after setting  $v_n^m = \exp(X_n^m/\epsilon)$ ,  $w_n^m = \exp(Y_n^m/\epsilon)$ ,  $\alpha_1 = \exp(c_1/\epsilon)$ ,  $\alpha_2 = \exp(c_2/\epsilon)$  where  $w_n^m = \alpha_1 v_{n+1}^{m-1} + \alpha_2 v_n^m$ ,  $\alpha_1 = a_1(a_2 - a_3)$ ,  $\alpha_2 = a_2(a_3 - a_1)$ .

### 3.2. The lattice potential Boussinesq equation

Singularity confinement (SC) test was proposed by Grammaticos et al. as a detector of integrability in discrete systems[25]. This property has been applied to several problems [26–30]. The SC test is also powerful tool for constructing solutions for discrete integrable systems[28–30]. In this section, we apply the SC test to the lattice potential BSQ equation and obtain bilinear equations using the result of SC test.

We start from the slightly simplified form of the lattice potential BSQ equation

$$\begin{aligned} \frac{p^3 - q^3}{p - q + u_{n+1}^{m+1} - u_{n+2}^m} - \frac{p^3 - q^3}{p - q + u_n^{m+2} - u_{n+1}^{m+1}} \\ = (p + 2q + u_{n+1}^m - u_{n+2}^{m+2})(p - q + u_{n+1}^{m+2} - u_{n+2}^{m+1}) \\ - (p + 2q + u_n^m - u_{n+1}^{m+2})(p - q + u_n^{m+1} - u_{n+1}^m). \end{aligned} \quad (47)$$

Introducing new variables

$$I_n^m = p - q + u_{n-1}^{m+1} - u_n^m, \quad V_n^m = p + 2q + u_{n-1}^{m-2} - u_n^m, \quad (48)$$

Eq.(47) is written as

$$\frac{p^3 - q^3}{I_{n+2}^{m+2}} - \frac{p^3 - q^3}{I_{n+1}^{m+1}} = V_{n+2}^{m+2} I_{n+2}^{m+1} - V_{n+1}^{m+2} I_{n+1}^m, \quad (49)$$

$$I_n^m + V_{n+1}^{m+2} = I_{n+1}^{m+2} + V_n^{m+3}. \quad (50)$$

After the independent variable transformation  $m + n \rightarrow m$ , Eqs.(47),(48) and (49), (50) are rewritten in the following form:

$$\begin{aligned} & \frac{p^3 - q^3}{p - q + u_{n+1}^{m+2} - u_{n+2}^{m+2}} - \frac{p^3 - q^3}{p - q + u_n^{m+2} - u_{n+1}^{m+2}} \\ &= (p + 2q + u_{n+1}^{m+1} - u_{n+2}^{m+4})(p - q + u_{n+1}^{m+3} - u_{n+2}^{m+3}) \\ & - (p + 2q + u_n^m - u_{n+1}^{m+3})(p - q + u_n^{m+1} - u_{n+1}^{m+1}), \end{aligned} \quad (51)$$

and

$$I_n^m = p - q + u_{n-1}^m - u_n^m, \quad V_n^m = p + 2q + u_{n-1}^{m-3} - u_n^m, \quad (52)$$

and

$$\frac{p^3 - q^3}{I_{n+2}^{m+2}} - \frac{p^3 - q^3}{I_{n+1}^{m+2}} = V_{n+2}^{m+4} I_{n+2}^{m+3} - V_{n+1}^{m+3} I_{n+1}^{m+1}, \quad (53)$$

$$I_n^m + V_{n+1}^{m+3} = I_{n+1}^{m+3} + V_n^{m+3}. \quad (54)$$

After some calculation, we get the form which is suitable to perform the SC test:

