

Analytic solutions and integrability for bilinear recurrences of order six

Andrew N.W. Hone

(Received 00 Month 200x; in final form 00 Month 200x)

Somos sequences are integer sequences generated by recurrence relations that are in bilinear form, meaning that they can be written as a quadratic relation between adjacent sets of iterates. Such sequences have appeared in number theory, statistical mechanics, and algebraic combinatorics, as well as arising from reductions of bilinear partial difference equations in the theory of discrete integrable systems. This article is concerned with the general form of the Somos-6 recurrence, which is a three-parameter family of bilinear recurrences of order six. After explaining how it arises by reduction from the bilinear discrete BKP equation (Miwa's equation), an invariant Poisson bracket for Somos-6 is presented. Four independent Casimirs of this bracket, which are the invariants under the action of a group of gauge transformations, lead to an associated map on a four-dimensional reduced phase space. Two rational first integrals for this map are constructed, and (for certain parameter choices) these are found to be in involution for a non-degenerate Poisson bracket associated with a symplectic form on the reduced phase space, so that the map is Liouville integrable. For generic parameter values the explicit analytic solution of the Somos-6 recurrence is given in terms of the Kleinian sigma function for a curve of genus two.

Keywords: Somos sequences, Laurent phenomenon, integrable maps, Poisson bracket, theta function, Kleinian sigma function

1. Introduction

The properties of the integer sequence defined by the linear recurrence relation

$$F_{n+1} = F_n + F_{n-1}, \quad \text{with } F_1 = F_2 = 1, \quad (1.1)$$

have been studied for centuries. One particularly striking property of the Fibonacci numbers is that they form a divisibility sequence:

$$F_m | F_n \quad \text{whenever} \quad m | n. \quad (1.2)$$

Lucas sequences, defined by

$$F_{n+1} = \alpha F_n + \beta F_{n-1}, \quad \text{with } F_1 = 1, \quad F_2 = \alpha \in \mathbb{Z}, \quad \beta \in \mathbb{Z}, \quad (1.3)$$

are a natural generalization of Fibonacci numbers that have the same divisibility property (1.2). Being defined by linear recurrences of order two, the general term in such sequences can be expressed explicitly using exponential/trigonometric/hyperbolic functions. To be precise, the n th term of the sequence defined by (1.3) is given by the explicit formula

$$F_n = \rho^{n-1} \frac{\sin n\vartheta}{\sin \vartheta}, \quad \alpha = 2\rho \cos \vartheta, \quad \beta = -\rho^2, \quad (1.4)$$

Institute of Mathematics, Statistics and Actuarial Science, University of Kent, Canterbury CT2 7NF, U.K.
Email: A.N.W.Hone@kent.ac.uk

ISSN: 0003-6811 print/ISSN 1563-504X online
© 2009 Taylor & Francis
DOI: 10.1080/0003681Yxxxxxxx
<http://www.informaworld.com>

where $\rho, \vartheta \in \mathbb{C}$ for generic integers $\alpha, \beta \in \mathbb{Z}$.

In the 1940s, Morgan Ward sought an elliptic function generalization of Lucas sequences, and was led to define elliptic divisibility sequences [50], which are specified by a quadratic recurrence of fourth order, namely

$$\tau_{n+4} \tau_n = \alpha \tau_{n+3} \tau_{n+1} + \beta (\tau_{n+2})^2, \quad (1.5)$$

with the integer parameters and initial data related by

$$\alpha = (\tau_2)^2, \quad \beta = -\tau_1 \tau_3, \quad \tau_1 = 1, \quad \tau_2, \tau_3, \tau_4 \in \mathbb{Z} \quad \text{with} \quad \tau_2 | \tau_4.$$

Observe that the recurrence relation (1.5) is rational [28], in the sense that each new iterate τ_{n+4} is given by a rational function of the four previous terms; hence the recurrence gives rise to a rational map in four dimensions. With these constraints on the initial values and parameters, the nonlinear recurrence (1.5) produces a sequence of integers $\tau_n \in \mathbb{Z}$ satisfying the same divisibility property as in (1.2). The general term is given by

$$\tau_n = \frac{\sigma(nv)}{\sigma(v)^{n^2}}, \quad (1.6)$$

with $\sigma(z) = \sigma(z; g_2, g_3)$ denoting the Weierstrass sigma function associated with the elliptic curve E defined by the cubic equation $y^2 = 4x^3 - g_2x - g_3$. The value $v \in \mathbb{C}$ is defined up to the periods of the curve, and fixes a point $P \in E$, so that the n th term of the sequence corresponds to $[n]P \in E$, where $[n]P = P + P + \dots + P$ (n times) denotes multiple addition of P to itself in the group law of the curve.

If one is prepared to relinquish the divisibility property, then one can consider linear recurrence relations like (1.3) with arbitrary values of the coefficients and initial data, and provided that these values are integers it is clear that $F_n \in \mathbb{Z}$ for all n . However, for nonlinear recurrences yielding rational maps it is no longer obvious under what circumstances a sequence of integers can be produced. While investigating the properties of elliptic theta functions, Michael Somos [47] made the surprising empirical observation that if one takes the quadratic relation

$$\tau_{n+6} \tau_n = \tau_{n+5} \tau_{n+1} + \tau_{n+4} \tau_{n+2} + (\tau_{n+3})^2 \quad (1.7)$$

with initial values $\tau_0 = \tau_1 = \dots = \tau_5 = 1$, then an integer sequence beginning

$$1, 1, 1, 1, 1, 1, 3, 5, 9, 23, 75, 421, 1103, 5047, 41783, 281527, 2534423, \dots \quad (1.8)$$

is generated. This observation, which initially was based on purely numerical evidence, led to a series of conjectures concerning similar recurrences defined by quadratic relations [20]. For instance, for the recurrence (1.5), if the coefficients are $\alpha = \beta = 1$, and the four initial values are chosen to be $\tau_0 = \tau_1 = \tau_2 = \tau_3 = 1$, then an integer sequence beginning

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, \dots \quad (1.9)$$

is produced [46]. Various proofs of this observation were soon found [28], but a further valuable insight (see [20]) was the fact that if the initial data are treated as variables, then the iterates of the recurrence (1.5) are polynomials in these variables and their inverses, or in other words, Laurent polynomials. The fact that $\tau_n \in \mathbb{Z}[\alpha, \beta, \tau_0^{\pm 1}, \tau_1^{\pm 1}, \tau_2^{\pm 1}, \tau_3^{\pm 1}]$ for all $n \in \mathbb{Z}$ is known as the Laurent property

for the recurrence (1.5), and the integrality of the particular sequence (1.9) is an immediate corollary of this more general fact.

The Laurent property has subsequently become highly relevant to algebraic combinatorics, because it is an essential feature of Fomin and Zelevinsky's theory of cluster algebras [17]. A cluster algebra of rank m is a commutative algebra produced by distinguished sets of generators, called clusters, that live on the vertices of an m -tree. Each cluster contains m generators x_0, \dots, x_{m-1} , and for any cluster at an adjacent vertex the generators x'_0, \dots, x'_{m-1} can be ordered as $x'_{j-1} = x_j$ for $j = 1, \dots, m$, so that $n-1$ of them are the same, but x_0 is replaced by $x'_{m-1} = x_m$ which is defined by an exchange relation of the form

$$x_m x_0 = c_1 M_1(\mathbf{x}) + c_2 M_2(\mathbf{x}), \quad (1.10)$$

where c_1, c_2 are coefficients and M_1 and M_2 are certain monomials in the other variables $\mathbf{x} = (x_1, \dots, x_{m-1})$. In a cluster algebra, the variables in each cluster are Laurent polynomials in the variables of any initial cluster. Recurrence sequences can be generated by iterating (1.10) along a particular sequence of vertices that share the same exchange relation [19]. Hence if a recurrence comes from a cluster algebra, then it has the Laurent property.

The general Somos- k recurrence is a quadratic recurrence of order k of the form

$$\tau_{n+k} \tau_n = \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \alpha_j \tau_{n+k-j} \tau_{n+j}, \quad (1.11)$$

where α_j are coefficients. Hickerson used computer algebra to prove that the original Somos-6 recurrence (1.7) has the Laurent property [20]. Some time later, Fomin and Zelevinsky used the machinery of cluster algebras to prove that this property holds for the general Somos- k recurrence when $k = 4, 5, 6$ or 7 [16]. However, note that general Somos-6 and Somos-7 recurrences have three terms on the right hand side, so they do not fit within the framework of cluster algebras, which are defined by exchange relations with a sum of two monomials, as in (1.10). It appears that for a generic choice of parameters α_j the Laurent property never holds for $k \geq 8$. To see this for the particular case $k = 8$, it is sufficient to choose the four parameters $\alpha_1 = \dots = \alpha_4 = 1$ and the eight initial values $\tau_0 = \dots = \tau_7 = 1$ for Somos-8, which at the tenth step yields the non-integer rational value $\tau_{17} = 420514/7$.

