GALOIS-THEORETIC CHARACTERIZATION OF ISOMORPHISM CLASSES OF MONODROMICALLY FULL HYPERBOLIC CURVES OF GENUS ZERO

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Abstract. Let l be a prime number. In the present paper, we prove that the isomorphism class of an *l-monodromically full hyperbolic curve of genus zero* over a finitely generated extension of the field of rational numbers is completely determined by the kernel of the natural pro-l outer Galois representation associated to the hyperbolic curve. This result can be regarded as a *genus zero analogue* of a result due to S. Mochizuki which asserts that the isomorphism class of an *elliptic curve which does not admit complex multiplication* over a number field is completely determined by the kernels of the natural Galois representations on the various finite quotients of its Tate module.

Contents

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Introduction

Throughout the present paper, let k be a *field of characteristic zero*, $\overline{k}$ an algebraic closure of k, and $G_k \stackrel{\text{def}}{=} \text{Gal}(\overline{k}/k)$. In the present paper, we prove that if l is a prime number, then the isomorphism class of an *l-monodromically full hyperbolic curve of genus zero* over a finitely generated extension of the field of rational numbers is *completely determined* by the kernel of the associated pro-l outer Galois representation.

In [14], §1, S. Mochizuki proved the following theorem (cf. [14], Theorem 1.1):

> Let $(E_1, o_1 \in E_1(k))$, $(E_2, o_2 \in E_2(k))$ be elliptic curves over k which do *not admit complex multiplication over* $\overline{k}$. Suppose that k is a number field — i.e., a finite extension of the field of rational numbers. Then the following conditions are *equivalent*:
> (i) (E_1, o_1) is *isomorphic to* (E_2, o_2) over k.
> (ii) For $i = 1, 2$, write $T(E_i, o_i)$ for the *full Tate module* of (E_i, o_i) and
> $$\rho^{(n)}_{(E_i,o_i)/k} \colon G_k \longrightarrow \text{Aut}\Big(T(E_i, o_i) \otimes_{\widehat{\mathbb{Z}}} (\mathbb{Z}/n\mathbb{Z})\Big)$$
> for the natural Galois representation on $T(E_i, o_i) \otimes_{\widehat{\mathbb{Z}}} (\mathbb{Z}/n\mathbb{Z})$. Then $\text{Ker}(\rho^{(n)}_{(E_1,o_1)/k}) = \text{Ker}(\rho^{(n)}_{(E_2,o_2)/k})$ for any positive integer n.

In the present paper, we prove a *genus zero analogue* of the above result of Mochizuki. The main theorem of the present paper is as follows (cf. Theorem 6.1):

Theorem A (**Galois-theoretic characterization of isomorphism classes of monodromically full hyperbolic curves of genus zero**). *Let l be a prime number; k a* **finitely generated field of characteristic zero**, *i.e., a finitely generated extension of the field of rational numbers; $X_1 = (C_1, D_1 \subseteq C_1)$, $X_2 = (C_2, D_2 \subseteq C_2)$ hyperbolic curves* (*cf. Definition 1.1,* (ii)) **of genus zero** *over k which are* **l-monodromically full** (*cf. Definition 2.2,* (i)). *Suppose that the following condition* $(\dagger)^{\text{prime}}$ *is satisfied:*

> $(\dagger)^{\text{prime}}$: *There exists a finite Galois extension $k' \subseteq \overline{k}$ of k of extension degree* **prime to** *l such that $X_1 \otimes_k k'$ and $X_2 \otimes_k k'$ are* **split** (*cf. Definition 1.5,* (i)).

(*For example, if one of the following conditions is satisfied, then the above condition* $(\dagger)^{\text{prime}}$ *is satisfied:*

- *X_1 and X_2 are* **split**.
- *If we write r_i for the number of the cusps of X_i — i.e., if X_i is of type $(0, r_i)$ — then l is* **prime to** *$r_1!$ and $r_2!$ — or, equivalently, r_1, $r_2 < l$.*)

Then the following conditions are **equivalent***:*

(i) X_1 *is* **isomorphic to** X_2 *over* k*.*
(ii) *For* $i = 1, 2$*, write*

$$\rho^{\{l\}}_{X_i/k} \colon G_k \longrightarrow \mathrm{Out}\Big(\pi_1((C_i \setminus D_i) \otimes_k \overline{k})^{(l)}\Big)$$

for the natural pro-l *outer Galois representation associated to* X_i*. Then* $\mathrm{Ker}(\rho^{\{l\}}_{X_1/k}) = \mathrm{Ker}(\rho^{\{l\}}_{X_2/k})$*.*

The term "*l-monodromically full*" is a term introduced in the present paper, but the corresponding notion was studied by M. Matsumoto and A. Tamagawa in [11]. It is known (cf. [11], Theorem 1.2, as well as Corollary 2.6 of the present paper) that *many* hyperbolic curves are *l-monodromically full*. This property of *being l-monodromically full* may be regarded as an analogue for hyperbolic curves of the property of *not admitting complex multiplication* for elliptic curves. In fact, if a hyperbolic curve X of type (g, r) over a finitely generated extension k of the field of rational numbers is *l-monodromically full*, then the following hold:

- *X has no special symmetry* (i.e., roughly speaking, the automorphism group of X over $\overline{k}$ is isomorphic to the automorphism group of a *general* hyperbolic curve of type (g, r) over $\overline{k}$ — cf. Definition 3.3, Proposition 3.4).
- X is *of $\{l\}$-AIJ-type* (i.e., roughly speaking, the l-adic Tate module of the Jacobian variety of the compactification of X is, as a Galois module, *absolutely irreducible* — cf. Definition 3.5, Proposition 3.6).
- *X does not have a JCM-component* (i.e., roughly speaking, there is *no subabelian variety with complex multiplication over* $\overline{k}$ of the Jacobian variety of the compactification of X — cf. Definition 3.7, Proposition 3.8).

In the present paper, as an example, we consider *hyperbolic curves of type* $(0, 4)$ and obtain results concerning sufficient conditions for such a hyperbolic curve to be monodromically full (cf. Theorem 7.8, Corollaries 7.10, 7.11, 8.2). These results, together with Theorem A, imply the following result (cf. Corollaries 7.12, 8.3):

Theorem B (Galois-theoretic characterization of isomorphism classes of certain hyperbolic curves of type $(0, 4)$). *Let k be a* **finitely generated field of characteristic zero***, i.e., a finitely generated extension of the field of rational numbers;* $X_1 = (C_1, D_1 \subseteq C_1)$*,* $X_2 = (C_2, D_2 \subseteq C_2)$ *hyperbolic curves (cf. Definition 1.1,* (ii)*)* **of type $(0, 4)$** *over k. Suppose that one of the following conditions is satisfied:*

- *The field k is a* **number field**, *i.e., a finite extension of the field of rational numbers, and, moreover, if we write $\mathfrak{o}_{\overline{k}}$ for the ring of integers of $\overline{k}$, then $\mathfrak{m}_{X_1} \cap \mathfrak{o}^*_{\overline{k}} = \mathfrak{m}_{X_2} \cap \mathfrak{o}^*_{\overline{k}} = \emptyset$ (cf. Definition 7.9).*
- *The hyperbolic curves X_1 and X_2 are* **not NF-isotrivial** *(cf. Definition 8.1).*

Then the following conditions are **equivalent**:

(i) *X_1 is* **isomorphic to** *X_2 over k.*
(ii) *There exists an* **infinite** *set Σ of prime numbers such that, for any $l \in \Sigma$, if we write*

$$\rho^{\{l\}}_{X_i/k} \colon G_k \longrightarrow \mathrm{Out}\Big(\pi_1((C_i \setminus D_i) \otimes_k \overline{k})^{(l)}\Big)$$

for the natural pro-l outer Galois representation associated to X_i, then $\mathrm{Ker}(\rho^{\{l\}}_{X_1/k}) = \mathrm{Ker}(\rho^{\{l\}}_{X_2/k})$.

On the other hand, one may also take the point of view that Theorems A and B serve to highlight the *difference* between the *profinite* and *pro-l* outer Galois representations associated to a hyperbolic curve. In [11], Matsumoto and Tamagawa compared the *profinite* and *pro-l* outer Galois representations associated to hyperbolic curves. One result obtained in [11] which shows the difference between the *profinite* and *pro-l* outer Galois representations is the following:

> The image of the **profinite** outer Galois representation associated to any hyperbolic curve of type (g, r) over a number field k has *trivial intersection* with the image of the outer **profinite** geometric universal monodromy representation of $\pi_1(\mathcal{M}_{g,r} \otimes_k \overline{k})$ (cf. [11], Theorem 1.1 and [8], Corollary 6.4). On the other hand, there exist *many* hyperbolic curves of type (g, r) over number fields k for which the image of the associated **pro-*l*** outer Galois representation *contains* the image of the outer **pro-*l*** geometric universal monodromy representation of $\pi_1(\mathcal{M}_{g,r} \otimes_k \overline{k})$ (cf. [11], Theorem 1.2).

By Theorems A, B (cf. also Theorem C below), one obtains another result which highlights the difference between the *profinite* and *pro-l* outer Galois representations:

> The kernel of the **profinite** outer Galois representation associated to any hyperbolic curve over a number field is always *trivial*, namely, the kernel does *not depend on* the given hyperbolic curve (cf. [8], Theorem C). On the other hand, the kernel of the **pro-*l*** outer Galois representation associated to a hyperbolic curve over a number field *depends strongly on* the given hyperbolic curve (cf. Theorems A, B, also Theorem C below).

Finally, in the Appendix, we prove the following *finiteness result*, which is related to the main result of the present paper (cf. Corollary A.4):

Theorem C (Finiteness of the set of isomorphism classes of certain hyperbolic curves). *Let l be a prime number, k a* **number field***, i.e., a finite extension of the field of rational numbers, (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$, and $N \subseteq G_k$ a normal closed subgroup of G_k. Then there are* **only finitely many** *isomorphism classes over k of hyperbolic curves X of type (g, r) over k for which the kernel of the natural pro-l outer Galois representation associated to X* **coincides with** *N.*

This result follows immediately from *various well-known finiteness theorems* in number theory and arithmetic geometry, together with the criterion of Oda-Tamagawa for good reduction of hyperbolic curves. It seems to the author that this result is likely to be well-known. Since, however, this result could not be found in the literature, the author decided to give a proof of it in the Appendix of the present paper.

The present paper is organized as follows: In §1, we review some generalities concerning *outer monodromy representations* arising from hyperbolic curves. In §2, we define the notion of a Σ-*monodromically full hyperbolic curve*, as well as the related notion of a Σ-*monodromically full point*. In §3, we consider the relationship between monodromic fullness and certain properties of hyperbolic curves. In §4, we consider the moduli stacks of hyperbolic curves *of genus zero*. In §5, we prove a Grothendieck conjecture-type lemma for certain images of the universal monodromy. In §6, we derive Theorem A from the results obtained in §4 and §5. In §7 and §8, we consider the monodromic fullness of *hyperbolic curves of type* $(0, 4)$. In particular, we obtain results concerning sufficient conditions for such a hyperbolic curve to be monodromically full and prove Theorem B. In the Appendix, we derive Theorem C as a consequence of *various well-known finiteness theorems* in number theory and arithmetic geometry, together with the criterion of Oda-Tamagawa for good reduction of hyperbolic curves.

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0. Notations and Conventions

Numbers: A finite extension (respectively, finitely generated extension) of the field of rational numbers will be referred to as a *number field* (respectively, *finitely generated field of characteristic zero*). If p is a prime number, then a field which may be embedded as a subfield of a finitely generated extension of the field of fractions of the ring of Witt vectors with coefficients in an algebraic closure of the finite field of p elements will be referred to as a *generalized sub-p-adic field* (cf. [14], Definition 4.11).

Topological groups: Let G be a topological group and $\mathbf{P}$ a property for a topological group (e.g., "abelian" or "pro-l" for some prime number l). Then we shall say that G is *almost* $\mathbf{P}$ if there exists an open subgroup of G that is $\mathbf{P}$.

If G is a topological group, then we shall write G^{ab} for the *abelianization* of G, i.e., the quotient of G by the closure of the commutator subgroup of G.

If G is a topological group, and $H \subseteq G$ is a closed subgroup of G, then we shall write $Z_G(H)$ for the *centralizer* of H in G, i.e.,

$$Z_G(H) \stackrel{\mathrm{def}}{=} \{\, g \in G \mid ghg^{-1} = h \text{ for any } h \in H \,\} \subseteq G\,,$$

$Z_G^{\mathrm{loc}}(H)$ for the *local centralizer* of H in G, i.e.,

$$Z_G^{\mathrm{loc}}(H) \stackrel{\mathrm{def}}{=} \varinjlim_{H' \subseteq H} Z_G(H') \subseteq G$$

— where $H' \subseteq H$ ranges over the open subgroups of H — $Z(G) \stackrel{\mathrm{def}}{=} Z_G(G)$ for the *center* of G, and $Z^{\mathrm{loc}}(G) \stackrel{\mathrm{def}}{=} Z_G^{\mathrm{loc}}(G)$ for the *local center* of G. It is immediate from the various definitions involved that $Z_G(H) \subseteq Z_G^{\mathrm{loc}}(H)$ and that if H_1, $H_2 \subseteq G$ are closed subgroups of G such that $H_1 \subseteq H_2$ (respectively, $H_1 \subseteq H_2$; $H_1 \cap H_2$ is *open* in H_1 and H_2), then $Z_G(H_2) \subseteq Z_G(H_1)$ (respectively, $Z_G^{\mathrm{loc}}(H_2) \subseteq Z_G^{\mathrm{loc}}(H_1)$; $Z_G^{\mathrm{loc}}(H_1) = Z_G^{\mathrm{loc}}(H_2)$).

We shall say that a topological group G is *center-free* (respectively, *slim*) if $Z(G) = \{1\}$ (respectively, $Z^{\mathrm{loc}}(G) = \{1\}$). Note that it follows from [15], Remark 0.1.3, that a profinite group G is *slim* if and only if every open subgroup of G has *trivial center*.