$$\begin{aligned} u_n^m &= p + 2q + u_{n-1}^{m-3} - \frac{1}{p - q + u_{n-1}^{m-1} - u_n^{m-1}} \\ &\times \left( \frac{p^3 - q^3}{p - q + u_{n-1}^{m-2} - u_n^{m-2}} - \frac{p^3 - q^3}{p - q + u_{n-2}^{m-2} - u_{n-1}^{m-2}} \right. \\ & \left. + (p + 2q + u_{n-2}^{m-4} - u_{n-1}^{m-1})(p - q + u_{n-2}^{m-3} - u_{n-1}^{m-3}) \right), \end{aligned} \quad (55)$$

$$V_n^m = \frac{1}{I_n^{m-1}} \left( V_{n-1}^{m-1} I_{n-1}^{m-3} + \frac{p^3 - q^3}{I_n^{m-2}} - \frac{p^3 - q^3}{I_{n-1}^{m-2}} \right), \quad (56)$$

$$I_n^m = I_{n-1}^{m-3} + V_n^m - V_{n-1}^m. \quad (57)$$

Performing the SC test, we obtain the following result:

### Pattern 1

We have the following pattern:

$$\begin{aligned} \{I_n^{m-1}, I_n^m, I_{n+1}^m, I_{n+1}^{m+1}\} &\rightarrow \{0, \infty, \infty, 0\}, \\ \{V_n^m, V_{n+1}^{m+3}\} &\rightarrow \{\infty, \infty\}. \end{aligned}$$

Note that  $u_n^m$  takes  $\infty$ . One can see finite values for all dependent variables in further steps, so the singularity is confined. Suppose that this singularity was created by a function  $F_n^m$  which has a zero at  $(m, n)$ .

If we see Eq.(55), we notice that there is a possibility to have a singularity when  $u_n^m \rightarrow 0$ . However, this will not make singularity. Thus

**Pattern 2**

$$\{u_n^m\} \rightarrow \{0\},$$

and one can see finite values for all dependent variables in further steps, so the singularity is confined. Suppose that this singularity pattern was created by a function  $G_n^m$  which has a zero at  $(m, n)$ .

Using the singularity pattern 1, we obtain the independent variable transformation

$$I_n^m = \alpha \frac{F_n^{m+1} F_{n-1}^{m-1}}{F_n^m F_{n-1}^m}. \tag{58}$$

Since  $u_n^m$  and  $I_n^m$  are related by Eq.(52),  $u_n^m$  should have  $F_n^m$  in the denominator. Thus from the singularity pattern 2 we have

$$u_n^m = \beta \frac{G_n^m}{F_n^m}. \tag{59}$$

Note

$$I_n^m = p - q + \beta \frac{G_{n-1}^m}{F_{n-1}^m} - \beta \frac{G_n^m}{F_n^m}, \quad V_n^m = p + 2q + \beta \frac{G_{n-1}^{m-3}}{F_{n-1}^{m-3}} - \beta \frac{G_n^m}{F_n^m}. \tag{60}$$

Using Eq.(60), we obtain the following equations:

$$(p - q)F_{n-1}^{m-1} F_n^{m-1} + \beta G_{n-1}^{m-1} F_n^{m-1} - \beta F_{n-1}^{m-1} G_n^{m-1} = \alpha F_n^m F_{n-1}^{m-2}, \tag{61}$$

$$\frac{p^3 - q^3}{\alpha} F_{n+1}^{m+1} F_{n+1}^m - \alpha(p + 2q) F_{n+1}^{m+2} F_n^{m-1} - \alpha \beta G_{n+1}^{m-1} F_{n+1}^{m+2} + \alpha \beta F_n^{m-1} G_{n+1}^{m+2} = \gamma F_n^m F_{n+1}^{m+1}, \tag{62}$$