In fact, the Laurent property for the general Somos-6 and Somos-7 recurrences came about in [16] as a special case of that property for the four-term Gale-Robinson recurrences, that have the form

$$\tau_{n+k} \tau_n = \alpha \tau_{n+p} \tau_{n+k-p} + \beta \tau_{n+q} \tau_{n+k-q} + \gamma \tau_{n+r} \tau_{n+k-r} \quad (1.12)$$

for distinct positive integers p, q, r with $p + q + r = k$. Moreover, Fomin and Zelevinsky's proof of the Laurent property for (1.12) (Theorem 1.4 in [16]) comes about as a consequence of the same property for the three-dimensional lattice equation

$$T_{\ell+1, m+1, n+1} T_{\ell, m, n} = T_{\ell+1, m, n} T_{\ell, m+1, n+1} - T_{\ell, m+1, n} T_{\ell+1, m, n+1} + T_{\ell, m, n+1} T_{\ell+1, m+1, n}, \quad (1.13)$$

with $(\ell, m, n) \in \mathbb{Z}^3$. Indeed, every four-term Gale-Robinson recurrence (1.12) arises

as a one-dimensional reduction of (1.13), by taking

$$T_{\ell,m,n} = e^{Q(\ell,m,n)} \tau_N, \quad N = n_0 + p\ell + qm + rn, \quad (1.14)$$

where $Q = Q(\ell, m, n)$ is an arbitrary quadratic form in ℓ, m, n and n_0 is arbitrary; the coefficients α, β, γ fix the off-diagonal terms in Q . The partial difference equation (1.13) is well known in the literature on discrete integrable systems, where it goes by the name of Miwa's equation [31], or the discrete BKP equation; in the combinatorics literature it is known as the cube recurrence [8]. Similarly, the three-term Gale-Robinson recurrences (including Somos-4 and Somos-5), which are obtained from (1.12) by setting one of the parameters α, β, γ to zero, provide one-dimensional (ordinary difference) reductions of the Hirota-Miwa equation [53],

$$T_{\ell+1,m,n}T_{\ell-1,m,n} = T_{\ell,m+1,n}T_{\ell,m-1,n} + T_{\ell,m,n+1}T_{\ell,m,n-1}, \quad (1.15)$$

which is also known as the bilinear discrete KP equation in the theory of integrable systems, and as the octahedron recurrence in the combinatorial literature [38, 48].

Somos recurrences and related sequences also appear in connection with solvable models in statistical mechanics, as mappings on the parameter spaces of such models. For instance, in the last example of [40] it is mentioned that the Somos-4 recurrence (1.5) with parameters $\alpha = \delta^2, \beta = -\delta$ is the equation for Boltzmann weights in the hard hexagon model, where it is required that the solutions should have period five: $\tau_{n+5} = \tau_n$. Similarly, by considering more general transformations on the parameters in the sixteen vertex model, Boukraa *et al.* came up with a general framework for generating birational maps by compositions of elementary involutive transformations on matrices (see [3] and references). Certain classes of the maps obtained in the latter framework are integrable, and some in particular lead to a fibration of the phase space by elliptic curves, as in the case of the QRT family of maps [10, 41]. Furthermore, the determinants of the matrices that appear in [3] satisfy multilinear homogeneous recurrence relations, which can be viewed as higher degree analogues of (1.11).

The original interest in elliptic divisibility sequences stemmed from their arithmetical properties, and in particular the appearance of primes and new prime divisors in such sequences [51]. There has been a considerable amount of further interest in them recently [11–13, 43, 45], especially because they have been used to resolve Hilbert's tenth problem for larger subrings of \mathbb{Q} than the integers [15]. As observed by Robinson [42], more general Somos sequences, such as the Somos-4 sequence (1.9), that do not have the divisibility property (1.2), still have special arithmetical properties e.g. when taken modulo a prime. Some of Robinson's conjectures for Somos-4 were explained in the thesis of Swart [49], who gave an explicit algebraic construction of the correspondence between a general Somos-4 sequence (defined over the integers) and a sequence of points $P_0 + [n]P \in E$, for an elliptic curve E , making use of unpublished work of Stephens. Somewhat earlier, the fact that the general term of a Somos-4 sequence can be expressed analytically using elliptic theta functions was known privately to several people: Malouf mentions results of Bombieri and Granville [28], while Robinson refers to formulae of Gardner [42]. For the case of the original Somos-5 sequence 1, 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, 274, 1217, . . . , which is generated by (1.11) with $k = 5$ and the coefficients set to 1, the relation with elliptic curves and theta functions was given by Zagier [54], and was also explained by Elkies when it arose in connection with a Diophantine problem in geometry [5].

More recently [23], this author obtained the explicit solution to the initial value

problem for the general Somos-4 recurrence, given in terms of the Weierstrass sigma function as

$$\tau_n = AB^n \sigma(v_0 + nv) / \sigma(v)^{n^2} \quad (1.16)$$

(for suitable parameters A, B, v_0, v and invariants g_2, g_3 for the curve), and found an analogous formula for Somos-5, which depends on the parity of n [24]. Both Somos-4 and Somos-5 can be understood in terms of associated integrable birational maps of the plane whose orbits lie on biquadratic curves of genus one, corresponding to particular symmetric cases of the QRT family of maps.

The purpose of this article is to present explicit analytic formulae for the simplest Somos recurrence that is beyond genus one, namely the general Somos-6 recurrence

$$\tau_{n+6}\tau_n = \alpha\tau_{n+5}\tau_{n+1} + \beta\tau_{n+4}\tau_{n+2} + \gamma(\tau_{n+3})^2 \quad (1.17)$$

with three arbitrary coefficients α, β, γ . Although the original recurrence (1.7) was taken with integer initial values, here all coefficients and initial data are taken in \mathbb{C} . In the next section Poisson brackets and invariant differential forms associated with Somos recurrences are presented, and it is shown how the iteration of (1.17) can be projected down to an iteration of a birational map with an invariant meromorphic volume form in four dimensions. A direct method to construct conserved quantities for Somos sequences is described, and as a result two independent rational functions are found that are invariant under the iteration of the Somos-6 recurrence. These two invariants project down to two independent conserved quantities for the corresponding four-dimensional map, and in each of the cases where one of the parameters vanishes (i.e. for $\alpha\beta\gamma = 0$) an explicit non-degenerate Poisson bracket is found which means that the map is symplectic and also integrable in the Liouville-Arnold sense [52]. In the third section, Baker's addition formula [2] is used to derive an analytic expression for the solution of the recurrence (1.17) in terms of the Kleinian sigma function $\sigma(\mathbf{z})$ associated with a genus two curve, which is a quasiperiodic function of $\mathbf{z} \in \mathbb{C}^2$. The main result is the following.

Theorem 1.1: *Given an algebraic curve X of genus two defined by the affine model*

$$X := \left\{ (\nu, \mu) \in \mathbb{C}^2 \mid \mu^2 = f(\nu) \equiv 4\nu^5 + \sum_{j=0}^3 c_j \nu^j, \right\}, \quad (1.18)$$

let $\sigma(\mathbf{z})$ denote the associated Kleinian sigma function, with $\wp_{jk}(\mathbf{z}) = -\partial_j \partial_k \log \sigma(\mathbf{z})$ for $j, k = 1, 2$ being the associated Kleinian \wp -functions. For arbitrary $A, B \in \mathbb{C}^*$, and $\mathbf{v}_0 \in \mathbb{C}^2$, the sequence with n th term

$$\tau_n = AB^n \frac{\sigma(\mathbf{v}_0 + n\mathbf{v})}{\sigma(\mathbf{v})^{n^2}} \quad (1.19)$$

satisfies a Somos-6 recurrence (1.17) where the parameters are given by

$$\alpha = \frac{\sigma(3\mathbf{v})^2 \hat{\alpha}}{\sigma(2\mathbf{v})^2 \sigma(\mathbf{v})^{10}}, \quad \beta = \frac{\sigma(3\mathbf{v})^2 \hat{\beta}}{\sigma(\mathbf{v})^{18}}, \quad \gamma = \frac{\sigma(3\mathbf{v})^2}{\sigma(\mathbf{v})^{18}} \left(\wp_{11}(3\mathbf{v}) - \hat{\alpha} \wp_{11}(2\mathbf{v}) - \hat{\beta} \wp_{11}(\mathbf{v}) \right),$$

with

$$\hat{\alpha} = \frac{\wp_{12}(3\mathbf{v}) - \wp_{12}(\mathbf{v})}{\wp_{12}(2\mathbf{v}) - \wp_{12}(\mathbf{v})}, \quad \hat{\beta} = \frac{\wp_{12}(2\mathbf{v}) - \wp_{12}(3\mathbf{v})}{\wp_{12}(2\mathbf{v}) - \wp_{12}(\mathbf{v})} = 1 - \hat{\alpha}, \quad (1.20)$$

provided that $\mathbf{v} \in \mathbb{C}^2$ obeys the constraint

$$\begin{vmatrix} 1 & 1 & 1 \\ \wp_{12}(\mathbf{v}) & \wp_{12}(2\mathbf{v}) & \wp_{12}(3\mathbf{v}) \\ \wp_{22}(\mathbf{v}) & \wp_{22}(2\mathbf{v}) & \wp_{22}(3\mathbf{v}) \end{vmatrix} = 0. \quad (1.21)$$