If G is a profinite group, then we shall denote the group of automorphisms of G by $\mathrm{Aut}(G)$ and the group of inner automorphisms of G by $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$. Conjugation by elements of G determines a surjection $G \twoheadrightarrow \mathrm{Inn}(G)$. Thus, we have a homomorphism $G \to \mathrm{Aut}(G)$ whose image is $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$. We shall denote by $\mathrm{Out}(G)$ the quotient of $\mathrm{Aut}(G)$ by the normal subgroup $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ and refer to an element of $\mathrm{Out}(G)$ as an *outomorphism* of G. In particular, if G is *center-free*, then the natural homomorphism $G \to \mathrm{Inn}(G)$ is an *isomorphism*; thus, we have an exact sequence of groups

$$1 \longrightarrow G \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1\,.$$

If, moreover, G is *topologically finitely generated*, then one verifies easily that the topology of G admits a basis of *characteristic open subgroups*, which thus induces a *profinite topology* on the groups $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ with respect to which the above exact sequence determines an exact sequence of *profinite groups*. If J is a profinite group, and $\rho\colon J \to \mathrm{Out}(G)$ is a continuous homomorphism, then we shall denote by $G \overset{\mathrm{out}}{\rtimes} J$ the *profinite group* obtained by pulling back the above exact sequence of profinite groups via ρ. Thus, we have a *natural exact sequence* of profinite groups

$$1 \longrightarrow G \longrightarrow G \overset{\mathrm{out}}{\rtimes} J \longrightarrow J \longrightarrow 1\,.$$

1. Outer monodromy representations

Throughout the present paper, let k be a *field of characteristic zero* and $\overline{k}$ an algebraic closure of k. If $k' \subseteq \overline{k}$ is a(n) (possibly infinite) algebraic extension of k, then we shall write $G_{k'} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k')$.

In the present §, we review some generalities concerning *outer monodromy representations* arising from hyperbolic curves. In the present §, let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$ and Σ a nonempty set of prime numbers.

Definition 1.1. Let S be a scheme.

(i) Let C be a scheme over S and $s_i\colon S \to C$ a section of the structure morphism of C — where $i = 1, \cdots, r$. Then we shall say that $(C, (s_1, \cdots, s_r))$ is an *r-pointed smooth curve of genus g over S whose marked points are equipped with an ordering* if C is smooth and proper over S, any geometric fiber of $C \to S$ is a (necessarily smooth and proper) connected curve of genus g, and the image of s_i does not intersect the image of s_j if $i \neq j$.

(ii) Let C be a scheme over S and $D \subseteq C$ a closed subscheme of C. Then we shall say that $(C, D \subseteq C)$ is a *hyperbolic curve of type (g, r) over S* if C is smooth and proper over S, any geometric fiber of $C \to S$ is a (necessarily smooth and proper) connected curve of genus g, and the composite $D \hookrightarrow C \to S$ is a finite étale covering over S of degree r.

Definition 1.2.

(i) We shall denote by $\mathcal{M}_{g,r} \to \mathrm{Spec}\, k$ the moduli stack (cf. [5], [10]) of *r-pointed smooth curves of genus g over k whose marked points are equipped with orderings* (cf. Definition 1.1, (i)) and by $(\mathcal{C}_{g,r} \to \mathcal{M}_{g,r}, (s_1^{\mathcal{M}}, \cdots, s_r^{\mathcal{M}}))$ the universal curve over $\mathcal{M}_{g,r}$.

(ii) We shall denote by $\mathcal{M}_{g,[r]} \to \operatorname{Spec} k$ the moduli stack of *hyperbolic curves of type* (g, r) *over* k (cf. Definition 1.1, (ii)) and by $(\mathcal{C}_{g,[r]} \to \mathcal{M}_{g,[r]}, D^{\mathcal{M}}_{g,[r]} \subseteq \mathcal{C}_{g,[r]})$ the universal curve over $\mathcal{M}_{g,[r]}$.

It follows from the various definitions involved that we have a commutative diagram

$$\begin{array}{ccccc}
\mathcal{C}_{g,r} \setminus \bigcup_{i=1}^{r} \operatorname{Im}(s_i^{\mathcal{M}}) & \xrightarrow{\subseteq} & \mathcal{C}_{g,r} & \longrightarrow & \mathcal{M}_{g,r} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{C}_{g,[r]} \setminus D^{\mathcal{M}}_{g,[r]} & \xrightarrow{\subseteq} & \mathcal{C}_{g,[r]} & \longrightarrow & \mathcal{M}_{g,[r]}
\end{array}$$

such that the two squares in this diagram are *cartesian*; moreover, as is well-known, in this commutative diagram, the right-hand vertical arrow $\mathcal{M}_{g,r} \to \mathcal{M}_{g,[r]}$ is a *finite étale Galois covering* whose Galois group is isomorphic to the symmetric group on r letters $\mathfrak{S}_r$. In particular, we obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & N_{g,r} & \longrightarrow & \pi_1(\mathcal{C}_{g,r} \setminus \bigcup_{i=1}^{r} \operatorname{Im}(s_i^{\mathcal{M}})) & \longrightarrow & \pi_1(\mathcal{M}_{g,r}) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & N_{g,r} & \longrightarrow & \pi_1(\mathcal{C}_{g,[r]} \setminus D^{\mathcal{M}}_{g,[r]}) & \longrightarrow & \pi_1(\mathcal{M}_{g,[r]}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & 1 & \longrightarrow & \mathfrak{S}_r & = & \mathfrak{S}_r & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

— where $N_{g,r}$ is the kernel of the surjection $\pi_1(\mathcal{C}_{g,r} \setminus \bigcup_{i=1}^{r} \operatorname{Im}(s_i^{\mathcal{M}})) \twoheadrightarrow \pi_1(\mathcal{M}_{g,r})$, and the vertical and horizontal sequences are *exact*. (See [20] for the fundamental groups of stacks.)

Definition 1.3.

(i) We shall write

$$\Delta^{\Sigma}_{g,r}$$

for the maximal pro-Σ quotient of the kernel $N_{g,r}$ of the surjection $\pi_1(\mathcal{C}_{g,r} \setminus \bigcup_{i=1}^{r} \operatorname{Im}(s_i^{\mathcal{M}})) \twoheadrightarrow \pi_1(\mathcal{M}_{g,r})$ (cf. Remark 1.3.1 below).

(ii) We shall write

$$\rho^{\Sigma}_{g,r} \ \ (\text{respectively, } \rho^{\Sigma}_{g,[r]} \ ; \ \rho^{\Sigma\text{-geom}}_{g,r} \ ; \ \rho^{\Sigma\text{-geom}}_{g,[r]})$$

for the natural homomorphism determined by the above commutative diagram

$$\pi_1(\mathcal{M}_{g,r}) \longrightarrow \mathrm{Out}(\Delta^{\Sigma}_{g,r})$$

$$(\text{respectively, } \pi_1(\mathcal{M}_{g,[r]}) \longrightarrow \mathrm{Out}(\Delta^{\Sigma}_{g,r})\,;$$

$$\pi_1(\mathcal{M}_{g,r} \otimes_k \overline{k}) \longrightarrow \mathrm{Out}(\Delta^{\Sigma}_{g,r})\,;$$

$$\pi_1(\mathcal{M}_{g,[r]} \otimes_k \overline{k}) \longrightarrow \mathrm{Out}(\Delta^{\Sigma}_{g,r}))\,.$$

(iii) Let S be a scheme that is connected and of finite type over k, and $X = (C, D \subseteq C)$ a hyperbolic curve of type (g, r) over S. Then the classifying morphism $S \to \mathcal{M}_{g,[r]}$ of X determines — up to $\pi_1(\mathcal{M}_{g,[r]} \otimes_k \overline{k})$-inner automorphism — a section $s_{X/S}$ of the natural exact sequence

$$1 \longrightarrow \pi_1(\mathcal{M}_{g,[r]} \otimes_k \overline{k}) \longrightarrow \pi_1(\mathcal{M}_{g,[r]}) \times_{G_k} \pi_1(S) \longrightarrow \pi_1(S) \longrightarrow 1\,.$$

Thus, by considering the composite of $s_{X/S}$ and $\rho^{\Sigma}_{g,[r]}$, we obtain a homomorphism

$$\rho^{\Sigma}_{X/S} \colon \pi_1(S) \longrightarrow \mathrm{Out}(\Delta^{\Sigma}_{g,r})$$

which is determined up to $\mathrm{Im}(\rho^{\Sigma\text{-geom}}_{g,[r]})$-inner automorphism.

Remark 1.3.1. If follows immediately from, for example, [11], Lemma 2.1, that $\Delta^{\Sigma}_{g,r}$ is *naturally isomorphic to the maximal pro-*Σ *quotient* of the fundamental group of the geometric fiber of the universal curve $\mathcal{C}_{g,r} \setminus \bigcup_{i=1}^{r} \mathrm{Im}(s_i^{\mathcal{M}}) \to \mathcal{M}_{g,r}$ at a geometric point of $\mathcal{M}_{g,r}$. In particular, it follows immediately from, for example, [17], Corollary 1.3.4, that $\Delta^{\Sigma}_{g,r}$ is *slim* (cf. the discussion entitled "Topological groups" in §0); moreover, there exists a *natural bijection* between the following two sets:

- The set of the cusps of the geometric fiber of the universal curve $\mathcal{C}_{g,r} \setminus \bigcup_{i=1}^{r} \mathrm{Im}(s_i^{\mathcal{M}}) \to \mathcal{M}_{g,r}$ at a geometric point of $\mathcal{M}_{g,r}$.
- The set of the conjugacy classes of the cuspidal inertia subgroups of $\Delta^{\Sigma}_{g,r}$ associated to the cusps of the geometric fiber of the universal curve $\mathcal{C}_{g,r} \setminus \bigcup_{i=1}^{r} \mathrm{Im}(s_i^{\mathcal{M}}) \to \mathcal{M}_{g,r}$ at a geometric point of $\mathcal{M}_{g,r}$.

Lemma 1.4 (Kernels of the universal outer monodromy representations).

(i) *The action of* $\pi_1(\mathcal{M}_{g,[r]})$ *on the set of the conjugacy classes of the cuspidal inertia subgroups of* $\Delta^{\Sigma}_{g,r}$ *induced by* $\rho^{\Sigma}_{g,[r]}$ **factors through** *the quotient* $\pi_1(\mathcal{M}_{g,[r]}) \twoheadrightarrow \pi_1(\mathcal{M}_{g,[r]})/\pi_1(\mathcal{M}_{g,r}) \simeq \mathfrak{S}_r$, *and the resulting action of* $\mathfrak{S}_r$ *on the set of the conjugacy classes of the cuspidal inertia subgroups of* $\Delta^{\Sigma}_{g,r}$ *is* **faithful**.

(ii) *The kernel of* $\rho^{\Sigma}_{g,[r]}$ *is* **contained in** $\pi_1(\mathcal{M}_{g,r})$ *and* **coincides with** *the kernel of* $\rho^{\Sigma}_{g,r}$.

Proof. Assertion (i) follows immediately from the various definitions involved, together with Remark 1.3.1. Assertion (ii) follows immediately from assertion (i), together with Remark 1.3.1. □

Definition 1.5. Let S be a scheme and $X = (C, D \subseteq C)$ a hyperbolic curve of type (g, r) over S.

(i) We shall say that the hyperbolic curve X is *split* if the finite étale covering obtained as the composite $D \hookrightarrow C \to S$ (cf. Definition 1.1, (ii)) is trivial, i.e., D is isomorphic to the disjoint union of r copies of S over S.

(ii) Let $X_0 = (C_0, D_0 \subseteq C_0)$ be a hyperbolic curve over S. Then we shall say that X_0 is a *hyperbolic partial compactification* of X if there exists an open immersion $C \setminus D \hookrightarrow C_0 \setminus D_0$ over S.

(iii) Suppose that $g \geq 2$. Then it is immediate that the pair $(C, \emptyset \subseteq C)$ is a hyperbolic partial compactification of the hyperbolic curve X. We shall write $X^{\mathrm{cpt}} = (C, D \subseteq C)^{\mathrm{cpt}} \stackrel{\mathrm{def}}{=} (C, \emptyset \subseteq C)$ and refer to as the *compactification* of X.

Remark 1.5.1. Let S be a scheme that is connected and of finite type over k, and X a hyperbolic curve of type (g, r) over S.

(i) It follows immediately from Lemma 1.4, (i), that the hyperbolic curve X is *split* if and only if the image $\mathrm{Im}(\rho^{\Sigma}_{X/S})$ is *contained in* the image $\mathrm{Im}(\rho^{\Sigma}_{g,r})$.

(ii) Let X_0 be a hyperbolic partial compactification of X. Then it follows immediately from the various definitions involved that the homomorphism $\rho^{\Sigma}_{X_0/S}$ factors through the homomorphism $\rho^{\Sigma}_{X/S}$; thus, we obtain natural surjections

$$\pi_1(S) \twoheadrightarrow \mathrm{Im}(\rho^{\Sigma}_{X/S}) \twoheadrightarrow \mathrm{Im}(\rho^{\Sigma}_{X_0/S}) .$$

In particular, if $g \geq 2$, then we obtain natural surjections

$$\pi_1(S) \twoheadrightarrow \mathrm{Im}(\rho^{\Sigma}_{X/S}) \twoheadrightarrow \mathrm{Im}(\rho^{\Sigma}_{X^{\mathrm{cpt}}/S}) .$$

Lemma 1.6 (Universal pro-l outer monodromy representations). *Suppose that* Σ *is* **of cardinality one**. *Then the following hold:*

(i) *The natural surjection* $\pi_1(\mathcal{M}_{g,r}) \twoheadrightarrow G_k = \pi_1(\mathcal{M}_{0,3})$ *induces a* **surjection** $\mathrm{Ker}(\rho^{\Sigma}_{g,r}) \twoheadrightarrow \mathrm{Ker}(\rho^{\Sigma}_{0,3})$. *In particular, we obtain a commutative diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\mathcal{M}_{g,r} \otimes_k \overline{k}) & \longrightarrow & \pi_1(\mathcal{M}_{g,r}) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \rho^{\Sigma\text{-geom}}_{g,r}} & & \downarrow{\scriptstyle \rho^{\Sigma}_{g,r}} & & \downarrow{\scriptstyle \rho^{\Sigma}_{0,3}} & & \\ 1 & \longrightarrow & \mathrm{Im}(\rho^{\Sigma\text{-geom}}_{g,r}) & \longrightarrow & \mathrm{Im}(\rho^{\Sigma}_{g,r}) & \longrightarrow & \mathrm{Im}(\rho^{\Sigma}_{0,3}) & \longrightarrow & 1 \end{array}$$

— *where the horizontal sequences are* **exact**.