$$\alpha \frac{F_n^m F_{n-1}^{m-2}}{F_n^{m-1} F_{n-1}^{m-1}} - \frac{p^3 - q^3}{\alpha^2} \frac{F_{n-1}^{m+1} F_n^m}{F_n^{m+2} F_{n-1}^{m-1}} + \frac{\gamma}{\alpha} \frac{F_{n-1}^m F_n^{m+1}}{F_n^{m+2} F_{n-1}^{m-1}} = \alpha \frac{F_{n+1}^{m+3} F_n^{m+1}}{F_{n+1}^{m+2} F_n^{m+2}} - \frac{p^3 - q^3}{\alpha^2} \frac{F_{n+1}^{m+1} F_{n+1}^m}{F_{n+1}^{m+2} F_n^{m-1}} + \frac{\gamma}{\alpha} \frac{F_n^m F_{n+1}^{m+1}}{F_{n+1}^{m+2} F_n^{m-1}}, \tag{63}$$

where  $\gamma$  is a decoupling constant. Note that  $V$  is written in the following form:

$$V_n^m = (p + 2q) + \beta \frac{G_{n-1}^{m-2}}{F_{n-1}^{m-2}} - \beta \frac{G_n^{m+1}}{F_n^{m+1}} = \frac{p^3 - q^3}{\alpha^2} \frac{F_{n-1}^m F_n^{m-1}}{F_n^{m+1} F_{n-1}^{m-2}} - \frac{\gamma}{\alpha} \frac{F_{n-1}^{m-1} F_n^m}{F_n^{m+1} F_{n-1}^{m-2}}. \tag{64}$$

Assuming  $\gamma = 0$ , we obtain

$$(p - q)F_{n-1}^{m-1}F_n^{m-1} + \beta G_{n-1}^{m-1}F_n^{m-1} - \beta F_{n-1}^{m-1}G_n^{m-1} = \alpha F_n^m F_{n-1}^{m-2}, \quad (65)$$

$$\begin{aligned} \frac{p^3 - q^3}{\alpha} F_n^{m+1} F_{n+1}^m - \alpha(p + 2q)F_{n+1}^{m+2} F_n^{m-1} - \alpha\beta G_n^{m-1} F_{n+1}^{m+2} \\ + \alpha\beta F_n^{m-1} G_{n+1}^{m+2} = 0, \end{aligned} \quad (66)$$

$$\alpha F_n^{m-1} F_{n+1}^{m+3} - \frac{p^3 - q^3}{\alpha^2} F_n^{m+2} F_{n+1}^m = \delta F_n^m F_{n+1}^{m+2}, \quad (67)$$

where  $\delta$  is a decoupling constant. After changing back to original independent variables ( $m - n \rightarrow m$ ), we have

$$(p - q)F_{n-1}^m F_n^{m-1} + \beta G_{n-1}^m F_n^{m-1} - \beta F_{n-1}^m G_n^{m-1} = \alpha F_n^m F_{n-1}^{m-1}, \quad (68)$$

$$\begin{aligned} \frac{p^3 - q^3}{\alpha} F_n^{m+1} F_{n+1}^{m-1} - \alpha(p + 2q)F_{n+1}^{m+1} F_n^{m-1} - \alpha\beta G_n^{m-1} F_{n+1}^{m+1} \\ + \alpha\beta F_n^{m-1} G_{n+1}^{m+1} = 0, \end{aligned} \quad (69)$$

$$\alpha F_n^{m-1} F_{n+1}^{m+2} - \frac{p^3 - q^3}{\alpha^2} F_n^{m+2} F_{n+1}^{m-1} = \delta F_n^m F_{n+1}^{m+1}. \quad (70)$$

Note that Eq.(70) can be derived by vanishing  $G$  from Eqs.(68) and (69). This is a discrete analogue of a bilinear form of the potential BSQ equation (3). Thus we have the following theorem.