Before proceeding with further details, it is worth comparing the above result with some of the existing literature on recurrences associated with addition formulae in genus two. The division polynomials corresponding to the multiples of a generic point P on an elliptic curve E are well known, and the recurrence relations they satisfy are most easily proved using the analytic formula (1.6) (see chapter II in [27], or Exercise 3.7 in [44]). Morgan Ward's elliptic divisibility sequences provide a specific arithmetical realization of the division polynomials, since (modulo some scaling) they are obtained by substituting numerical values into the latter. Cantor constructed the analogue of the division polynomials for hyperelliptic curves, corresponding to multiples of a single point on the curve [7], and derived recurrence formulae for them. Onishi made use of the Kleinian sigma function, and obtained further identities for hyperelliptic division polynomials defined analytically in terms of the so called psi-function, which in genus two takes the form

$$\psi_n(\mathbf{v}) = \frac{\sigma(n\mathbf{v})}{\sigma_2(\mathbf{v})n^2}. \quad (1.22)$$

In the above, $\mathbf{v} \in \mathbb{C}^2$ is on the theta divisor, being the image under the Abel map of a single point on a genus two curve; the sigma function vanishes on the theta divisor, so $\sigma(\mathbf{v}) = 0$, but $\sigma_2(\mathbf{v}) \neq 0$ for $\mathbf{v} \neq 0$ (see Proposition 6.5 in [33]). Matsutani considered the genus two psi-function (1.22), and derived associated higher order difference equations [29, 30]. In [4] we considered the generalization of the genus two psi-function given by the formula (1.19) but with \mathbf{v} being on the theta divisor, and showed that it satisfies a Somos-8 recurrence. Kanayama presented a different analogue of the division polynomials in genus two, by generalizing the formula (1.6) to

$$\phi_n(\mathbf{v}) = \frac{\sigma(n\mathbf{v})}{\sigma(\mathbf{v})n^2}. \quad (1.23)$$

where $\mathbf{v} \in \mathbb{C}^2$ corresponds to a generic element of the Jacobian of a genus two curve (where generic means not on the theta divisor, so $\sigma(\mathbf{v}) \neq 0$). Recurrence relations for ϕ_n provided an effective way to construct multiplication formulae in genus two [26].

We shall return to Kanayama's results in the third section, with the proof of Theorem 1.1. The final section of the paper is devoted to some conclusions and further open problems.

2. Poisson brackets and conserved quantities

Sequences generated by iteration of the general Somos- k recurrence (1.11) are equivalent to the orbits of the birational map φ on \mathbb{C}^k , with coordinates $(\tau_0, \dots, \tau_{k-1})$, defined by

$$\varphi : \begin{pmatrix} \tau_0 \\ \tau_1 \\ \vdots \\ \tau_{k-2} \\ \tau_{k-1} \end{pmatrix} \mapsto \begin{pmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_{k-1} \\ \tau_k \end{pmatrix}, \quad \tau_k = \frac{\sum_{\ell=1}^{\lfloor \frac{k}{2} \rfloor} \alpha_\ell \tau_{k-\ell} \tau_\ell}{\tau_0}. \quad (2.1)$$

This birational map preserves a very simple Poisson bracket, which in these coordinates takes the log-canonical form $\{\tau_m, \tau_n\} = f_{mn} \tau_m \tau_n$. The particular form of the recurrence also means that for any k there is a natural volume form which is invariant, up to a sign. The following result is easily verified by direct calculation.

Lemma 2.1: *The meromorphic (rational) k -form*

$$V = \frac{1}{\tau_0 \tau_1 \dots \tau_{k-1}} d\tau_0 \wedge d\tau_1 \wedge \dots \wedge d\tau_{k-1}$$

*is preserved by the birational map (2.1) for even k , and anti-preserved for odd k , i.e. $\varphi^*V = (-1)^k V$. This map is a Poisson map with respect to a log-canonical bracket of rank two,*

$$\{\tau_m, \tau_n\}_0 = (n - m) \tau_m \tau_n, \quad (2.2)$$

unique up to rescaling, with $k - 2$ independent Casimirs given by

$$x_j = \frac{\tau_j \tau_{j+2}}{(\tau_{j+1})^2}, \quad j = 0, \dots, k - 2.$$

Remark: In [21] it was shown that the exchange relations of a cluster algebra are Poisson maps with respect to a suitable log-canonical Poisson bracket. However, the general Somos- k recurrence (with more than two terms on the right hand side) is not of the correct form (1.10) for a cluster algebra.

There is another natural way to understand the Casimir functions for the Poisson bracket $\{\cdot, \cdot\}_0$. The Somos- k recurrence has the form of a discrete Hirota bilinear equation, in the sense that if (1.11) is viewed as an analytic difference equation for the tau-function $\tau_n \equiv \tau(n)$ then it can be rewritten as

$$\exp\left(\frac{k}{2} D_n\right) \tau \cdot \tau = \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \alpha_j \exp\left(\frac{k-2j}{2} D_n\right) \tau \cdot \tau, \quad (2.3)$$

where D_n is the Hirota derivative [22], defined by

$$D_n^N F \cdot G(n) = \left(\frac{\partial}{\partial n} - \frac{\partial}{\partial n'}\right)^N F(n) G(n') \Big|_{n'=n}$$

for $N \in \mathbb{N}$. Hirota bilinear equations are invariant under gauge transformations whereby the tau-function is multiplied by the exponential of an affine linear function of independent variables. In this case the gauge transformations correspond to an action of $(\mathbb{C}^*)^2$ on the phase space, such that $\tau_n \mapsto A B^n \tau_n$ for parameters A, B with $AB \neq 0$. The Poisson bracket (2.2) is equivariant under this group action, and its Casimirs x_j are invariants of the action. Any Casimir of a Poisson bracket is invariant under Hamiltonian flows; but in general a Casimir need not be preserved by a Poisson map, although Casimirs are always mapped to Casimirs. In the case of Somos sequences, the interesting dynamics takes place on the space of the Casimirs, where it can be described in the following terms.

Lemma 2.2: *The map (2.1) induces a birational map $\hat{\varphi} : \mathbb{C}^{k-2} \rightarrow \mathbb{C}^{k-2}$ on the space of Casimirs (x_0, \dots, x_{k-3}) , so that the projection*

$$\varpi : \begin{array}{ccc} \mathbb{C}^k & \longrightarrow & \mathbb{C}^{k-2} \\ (\tau_0, \dots, \tau_{k-1})^T & \longmapsto & (x_0, \dots, x_{k-3})^T \end{array}$$

intertwines φ and $\hat{\varphi}$, i.e. $\hat{\varphi} \cdot \varpi = \varpi \cdot \varphi$. The orbits of $\hat{\varphi}$ are obtained by iteration of the recurrence

$$\begin{aligned} (x_{n+(k-2)/2})^{k/2} & \prod_{j=0}^{(k-4)/2} (x_{n+j} x_{n+k-j-2})^{j+1} \\ & = \sum_{\ell=1}^{k/2} \alpha_\ell (x_{n+(k-2)/2})^{k/2-\ell} \prod_{j=\ell}^{(k-4)/2} (x_{n+j} x_{n+k-j-2})^{j-\ell+1} \end{aligned} \tag{2.4}$$

for k even, and

$$\prod_{j=0}^{(k-3)/2} (x_{n+j} x_{n+k-j-2})^{j+1} = \sum_{\ell=1}^{(k-1)/2} \alpha_\ell \prod_{j=\ell}^{(k-3)/2} (x_{n+j} x_{n+k-j-2})^{j-\ell+1} \tag{2.5}$$

for k odd. The meromorphic $(k-2)$ -form

$$\hat{V} = \frac{1}{\prod_{j=0}^{k-3} x_j} dx_0 \wedge \dots \wedge dx_{k-3} \tag{2.6}$$

is preserved/anti-preserved by $\hat{\varphi}$ for even/odd k respectively, and the pullback of \hat{V} is $\varpi^ \hat{V} = J_0 \lrcorner V$ where J_0 is the Poisson bivector field for the bracket $\{, \}_0$.*

Remark: The generic fibre of the projection ϖ is the gauge group $(\mathbb{C}^*)^2$, so that after removing the coordinate hyperplanes $\{\tau_j = 0\}$ the original phase space has the structure of a principal fibre bundle. In fact, Somos- k is covariant under the action of the larger scaling group $\tau_n \rightarrow A B^n C^{n^2} \tau_n$ with non-zero A, B, C , which changes the coefficients α_j for $C \neq 1$. This scaling group also appeared as a symmetry of the determinantal variables in [3]. In the case that k is odd, there is a further freedom to rescale τ_n differently for odd/even n , and φ projects down to a map on a $(k-3)$ -dimensional space of Casimirs, with coordinates $(x_0 x_1, x_1 x_2, \dots, x_{k-4} x_{k-3})$ (cf. the results for $k = 5$ in [24]).

Using recurrences due to Cantor [7], Matsutani showed that suitable ratios of the genus two psi-function (1.22) satisfy a recurrence of the form (2.4) in the case $k = 8$ (cf. equation (3.15) in [29]). Henceforth we restrict ourselves to the case $k = 6$. In that case the recurrence (2.4) is of order four, and with the coefficients

α, β, γ as in (1.17) it is given by

$$x_{n+4} (x_{n+3})^2 (x_{n+2})^3 (x_{n+1})^2 x_n = \alpha x_{n+3} (x_{n+2})^2 x_{n+1} + \beta x_{n+2} + \gamma, \quad (2.7)$$

which corresponds to a birational map in \mathbb{C}^4 with coordinates (x_0, x_1, x_2, x_3) . The special case of the recurrence (2.7) with $\beta = 0$ was found by van der Poorten (Theorem 3.1 in [36]), based on the continued fraction expansion of the square root of a sextic polynomial, defining a genus two curve; the corresponding result in the elliptic case, related to Somos-4, was given in [35].