(ii) *The natural surjection* $\pi_1(\mathcal{M}_{g,[r]}) \twoheadrightarrow G_k = \pi_1(\mathcal{M}_{0,3})$ *induces a* **surjection** $\mathrm{Ker}(\rho^{\Sigma}_{g,[r]}) \twoheadrightarrow \mathrm{Ker}(\rho^{\Sigma}_{0,3})$. *In particular, we obtain a commutative diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\mathcal{M}_{g,[r]} \otimes_k \overline{k}) & \longrightarrow & \pi_1(\mathcal{M}_{g,[r]}) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \rho^{\Sigma\text{-geom}}_{g,[r]}} & & \downarrow{\scriptstyle \rho^{\Sigma}_{g,[r]}} & & \downarrow{\scriptstyle \rho^{\Sigma}_{0,3}} & & \\ 1 & \longrightarrow & \mathrm{Im}(\rho^{\Sigma\text{-geom}}_{g,[r]}) & \longrightarrow & \mathrm{Im}(\rho^{\Sigma}_{g,[r]}) & \longrightarrow & \mathrm{Im}(\rho^{\Sigma}_{0,3}) & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are **exact**.

(iii) *The commutative diagram*

$$\begin{array}{ccc} \mathcal{M}_{g,r} \otimes_k \overline{k} & \longrightarrow & \mathcal{M}_{g,[r]} \otimes_k \overline{k} \\ \downarrow & & \downarrow \\ \mathcal{M}_{g,r} & \longrightarrow & \mathcal{M}_{g,[r]} \end{array}$$

induces a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Im}(\rho^{\Sigma\text{-geom}}_{g,r}) & \longrightarrow & \mathrm{Im}(\rho^{\Sigma\text{-geom}}_{g,[r]}) & \longrightarrow & \mathfrak{S}_r & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathrm{Im}(\rho^{\Sigma}_{g,r}) & \longrightarrow & \mathrm{Im}(\rho^{\Sigma}_{g,[r]}) & \longrightarrow & \mathfrak{S}_r & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are **exact**, *and the vertical arrows are* **injective**.

Proof. Assertion (i) is a consequence of a result concerning *Oda's problem*: If $r \neq 0$, then the desired surjectivity was proven in [9], Corollary 4.2.2; on the other hand, if $r = 0$, then the desired surjectivity follows from [9], Theorem 3B, together with [8], Theorem C, or a result obtained in [24].

Assertion (ii) follows immediately from assertion (i), together with Lemma 1.4, (ii). Assertion (iii) follows immediately from Lemma 1.4, (ii). □

In the rest of the present §, we consider the *almost slimness* (cf. the discussion entitled "Topological groups" in §0) of the images of outer monodromy representations.

Proposition 1.7 (Almost slimness of the images of outer monodromy representations). *Let* $H \subseteq \mathrm{Im}(\rho^{\Sigma}_{g,[r]})$ *be a closed subgroup of the image* $\mathrm{Im}(\rho^{\Sigma}_{g,[r]})$. *Then the following hold:*

(i) *If* Σ *consists of* **exactly one** *prime number* l, *then* H *is* **almost pro-*l*** (*cf. the discussion entitled "Topological groups" in §0*).

(ii) *Suppose that* k *is a* **generalized sub-*l*-adic field** (*cf. the discussion entitled "Numbers" in §0*) *for some* $l \in \Sigma$ *and that there exists a hyperbolic curve* X *of type* (g, r) *over a finite extension* $k' \subseteq \overline{k}$ *of* k *such that* H **contains** *the image* $\mathrm{Im}(\rho^{\Sigma}_{X/k'})$.

Then H is **almost slim** (*cf. the discussion entitled "Topological groups" in* §*0*)*. In particular, the images* $\mathrm{Im}(\rho^{\Sigma}_{g,r})$, $\mathrm{Im}(\rho^{\Sigma}_{g,[r]})$, *and* $\mathrm{Im}(\rho^{\Sigma}_{X/k'})$ *— where X is a hyperbolic curve of type (g,r) over a finite extension $k' \subseteq \overline{k}$ of k — are* **almost slim**.

Proof. First, we consider assertion (i). It follows from [2], Corollary 7, together with the fact that $\Delta^{\Sigma}_{g,r}$ is *topologically finitely generated* (cf. Remark 1.3.1) and *pro-l*, that the image $\mathrm{Im}(\rho^{\Sigma}_{g,[r]})$ is *almost pro-l*. Thus, H is *almost pro-l*, as desired. This completes the proof of assertion (i).

Next, we consider assertion (ii). Suppose that there exists a hyperbolic curve X of type (g,r) over a finite extension $k' \subseteq \overline{k}$ of k such that H *contains* the image $\mathrm{Im}(\rho^{\Sigma}_{X/k'})$. Then since $\Delta^{\Sigma}_{g,r}$ is *center-free* (cf. Remark 1.3.1), it follows from [14], Theorem 4.12, together with [17], Corollary 1.5.7, that there exists a natural bijection

$$\mathrm{Aut}_{\overline{k}}(X \otimes_{k'} \overline{k}) \xrightarrow{\sim} Z^{\mathrm{loc}}_{\mathrm{Out}(\Delta^{\Sigma}_{g,r})}(\mathrm{Im}(\rho^{\Sigma}_{X/k'}))$$

(cf. the discussion entitled "Topological groups" in §0); in particular, $Z^{\mathrm{loc}}_{\mathrm{Out}(\Delta^{\Sigma}_{g,r})}(\mathrm{Im}(\rho^{\Sigma}_{X/k'}))$ is *finite*. On the other hand, since $\mathrm{Im}(\rho^{\Sigma}_{X/k'}) \subseteq H$, it follows that $Z^{\mathrm{loc}}_{\mathrm{Out}(\Delta^{\Sigma}_{g,r})}(H) \subseteq Z^{\mathrm{loc}}_{\mathrm{Out}(\Delta^{\Sigma}_{g,r})}(\mathrm{Im}(\rho^{\Sigma}_{X/k'}))$ (cf. the discussion entitled "Topological groups" in §0) is *finite*. Therefore, it follows from Lemma 1.8 below that H is *almost slim*. This completes the proof of assertion (ii). □

Lemma 1.8 (**Almost slimness and the finiteness of local center**)**.** *Let G be a profinite group. Then the following conditions are* **equivalent***:*

(i) *G is* **almost slim** (*cf. the discussion entitled "Topological groups" in* §*0*)*.*

(ii) *The local centre $Z^{\mathrm{loc}}(G)$* (*cf. the discussion entitled "Topological groups" in* §*0*) *is* **finite**.

Proof. First, to prove the *implication*

$$\text{(i)} \Longrightarrow \text{(ii)}\,,$$

suppose that condition (i) is satisfied, i.e., there exists an *open* subgroup $H \subseteq G$ of G that is *slim*. By replacing H by a suitable open subgroup of H, we may assume without loss of generality that H is *normal* in G. Now since H is *slim*, it follows that $Z^{\mathrm{loc}}(H) = Z^{\mathrm{loc}}(G) \cap H = \{1\}$. Thus, the composite $Z^{\mathrm{loc}}(G) \hookrightarrow G \twoheadrightarrow G/H$ is *injective*; in particular, $Z^{\mathrm{loc}}(G)$ is *finite*. This completes the proof of the above *implication*. Finally, to prove the *implication*

$$\text{(ii)} \Longrightarrow \text{(i)}\,,$$

suppose that condition (ii) is satisfied. Since $Z^{\mathrm{loc}}(G) \subseteq G$ is *finite*, there exists an *open* subgroup $H \subseteq G$ of G such that $Z^{\mathrm{loc}}(G) \cap H = \{1\}$. On the other hand, since $Z^{\mathrm{loc}}(H) = Z^{\mathrm{loc}}(G) \cap H$, it follows that

$Z^{\mathrm{loc}}(H) = \{1\}$, i.e., H is *slim*. This completes the proof of the above *implication*. □

2. Monodromically full points and curves

In the present §, we define the notion of a Σ-*monodromically full hyperbolic curve* (cf. Definition 2.2 below), as well as the related notion of a Σ-*monodromically full point* (cf. Definition 2.1 below). In the present §, let (g, r) be a pair of nonnegative integers such that $2g-2+r > 0$ and Σ a nonempty set of prime numbers.

First, we define the notion of a Σ-monodromically full, strictly Σ-monodromically full, and quasi-Σ-monodromically full point.

Definition 2.1. Let S be a scheme that is connected and of finite type over k, $X = (C, D \subseteq C)$ a hyperbolic curve of type (g, r) over S, and $s \in S$ a closed point of S. Write X_s for the hyperbolic curve over the residue field $k(s)$ of S at s obtained as the fiber of $X \to S$ at $s \in S$, i.e., $X_s = (C \times_S \operatorname{Spec} k(s), D \times_S \operatorname{Spec} k(s))$.

(i) We shall say that $s \in S$ is a Σ-*monodromically full point with respect to* X/S if, for any $l \in \Sigma$, the closed subgroup $\operatorname{Im}(\rho^{\{l\}}_{X_s/k(s)})$ of $\operatorname{Im}(\rho^{\{l\}}_{X/S})$ — here, $\operatorname{Im}(\rho^{\{l\}}_{X_s/k(s)})$ and $\operatorname{Im}(\rho^{\{l\}}_{X/S})$ are determined up to $\operatorname{Im}(\rho^{\{l\}\text{-geom}}_{g,[r]})$-conjugation — contains $\operatorname{Im}(\rho^{\{l\}}_{X/S}) \cap \operatorname{Im}(\rho^{\{l\}}_{g,r})$.

(ii) We shall say that $s \in S$ is a *strictly* Σ-*monodromically full point with respect to* X/S if, for any $l \in \Sigma$, the closed subgroup $\operatorname{Im}(\rho^{\{l\}}_{X_s/k(s)})$ of $\operatorname{Im}(\rho^{\{l\}}_{X/S})$ — here, $\operatorname{Im}(\rho^{\{l\}}_{X_s/k(s)})$ and $\operatorname{Im}(\rho^{\{l\}}_{X/S})$ are determined up to $\operatorname{Im}(\rho^{\{l\}\text{-geom}}_{g,[r]})$-conjugation — coincides with $\operatorname{Im}(\rho^{\{l\}}_{X/S})$.

(iii) We shall say that $s \in S$ is a *quasi*-Σ-*monodromically full point with respect to* X/S if, for any $l \in \Sigma$, the closed subgroup $\operatorname{Im}(\rho^{\{l\}}_{X_s/k(s)})$ of $\operatorname{Im}(\rho^{\{l\}}_{X/S})$ — here, $\operatorname{Im}(\rho^{\{l\}}_{X_s/k(s)})$ and $\operatorname{Im}(\rho^{\{l\}}_{X/S})$ are determined up to $\operatorname{Im}(\rho^{\{l\}\text{-geom}}_{g,[r]})$-conjugation — is an open subgroup of $\operatorname{Im}(\rho^{\{l\}}_{X/S})$.

If l is a prime number, then for simplicity, we write l-monodromically full (respectively, strictly l-monodromically full; quasi-l-monodromically full) instead of $\{l\}$-monodromically full (respectively, strictly $\{l\}$-monodromically full; quasi-$\{l\}$-monodromically full).

Remark 2.1.1. Let S be a scheme that is connected and of finite type over k, X a hyperbolic curve over S, and $s \in S$ a closed point of S. Consider the following conditions:

(i) $s \in S$ is *strictly* Σ-*monodromically full* with respect to X/S.
(ii) $s \in S$ is Σ-*monodromically full* with respect to X/S.
(iii) $s \in S$ is *quasi*-Σ-*monodromically full* with respect to X/S.

Then, as the terminologies suggest, it follows immediately from the various definitions involved that the implications

$$\text{(i)} \Longrightarrow \text{(ii)} \Longrightarrow \text{(iii)}$$

hold.

Next, we define the notion of a Σ-monodromically full, strictly Σ-monodromically full, and quasi-Σ-monodromically full hyperbolic curve. Roughly speaking, a Σ-monodromically full (respectively, strictly Σ-monodromically full; quasi-Σ-monodromically full) hyperbolic curve is a hyperbolic curve corresponding to a Σ-monodromically full (respectively, strictly Σ-monodromically full; quasi-Σ-monodromically full) point of the moduli stack with respect to the universal curve.

Definition 2.2. Let X be a hyperbolic curve of type (g, r) over k.

(i) We shall say that X is Σ-*monodromically full* if, for any $l \in \Sigma$, the closed subgroup $\mathrm{Im}(\rho^{\{l\}}_{X/k})$ — which is determined up to $\mathrm{Im}(\rho^{\{l\}\text{-geom}}_{g,[r]})$-conjugation — of $\mathrm{Im}(\rho^{\{l\}}_{g,[r]})$ contains $\mathrm{Im}(\rho^{\{l\}}_{g,r})$.

(ii) We shall say that X is *strictly* Σ-*monodromically full* if, for any $l \in \Sigma$, the closed subgroup $\mathrm{Im}(\rho^{\{l\}}_{X/k})$ — which is determined up to $\mathrm{Im}(\rho^{\{l\}\text{-geom}}_{g,[r]})$-conjugation — of $\mathrm{Im}(\rho^{\{l\}}_{g,[r]})$ contains $\mathrm{Im}(\rho^{\{l\}\text{-geom}}_{g,[r]})$, or, equivalently, the closed subgroup $\mathrm{Im}(\rho^{\{l\}}_{X/k})$ of $\mathrm{Im}(\rho^{\{l\}}_{g,[r]})$ coincides with $\mathrm{Im}(\rho^{\{l\}}_{g,[r]})$.

(iii) We shall say that X is *quasi*-Σ-*monodromically full* if, for any $l \in \Sigma$, the closed subgroup $\mathrm{Im}(\rho^{\{l\}}_{X/k})$ — which is determined up to $\mathrm{Im}(\rho^{\{l\}\text{-geom}}_{g,[r]})$-conjugation — of $\mathrm{Im}(\rho^{\{l\}}_{g,[r]})$ is an open subgroup of $\mathrm{Im}(\rho^{\{l\}}_{g,[r]})$.

If l is a prime number, then for simplicity, we write l-monodromically full (respectively, strictly l-monodromically full; quasi-l-monodromically full) instead of $\{l\}$-monodromically full (respectively, strictly $\{l\}$-monodromically full; quasi-$\{l\}$-monodromically full).

Remark 2.2.1. Let X be a hyperbolic curve over k. Consider the following conditions:

(i) X is *strictly* Σ-*monodromically full*.
(ii) X is Σ-*monodromically full*.
(iii) X is *quasi*-Σ-*monodromically full*.

Then, as the terminologies suggest, it follows immediately from the various definitions involved that the implications

$$\text{(i)} \Longrightarrow \text{(ii)} \Longrightarrow \text{(iii)}$$

hold.