**Theorem 3.3:** *Solutions of Eq.(47) (Eqs.(49) and (50)) are expressed in the following form using the  $\tau$ -function:*

$$\begin{aligned} u_n^m &= \frac{G_n^m}{F_n^m}, \quad I_n^m = \frac{F_n^{m+1} F_{n-1}^{m-1}}{F_n^m F_{n-1}^m} = p - q + \frac{G_{n-1}^{m+1}}{F_{n-1}^{m+1}} - \frac{G_n^m}{F_n^m}, \\ V_n^m &= p + 2q + \frac{G_{n-1}^{m-2}}{F_{n-1}^{m-2}} - \frac{G_n^m}{F_n^m}. \end{aligned}$$

Moreover,  $F_n^m$  is given by  $\tau_n^m$  of the discrete potential BSQ equation (22) in Theorem 3.1 when parameters have relations

$$\frac{\alpha}{\delta} = \frac{a_3(a_1 - a_2)}{a_1(a_3 - a_2)}, \quad \frac{p^3 - q^3}{\alpha^2 \delta} = \frac{a_2(a_1 - a_3)}{a_1(a_3 - a_2)}.$$

Note that Hietarinta and Zhang gave the Casorati determinant form of the  $\tau$ -functions  $F_n^m$  and  $G_n^m$  [6].

**Remark:**

In the case of  $p^3 - q^3 = 0$ , Eq.(62) is

$$-\alpha(p + 2q)F_{n+1}^{m+1} F_n^{m-1} - \alpha\beta G_n^{m-1} F_{n+1}^{m+1} + \alpha\beta F_n^{m-1} G_{n+1}^{m+1} = \gamma F_n^m F_{n+1}^m. \quad (71)$$

Equation (63) is written as

$$\alpha \frac{F_n^m F_{n-1}^{m-1}}{F_n^{m-1} F_{n-1}^m} + \frac{\gamma}{\alpha} \frac{F_{n-1}^{m+1} F_n^{m+1}}{F_n^{m+2} F_{n-1}^m} = \alpha \frac{F_{n+1}^{m+2} F_n^{m+1}}{F_{n+1}^{m+1} F_n^{m+2}} + \frac{\gamma}{\alpha} \frac{F_n^m F_{n+1}^m}{F_{n+1}^{m+1} F_n^{m-1}}. \quad (72)$$

After some calculation, we have

$$\frac{\alpha F_{n-1}^{m-1} F_{n+1}^{m+1} - \frac{\gamma}{\alpha} F_{n+1}^m F_{n-1}^m}{F_n^{m-1} F_n^{m+1}} = \frac{\alpha F_{n+1}^{m+2} F_{n-1}^m - \frac{\gamma}{\alpha} F_{n-1}^{m+1} F_{n+1}^{m+1}}{F_n^m F_n^{m+2}}. \quad (73)$$

After decoupling, we obtain

$$\alpha F_{n-1}^{m-1} F_{n+1}^{m+1} - \frac{\gamma}{\alpha} F_{n+1}^m F_{n-1}^m = \delta F_n^{m-1} F_n^{m+1}, \quad (74)$$

which is nothing but Eq.(43). Thus we conclude that this special case gives a discrete potential BSQ equation which has the same  $\tau$ -function to Eq.(42).

#### 4. Conclusion

We have proposed a new discrete potential BSQ equation. We have constructed the bilinear equations and multisoliton solutions for the discrete potential BSQ equation. The bilinear equations and multisoliton solutions have been constructed by one of 3-reductions of the Hirota-Miwa equation. Using the discrete potential BSQ equation, we have presented the ultradiscrete potential BSQ equation. We have also studied the lattice potential Boussinesq equation using the singularity confinement test. Although the lattice potential Boussinesq equation is in very complicated form, we can find bilinear equations easily by using singularity confinement test. We have investigated the relationships among our new discrete potential BSQ equation, the discrete potential BSQ equation of Date et al. and the lattice BSQ equation by Nijhoff et al.

An interesting problem is to present explicit forms of soliton solutions of the ultradiscrete potential BSQ equation. Since 3-reduction condition is very complicated, it is not easy to see which solutions can survive in the ultradiscrete limit. We will address this problem in the near future.

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