For what follows it will be more convenient to use the alternative coordinate system $(u, x, y, v) := (x_0 x_1 x_2, x_1, x_2, x_1 x_2 x_3)$, in which the map corresponding to (2.7) takes the form

$$\hat{\varphi} : \begin{pmatrix} u \\ x \\ y \\ v \end{pmatrix} \mapsto \begin{pmatrix} v \\ y \\ \tilde{y} \\ \tilde{v} \end{pmatrix}, \quad \tilde{y} = \frac{v}{xy}, \quad \tilde{v} = \frac{(\alpha v + \beta)y + \gamma}{uv} \quad (2.8)$$

To be more precise, the above map and the original one defined by (2.7) are conjugate to one another via the change of coordinates in \mathbb{C}^4 , but we use the same symbol $\hat{\varphi}$ to denote both. This map decomposes as the product $\hat{\varphi} = \iota_1 \cdot \iota_2$ of two involutions that do not commute with one another:

$$\begin{aligned} \iota_1 : & \quad (u, x, y, v)^T \mapsto (v, y, x, u)^T, \\ \iota_2 : & \quad (u, x, y, v)^T \mapsto [(\alpha v + \beta)y + \gamma]u^{-1}v^{-1}, vx^{-1}y^{-1}, y, v)^T, \end{aligned}$$

with $\iota_j^2 = id$, $j = 1, 2$. Each of these involutions is itself a product of two commuting involutions: the permutation ι_1 decomposes as the product of the transpositions $u \leftrightarrow v$ and $x \leftrightarrow y$, and ι_2 is the composition of the replacement $u \rightarrow [(\alpha v + \beta)y + \gamma]u^{-1}v^{-1}$ and the replacement $x \rightarrow vx^{-1}y^{-1}$ (where the other variables are held fixed).

The rest of this section is concerned with proving the following result.

Theorem 2.3: *For arbitrary coefficients α, β, γ the volume-preserving map (2.8) in \mathbb{C}^4 has two independent rational first integrals H_1, H_2 . It is an integrable map in the Liouville-Arnold sense for $\alpha\beta\gamma = 0$.*

Conjecturally, the second part of the above statement should hold for arbitrary α, β, γ , and not just in the cases where one or more of the coefficients vanishes. For Liouville integrability, the map should be symplectic with H_1, H_2 being in involution with respect to the associated nondegenerate Poisson bracket. The technical obstacle when $\alpha\beta\gamma \neq 0$ is that there is no general method to construct a Poisson bracket for an arbitrary birational map. Yet when $\alpha\beta\gamma = 0$ it is straightforward to obtain a log-canonical Poisson bracket.

Proposition 2.4: *The map $\hat{\varphi}$ defined by (2.8) preserves a 4-form \hat{V} . It is a Poisson map with respect to a non-trivial bracket of log-canonical type if and only if $\alpha\beta\gamma = 0$. In each of the three cases $\alpha = 0$, $\beta = 0$, $\gamma = 0$ separately there is a distinct non-degenerate log-canonical bracket.*

Proof: In the case $k = 6$ the formula (2.6) gives an invariant 4-form, which is $\hat{V} = (uxyv)^{-1} du \wedge dx \wedge dy \wedge dv$ in the alternative set of coordinates. The rest of the proof follows from a direct calculation, which is most easily carried out in the coordinates x_j , $j = 0, 1, 2, 3$. Since the map defines the recurrence (2.7) in these variables, the log-canonical bracket must take the form $\{x_m, x_n\} = \tilde{c}_{n-m} x_m x_n$ for

some constant coefficients \tilde{c}_n with $\tilde{c}_n = -\tilde{c}_{-n}$. Clearly $\tilde{c}_0 = 0$ and the whole bracket is determined by $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$. Taking the Poisson bracket of each side of (2.7) for $n = 0$ with x_1, x_2, x_3 successively gives a system of homogeneous linear equations for these coefficients. For $\alpha\beta\gamma \neq 0$ one finds $\tilde{c}_n = 0$ for all n , so there is no non-trivial bracket of this type, while for each case $\alpha = 0, \beta = 0, \gamma = 0$ separately there is a different solution of the linear system for the \tilde{c}_n , unique up to overall rescaling, giving a non-degenerate Poisson bracket. The property of being log-canonical is preserved upon changing to the variables u, x, y, v . For completeness the explicit forms of the brackets are presented here in terms of these coordinates (only non-vanishing brackets are given):

$$\begin{aligned} \alpha = 0 : & \quad \{u, y\} = uy, & \{x, v\} = xv; \\ \beta = 0 : & \quad \{u, y\} = uy, & \{x, y\} = -xy, & \{x, v\} = xv; \\ \gamma = 0 : & \quad \{u, v\} = uv, & \{x, y\} = xy. \end{aligned}$$

□

In order to obtain first integrals (i.e. conserved quantities) for the map $\hat{\phi}$, it is instructive to present an *ad hoc* method for finding such quantities directly for bilinear recurrences, which seems to work whenever the recurrence is associated with an integrable map. The method is based on the observation that Somos-4 sequences satisfy infinitely many independent bilinear relations of higher order, with coefficients that are constant along each orbit, but can depend on the choice of orbit. To be precise, each Somos-4 sequence satisfies an independent Somos- k relation for every $k > 4$. This phenomenon is neatly encoded into the slogan “every Somos-4 is a Somos- k ” [37], and it persists for Somos-5,-6,-7 (subject to suitable constraints on the higher values of k that are allowed). As is explained in the next section, a formula for the iterates in terms of theta functions implies the existence of infinitely many relations of higher order. By regarding the coefficients of these higher recurrences as functions of the initial data, they furnish conserved quantities for the original map, which are non-trivial provided that they are not just functions of the α_j . The existence of these relations of higher order should be related to the notion of the Hirota-Kimura basis introduced in [34].

In practice one can begin to search for first integrals by looking for bilinear relations along an orbit with specific numerical values. For example, starting with an even order Somos- $2M$ recurrence one can use it to generate sufficiently many terms of a particular sequence, starting with fixed numerical values for the initial data and α_j , in order to calculate the $N \times N$ determinant

$$\begin{vmatrix} \tau_{2N}\tau_0 & \tau_{2N-1}\tau_1 & \cdots & \widehat{\tau_{N+M}\tau_{N-M}} & \cdots & \tau_{N+1}\tau_{N-1} & (\tau_N)^2 \\ \tau_{2N+1}\tau_1 & \tau_{2N}\tau_2 & \cdots & \cdots & \cdots & \tau_{N+2}\tau_N & (\tau_{N+1})^2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ \tau_{3N-1}\tau_{N-1} & \tau_{3N-2}\tau_N & \cdots & \widehat{\tau_{2N+M-1}\tau_{2N-M-1}} & \cdots & \tau_{2N}\tau_{2N-2} & (\tau_{2N-1})^2 \end{vmatrix}$$

with $N > M$, where the hat denotes that the column is deleted. If this determinant vanishes then the particular sequence in question also satisfies a Somos- $2N$ relation, and by calculating a non-zero vector in the kernel of the matrix one finds the coefficients of such a relation. Moreover, the existence of this higher order relation for a single orbit suggests that the same might hold for a generic orbit. One can check other numerical sequences, and then having found the smallest $N > M$ for which the determinant vanishes on these particular orbits, one can redo the calculation symbolically in order to find the coefficients of the Somos- $2N$ recurrence

as functions of the α_j and an arbitrary set of initial data τ_j , $j = 0, \dots, 2M - 1$ for the original Somos- $2M$ recurrence.

In the case of interest here, starting from the Somos-6 recurrence (1.17), one can take the particular numerical sequence (1.8), and one finds that the first even-width sequence of higher order is a Somos-10. Indeed, in that case the above determinant with $M = 3$ and $N = 5$ becomes

$$\begin{vmatrix} \tau_{10}\tau_0 & \tau_9\tau_1 & \tau_7\tau_3 & \tau_6\tau_4 & (\tau_5)^2 \\ \tau_{11}\tau_1 & \tau_{10}\tau_2 & \tau_8\tau_4 & \tau_7\tau_5 & (\tau_6)^2 \\ \tau_{12}\tau_2 & \tau_{11}\tau_3 & \tau_9\tau_5 & \tau_8\tau_6 & (\tau_7)^2 \\ \tau_{13}\tau_3 & \tau_{12}\tau_4 & \tau_{10}\tau_6 & \tau_9\tau_7 & (\tau_8)^2 \\ \tau_{14}\tau_4 & \tau_{13}\tau_5 & \tau_{11}\tau_7 & \tau_{10}\tau_8 & (\tau_9)^2 \end{vmatrix} = \begin{vmatrix} 75 & 23 & 5 & 3 & 1 \\ 421 & 75 & 9 & 5 & 9 \\ 1103 & 421 & 23 & 27 & 25 \\ 5047 & 1103 & 225 & 115 & 81 \\ 41783 & 5047 & 2105 & 675 & 529 \end{vmatrix} = 0.$$

The kernel of the 5×5 matrix here is spanned by the vector $(-1, 1, 15, -19, 34)^T$, which suggests that the sequence (1.8) should satisfy the Somos-10 recurrence

$$\tau_{n+5}\tau_{n-5} = \tau_{n+4}\tau_{n-4} + 15\tau_{n+2}\tau_{n-2} - 19\tau_{n+1}\tau_{n-1} + 34(\tau_n)^2.$$

To prove this for all n , and generalize it to the case of arbitrary initial data and arbitrary coefficients α, β, γ in (1.17), one should assume that in general the kernel is spanned by $(-1, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4)^T$, and then use the first four rows of the above matrix to obtain a linear system for the $\tilde{\alpha}_j$. In general, the terms τ_6, \dots, τ_{13} that appear in this system are certain Laurent polynomials in τ_0, \dots, τ_5 and α, β, γ , determined by iterating (1.17). Upon solving this linear system, it is found that $\tilde{\alpha}_1 = \gamma$, so it is independent of the τ_j , while $\tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4$ are given by more complicated expressions in the coefficients and initial data, detailed in (2.11) below. Having found these $\tilde{\alpha}_j$, one can then verify directly that they are invariant under the map φ given by (2.1) with $k = 6$, and hence are constant along each orbit, as required. Subsequently one must then check for dependencies between the $\tilde{\alpha}_j$, in order to count the number of independent first integrals so obtained.