Remark 2.2.2. Let X be a hyperbolic curve over k and Σ_1, Σ_2 nonempty sets of prime numbers. Suppose that $\Sigma_2 \subseteq \Sigma_1$. Consider the following conditions:

(i) X is Σ_1-*monodromically full* (respectively, *strictly* Σ_1-*monodromically full*; *quasi*-Σ_1-*monodromically full*).

(ii) X is Σ_2-*monodromically full* (respectively, *strictly* Σ_2-*monodromically full*; *quasi*-Σ_2-*monodromically full*).

Then it follows immediately from the various definitions involved that the implication

$$\text{(i)} \Longrightarrow \text{(ii)}$$

holds.

Remark 2.2.3. Let X be a hyperbolic curve of type (g, r) over k. Suppose that $r \leq 1$. Consider the following conditions:

(i) X is Σ-*monodromically full*.

(ii) X is *strictly* Σ-*monodromically full*.

Then it follows immediately from the various definitions involved that the equivalence

$$\text{(i)} \Longleftrightarrow \text{(ii)}$$

holds.

Remark 2.2.4. Let X be a hyperbolic curve of type (g, r) over k. Suppose that $r \geq 2$. Consider the following conditions:

(i) X is *strictly* Σ-*monodromically full*.

(ii) X is *not split* (cf. Definition 1.5, (i)).

Then it follows immediately from Remark 1.5.1, (i), that the implication

$$\text{(i)} \Longrightarrow \text{(ii)}$$

holds.

Remark 2.2.5. Let X_1 be a hyperbolic curve over k and X_2 a hyperbolic partial compactification of X_1 (cf. Definition 1.5, (ii)). Consider the following conditions:

(i) X_1 is Σ-*monodromically full* (respectively, *strictly* Σ-*monodromically full*; *quasi*-Σ-*monodromically full*).

(ii) X_2 is Σ-*monodromically full* (respectively, *strictly* Σ-*monodromically full*; *quasi*-Σ-*monodromically full*).

Then it follows immediately from Remark 1.5.1, (ii), that the implication

$$\text{(i)} \Longrightarrow \text{(ii)}$$

holds.

Remark 2.2.6. Let X be a hyperbolic curve over k, and $k' \subseteq \overline{k}$ a finite extension of k. Consider the following conditions:

(i) X is a *quasi*-Σ-*monodromically full hyperbolic curve over* k.

(ii) $X \otimes_k k'$ is a *quasi-Σ-monodromically full hyperbolic curve over k'.*

Then it follows immediately from the various definitions involved that the equivalence

$$\text{(i)} \iff \text{(ii)}$$

holds.

Remark 2.2.7. Let X be a hyperbolic curve of type (g, r) over k. Consider the following conditions:

(i) X is *split* and Σ-*monodromically full*.

(ii) For any $l \in \Sigma$, the closed subgroup $\mathrm{Im}(\rho^{\{l\}}_{X/k})$ — which is determined up to $\mathrm{Im}(\rho^{\{l\}\text{-geom}}_{g,[r]})$-conjugation — of $\mathrm{Im}(\rho^{\{l\}}_{g,[r]})$ *coincides with* $\mathrm{Im}(\rho^{\{l\}}_{g,r})$.

Then it follows immediately from Remark 1.5.1, (i), together with the various definition involved, that the equivalence

$$\text{(i)} \iff \text{(ii)}$$

holds.

Remark 2.2.8. Let S be a scheme that is connected and of finite type over k, X a hyperbolic curve over S, and $s \in S$ a closed point of S. Write $k(s)$ for the residue field of S at s and X_s for the hyperbolic curve over $k(s)$ obtained as the fiber of $X \to S$ at $s \in S$ (cf. Definition 2.1). Consider the following conditions:

(i) X_s is a Σ-*monodromically full* (respectively, *strictly* Σ-*monodromically full*; *quasi*-Σ-*monodromically full*) hyperbolic curve over $k(s)$.

(ii) $s \in S$ is a Σ-*monodromically full* (respectively, *strictly* Σ-*monodromically full*; *quasi*-Σ-*monodromically full*) point with respect to X/S.

Then it follows immediately from the various definitions involved that the implication

$$\text{(i)} \implies \text{(ii)}$$

holds.

The following result is a result essentially obtained in [11] (cf. [11], Theorem 1.2). Note that in [11], the following theorem in the case where Σ is *of cardinality one*, and k is a *number field* was proven. However, by a similar argument used in the proof of [11], Theorem 1.2, one may prove the following theorem.

Theorem 2.3 (**Existence of many monodromically full points**). *Let k be a* **finitely generated field of characteristic zero** *(cf. the discussion entitled "Numbers" in §0); $\overline{k}$ an algebraic closure of k; S a scheme that is* **connected**, **regular**, **of finite type**, *and* **separated**

over k; X a hyperbolic curve over S (cf. Definition 1.1, (ii)*); Σ a nonempty* **finite** *set of prime numbers; $S^{\mathrm{MF}} \subseteq S(\overline{k})$ the subset of $S(\overline{k})$ consisting of closed points of S which are* **strictly Σ-monodromically full** *with respect to X/S (cf. Definition 2.1,* (ii)*). Fix an inclusion $\overline{k} \hookrightarrow \mathbb{C}$; in particular, we obtain an inclusion $S(\overline{k}) \hookrightarrow S(\mathbb{C})$. Then the subset $S^{\mathrm{MF}} \subseteq (S(\overline{k}) \subseteq)\ S(\mathbb{C})$ is* **dense with respect to the complex topology** *of $S(\mathbb{C})$. If, moreover, S is* **rational** *(i.e., there exists an open subscheme of S which is isomorphic to an open subscheme of $\mathbb{P}^n_k$ for some positive integer n), then the complement $S(k) \setminus (S(k) \cap S^{\mathrm{MF}})$ in $S(k)$ of $S(k) \cap S^{\mathrm{MF}}$ forms a* **thin set** *in $S(k)$ in the sense of Hilbert's irreducibility theorem.*

Proof. This follows from the fact that a finitely generated field of characteristic zero is *Hilbertian*, together with a similar argument to the argument used in the proof of [11], Theorem 1.2, by replacing [11], Lemma 3.1 (respectively, [11], Lemma 3.3) by Lemma 2.4 (respectively, Lemma 2.5) below. □

Lemma 2.4 (**Existence of a certain open subgroup**). *Let G be a profinite group, Σ a nonempty* **finite** *set of prime numbers, and for each $l \in \Sigma$, $G \twoheadrightarrow Q_l$ a quotient of G which is* **topologically finitely generated** *and* **almost pro-l** *(cf. the discussion entitled "Topological groups" in §0). Then there exists a normal* **open** *subgroup $N \subseteq G$ of G satisfying the following condition: If H is a profinite group and $H \to G$ is a continuous homomorphism such that the composite $H \to G \twoheadrightarrow G/N$ is* **surjective***, then the composite $H \to G \twoheadrightarrow Q_l$ is* **surjective** *for each $l \in \Sigma$.*

Proof. If Σ is *of cardinality one*, then Lemma 2.4 follows from [11], Lemma 3.1; in particular, for each $l \in \Sigma$, there exists a normal open subgroup $N_l \subseteq G$ satisfying the following condition: If H is a profinite group and $H \to G$ is a continuous homomorphism such that the composite $H \to G \to G/N_l$ is *surjective*, then the composite $H \to G \to Q_l$ is *surjective*. Now write $N \stackrel{\mathrm{def}}{=} \bigcap_{l \in \Sigma} N_l \subseteq G$. Then it is immediate that this normal *open* subgroup N of G satisfies the condition in the statement of Lemma 2.4. This completes the proof of Lemma 2.4. □

Lemma 2.5 (**Finitely generatedness of the images of outer monodromy representations**). *Let k be a* **finitely generated field of characteristic zero** *(cf. the discussion entitled "Numbers" in §0), S a scheme that is connected and of finite type over k, X a hyperbolic curve over S, and l a prime number. Suppose that S is* **regular** *and* **separated** *over k. Then the quotient $\mathrm{Im}(\rho^{\{l\}}_{X/S})$ of $\pi_1(S)$ is* **topologically finitely generated**.

Proof. To verify Lemma 2.5, it is immediate that by replacing k by a finite extension of k, we may assume without loss of generality that

S is *geometrically connected* over k and that S *has a k-rational point* $s \in S(k)$. Then we have an exact sequence

$$1 \longrightarrow \pi_1(S \otimes_k \overline{k}) \longrightarrow \pi_1(S) \longrightarrow G_k \longrightarrow 1\,.$$

Since $\pi_1(S \otimes_k \overline{k})$ is *topologically finitely generated* (cf. [7], Exposé II, Théorème 2.3.1), to verify Lemma 2.5, it suffices to show that the image of the composite

$$G_k \longrightarrow \pi_1(S) \xrightarrow{\rho^{\{l\}}_{X/S}} \mathrm{Out}(\Delta^{\{l\}}_{g,r})$$

— where the first arrow is the homomorphism (which is determined up to $\pi_1(S \otimes_k \overline{k})$-inner automorphism) induced by $s \in S(k)$, and (g, r) is the type of the hyperbolic curve X over S — is *topologically finitely generated*; in particular, since the above composite *coincides with* the pro-l outer monodromy representation $\rho^{\{l\}}_{X_s/k}$ associated to the hyperbolic curve X_s over k obtained as the fiber of $X \to S$ at $s \in S(k)$, to verify Lemma 2.5 — by replacing X by X_s — we may assume without loss of generality that $S = \mathrm{Spec}\, k$.

Since k is *finitely generated field of characteristic zero*, there exist a finite extension $k' \subseteq \overline{k}$ of k, a subfield $k_0 \subseteq k'$ of k', and a scheme V_0 over k_0 satisfying the following conditions:

(i) k_0 is a number field (cf. the discussion entitled "Numbers" in §0).

(ii) V_0 is regular, separated, geometrically connected, and of finite type over k_0.

(iii) V_0 has a k_0-rational point $v \in V_0(k_0)$.

(iv) The function field of V_0 is isomorphic to k'.

(v) The hyperbolic curve $X \otimes_k k'$ over k' extends to a hyperbolic curve X_0 over V_0.

Now since the natural homomorphism $\pi_1(\mathrm{Spec}\, k') \to \pi_1(V_0)$ (cf. (iv)) is *surjective* (cf. (ii)), and the pro-l outer monodromy representation $\rho^{\{l\}}_{X \otimes_k k'/k'}$ factors through $\rho^{\{l\}}_{X_0/V_0}$ (cf. (v)), to verify Lemma 2.5, it suffices to show that the image $\mathrm{Im}(\rho^{\{l\}}_{X_0/V_0})$ is *topologically finitely generated*. Moreover, by the existence of the exact sequence (cf. (ii))

$$1 \longrightarrow \pi_1(V_0 \otimes_{k_0} \overline{k}_0) \longrightarrow \pi_1(V_0) \longrightarrow \mathrm{Gal}(\overline{k}_0/k_0) \longrightarrow 1$$

— where $\overline{k}_0$ is the algebraic closure of k_0 determined by $\overline{k}$ — together with the fact that $\pi_1(V_0 \otimes_{k_0} \overline{k}_0)$ is *topologically finitely generated* (cf. [7], Exposé II, Théorème 2.3.1), to verify Lemma 2.5, it suffices to show that the image of the composite

$$\mathrm{Gal}(\overline{k}_0/k_0) \longrightarrow \pi_1(V_0) \xrightarrow{\rho^{\{l\}}_{X_0/V_0}} \mathrm{Out}(\Delta^{\{l\}}_{g,r})$$

— where the first arrow is the homomorphism (which is determined up to $\pi_1(V_0 \otimes_{k_0} \overline{k}_0)$-inner automorphism) induced by $v \in V_0(k_0)$ (cf. (iii))

— is *topologically finitely generated*. On the other hand, since k_0 is a *number field* (cf. (i)), it follows from [11], Lemma 3.1, that the image of the above composite is *topologically finitely generated*, as desired. This completes the proof of Lemma 2.5. □

By Theorem 2.3, we obtain the following result.

Corollary 2.6 (**Existence of many monodromically full hyperbolic curves**). *Let* k *be a* **finitely generated field of characteristic zero** (*cf. the discussion entitled "Numbers" in §0*)*,* $\overline{k}$ *an algebraic closure of* k*,* (g, r) *a pair of nonnegative integers such that* $2g-2+r > 0$*,* $\mathcal{M}_{g,[r]}$ *the moduli stack of hyperbolic curves of type* (g, r) *over* k (*cf. Definition 1.2,* (ii))*,* $M_{g,[r]}$ *the coarse moduli space associated to* $\mathcal{M}_{g,[r]}$*, and* Σ *a nonempty* **finite** *set of prime numbers. Fix an inclusion* $\overline{k} \hookrightarrow \mathbb{C}$*. Then the subset of* $M_{g,[r]}(\mathbb{C})$ *of* $\mathbb{C}$*-valued points* $s \in M_{g,[r]}(\mathbb{C})$ *satisfying the following condition* $(*)^{\mathrm{MF}}$ *is* **dense with respect to the complex topology** *of* $M_{g,[r]}(\mathbb{C})$*:*

> $(*)^{\mathrm{MF}}$: *There exists a subfield* $k' \subseteq \overline{k}$ $(\subseteq \mathbb{C})$ *containing* k *and a morphism* $s_{k'}\colon \operatorname{Spec} k' \to \mathcal{M}_{g,[r]}$ *such that the hyperbolic curve corresponding to* $s_{k'}$ *is a* **Σ-monodromically full** *hyperbolic curve over* k' (*cf. Definition 2.2,* (i))*, and, moreover,* $s\colon \operatorname{Spec}\mathbb{C} \to M_{g,[r]}$ **factors through** *the composite* $\operatorname{Spec} k' \xrightarrow{s_{k'}} \mathcal{M}_{g,[r]} \to M_{g,[r]}$.

3. Relationship between monodromic fullness and certain properties of hyperbolic curves

In the present §, we consider the relationship between monodromic fullness and certain properties of hyperbolic curves (cf. Propositions 3.4, 3.6, 3.8 below). In the present §, let (g, r) be a pair of nonnegative integers such that $2g-2+r > 0$.

Definition 3.1. We shall write

$$G_{g,r} \stackrel{\mathrm{def}}{=} \begin{cases} \{1\} & (\text{if } 2g-2+r \geq 3) \\ \mathbb{Z}/2\mathbb{Z} & (\text{if } (g,r) = (1,1)\,,\ (1,2)\,,\ \text{or } (2,0)) \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & (\text{if } (g,r) = (0,4)) \\ \mathfrak{S}_3 & (\text{if } (g,r) = (0,3))\,. \end{cases}$$

It seems to the author that the following proposition is likely to be well-known.

Proposition 3.2 (**Automorphisms of general hyperbolic curves**). *Suppose that* k *is* **algebraically closed**. *Then the following hold:*

(i) *If* $X = (C, D \subseteq C)$ *is a hyperbolic curve of type* (g, r) *over* k*, then* $G_{g,r}$ *is* **isomorphic to** *a subgroup of the group* $\mathrm{Aut}_k(X)$ *of automorphisms of* X *over* k.

(ii) *There exists a hyperbolic curve $X = (C, D \subseteq C)$ of type (g, r) over k such that the group* $\mathrm{Aut}_k(X)$ *of automorphisms of X over k is* **isomorphic to** $G_{g,r}$.

Proof. First, we verify assertion (i). If $2g - 2 + r \geq 3$, then assertion (i) is immediate. If $(g, r) = (0, 3)$ or $(0, 4)$, then assertion (i) may be verified by the fact that $\mathrm{Aut}_k(C)$ — note that C is isomorphic to $\mathbb{P}^1_k$ over k — is isomorphic to $\mathrm{PGL}_2(k)$, together with a straightforward calculation. (Note that if $(g, r) = (0, 4)$, i.e., $X = (C, D \subseteq C)$ is isomorphic to $(\mathbb{P}^1_k, \{0, 1, \infty, x\} \subseteq \mathbb{P}^1_k)$ for some $x \in k \setminus \{0, 1\}$, then the following two automorphisms generate a subgroup of $\mathrm{Aut}_k(X)$ which is isomorphic to $G_{0,4} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

$$C \simeq \begin{array}{ccc} \mathbb{P}^1_k & \stackrel{\sim}{\longrightarrow} & \mathbb{P}^1_k \\ t/s & \mapsto & sx/t \end{array} \simeq C \quad ;$$

$$C \simeq \begin{array}{ccc} \mathbb{P}^1_k & \stackrel{\sim}{\longrightarrow} & \mathbb{P}^1_k \\ t/s & \mapsto & x(t-s)/(t-sx) \end{array} \simeq C \quad .)$$

Next, suppose that $(g, r) = (1, 1)$ or $(1, 2)$. If $(g, r) = (1, 1)$ (respectively, $(1, 2)$), then write $\{o\}$ (respectively, $\{o, x\}$) $\subseteq C(k)$ for the set of the marked divisor "D" of the hyperbolic curve $X = (C, D \subseteq C)$. Then since $g = 1$, by regarding the marked k-rational point o of C as an *origin*, one may regard C as an *abelian group scheme* over k whose identity section is the section determined by the k-rational point o. Thus, we have an automorphism

$$\begin{cases} C \ni t \mapsto -t \in C & (\text{if } r = 1) \\ C \ni t \mapsto x - t \in C & (\text{if } r = 2) \end{cases}$$

over k of order 2 that *preserves* $D = \{o\}$ (respectively, $= \{o, x\}$) $\subseteq C$; in particular, $G_{g,r} = \mathbb{Z}/2\mathbb{Z}$ is *isomorphic to* a subgroup of $\mathrm{Aut}_k(X)$. Next, suppose that $(g, r) = (2, 0)$. Then since the proper curve C is *hyperelliptic*, we have an automorphism of C of order 2; in particular, $G_{2,0} = \mathbb{Z}/2\mathbb{Z}$ is *isomorphic to* a subgroup of $\mathrm{Aut}_k(X) = \mathrm{Aut}_k(C)$. This completes the proof of assertion (i).

Finally, we verify assertion (ii). If $2g - 2 + r \geq 3$, then assertion (ii) follows immediately from [12], Theorem C. If $(g, r) = (0, 3)$ or $(0, 4)$, then assertion (ii) may be verified by the fact that $\mathrm{Aut}_k(C)$ — note that C is isomorphic to $\mathbb{P}^1_k$ over k — is isomorphic to $\mathrm{PGL}_2(k)$, together with a straightforward calculation. Next, suppose that $(g, r) = (1, 1)$ or $(1, 2)$. Then since $g = 1$, one may regard C as an *abelian group scheme* over k. Moreover, as is well-known, there exists a hyperbolic curve $X = (C, D \subseteq C)$ of type $(1, 1)$ (respectively, $(1, 2)$) over k such that $\mathrm{Aut}_k(C)$ is isomorphic to $C(k) \rtimes \{\pm 1\}$ — where the action of $\{\pm 1\}$ on $C(k)$ is the natural action of $\{\pm 1\}$ on an *abelian group* $C(k)$. Now assertion (ii) in the case where $(g, r) = (1, 1)$ or $(1, 2)$ follows from this fact that $\mathrm{Aut}_k(C)$ is isomorphic to $C(k) \rtimes \{\pm 1\}$, together with a straightforward calculation. Next, suppose that $(g, r) = (2, 0)$.

Then the assertion follows from, for example, [21], Theorem 1. This completes the proof of assertion (ii). □

Definition 3.3. Let X be a hyperbolic curve of type (g, r) over k. Then we shall say that X *has no special symmetry* if the group $\mathrm{Aut}_{\overline{k}}(X \otimes_k \overline{k})$ of automorphisms of $X \otimes_k \overline{k}$ over $\overline{k}$ is isomorphic to $G_{g,r}$.

Proposition 3.4 (**Quasi-monodromic fullness and automorphisms of hyperbolic curves**)**.** *Let X be a hyperbolic curve of type (g, r) over k. Suppose that X is* **quasi-Σ-monodromically full** *for a nonempty set of prime numbers Σ and that k is a* **generalized sub-l-adic field** (*cf. the discussion entitled "Numbers" in §0*) *for some $l \in \Sigma$. Then X* **has no special symmetry**.

Proof. Let X_0 be a hyperbolic curve of type (g, r) over a finite extension $k_0 \subseteq \overline{k}$ of k such that $\mathrm{Aut}_{\overline{k}}(X_0 \otimes_{k_0} \overline{k}) \simeq G_{g,r}$ (cf. Lemma 3.2, (ii)). Then since $\Delta^{\Sigma}_{g,r}$ is *center-free* (cf. Remark 1.3.1), it follows from [14], Theorem 4.12, together with [17], Corollary 1.5.7, that there exist natural bijections

$$\mathrm{Aut}_{\overline{k}}(X \otimes_k \overline{k}) \xrightarrow{\sim} Z^{\mathrm{loc}}_{\mathrm{Out}(\Delta^{\Sigma}_{g,r})}(\mathrm{Im}(\rho^{\Sigma}_{X/k})) \ ;$$

$$G_{g,r} \xrightarrow{\sim} \mathrm{Aut}_{\overline{k}}(X_0 \otimes_k \overline{k}) \xrightarrow{\sim} Z^{\mathrm{loc}}_{\mathrm{Out}(\Delta^{\Sigma}_{g,r})}(\mathrm{Im}(\rho^{\Sigma}_{X_0/k_0}))\, .$$

On the other hand, since X is *quasi-Σ-monodromically full*, it follows immediately from the definition of the term "quasi-Σ-monodromically full" that

$$Z^{\mathrm{loc}}_{\mathrm{Out}(\Delta^{\Sigma}_{g,r})}(\mathrm{Im}(\rho^{\Sigma}_{X/k})) = Z^{\mathrm{loc}}_{\mathrm{Out}(\Delta^{\Sigma}_{g,r})}(\mathrm{Im}(\rho^{\Sigma}_{g,[r]}))$$

(cf. the discussion entitled "Topological groups" in §0). Thus, since $\mathrm{Im}(\rho^{\Sigma}_{X_0/k_0}) \subseteq \mathrm{Im}(\rho^{\Sigma}_{g,[r]})$, we obtain that

$$\mathrm{Aut}_{\overline{k}}(X \otimes_k \overline{k}) \xrightarrow{\sim} Z^{\mathrm{loc}}_{\mathrm{Out}(\Delta^{\Sigma}_{g,r})}(\mathrm{Im}(\rho^{\Sigma}_{X/k})) = Z^{\mathrm{loc}}_{\mathrm{Out}(\Delta^{\Sigma}_{g,r})}(\mathrm{Im}(\rho^{\Sigma}_{g,[r]}))$$

$$\subseteq Z^{\mathrm{loc}}_{\mathrm{Out}(\Delta^{\Sigma}_{g,r})}(\mathrm{Im}(\rho^{\Sigma}_{X_0/k_0})) \simeq G_{g,r}$$

(cf. the discussion entitled "Topological groups" in §0); in particular, it follows immediately from Lemma 3.2, (i), that X *has no special symmetry.* This completes the proof of Proposition 3.4. □

Definition 3.5. Let X be a hyperbolic curve of type (g, r) over k and Σ a nonempty set of prime numbers. Suppose that $g \neq 0$. Then we shall say that X is *of Σ-AIJ-type* (where the "AIJ" stands for "absolutely irreducible Jacobian") if the following condition is satisfied: For any prime number $l \in \Sigma$ and finite extension $k' \subseteq \overline{k}$ of k such that $X(k') \neq \emptyset$, the l-adic Tate module of the Jacobian variety of the compactification of the hyperbolic curve $X \otimes_k k'$ is *irreducible* as a $G_{k'}$-module.

Remark 3.5.1. It follows immediately from the definition of the term "of AIJ-type" that if a hyperbolic curve X over k is *of* Σ*-AIJ-type* for some nonempty set of prime numbers Σ, then the Jacobian variety of the compactification of the hyperbolic curve $X \otimes_k \overline{k}$ is *simple*.

Proposition 3.6 (**Quasi-monodromic fullness and the absolute irreducibility of Jacobian variety**)**.** *Let X be a hyperbolic curve of type (g, r) over k and Σ a nonempty set of prime numbers. Suppose that k is* **finitely generated field of characteristic zero** (*cf. the discussion entitled "Numbers" in §0*)*, that $g \neq 0$, and that X is* **quasi-Σ-monodromically full**. *Then X is* **of Σ-AIJ-type**. *In particular, the Jacobian variety of the compactification of the hyperbolic curve $X \otimes_k \overline{k}$ is* **simple** (*cf. Remark 3.5.1*)*.*

Proof. To prove Proposition 3.6, it follows from the definition of the term "of AIJ-type" that we may assume without loss of generality that Σ is *of cardinality one.* Write $H^{\Sigma}_{g,r}$ for the abelian quotient of $\Delta^{\Sigma}_{g,r}$ by the normal closed subgroup generated by the cuspidal inertia subgroups of $\Delta^{\Sigma}_{g,r}$ and the closure of the commutator subgroup of $\Delta^{\Sigma}_{g,r}$. (Thus, if $g \geq 2$, then $H^{\Sigma}_{g,r}$ is *naturally isomorphic to* $(\Delta^{\Sigma}_{g,0})^{\mathrm{ab}}$.) Now it follows from a similar argument to the argument used in Remark 1.5.1, (ii), that the pro-Σ outer representation $\rho^{\Sigma}_{g,[r]} \colon \pi_1(\mathcal{M}_{g,[r]}) \to \mathrm{Out}(\Delta^{\Sigma}_{g,r})$ *induces* a pro-Σ representation $\rho \colon \pi_1(\mathcal{M}_{g,[r]}) \to \mathrm{Aut}(H^{\Sigma}_{g,r})$. Moreover, as is well-known, the following holds (cf. also Remark 1.3.1):

> Let $k' \subseteq \overline{k}$ be a finite extension of k such that $X(k') \neq \emptyset$. Then there exists an isomorphism of $H^{\Sigma}_{g,r}$ with the Σ-adic Tate module of the Jacobian variety of the compactification of $X \otimes_k k'$ such that, under this isomorphism, the action of $G_{k'}$ on $H^{\Sigma}_{g,r}$ determined by ρ, $s_{X/k}$ (cf. Definition 1.3, (iii)) and the natural action of $G_{k'}$ on the pro-Σ Tate module *coincide.*

Therefore, Proposition 3.6 follows from the definition of the term "quasi-monodromically full", together with the existence of a hyperbolic curve *of* Σ*-AIJ-type* over a number field (cf. e.g., [4], the proof of Proposition 4, also [4], Remark 5, (iv), (v)). ☐

Definition 3.7. Let X be a hyperbolic curve of type (g, r) over k. Suppose that $g \neq 0$. Then we shall say that X *has a JCM-component* (where the "JCM" stands for "Jacobian complex multiplication") if there exist a *nontrivial simple* abelian variety A over $\overline{k}$ such that $\mathrm{End}_{\overline{k}}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is *isomorphic to a number field of degree* $2\dim(A)$ and a *nontrivial* morphism over $\overline{k}$ from A to the Jacobian variety of the compactification of the hyperbolic curve $X \otimes_k \overline{k}$.

Remark 3.7.1. Let X be a hyperbolic curve *of type* $(1, 1)$ over k. Then it follows from the various definitions involved that X *has a JCM-component* if and only if the elliptic curve determined by X *admits* *complex multiplication over* $\overline{k}$ — i.e., the ring of endomorphisms of the elliptic curve determined by X over $\overline{k}$ is isomorphic to an order of an imaginary quadratic field.

Proposition 3.8 (**Quasi-monodromic fullness and complex multiplication**). *Let* X *be a hyperbolic curve of type* (g, r) *over* k. *Suppose that* k *is* **finitely generated field of characteristic zero** (*cf. the discussion entitled "Numbers" in §0*), *that* $g \neq 0$, *and that* X *is* **quasi-Σ-monodromically full** *for a nonempty set of prime numbers* Σ. *Then* X **does not have a JCM-component**.

Proof. This follows immediately from Proposition 3.6, together with [23], Corollary 2 to Theorem 5. □

4. Moduli stacks of hyperbolic curves of genus zero

In the present §, we consider the moduli stacks of hyperbolic curves *of genus zero*. In the present §, let $r \geq 3$ be an integer and l a prime number.

Lemma 4.1 (**Moduli stacks of hyperbolic curves of genus zero**).

(i) *The moduli stack* $\mathcal{M}_{0,r}$ *is* **isomorphic to** *the* $(r-3)$**-rd configuration space** *of* $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ *over* k, *i.e., the open subscheme of the fiber product over* k *of* $r-3$ *copies of* $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ *obtained as the complement of the various diagonal divisors.*

(ii) *The natural homomorphism* $\mathfrak{S}_r \to \mathrm{Aut}_k(\mathcal{M}_{0,r})$ *determined by the* $\mathfrak{S}_r$*-covering* $\mathcal{M}_{0,r} \to \mathcal{M}_{0,[r]}$ *is* **surjective**. *In particular, any automorphism* ϕ *of* $\mathcal{M}_{0,r}$ *over* k *is an* **automorphism over** $\mathcal{M}_{0,[r]}$, *i.e., there exists a commutative diagram*

$$\begin{array}{ccc} \mathcal{M}_{0,r} & \xrightarrow{\phi} & \mathcal{M}_{0,r} \\ \downarrow & & \downarrow \\ \mathcal{M}_{0,[r]} & = & \mathcal{M}_{0,[r]} \end{array}$$

— where the vertical arrows are natural morphisms, and the lower horizontal arrow is the identity automorphism of $\mathcal{M}_{0,[r]}$.