Although we have explained how to look for higher relations of even order (Somos- $2N$), one should also check for relations of odd order, again by calculating suitable determinants. It turns out that for Somos-6 sequences the first higher relation one finds is actually a Somos-9 recurrence. Before we state the exact results for this case, it is helpful to present certain polynomials $p_j = p_j(u, x, y, v)$ for $j = 0, 1, 2$, which are the building blocks for the first integrals:

$$\begin{aligned} p_0 &= uxyv, \\ p_1 &= \alpha\beta uv(x^2y^2 + xu + yv) + \alpha\gamma xy(xu^2 + yv^2) + \gamma xyu^2v^2 \\ &\quad + \beta\gamma(x^2yu + xy^2v + uv) + \gamma^2xy(u + v) + \alpha^2\beta xy(u + v) \\ &\quad + \alpha\beta^2xy + \alpha^2\gamma(xu + yv) + \alpha\beta\gamma(x + y) + \alpha\gamma^2, \\ p_2 &= \alpha\beta\left(uv(x + y) + x^2y^2(u + v)\right) + \gamma uxyv(u + v) + \beta u^2v^2 \\ &\quad + \alpha\gamma\left(xy(ux + vy) + uv\right) + \beta^2x^2y^2 + \beta\gamma xy(x + y) + \gamma^2xy. \end{aligned} \tag{2.9}$$

Observe that these polynomials are invariant under the involution ι_1 .

Proposition 2.5: *The iterates of the Somos-6 recurrence (1.17) also satisfy the Somos-9 recurrence*

$$\begin{aligned} \tau_{n+5}\tau_{n-4} = & -\beta\tau_{n+4}\tau_{n-3} - \alpha\gamma\tau_{n+3}\tau_{n-2} \\ & + \gamma^2\tau_{n+2}\tau_{n-1} + (\alpha^4 + \alpha H_1 + \beta H_2)\tau_{n+1}\tau_n, \end{aligned} \quad (2.10)$$

as well as the Somos-10 recurrence

$$\begin{aligned} \tau_{n+5}\tau_{n-5} = & \gamma\tau_{n+4}\tau_{n-4} + (\alpha\beta^2 + \gamma H_2)\tau_{n+2}\tau_{n-2} \\ & + (\beta^3 - \alpha^3\gamma - \gamma H_1)\tau_{n+1}\tau_{n-1} + \alpha(\alpha^4 + \alpha H_1 + \beta H_2)(\tau_n)^2, \end{aligned} \quad (2.11)$$

where the quantities H_1 and H_2 are constant along each orbit. These H_j are gauge-invariant Laurent polynomials in the variables τ_0, \dots, τ_5 , and each of them is the ratio of two homogeneous polynomials of degree six. In terms of the variables u, x, y, v they are given explicitly by

$$H_1 = \frac{p_1(u, x, y, v)}{p_0(u, x, y, v)}, \quad H_2 = \frac{p_2(u, x, y, v)}{p_0(u, x, y, v)}, \quad (2.12)$$

with the p_j as defined in (2.9) above.

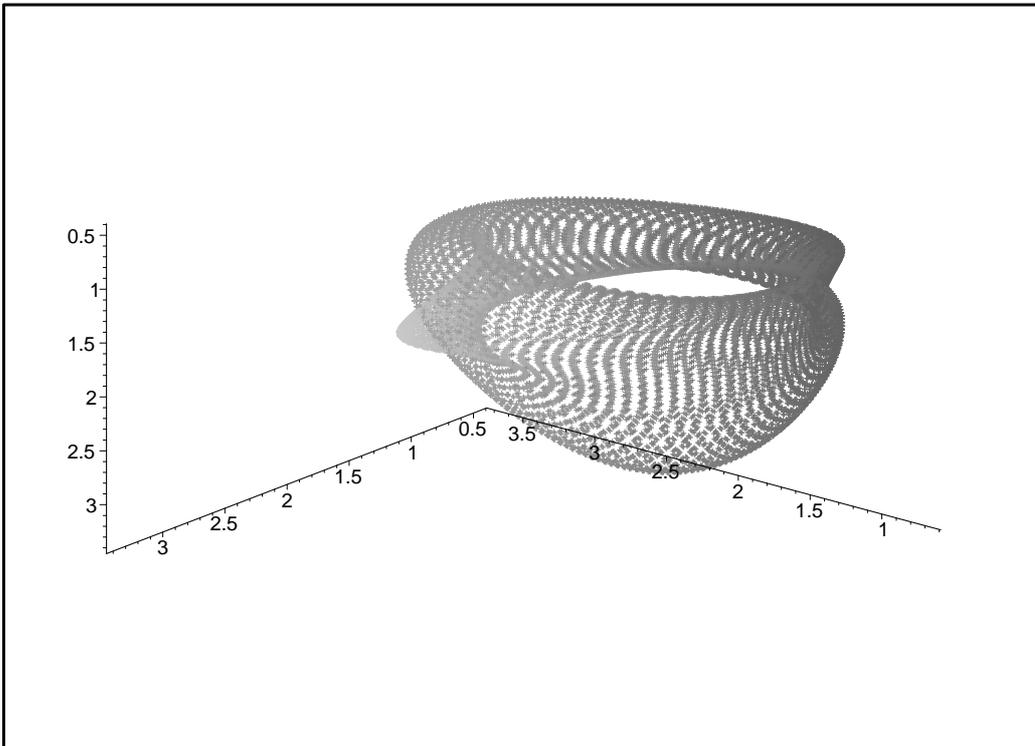


Figure 1. A plot of 25,000 points on the orbit of the point $(u, x, y, v) = (1, 1, 1, 1)$ for the map (2.8) with $\alpha = \beta = \gamma = 1$, projected onto three dimensions, namely the (u, x, y) components of each iterate.

The proof of Theorem 2.3 is now completed by noting that $H_j = H_j(u, x, y, v)$ for $j = 1, 2$ are two independent first integrals for the map $\hat{\varphi}$ defined by (2.8), and then one can verify directly that $\{H_1, H_2\} = 0$ for each of the three different cases $\alpha = 0, \beta = 0, \gamma = 0$ where there is a non-degenerate log-canonical bracket, so the map is integrable in the Liouville-Arnold sense in those cases. The intersection of the level sets for the two first integrals (2.12) defines a surface in \mathbb{C}^4 . The real version of such a surface, in the case $\alpha = \beta = \gamma = 1, H_1 = 19, H_2 = 14$, which

corresponds to the sequence (1.8), can be seen in Figure 1, where a real orbit of the map $\hat{\varphi}$ has been projected onto \mathbb{R}^3 ; in accordance with the Liouville-Arnold theorem, the structure of a real 2-torus is visible. From (2.9) the defining equations of the surface are quadratic in each of the coordinates u, x, y, v . The orbits of QRT maps lie on biquadratic curves in \mathbb{C}^2 , so the map $\hat{\varphi}$ can be viewed as a four-dimensional analogue of the QRT family. Other analogues of QRT maps in higher dimensions were constructed in [18].

3. Sigma function formulae

The connection of higher order Somos sequences with multidimensional theta functions was explained by Elkies in posts to Propp's "bilinear" forum [39]. The crucial observation of Elkies is that while suitable expressions in theta functions satisfy Somos recurrences, the number of available parameters appears insufficient to account for a general Somos- k sequence when k is large. To be precise, suppose that we are given a complex torus \mathbb{C}^g/\mathcal{L} , where the lattice $\mathcal{L} = \mathbb{Z}^g \oplus \Omega\mathbb{Z}^g$ is specified by the symmetric $g \times g$ complex matrix Ω lying in the Siegel upper half-space (i.e. the imaginary part of Ω is positive definite). The standard Riemann theta function is an entire function defined by the Fourier series

$$\Theta(\mathbf{z}; \Omega) = \sum_{\mathbf{n} \in \mathbb{Z}^g} e^{\pi i(\mathbf{n}, \Omega \mathbf{n}) + 2\pi i(\mathbf{n}, \mathbf{z})}, \quad (3.1)$$

where $(,)$ denotes the standard scalar product in dimension g . For a positive integer w , an entire function $f = f(\mathbf{z})$ on \mathbb{C}^g is said to be \mathcal{L} -quasiperiodic of weight w if for all \mathbf{z} it satisfies

$$f(\mathbf{z} + \mathbf{m}) = f(\mathbf{z}) \quad \text{and} \quad f(\mathbf{z} + \Omega \mathbf{m}) = e^{-\pi i w(\mathbf{m}, \Omega \mathbf{m}) - 2\pi i w(\mathbf{m}, \mathbf{z})} f(\mathbf{z}),$$

for all $\mathbf{m} \in \mathbb{Z}^g$. A standard result says that the vector space of such functions has dimension w^g (see Proposition 1.3 in chapter II of [32]). Up to multiplication by a scalar, the standard theta function (3.1) is the unique \mathcal{L} -quasiperiodic entire function of weight one.