Proof. Assertions (i), (ii) are well-known. (Concerning assertion (ii), see [17], discussion following Theorem A in §0.) □

Lemma 4.2 (**Universal geometric monodromy outer representations of genus zero**).

(i) *The quotient* $\pi_1(\mathcal{M}_{0,r} \otimes_k \overline{k}) \twoheadrightarrow \mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}})$ *of* $\pi_1(\mathcal{M}_{0,r} \otimes_k \overline{k})$ **coincides with** *the* **maximal pro-*l* quotient** *of* $\pi_1(\mathcal{M}_{0,r} \otimes_k \overline{k})$. *In particular, there exists a natural homomorphism*

$$\mathrm{Aut}_{G_k}(\pi_1(\mathcal{M}_{0,r})) \longrightarrow \mathrm{Aut}_{\mathrm{Im}(\rho_{0,3}^{\{l\}})}(\mathrm{Im}(\rho_{0,r}^{\{l\}}))$$

(*cf. Lemma 1.6,* (i)).

(ii) *The abelianization of* $\mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}})$ *is a* **free** $\mathbb{Z}_l$*-module* **of rank** $\boldsymbol{(r-2)(r+1)/2}$.

Proof. Assertion (i) follows from [3], Remark following the proof of Theorem 1, together with Lemma 4.1, (i). Assertion (ii) follows immediately from [18], Corollary 2.5, together with Lemma 4.1, (i). (Indeed, it follows [18], Corollary 2.5, together with Lemma 4.1, (i), that $\mathrm{rank}_{\mathbb{Z}_l}(\mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}})^{\mathrm{ab}}) = \sum_{i=3}^{r} \mathrm{rank}_{\mathbb{Z}_l}((\Delta_{0,i}^{\{l\}})^{\mathrm{ab}}) = \sum_{i=3}^{r}(i-1) = (r-2)(r+1)/2$.) □

Lemma 4.3 (**Universal monodromy outer representations of genus zero**)**.** *Suppose that* k *is a* **finitely generated field of characteristic zero** (*cf. the discussion entitled "Numbers" in §0*). *Then the following hold:*

(i) *The image* $\mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}})$ *is* **pro-*l*** *and* **slim**.

(ii) *The image* $\mathrm{Im}(\rho_{0,r}^{\{l\}})$ *is* **slim**. *If, moreover,* k *contains a* **primitive *l*-th root of unity**, *then the image* $\mathrm{Im}(\rho_{0,r}^{\{l\}})$ *is* **pro-*l***.

(iii) *The composite of natural homomorphisms*

$$\mathrm{Aut}_k(\mathcal{M}_{0,r}) \longrightarrow \mathrm{Aut}_{G_k}(\pi_1(\mathcal{M}_{0,r}))/\mathrm{Inn}(\pi_1(\mathcal{M}_{0,r}\otimes_k \overline{k})) \longrightarrow \mathrm{Aut}_{\mathrm{Im}(\rho_{0,3}^{\{l\}})}(\mathrm{Im}(\rho_{0,r}^{\{l\}}))/\mathrm{Inn}(\mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}}))$$

(*cf. Lemma 4.2,* (i)) *is* **bijective** (*cf. Remark 4.3.1*).

(iv) *The composite of natural maps*

$$\mathcal{M}_{0,r}(k) \longrightarrow \mathrm{Hom}_{G_k}(G_k, \pi_1(\mathcal{M}_{0,r}))/\mathrm{Inn}(\pi_1(\mathcal{M}_{0,r}\otimes_k \overline{k})) \longrightarrow \mathrm{Hom}_{\mathrm{Im}(\rho_{0,3}^{\{l\}})}(G_k, \mathrm{Im}(\rho_{0,r}^{\{l\}}))/\mathrm{Inn}(\mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}}))$$

(*cf. Lemma 4.2,* (i)) *is* **injective**.

(v) *The composite of natural maps*

$$\mathcal{M}_{0,[r]}(k) \longrightarrow \mathrm{Hom}_{G_k}(G_k, \pi_1(\mathcal{M}_{0,[r]}))/\mathrm{Inn}(\pi_1(\mathcal{M}_{0,[r]}\otimes_k \overline{k})) \longrightarrow \mathrm{Hom}_{\mathrm{Im}(\rho_{0,3}^{\{l\}})}(G_k, \mathrm{Im}(\rho_{0,[r]}^{\{l\}}))/\mathrm{Inn}(\mathrm{Im}(\rho_{0,[r]}^{\{l\}\text{-geom}}))$$

(*cf. Lemma 4.2,* (i)) *is* **injective**.

Proof. Assertion (i) follows from [16], Proposition 2.2, (ii), together with Lemmas 4.1, (i); 4.2, (i).

Next, we verify assertion (ii). Since we have an exact sequence

$$1 \longrightarrow \mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}}) \longrightarrow \mathrm{Im}(\rho_{0,r}^{\{l\}}) \longrightarrow \mathrm{Im}(\rho_{0,3}^{\{l\}}) \longrightarrow 1$$

(cf. Lemma 1.6, (i)), it follows from assertion (i) that to verify the fact that $\mathrm{Im}(\rho_{0,r}^{\{l\}})$ is *slim* (respectively, *pro-l*), it suffices to show that $\mathrm{Im}(\rho_{0,3}^{\{l\}})$ is *slim* (respectively, *pro-l*). Now we prove the fact that $\mathrm{Im}(\rho_{0,3}^{\{l\}})$ is *slim*. It follows from a similar argument to the argument used in the proof of Proposition 1.7, (ii), together with Lemma 4.1, (i), that we obtain a natural bijection

$$\mathrm{Aut}_{\overline{k}}(\mathbb{P}_{\overline{k}} \setminus \{0,1,\infty\}) \xrightarrow{\sim} Z^{\mathrm{loc}}_{\mathrm{Out}(\Delta^{\{l\}}_{0,3})}(\mathrm{Im}(\rho^{\{l\}}_{0,3}))\,.$$

Therefore, *by comparing the natural actions of* $\mathrm{Aut}_{\overline{k}}(\mathbb{P}_{\overline{k}} \setminus \{0,1,\infty\})$ *and* $\mathrm{Im}(\rho^{\{l\}}_{0,3})$ *on the set of the conjugacy classes of the cuspidal inertia subgroups of* $\Delta^{\{l\}}_{0,3}$ (cf. Remark 1.3.1), it follows that the intersection

$$Z^{\mathrm{loc}}_{\mathrm{Out}(\Delta^{\{l\}}_{0,3})}(\mathrm{Im}(\rho^{\{l\}}_{0,3})) \cap \mathrm{Im}(\rho^{\{l\}}_{0,3})$$

is *trivial*; in particular, the local center $Z^{\mathrm{loc}}(\mathrm{Im}(\rho^{\{l\}}_{0,3}))$ of $\mathrm{Im}(\rho^{\{l\}}_{0,3})$ is *trivial*. This completes the proof of the fact that $\mathrm{Im}(\rho^{\{l\}}_{0,3})$ is *slim*. On the other hand, it follows immediately from [1], Theorems A, B, that if k contains a *primitive* l*-th root of unity*, then $\mathrm{Im}(\rho^{\{l\}}_{0,3})$ is *pro-*l. This completes the proof of assertion (ii).

Next, we prove assertion (iii). *By considering the action of* $\mathrm{Aut}_k(\mathcal{M}_{0,r})$ *on the set of the conjugacy classes of the cuspidal inertia subgroups of* $\mathrm{Im}(\rho^{\{l\}\text{-geom}}_{0,r})$, the *injectivity* of the composite in question follows immediately from Lemmas 4.1, (i), (ii); 4.2, (i), together with Remark 1.3.1. Now we verify the *surjectivity* of the composite in question by *induction on* r. If $r = 3$, 4, then the *surjectivity* of the composite in question follows from [14], Theorem 4.12, together with Lemmas 4.1, (i); 4.2, (i). Suppose that $r \geq 5$ and that the composite of natural homomorphisms

$$\mathrm{Aut}_k(\mathcal{M}_{0,r-1}) \longrightarrow \mathrm{Aut}_{G_k}(\pi_1(\mathcal{M}_{0,r-1}))/\mathrm{Inn}(\pi_1(\mathcal{M}_{0,r-1} \otimes_k \overline{k}))$$

$$\longrightarrow \mathrm{Aut}_{\mathrm{Im}(\rho^{\{l\}}_{0,3})}(\mathrm{Im}(\rho^{\{l\}}_{0,r-1}))/\mathrm{Inn}(\mathrm{Im}(\rho^{\{l\}\text{-geom}}_{0,r-1}))$$

is *bijective*. Let α be an automorphism of $\mathrm{Im}(\rho^{\{l\}}_{0,r})$ over $\mathrm{Im}(\rho^{\{l\}}_{0,3})$. Then it follows immediately from [17], Theorem 3.1.13 (note that [17], Theorem 3.1.13, is valid for a *finitely generated field of characteristic zero*, even though in [17], this result for a *number field* is only stated), that — by compositing a suitable automorphism of $\mathrm{Im}(\rho^{\{l\}}_{0,r})$ over $\mathrm{Im}(\rho^{\{l\}}_{0,3})$ *arising from an element of* $\mathrm{Aut}_k(\mathcal{M}_{0,r})$ — we may assume without loss of generality that α *preserves* the kernel $\Delta^{\{l\}}_{0,r-1} \subseteq \mathrm{Im}(\rho^{\{l\}\text{-geom}}_{0,r})$ of the natural surjection $\mathrm{Im}(\rho^{\{l\}}_{0,r}) \twoheadrightarrow \mathrm{Im}(\rho^{\{l\}}_{0,r-1})$ (cf. Lemmas 4.1, (i); 4.2, (i)). Moreover, it follows immediately from the above induction hypothesis that — again by compositing a suitable automorphism of $\mathrm{Im}(\rho^{\{l\}}_{0,r})$ over $\mathrm{Im}(\rho^{\{l\}}_{0,3})$ *arising from an element of* $\mathrm{Aut}_k(\mathcal{M}_{0,r})$ — we may assume without loss of generality that the automorphism of $\mathrm{Im}(\rho^{\{l\}}_{0,r-1})$ induced by α is the *identity automorphism* of $\mathrm{Im}(\rho^{\{l\}}_{0,r-1})$, i.e., we obtain a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta^{\{l\}}_{0,r-1} & \longrightarrow & \mathrm{Im}(\rho^{\{l\}}_{0,r}) & \longrightarrow & \mathrm{Im}(\rho^{\{l\}}_{0,r-1}) & \longrightarrow & 1 \\ & & \downarrow & & \alpha\downarrow & & \| & & \\ 1 & \longrightarrow & \Delta^{\{l\}}_{0,r-1} & \longrightarrow & \mathrm{Im}(\rho^{\{l\}}_{0,r}) & \longrightarrow & \mathrm{Im}(\rho^{\{l\}}_{0,r-1}) & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are *exact*, and the right-hand vertical arrow is the *identity automorphism*. Therefore, it follows immediately from [14], Theorem 4.12, together with Lemma 4.4, (ii), below, that α *arises from an automorphism of* $\mathcal{M}_{0,r}$ over k. This completes the proof of assertion (iii).

Assertion (iv) follows immediately from [13], Theorem C, together with Lemmas 4.1, (i); 4.2, (i). Assertion (v) follows from [14], Remark following Theorem 4.12 (cf. also the proof of [13], Theorem C). ☐

Remark 4.3.1. In [17], Theorem A, the *bijectivity* of the composite of natural homomorphisms

$$\mathrm{Aut}_k(\mathcal{M}_{0,r}) \longrightarrow \mathrm{Aut}_{G_k}(\pi_1(\mathcal{M}_{0,r}))/\mathrm{Inn}(\pi_1(\mathcal{M}_{0,r} \otimes_k \overline{k}))$$
$$\longrightarrow \mathrm{Aut}_{\mathrm{Im}(\rho^{\{l\}}_{0,3})}(\mathrm{Im}(\rho^{\{l\}}_{0,r}))/\mathrm{Inn}(\mathrm{Im}(\rho^{\{l\}\text{-geom}}_{0,r}))$$

in the case where p is *odd* was proven.

Lemma 4.4. *Let S be a connected* **normal** *scheme and $\eta_S \to S$ the generic point of S. Then the following hold:*

(i) *Let $T \to S$ be a scheme that is* **finite** *over S. Then the natural morphism* $\mathrm{Hom}_S(S, T) \to \mathrm{Hom}_S(\eta_S, T)$ *is* **bijective**.

(ii) *Let X_1, X_2 be hyperbolic curves over S. Then the natural morphism*

$$\mathrm{Isom}_S(X_1, X_2) \longrightarrow \mathrm{Isom}_{\eta_S}(X_1 \times_S \eta_S, X_2 \times_S \eta_S)$$

is **bijective**.

Proof. First, we consider assertion (i). The *injectivity* of the morphism in question follows immediately from the fact that the natural morphism $\eta_S \to S$ is *scheme-theoretically dense*. To verify the *surjectivity* of the morphism in question, let $\phi\colon \eta_S \to T$ be a morphism over S. Write $F \subseteq T$ for the scheme-theoretic image of ϕ. Then it follows immediately from the various definitions involved that F is *integral*, and the composite $F \hookrightarrow T \to S$ is *birational* and *finite*. Thus, since S is *normal*, it follows from Zariski's main theorem (cf. [6], Corollaire 4.4.9) that the composite $F \hookrightarrow T \to S$ is an *isomorphism*; in particular, ϕ extends to a morphism $S \to T$ over S.

Finally, we consider assertion (ii). It follows from, for example, [5], Theorem 1.11, that the functor $\mathrm{Isom}_S(X_1, X_2)$ is represented by a *scheme* that is *finite* and *unramified* over S. Thus, assertion (ii) follows from assertion (i). ☐

5. A Grothendieck conjecture-type lemma for certain images of the universal monodromy

In the present §, we prove a Grothendieck conjecture-type lemma for certain images of the universal monodromy (cf. Lemma 5.2 below). In the present §, let $r \geq 3$ be an integer and l a prime number. Suppose, moreover, that k is a *finitely generated field of characteristic zero* (cf. the discussion entitled "Numbers" in §0). Let us fix an isomorphism $\mathrm{Im}(\rho_{0,[r]}^{\{l\}})/\mathrm{Im}(\rho_{0,r}^{\{l\}}) \xrightarrow{\sim} \mathfrak{S}_r$ (cf. Lemma 1.6, (iii)).

For $i = 1, 2$, let

$$H_i \subseteq \mathrm{Im}(\rho_{0,[r]}^{\{l\}})$$

be an open subgroup of $\mathrm{Im}(\rho_{0,[r]}^{\{l\}})$ that *contains* the normal open subgroup $\mathrm{Im}(\rho_{0,r}^{\{l\}}) \subseteq \mathrm{Im}(\rho_{0,[r]}^{\{l\}})$,

$$(H_i \twoheadrightarrow)\, Q_i$$

the image of the composite $H_i \hookrightarrow \mathrm{Im}(\rho_{0,[r]}^{\{l\}}) \twoheadrightarrow \mathrm{Im}(\rho_{0,[r]}^{\{l\}})/\mathrm{Im}(\rho_{0,r}^{\{l\}}) \simeq \mathfrak{S}_r$, and

$$H_i^{\mathrm{geom}} \subseteq H_i$$

the kernel of the composite $H_i \hookrightarrow \mathrm{Im}(\rho_{0,[r]}^{\{l\}}) \twoheadrightarrow \mathrm{Im}(\rho_{0,3}^{\{l\}})$, i.e., $H_i^{\mathrm{geom}} \stackrel{\mathrm{def}}{=} H_i \cap \mathrm{Im}(\rho_{0,[r]}^{\{l\}\text{-geom}})$ (cf. Lemma 1.6, (ii)). Thus, H_i fits into the following exact sequences:

$$1 \longrightarrow H_i^{\mathrm{geom}} \longrightarrow H_i \longrightarrow \mathrm{Im}(\rho_{0,3}^{\{l\}}) \longrightarrow 1\,;$$

$$1 \longrightarrow \mathrm{Im}(\rho_{0,r}^{\{l\}}) \longrightarrow H_i \longrightarrow Q_i\,(\subseteq \mathfrak{S}_r) \longrightarrow 1\,.$$

(Here, the surjectivity of $H_i \to \mathrm{Im}(\rho_{0,3}^{\{l\}})$ follows from Lemma 1.6, (i).) By the various definitions involved, this open subgroup $H_i \subseteq \mathrm{Im}(\rho_{0,[r]}^{\{l\}})$ *corresponds to the intermediate connected finite étale covering*

$$[\mathcal{M}_{0,r}/Q_i] \longrightarrow \mathcal{M}_{0,[r]}$$

of the $\mathfrak{S}_r$-covering $\mathcal{M}_{0,r} \to [\mathcal{M}_{0,r}/\mathfrak{S}_r] = \mathcal{M}_{0,[r]}$ — where "$[\mathcal{M}_{0,r}/(-)]$" is the quotient of $\mathcal{M}_{0,r}$ by "$(-)$" in the sense of stacks. Now we shall write

$$\mathrm{Aut}_k^{Q_1,Q_2}(\mathcal{M}_{0,r})$$

for the set of automorphisms of $\mathcal{M}_{0,r}$ over k which is compatible with the respective actions $Q_1 \hookrightarrow \mathfrak{S}_r \to \mathrm{Aut}_k(\mathcal{M}_{0,r})$ and $Q_2 \hookrightarrow \mathfrak{S}_r \to \mathrm{Aut}_k(\mathcal{M}_{0,r})$ relative to an isomorphism $Q_1 \xrightarrow{\sim} Q_2$ of finite groups, i.e., the subset of $\mathrm{Aut}_k(\mathcal{M}_{0,r})$ consisting of automorphisms ϕ of $\mathcal{M}_{0,r}$ over k which fit into a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{0,r} & \xrightarrow{\phi} & \mathcal{M}_{0,r} \\ \downarrow & & \downarrow \\ [\mathcal{M}_{0,r}/Q_1] & \longrightarrow & [\mathcal{M}_{0,r}/Q_2] \end{array}$$

— where the vertical arrows are natural morphisms, and the horizontal arrows are *isomorphisms* over k. Then we define a map

$$\Phi\colon \mathrm{Aut}_k^{Q_1,Q_2}(\mathcal{M}_{0,r}) \longrightarrow \mathrm{Isom}_{\mathrm{Im}(\rho_{0,3})}(H_1,H_2)/\mathrm{Inn}(H_2^{\mathrm{geom}})$$

as follows: Let $\phi \in \mathrm{Aut}_k^{Q_1,Q_2}(\mathcal{M}_{0,r})$. Then it follows from the definition of $\mathrm{Aut}_k^{Q_1,Q_2}(\mathcal{M}_{0,r})$ that ϕ induces a diagram

$$\begin{array}{ccc} \pi_1(\mathcal{M}_{0,r}) & \longrightarrow & \pi_1(\mathcal{M}_{0,r}) \\ \downarrow & & \downarrow \\ \pi_1([\mathcal{M}_{0,r}/Q_1]) & \longrightarrow & \pi_1([\mathcal{M}_{0,r}/Q_2]) \end{array}$$

— where the top horizontal arrow is the $\pi_1(\mathcal{M}_{0,r}\otimes_k \overline{k})$-conjugacy class of the automorphism of $\pi_1(\mathcal{M}_{0,r})$ induced by ϕ, and this diagram commutes up to $\pi_1([\mathcal{M}_{0,r}/Q_2]\otimes_k \overline{k})$-inner automorphism. Thus, by considering the H_2^{geom}-conjugacy class of the isomorphism

$$H_1 = \pi_1([\mathcal{M}_{0,r}/Q_1])/\mathrm{Ker}(\rho_{0,r}^{\{l\}}) \xrightarrow{\sim} \pi_1([\mathcal{M}_{0,r}/Q_2])/\mathrm{Ker}(\rho_{0,r}^{\{l\}}) = H_2$$

induced by the lower horizontal arrow in the above diagram (note that by Lemma 4.2, (i), the top horizontal arrow in the above diagram preserves $\mathrm{Ker}(\rho_{0,r}^{\{l\}}) \subseteq \pi_1(\mathcal{M}_{0,r})$), we obtain an element $\Phi(\phi)$ of $\mathrm{Isom}_{\mathrm{Im}(\rho_{0,3}^{\{l\}})}(H_1,H_2)/\mathrm{Inn}(H_2^{\mathrm{geom}})$, as desired.

The purpose of the present § is to prove the *surjectivity* of this map Φ under the assumption that

$(*)^{\mathrm{prime}}$: l is *prime to* the orders of Q_1 and Q_2.

In the rest of the present §, suppose that the above condition $(*)^{\mathrm{prime}}$ *is satisfies.*

Lemma 5.1 (Preserving the $\boldsymbol{\mathcal{M}_{0,r}}$-parts). *Let* $\phi\colon H_1 \xrightarrow{\sim} H_2$ *be an isomorphism over* $\mathrm{Im}(\rho_{0,3}^{\{l\}})$. *Then* $\phi(\mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}})) = \mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}})$. *If, moreover,* k *contains a* **primitive $\boldsymbol{l}$-th root of unity**, *then* $\phi(\mathrm{Im}(\rho_{0,r}^{\{l\}})) = \mathrm{Im}(\rho_{0,r}^{\{l\}})$.

Proof. It follows immediately from Lemma 4.3, (i), together with the assumption that the condition $(*)^{\mathrm{prime}}$ is satisfied (cf. the discussion preceding Lemma 5.1), that $\mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}}) \subseteq H_i^{\mathrm{geom}}$ is *the maximal pro-l closed subgroup of* H_i^{geom}; therefore, it follows that $\phi(\mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}})) = \mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}})$. If, moreover, k contains a *primitive l-th root of unity*, then it follows from Lemma 4.3, (ii), together with the assumption that the condition $(*)^{\mathrm{prime}}$ is satisfied (cf. the discussion preceding Lemma 5.1), that $\mathrm{Im}(\rho_{0,r})^{\{l\}} \subseteq H_i$ is *the maximal pro-l closed subgroup of* H_i; therefore, it follows that $\phi(\mathrm{Im}(\rho_{0,r}^{\{l\}})) = \mathrm{Im}(\rho_{0,r}^{\{l\}})$. □

Next, we shall write

$$\widetilde{\Phi}\colon \mathrm{Aut}_k^{Q_1,Q_2}(\mathcal{M}_{0,r}) \longrightarrow \mathrm{Isom}_{\mathrm{Im}(\rho_{0,3}^{\{l\}})}(H_1,H_2)/\mathrm{Inn}(\mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}}))$$

for the map defined as follows: Let $\phi \in \mathrm{Aut}_k^{Q_1,Q_2}(\mathcal{M}_{0,r})$. Then ϕ determines an $\mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}})$-conjugacy class of an automorphism of $\mathrm{Im}(\rho_{0,r}^{\{l\}})$ over $\mathrm{Im}(\rho_{0,3}^{\{l\}})$. Moreover, by the definition of $\mathrm{Aut}_k^{Q_1,Q_2}(\mathcal{M}_{0,r})$, this $\mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}})$-conjugacy class is compatible with the respective outer actions of Q_1 and Q_2 on $\mathrm{Im}(\rho_{0,r}^{\{l\}})$ relative to an isomorphism $Q_1 \xrightarrow{\sim} Q_2$. Therefore, since $\mathrm{Im}(\rho_{0,r}^{\{l\}})$ is *center-free* (cf. Lemma 4.3, (ii)), we obtain an $\mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}})$-conjugacy class $\Phi^{Q_1,Q_2}(\phi)$ of an isomorphism

$$H_1 \simeq \mathrm{Im}(\rho_{0,r}^{\{l\}}) \overset{\text{out}}{\rtimes} Q_1 \xrightarrow{\sim} \mathrm{Im}(\rho_{0,r}^{\{l\}}) \overset{\text{out}}{\rtimes} Q_2 \simeq H_2$$

(cf. the discussion entitled "Topological groups" in §0) over $\mathrm{Im}(\rho_{0,3}^{\{l\}})$.

Note that by the various definitions involved, the diagram

$$\begin{array}{ccc} \mathrm{Aut}_k^{Q_1,Q_2}(\mathcal{M}_{0,r}) & \xrightarrow{\widetilde{\Phi}} & \mathrm{Isom}_{\mathrm{Im}(\rho_{0,3}^{\{l\}})}(H_1,H_2)/\mathrm{Inn}(\mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}})) \\ \| & & \downarrow \\ \mathrm{Aut}_k^{Q_1,Q_2}(\mathcal{M}_{0,r}) & \xrightarrow[\Phi]{} & \mathrm{Isom}_{\mathrm{Im}(\rho_{0,3}^{\{l\}})}(H_1,H_2)/\mathrm{Inn}(H_2^{\mathrm{geom}}) \end{array}$$

— where the right-hand vertical arrow is the natural surjection — *commutes*.

Lemma 5.2 (A Grothendieck conjecture-type lemma for certain images of the universal monodromy). *In the above diagram, the following hold:*

(i) $\widetilde{\Phi}$ *is* **injective**.

(ii) $\widetilde{\Phi}$ *is* **surjective**.

(iii) Φ *is* **surjective**. *Moreover, for ϕ, $\phi' \in \mathrm{Aut}_k^{Q_1,Q_2}(\mathcal{M}_{0,r})$, it holds that $\Phi(\phi) = \Phi(\phi')$ if and only if $\phi' \circ \phi^{-1} \in \mathrm{Aut}_k(\mathcal{M}_{0,r})$ is an element of the image of the composite $Q_2 \hookrightarrow \mathfrak{S}_r \to \mathrm{Aut}(\mathcal{M}_{0,r})$.*

Proof. First, we consider assertion (i). To prove the injectivity of $\widetilde{\Phi}$ — by replacing k by a finite extension of k — we may assume without loss of generality that k contains a *primitive l-th root of unity* (cf. Remark 5.2.1 below). Now we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Aut}_k^{Q_1,Q_2}(\mathcal{M}_{0,r}) & \xrightarrow{\widetilde{\Phi}} & \mathrm{Isom}_{\mathrm{Im}(\rho_{0,3}^{\{l\}})}(H_1,H_2)/\mathrm{Inn}(\mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}})) \\ \downarrow & & \downarrow \\ \mathrm{Aut}_k(\mathcal{M}_{0,r}) & \longrightarrow & \mathrm{Aut}_{\mathrm{Im}(\rho_{0,3}^{\{l\}})}(\mathrm{Im}(\rho_{0,r}^{\{l\}}))/\mathrm{Inn}(\mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}})) \end{array}$$

— where the left-hand vertical arrow is the natural inclusion, the right-hand vertical arrow is the map obtained by restricting elements of $\mathrm{Isom}_{\mathrm{Im}(\rho_{0,3}^{\{l\}})}(H_1,H_2)/\mathrm{Inn}(\mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}}))$ to $\mathrm{Im}(\rho_{0,r}^{\{l\}}) \subseteq H_i$ (cf. Lemma 5.1), and the lower horizontal arrow is the homomorphism obtained in Lemma 4.2, (i). Thus, since the lower horizontal arrow is *injective* (cf. Lemma 4.3, (iii)), it follows that $\widetilde{\Phi}$ is *injective*. This completes the proof of assertion (i).