Now, given the sequence with n th term

$$\tau_n = AB^n C^{n^2} \Theta(\mathbf{z}_0 + n\mathbf{z}; \Omega), \quad (3.2)$$

it is straightforward to verify that (for fixed n) each of the products $\tau_{n+j}\tau_{n-j}$ for $j = 0, \dots, 2^g$ is an \mathcal{L} -quasiperiodic function of weight two with respect to shifts of the argument $\tilde{\mathbf{z}} = \mathbf{z}_0 + n\mathbf{z}$. Since the space of such functions has dimension 2^g , it follows that these products satisfy a linear relation, whose coefficients can depend on C and \mathbf{z} but are independent of n (and are also independent of the gauge factors A, B). As the same linear relation holds for all n , the sequence of these τ_n satisfies a Somos- k recurrence with $k \leq 2^{g+1}$. The formula (3.2) depends on $A, B, C, \mathbf{z}_0, \mathbf{z}$ and Ω , which means at most $3 + 2g + g(g+1)/2 = (g+2)(g+3)/2$ complex parameters, while the general recurrence (1.11) of order k has $\lfloor k/2 \rfloor$ coefficients and k initial data. So if we have a generic choice of parameters in (3.2) such that the minimum order relation is for $k = 2^{g+1}$, then the number of parameters required for the solution of the initial value problem for the general recurrence is $k + \lfloor \frac{k}{2} \rfloor = 3 \times 2^g > (g+2)(g+3)/2$ for $g > 1$. (However, note that for odd k the quantity A in (3.2) need not be constant, but can depend on the parity of n ; cf. [24].)

To formulate the explicit analytical results for Somos-6, it is convenient to work with the Kleinian sigma function in genus two rather than working directly with theta functions. Given a curve X of genus two defined by the quintic equation

$$\mu^2 = f(\nu) \equiv 4\nu^5 + \sum_{j=0}^4 c_j \nu^j, \tag{3.3}$$

it is always possible to remove the coefficient c_4 by making a shift in the ν coordinate, $\nu \rightarrow \nu + \text{const}$. By rescaling both μ and ν , one of the other four remaining coefficients (as long as it is non-zero) can be set to be 1, which leaves three moduli for genus two curves. However, having set $c_4 \rightarrow 0$, we prefer to leave the four coefficients c_0, c_1, c_2, c_3 . From (3.3) the curve is realized as a two-sheeted cover of the Riemann sphere with five branch points in the finite complex plane and a single branch point ∞ at infinity. The vector space of holomorphic differentials is two-dimensional, being generated by $\mu^{-1}d\nu$ and $\mu^{-1}\nu d\nu$, which are conveniently organized into a vector of canonical holomorphic differentials, denoted

$$d\mathbf{u} = \begin{pmatrix} \frac{d\nu}{\mu} \\ \frac{\nu d\nu}{\mu} \end{pmatrix}.$$

The period matrices of the curve are

$$2\omega = \begin{pmatrix} \oint_{a_1} \frac{d\nu}{\mu} & \oint_{a_2} \frac{d\nu}{\mu} \\ \oint_{a_1} \frac{\nu d\nu}{\mu} & \oint_{a_2} \frac{\nu d\nu}{\mu} \end{pmatrix}, \quad 2\omega' = \begin{pmatrix} \oint_{b_1} \frac{d\nu}{\mu} & \oint_{b_2} \frac{d\nu}{\mu} \\ \oint_{b_1} \frac{\nu d\nu}{\mu} & \oint_{b_2} \frac{\nu d\nu}{\mu} \end{pmatrix},$$

where $(a_1, a_2; b_1, b_2)$ is a canonical homology basis for the compact Riemann surface corresponding to X , with non-vanishing intersections $a_j \cdot b_k = \delta_{jk}$.

The Jacobian of X is the complex torus $\text{Jac}(X) = \mathbb{C}^2 / \Lambda$, where $\Lambda = 2\omega\mathbb{Z}^2 \oplus 2\omega'\mathbb{Z}^2$ is the period lattice generated by the a - and b -periods. The elements (P_1, P_2) of the symmetric product $\text{Sym}^2(X)$ can be identified with degree zero divisors $\mathcal{D} = (P_1 - \infty) + (P_2 - \infty)$, which are mapped to $\text{Jac}(X)$ by the Abel map:

$$\mathbf{u} = \int_{\infty}^{P_1} d\mathbf{u} + \int_{\infty}^{P_2} d\mathbf{u} \in \text{Jac}(X)$$

(where the map is based at ∞).

The Kleinian sigma function $\sigma(\mathbf{u})$ is an odd function of $(u_1, u_2)^T = \mathbf{u} \in \mathbb{C}^2$, quasiperiodic with respect to shifts by elements of the period lattice Λ . The sigma function is defined in terms of the standard Riemann theta function by $\sigma(\mathbf{u}) = e^{\tilde{Q}(\mathbf{u})} \Theta((2\omega)^{-1}\mathbf{u} - \mathbf{K}; \Omega)$, where $\Omega = \omega^{-1}\omega'$ is the normalized matrix of b -periods, \mathbf{K} is a normalized half-period vector of Riemann constants, and the function \tilde{Q} is a certain sum of quadratic, linear and constant terms in \mathbf{u} ; for precise details see [2, 6]. The Kleinian ζ and \wp functions are defined by

$$\zeta_j(\mathbf{u}) = \frac{\partial \log \sigma(\mathbf{u})}{\partial u_j}, \quad \wp_{jk}(\mathbf{u}) = -\frac{\partial^2 \log \sigma(\mathbf{u})}{\partial u_j \partial u_k}, \quad j, k = 1, 2.$$

The Kleinian \wp functions solve the Jacobi inversion problem: given $\mathbf{u} \in \text{Jac}(X)$ which is the image of the pair of points (P_1, P_2) under the Abel map, with

$P_j = (\nu_j, \mu_j)$ for $j = 1, 2$, the points $P_j \in X$ can be reconstructed from their ν coordinates via the formulae $\wp_{12}(\mathbf{u}) = -\nu_1\nu_2$, $\wp_{22}(\mathbf{u}) = \nu_1 + \nu_2$.

For our purposes, the most important property of the genus two sigma function is the addition formula

$$\frac{\sigma(\mathbf{u} + \mathbf{v})\sigma(\mathbf{u} - \mathbf{v})}{\sigma(\mathbf{u})^2\sigma(\mathbf{v})^2} = \wp_{22}(\mathbf{u})\wp_{12}(\mathbf{v}) - \wp_{12}(\mathbf{u})\wp_{22}(\mathbf{v}) + \wp_{11}(\mathbf{v}) - \wp_{11}(\mathbf{u}), \quad (3.4)$$

which was found by Baker [2]. Theorem 1.1 is a consequence of the following result, which is a direct application of Baker's formula.

Proposition 3.1: *For generic $A, B \in \mathbb{C}^*$ and $\mathbf{v}_0, \mathbf{v} \in \mathbb{C}^2$, the sequence of τ_n defined by (1.19), in terms of the genus two sigma function, satisfies a Somos-8 recurrence. It satisfies a Somos-6 recurrence if and only if the shift \mathbf{v} on the Jacobian $\text{Jac}(X)$ satisfies the constraint (1.21).*

Proof: The proof is essentially the same as the proof of Theorem 2 in [4], where the case of shifts \mathbf{v} on the theta divisor (where $\sigma(\mathbf{v}) = 0$) was considered. Upon substituting (1.19) into a general Somos-8 recurrence with four coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, one obtains an expression of the form

$$C_0(\mathbf{v}) + C_{11}(\mathbf{v})\wp_{11}(\mathbf{u}) + C_{12}(\mathbf{v})\wp_{12}(\mathbf{u}) + C_{22}(\mathbf{v})\wp_{22}(\mathbf{u}) = 0, \quad (3.5)$$

where $\mathbf{u} = \mathbf{v}_0 + n\mathbf{v}$, and the coefficients C_0, C_{jk} are functions of \mathbf{v} and linear in the α_j . Since this must hold for all n , and the functions $\wp_{jk}(\mathbf{u})$ for $j, k = 1, 2$ do not satisfy a linear relation, this means that C_0 and the C_{jk} must vanish, and this uniquely determines the coefficients α_j of the Somos-8 recurrence, which are obtained by solving a linear system. However, this linear system is degenerate if and only if the determinant (1.21) vanishes. In that case one finds that a Somos-6 relation (1.17) holds instead. \square

The preceding result can be understood more clearly by making a slight extension (as well as a correction) of some results derived by Kanayama. Upon differentiating the tau-function (1.19) with respect to the coordinates v_1, v_2 , one obtains the formula

$$\partial_j \partial_k \log \tau_n = n^2 \left(\wp_{jk}(\mathbf{v}) - \wp_{jk}(\mathbf{v}_0 + n\mathbf{v}) \right), \quad j, k = 1, 2, \quad (3.6)$$

where ∂_j denotes $\partial/\partial v_j$. In the special case that $A = B = 1$ and $\mathbf{v}_0 \rightarrow 0$, one has $\tau_n = \phi_n$ as given by (1.23), and then (3.6) reduces to an expression for the second logarithmic derivatives of ϕ_n which was found in [26]. The latter expression was the key to Kanayama's recursive method for calculating the multiplication formulae for the genus two Kleinian \wp functions in genus two, i.e. for calculating $\wp_{jk}(2\mathbf{v}), \wp_{jk}(3\mathbf{v}), \dots$ in terms of Abelian functions evaluated at the argument \mathbf{v} . Using the tau-function and the function ϕ_n , it is instructive to consider the combination $\Delta_{nm}(\mathbf{v}, \mathbf{v}_0)$ defined by

$$\begin{aligned} \Delta_{nm} &= [(\phi_1)^2 \tau_{n+m} \tau_{n-m} - (\phi_m)^2 \tau_{n+1} \tau_{n-1} + \phi_{m+1} \phi_{m-1} (\tau_n)^2] / (\phi_m \tau_n)^2 \\ &= \frac{1}{\phi_{m-1} (\phi_m)^2 (\tau_n)^3} \begin{vmatrix} \phi_1 \tau_{n+m} & \phi_0 \tau_{n+m-1} & \phi_{1-m} \tau_n \\ \phi_m \tau_{n+1} & \phi_{m-1} \tau_n & \phi_0 \tau_{n-m+1} \\ \phi_{m+1} \tau_n & \phi_m \tau_{n-1} & \phi_1 \tau_{n-m} \end{vmatrix}, \end{aligned} \quad (3.7)$$

for integers $n \in \mathbb{Z}, m \geq 1$ (and note that $\phi_{-n} = -\phi_n$, so $\phi_0 = 0$, and $\phi_1 = 1$, but

ϕ_1 is kept in (3.7) for the sake of homogeneity). Observe that Δ_{nm} is independent of A, B , and for each m the above determinant is a 3×3 minor of an infinite matrix of Casorati type. Another application of Baker's formula (3.4) allows Δ_{nm} to be rewritten as a determinant analogous to (1.21).

Lemma 3.2: *The combination (3.7) can be rewritten as the determinant*

$$\Delta_{nm} = \begin{vmatrix} 1 & 1 & 1 \\ \wp_{12}(\mathbf{v}) & \wp_{12}(m\mathbf{v}) & \wp_{12}(\mathbf{v}_0 + n\mathbf{v}) \\ \wp_{22}(\mathbf{v}) & \wp_{22}(m\mathbf{v}) & \wp_{22}(\mathbf{v}_0 + n\mathbf{v}) \end{vmatrix}. \quad (3.8)$$

Corollary 3.3: *The quantity Δ_{nm} can be written as a scalar product of two vectors in \mathbb{C}^2 , so that $\Delta_{nm} = (\mathbf{c}_m, \mathbf{l}_n)$, where*

$$\mathbf{c}_m(\mathbf{v}) = \begin{pmatrix} \wp_{22}(m\mathbf{v}) - \wp_{22}(\mathbf{v}) \\ \wp_{12}(m\mathbf{v}) - \wp_{12}(\mathbf{v}) \end{pmatrix}, \quad \mathbf{l}_n(\mathbf{v}, \mathbf{v}_0) = \frac{1}{n^2} \begin{pmatrix} \partial_1 \partial_2 \log \tau_n \\ -\partial_2^2 \log \tau_n \end{pmatrix}. \quad (3.9)$$

Proof: From the formula (3.6), the elements $\wp_{jk}(\mathbf{v}_0 + n\mathbf{v})$ in the third column of (3.8) can be replaced in terms of $\wp_{jk}(\mathbf{v})$ and logarithmic derivatives of τ_n . The scalar product of the vectors \mathbf{c}_m and \mathbf{l}_n is then obtained by expanding the determinant about its third column. \square

Remark: Proposition 2.3 in Kanayama's thesis [26] asserts that, for all m, n , the quantity $\Delta_{nm}(\mathbf{v}, 0)$ (defined by (3.7) with $\tau_k = \phi_k$ for every k) vanishes identically. However, this result is incorrect: indeed, such a recurrence characterizes elliptic divisibility sequences [50] (or equivalently, division polynomials for elliptic curves), so should not be satisfied by the sequence of $\phi_n(\mathbf{v})$, which are genus two Abelian functions.

It is clear that $\mathbf{c}_1 = 0$ and $\Delta_{n1} = 0$. Generically, \mathbf{c}_2 and \mathbf{c}_3 are two linearly independent vectors in \mathbb{C}^2 , so that $\mathbf{c}_4 \neq 0$ is given by a non-zero linear combination $\mathbf{c}_4 = \kappa_2 \mathbf{c}_2 + \kappa_3 \mathbf{c}_3$, and hence $\Delta_{n4} = \kappa_2 \Delta_{n2} + \kappa_3 \Delta_{n3}$ for all n , which implies that τ_n satisfies a Somos-8 recurrence. However, if \mathbf{c}_2 and \mathbf{c}_3 are linearly dependent then a non-trivial relation $\kappa_2 \Delta_{n2} + \kappa_3 \Delta_{n3} = 0$ must hold, and so τ_n satisfies a Somos-6 recurrence. The condition for the vectors \mathbf{c}_2 and \mathbf{c}_3 to be linearly dependent is precisely equivalent to the constraint (1.21). Thus we see that Corollary 3.3 yields an alternative proof of Proposition 3.1, albeit a less constructive one. The same arguments show that the solution (1.19) of a Somos-6 recurrence does not satisfy a Somos-8, but does satisfy a Somos-10 recurrence, in agreement with Proposition 2.5, as well as a Somos- $2k$ for all $k \geq 5$. Infinitely many odd-order relations also follow (Somos-9, 11, ...) by taking a suitably modified version of Δ_{nm} .

The proof of Theorem 1.1 is completed by substituting the sigma function formula (1.19) into (1.17) to obtain a linear system of the same form as (3.5), and then this gives an unique solution for the coefficients α, β, γ subject to the constraint (1.21).

Remark: Lemma 3.2 provides alternative ways to rewrite the constraint (1.21). In terms of Kanayama's phi-function, or the sigma function, it is $\Delta_{32} = [(\phi_1)^3 \phi_5 - (\phi_2)^3 \phi_4 + \phi_1 (\phi_3)^3] / (\phi_2 \phi_3)^2 = 0$, or

$$\Delta_{32} = \frac{\sigma(\mathbf{v})^3 \sigma(5\mathbf{v}) - \sigma(2\mathbf{v})^3 \sigma(4\mathbf{v}) + \sigma(\mathbf{v}) \sigma(3\mathbf{v})^3}{\sigma(\mathbf{v})^2 \sigma(2\mathbf{v})^2 \sigma(3\mathbf{v})^2} = 0.$$

The latter expression shows that on the Jacobian of a generic genus two curve X the constraint set does not intersect the theta divisor. Indeed, when $\sigma(\mathbf{v}) = 0$

the constraint equation generically has a double pole. For the above numerator to vanish at leading order requires that $\sigma(2\mathbf{v})^3\sigma(4\mathbf{v})=0$, but $\sigma(2\mathbf{v}) \neq 0$ since $2\mathbf{v}$ corresponds to the reduced divisor of two points, and so $4\mathbf{v}$ must also lie on the theta divisor in $\text{Jac}(X)$. To satisfy (1.21), the numerator must vanish up to third order, which puts constraints on the moduli of X .

4. Conclusions

An analytic solution for the general Somos-6 recurrence has been presented. Because it depends on nine complex parameters, namely $A, B \in \mathbb{C}^*$, the coefficients c_0, c_1, c_2, c_3 of the genus two curve, and two points \mathbf{v}_0, \mathbf{v} on its Jacobian with \mathbf{v} constrained by (1.21), we can assert that it represents the general solution of (1.17), since there are six initial data and three coefficients in the recurrence. However, compared with the elliptic case it is much more difficult to solve the initial value problem in genus two. The geometrical meaning of the constraint (1.21) is still not clear, but recently we learned from results of Atkinson that it is related to the lattice Schwarzian KP equation [1].

Recently we have used the Lax pair for the discrete BKP equation to obtain a 3×3 Lax pair for the map $\hat{\varphi}$ associated with Somos-6. This leads to a genus four trigonal spectral curve possessing an involution with two fixed points, and the two-dimensional Jacobian for the solutions corresponds to an associated Prym variety, as for the known algebro-geometric solutions of discrete BKP [9]. The details of this construction will be presented elsewhere. The problem of how to construct an invariant Poisson bracket for the map $\hat{\varphi}$ when $\alpha\beta\gamma \neq 0$ is still open, but it is hoped that the Lax pair will shed some light on this.

Acknowledgements: Some of this work was carried out with a Visiting Fellowship at the Isaac Newton Institute, Cambridge, during the Programme on Discrete Integrable Systems. I have benefited from various correspondence and conversations with several people, in particular J. Propp, N. Elkies, Y. Suris and A. Bobenko. Finally I am grateful to the referees for useful comments which have improved the final version of the paper.

References

- [1] J. Atkinson, *Schwarzian integrable systems and the Möbius group: a new connection*, talk at the Isaac Newton Institute, Cambridge, 4th March 2009.
- [2] H.F. Baker, *An Introduction to The Theory of Multiply Periodic Functions*, Cambridge Univ. Press (1907).
- [3] S. Boukraa, J.-M. Maillard and G. Rollet, *Determinantal identities on integrable mappings*, Int. J. Mod. Phys. B **8** (1994) 2157–2201.
- [4] H.W. Braden, V.Z. Enolskii and A.N.W. Hone, *Bilinear recurrences and addition formulae for hyperelliptic sigma functions*, J. Nonlin. Math. Phys. **12**, Supplement 2 (2005) 46–62.
- [5] R.H. Buchholz and R.L. Rathbun, *An Infinite Set of Heron Triangles with Two Rational Medians*, Amer. Math. Monthly **104** (1997) 107–115.
- [6] V.M. Buchstaber, V.Z. Enolskii and D.V. Leykin, *Hyperelliptic Kleinian Functions and Applications*, AMS Translations **179** (1997) 1–33.
- [7] D. Cantor, *On the analogue of the division polynomials for hyperelliptic curves*, J. reine angew. Math. **447** (1994) 91–145.
- [8] G. Carroll and D. Speyer, *The Cube Recurrence*, Elec. Jour. Comb. **11** (2004) #R73.
- [9] A. Doliwa, P. Grinevich, M. Nieszporski and P.M. Santini, *Integrable lattices and their sublattices: From the discrete Moutard (discrete Cauchy-Riemann) 4-point equation to the self-adjoint 5-point scheme*, J. Math. Phys. **48** (2007) 013513.
- [10] J.J. Duistermaat, *QRT and Elliptic Surfaces*, Springer-Verlag (2009).
- [11] M. Einsiedler, G. Everest and T. Ward, *Primes in elliptic divisibility sequences*, LMS Journal of Computation and Mathematics **4** (2001) 1–13.
- [12] G. Everest, V. Miller and N. Stephens, *Primes generated by elliptic curves*, Proc. Amer. Math. Soc. **132** (2003) 955–963.

- [13] G. Everest, A. van der Poorten, I. Shparlinski and T. Ward, *Recurrence Sequences*, AMS Mathematical Surveys and Monographs, vol. **104**, Providence, RI: Amer. Math. Soc. (2003).
- [14] G. Everest, S. Stevens, D. Tamsett and T. Ward, *Primes generated by recurrence sequences*, Amer. Math. Monthly **114** (2007) 417–431.
- [15] G. Everest and K. Eisentraeger, *Descent on elliptic curves and Hilbert’s tenth problem*, Proc. Amer. Math. Soc. **137** (2009) 1951–1959.
- [16] S. Fomin and A. Zelevinsky, *The Laurent Phenomenon*, Adv. Appl. Math. **28** (2002) 119–144.
- [17] S. Fomin and A. Zelevinsky, *Cluster algebras IV: Coefficients*, Compositio Math. **143** (2007) 112–164.
- [18] A.P. Fordy and P.G. Kassotakis, *Multidimensional maps of QRT type*, J. Phys. A Math. Gen. **39** (2006) 10773–10786.
- [19] A.P. Fordy and R.J. Marsh, *Cluster-mutation periodic quivers and associated Laurent sequences*, math.CO/0904.0200
- [20] D. Gale, *The strange and surprising saga of the Somos sequences*, Mathematical Intelligencer **13** (1) (1991), 40–42; *Somos sequence update*, Mathematical Intelligencer **13** (4) (1991), 49–50; reprinted in *Tracking the Automatic Ant*, Springer (1998).
- [21] M. Gekhtman, M. Shapiro and A. Vainshtein, *Cluster algebras and Poisson geometry*, Moscow Math. J. **3** (2003) 899–934.
- [22] R. Hirota, *Exact solution of the Korteweg–de Vries equation for multiple collision of solitons*, Phys. Rev. Lett. **27** (1972) 1192–1194.
- [23] A.N.W. Hone, *Elliptic curves and quadratic recurrence sequences*, Bull. Lond. Math. Soc. **37** (2005) 161–171; Corrigendum **38** (2006) 741–742.
- [24] A.N.W. Hone, *Sigma function solution of the initial value problem for Somos 5 sequences*, Trans. Amer. Math. Soc. **359** (2007) 5019–5034.
- [25] A.N.W. Hone and C.S. Swart, *Integrality and the Laurent phenomenon for Somos 4 sequences* Math. Proc. Camb. Phil. Soc. **145** (2008) 65–85.
- [26] N. Kanayama, *Division polynomials and multiplication formulae of Jacobian varieties of dimension 2*, Doctoral thesis, Waseda University (2003); Math. Proc. Camb. Phil. Soc. **139** (2005) 399–409.
- [27] S. Lang, *Elliptic curves: Diophantine analysis*, Berlin: Springer-Verlag (1978).
- [28] J.L. Malouf, *An integer sequence from a rational recursion*, Discrete Math. **110** (1992) 257–261.
- [29] S. Matsutani, *Elliptic and hyperelliptic solutions of discrete Painlevé I and its extensions to higher order difference equations*, Phys. Lett. A **300** (2002) 233–242.
- [30] S. Matsutani, *Recursion relation of hyperelliptic PSI-functions of genus two*, Int. Transforms Spec. Func. **14** (2003) 517–527.
- [31] T. Miwa, *On Hirota’s difference equations*, Proc. Jpn. Acad., Ser. A: Math. Sci. **58** (1982) 9–12.
- [32] D. Mumford, *Tata Lectures on Theta I*, Birkhäuser (1983).
- [33] Y. Onishi, *Determinant expressions for hyperelliptic functions (with an Appendix by S. Matsutani)*, Proc. Edinburgh Math. Soc. **48** (2005) 705–742.
- [34] M. Petrer, A. Pfadler and Yu. B. Suris, *On integrability of Hirota–Kimura type discretizations. Experimental study of the discrete Clebsch system*, Experimental Math. **18** (2009) 223–247.
- [35] A.J. van der Poorten, *Elliptic curves and continued fractions*, J. Integer Sequences **8** (2005) Article 05.2.5.
- [36] A.J. van der Poorten, *Curves of genus 2, continued fractions and Somos Sequences* J. Integer Sequences **8** (2005) Article 05.3.4.
- [37] A.J. van der Poorten and C.S. Swart, *Recurrence Relations for Elliptic Sequences: every Somos 4 is a Somos k*, Bull. Lond. Math. Soc. **38** (2006) 546–554.
- [38] J. Propp, *The many faces of alternating-sign matrices*, Discr. Math. Theor. Comp. Sci. Proc. AA (DM-CCG) (2001) 4358.
- [39] J. Propp, *The “bilinear” forum*, and *The Somos Sequence Site*, <http://www.math.wisc.edu/~propp/>
- [40] G.R.W. Quispel, J.A.G. Roberts and C.J. Thompson, *Integrable mappings and soliton equations I*, Physics Letters A **126** (1988) 419–421.
- [41] G.R.W. Quispel, J.A.G. Roberts and C.J. Thompson, *Integrable mappings and soliton equations II*, Physica D **34** (1989) 183–192.
- [42] R. Robinson, *Periodicity of Somos sequences*, Proc. Amer. Math. Soc. **116** (1992) 613–619.
- [43] R. Shipsey, *Elliptic divisibility sequences*, PhD thesis, Goldsmiths College, University of London (2000).
- [44] J.H. Silverman, *The Arithmetic of Elliptic Curves*, Springer (1986).
- [45] J.H. Silverman, *p-adic properties of division polynomials and elliptic divisibility sequences*, Math. Ann. **332** (2005) 443–471; Addendum 473–474.
- [46] N.J.A. Sloane, *On-Line Encyclopedia of Integer Sequences*, sequence A006720 <http://www.research.att.com/~njas/sequences>
- [47] M. Somos, Problem 1470, Crux Mathematicorum **15** (1989) 208.
- [48] D. Speyer, *Perfect Matchings and the Octahedron Recurrence*, J. Alg. Comb. **25** (2007) 309–348.
- [49] C.S. Swart, *Elliptic curves and related sequences*, PhD thesis, Royal Holloway, University of London (2003).
- [50] M. Ward, *Memoir on Elliptic Divisibility Sequences*, Amer. J. Math. **70** (1948) 31–74.
- [51] M. Ward, *The Law of Repetition of Primes in an Elliptic Divisibility Sequence*, Duke Math. J. **15** (1948) 941–946.
- [52] A.P. Veselov, *Integrable maps*, Russ. Math. Surveys **46** (1991) 1–51.
- [53] A. Zabrodin, *Teor. Mat. Fiz.* **113** (1997) 1347 (Russian); A survey of Hirota’s difference equations solv-int/9704001.
- [54] D. Zagier, *‘Problems posed at the St. Andrews Colloquium, 1996,’ Solutions, 5th day*; available at <http://www-groups.dcs.st-and.ac.uk/~john/Zagier/Problems.html>