Next, we consider assertion (ii). To prove the surjectivity of $\widetilde{\Phi}$, it follows from assertion (i), together with *Galois descent*, by replacing k by a finite extension of k, we may assume without loss of generality that k contains a *primitive* l*-th root of unity* (cf. Remark 5.2.1 below). Let $\phi\colon H_1 \xrightarrow{\sim} H_2$ be an isomorphism over $\mathrm{Im}(\rho_{0,3}^{\{l\}})$. Then it follows from Lemma 5.1 that we obtain a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Im}(\rho_{0,r}^{\{l\}}) & \longrightarrow & H_1 & \longrightarrow & Q_1 & \longrightarrow & 1 \\ & & \phi\downarrow\wr & & \phi\downarrow\wr & & \wr\downarrow\overline{\phi} & & \\ 1 & \longrightarrow & \mathrm{Im}(\rho_{0,r}^{\{l\}}) & \longrightarrow & H_2 & \longrightarrow & Q_2 & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are *exact*, and the vertical arrows are *isomorphisms*. Now it follows from Lemma 4.3, (iii), that the $\mathrm{Im}(\rho_{0,r}^{\{l\}\text{-geom}})$-conjugacy class of the left-hand vertical arrow *arises from an automorphism* $\widetilde{\phi}$ *of* $\mathcal{M}_{0,r}$ over k; moreover, since the above diagram *commutes*, it follows from assertion (i) that this automorphism $\widetilde{\phi}$ is *compatible with* the respective actions $Q_1 \hookrightarrow \mathfrak{S}_r \to \mathrm{Aut}_k(\mathcal{M}_{0,r})$ and $Q_2 \hookrightarrow \mathfrak{S}_r \to \mathrm{Aut}_k(\mathcal{M}_{0,r})$ relative to the isomorphism $\overline{\phi}\colon Q_1 \xrightarrow{\sim} Q_2$, i.e., $\widetilde{\phi}$ is an element of $\mathrm{Aut}_k^{Q_1,Q_2}(\mathcal{M}_{0,r})$. This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertions (i), (ii), together with the various definitions involved. □

Remark 5.2.1. Let $\zeta_l \in \overline{k}$ be a *primitive* l*-th root of unity*. Then it follows from [1], Theorems A, B, that $\mathrm{Ker}(G_k \twoheadrightarrow \mathrm{Im}(\rho_{0,3}^{\{l\}})) \subseteq G_{k(\zeta_l)}$. Therefore, if we write

$$(H_i)_{k(\zeta_l)} = H_i \cap \rho_{0,[r]}^{\{l\}}(\pi_1(\mathcal{M}_{0,[r]} \otimes_k k(\zeta_l))) \subseteq \mathrm{Im}(\rho_{0,[r]}^{\{l\}})$$

and

$$(H_i)_{k(\zeta_l)}^{\mathrm{geom}} \stackrel{\mathrm{def}}{=} (H_i)_{k(\zeta_l)} \cap H_i^{\mathrm{geom}}\,,$$

then $(H_i)_{k(\zeta_l)}^{\mathrm{geom}} = H_i^{\mathrm{geom}}$; in particular, $(H_i)_{k(\zeta_l)}$ fits into similar exact sequences

$$1 \longrightarrow (H_i)_{k(\zeta_l)}^{\mathrm{geom}} (= H_i^{\mathrm{geom}}) \longrightarrow (H_i)_{k(\zeta_l)} \longrightarrow \rho_{0,3}^{\{l\}}(\pi_1(\mathcal{M}_{0,3} \otimes_k k(\zeta_l))) \longrightarrow 1\,;$$

$$1 \longrightarrow \rho_{0,r}^{\{l\}}(\pi_1(\mathcal{M}_{0,r} \otimes_k k(\zeta_l))) \longrightarrow (H_i)_{k(\zeta_l)} \longrightarrow Q_i\,(\subseteq \mathfrak{S}_r) \longrightarrow 1$$

to the exact sequences

$$1 \longrightarrow H_i^{\mathrm{geom}} \longrightarrow H_i \longrightarrow \mathrm{Im}(\rho_{0,3}^{\{l\}}) \longrightarrow 1\,;$$

$$1 \longrightarrow \mathrm{Im}(\rho_{0,r}^{\{l\}}) \longrightarrow H_i \longrightarrow Q_i\,(\subseteq \mathfrak{S}_r) \longrightarrow 1\,.$$

6. Proof of the main result

In the present §, we prove that the isomorphism class of an *l-monodromically full* hyperbolic curve of genus zero over a finitely generated field of characteristic zero is *completely determined* by the kernel of the associated pro-l outer Galois representation (cf. Theorem 6.1 below).

Theorem 6.1 (**Galois-theoretic characterization of isomorphism classes of monodromically full hyperbolic curves of genus zero**). *Let l be a prime number; k a* **finitely generated field of characteristic zero** (*cf. the discussion entitled "Numbers" in* §*0*)*; $\overline{k}$ an algebraic closure of k; $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$; $X_1 = (C_1, D_1 \subseteq C_1)$, $X_2 = (C_2, D_2 \subseteq C_2)$ hyperbolic curves* (*cf. Definition 1.1,* (ii)) **of genus zero** *over k which are* **l-monodromically full** (*cf. Definition 2.2,* (i))*. Suppose that the following condition $(\dagger)^{\mathrm{prime}}$ is satisfied:*

> $(\dagger)^{\mathrm{prime}}$: *There exists a finite Galois extension $k' \subseteq \overline{k}$ of k of extension degree is* **prime to** *l such that $X_1 \otimes_k k'$ and $X_2 \otimes_k k'$ are* **split** (*cf. Definition 1.5,* (i)).

(*For example, if one of the following conditions is satisfied, then the above condition $(\dagger)^{\mathrm{prime}}$ is satisfied:*

- *X_1 and X_2 are* **split***.*
- *If we write r_i for the number of the cusps of X_i — i.e., if X_i is of type $(0, r_i)$ — then l is* **prime to** *$r_1!$ and $r_2!$ — or, equivalently, r_1, $r_2 < l$.*)

Then the following conditions are **equivalent***:*

(i) *X_1 is* **isomorphic to** *X_2 over k.*

(ii) *For $i = 1, 2$, write*

$$\rho^{\{l\}}_{X_i/k} \colon G_k \longrightarrow \mathrm{Out}\Big(\pi_1((C_i \setminus D_i) \otimes_k \overline{k})^{(l)}\Big)$$

for the pro-l outer Galois representation associated to X_i. Then $\mathrm{Ker}(\rho^{\{l\}}_{X_1/k}) = \mathrm{Ker}(\rho^{\{l\}}_{X_2/k})$.

Proof. The implication

$$\text{(i)} \Longrightarrow \text{(ii)}$$

is immediate; thus, to verify Theorem 6.1, it suffices to show the *implication*

$$\text{(ii)} \Longrightarrow \text{(i)}\,.$$

Suppose that condition (ii) is satisfied. Let us write $N \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\rho^{\{l\}}_{X_1/k}) = \mathrm{Ker}(\rho^{\{l\}}_{X_2/k}) \subseteq G_k$ (cf. condition (ii)) and r_i for the number of the cusps of the hyperbolic curve X_i, i.e., X_i is a hyperbolic curve *of type* $(0, r_i)$.

Now it follows immediately that the bijection of sets $\phi \colon \mathrm{Im}(\rho^{\{l\}}_{X_1/k}) \to \mathrm{Im}(\rho^{\{l\}}_{X_2/k})$ obtained as the composite

$$\mathrm{Im}(\rho^{\{l\}}_{X_1/k}) \stackrel{\sim}{\longleftarrow} G_k/N \stackrel{\sim}{\longrightarrow} \mathrm{Im}(\rho^{\{l\}}_{X_2/k})$$

— where the "$\overset{\sim}{\leftarrow}$" and "$\overset{\sim}{\rightarrow}$" are natural isomorphisms — is an *isomorphism of profinite groups*; moreover, it follows from Lemma 1.6, (ii), that this isomorphism ϕ is an *isomorphism over* $\mathrm{Im}(\rho_{0,3}^{\{l\}})$. Thus, since X_1 and X_2 are *l-monodromically full*, and the condition $(\dagger)^{\mathrm{prime}}$ is *satisfied*, it follows from a similar argument to the argument used in the proof of Lemma 5.1 (cf. the condition $(*)^{\mathrm{prime}}$ in the discussion preceding Lemma 5.1) that ϕ maps $\mathrm{Im}(\rho_{0,r_1}^{\{l\}\text{-geom}}) \subseteq \mathrm{Im}(\rho_{X_1/k}^{\{l\}})$ *bijectively onto* $\mathrm{Im}(\rho_{0,r_2}^{\{l\}\text{-geom}}) \subseteq \mathrm{Im}(\rho_{X_2/k}^{\{l\}})$. In particular, it follows immediately from Lemma 4.2, (ii), that $r_1 = r_2$.

Write $r \overset{\mathrm{def}}{=} r_1 = r_2$, Q_i for the image of the composite $\mathrm{Im}(\rho_{X_i/k}^{\{l\}}) \hookrightarrow \mathrm{Im}(\rho_{0,[r]}^{\{l\}}) \twoheadrightarrow \mathrm{Im}(\rho_{0,[r]}^{\{l\}})/\mathrm{Im}(\rho_{0,r}^{\{l\}})$ ($\simeq \mathfrak{S}_r$ — cf. Lemma 1.6, (iii)), and $[\mathcal{M}_{0,r}/Q_i] \to \mathcal{M}_{0,[r]}$ for the intermediate connected finite étale covering of the $\mathfrak{S}_r$-covering $\mathcal{M}_{0,r} \to [\mathcal{M}_{0,r}/\mathfrak{S}_r] = \mathcal{M}_{0,[r]}$ corresponding to the image $\mathrm{Im}(\rho_{X_i/k}^{\{l\}}) \subseteq \mathrm{Im}(\rho_{0,[r]}^{\{l\}})$. Then it follows from Lemma 5.2, (iii), together with the assumption that the condition $(\dagger)^{\mathrm{prime}}$ is *satisfied* (cf. the condition $(*)^{\mathrm{prime}}$ in the discussion preceding Lemma 5.1), that the isomorphism obtained as the composite

$$\pi_1([\mathcal{M}_{0,r}/Q_1])/\mathrm{Ker}(\rho_{0,r}^{\{l\}}) = \mathrm{Im}(\rho_{X_1/k}^{\{l\}})$$

$$\overset{\phi}{\underset{\sim}{\longrightarrow}} \mathrm{Im}(\rho_{X_2/k}^{\{l\}}) = \pi_1([\mathcal{M}_{0,r}/Q_2])/\mathrm{Ker}(\rho_{0,r}^{\{l\}})$$

arises from the lower horizontal arrow in a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{0,r} & \longrightarrow & \mathcal{M}_{0,r} \\ \downarrow & & \downarrow \\ [\mathcal{M}_{0,r}/Q_1] & \longrightarrow & [\mathcal{M}_{0,r}/Q_2] \end{array}$$

— where the vertical arrows are natural morphisms, and the horizontal arrows are *isomorphisms* over k. Therefore, it follows from Lemma 4.1, (ii), together with the various definitions involved, that if we write $\widetilde{s}_{X_i} \in \mathcal{M}_{0,[r]}(k)$ for the classifying morphism of X_i, then the elements of

$$\mathrm{Hom}_{\mathrm{Im}(\rho_{0,3}^{\{l\}})}(G_k, \mathrm{Im}(\rho_{0,[r]}^{\{l\}}))/\mathrm{Inn}(\mathrm{Im}(\rho_{0,[r]}^{\{l\}\text{-geom}}))$$

determined by $\widetilde{s}_{X_1}$ and $\widetilde{s}_{X_2}$, respectively, *coincide*. Thus, it follows from Lemma 4.3, (v), that X_1 is *isomorphic to* X_2 over k, as desired. This completes the proof of the above *implication*. □

7. Example I: Hyperbolic curves of type (0,4) over number fields

In the present §, we consider the monodromic fullness of *hyperbolic curves of type* $(0, 4)$ *over number fields*. In particular, we obtain sufficient conditions for such a hyperbolic curve to be monodromically full (cf. Theorem 7.8 and Corollaries 7.10, 7.11 below). Moreover, as an application of these sufficient conditions, we obtain a Galois-theoretic characterization of the isomorphism classes of certain hyperbolic curves of type $(0,4)$ over number fields (cf. Corollary 7.12 below). In the present §, suppose that k is a *number field* (cf. the discussion entitled "Numbers" in §0), and let $\mathfrak{o}_k \subseteq k$ be the ring of integers of k and $\lambda \in k \setminus \{0,1\}$ an element of $k \setminus \{0,1\}$. Moreover, in the present §, if $k' \subseteq \overline{k}$ is a finite extension of k, and p is a prime number, then write $\mathfrak{P}(k';p)$ for the set of nonarchimedean primes of k' whose residue characteristic are p.

Definition 7.1. Let l be an odd prime number and $\zeta_l \in \overline{k}$ a primitive l-th root of unity.

(i) We shall write $k_l \subseteq \overline{k}$ for the *finite* Galois extension of $k(\zeta_l)$ corresponding to the quotient

$$G_{k(\zeta_l)} \twoheadrightarrow \rho^{\{l\}}_{0,3}(G_{k(\zeta_l)}) \twoheadrightarrow \rho^{\{l\}}_{0,3}(G_{k(\zeta_l)})^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{F}_l$$

(cf. Lemma 7.2, (i), below).

(ii) We shall write $\pi_1(\mathbb{P}^1_{k(\zeta_l)} \setminus \{0,1,\infty\}) \twoheadrightarrow Q_l$ for the quotient of $\pi_1(\mathbb{P}^1_{k(\zeta_l)} \setminus \{0,1,\infty\})$ obtained as the composite

$$\pi_1(\mathbb{P}^1_{k(\zeta_l)} \setminus \{0,1,\infty\}) \simeq \pi_1(\mathcal{M}_{0,4} \otimes_k k(\zeta_l)) \twoheadrightarrow \rho^{\{l\}}_{0,4}(\pi_1(\mathcal{M}_{0,4} \otimes_k k(\zeta_l)))$$
$$\twoheadrightarrow \rho^{\{l\}}_{0,4}(\pi_1(\mathcal{M}_{0,4} \otimes_k k(\zeta_l)))^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{F}_l$$

— where the first arrow is the isomorphism obtained by an isomorphism $\mathbb{P}^1_k \setminus \{0,1,\infty\} \simeq \mathcal{M}_{0,4}$ over k (cf. Lemma 4.1, (i)).

(iii) We shall write $X_l \to \mathbb{P}^1_k \setminus \{0,1,\infty\}$ for the connected *finite* étale covering of $\mathbb{P}^1_k \setminus \{0,1,\infty\}$ corresponding to the *open* subgroup (cf. Lemma 7.2, (ii), below) obtained as the kernel of $\pi_1(\mathbb{P}^1_{k(\zeta_l)} \setminus \{0,1,\infty\}) \twoheadrightarrow Q_l$.

(iv) We shall write

$$Y_l \overset{\mathrm{def}}{=} \operatorname{Spec} k_l[s^{\pm 1}, t^{\pm 1}]/(s^l + t^l - 1)$$
$$\to \operatorname{Spec} k[u^{\pm 1}, 1/(u-1)] = \mathbb{P}^1_k \setminus \{0,1,\infty\}$$

— where s, t, and u are indeterminates — for the connected finite étale covering of $\mathbb{P}^1_k \setminus \{0,1,\infty\}$ determined by the homomorphism of algebras over k

$$\begin{array}{ccc} k[u^{\pm 1}, 1/(u-1)] & \longrightarrow & k_l[s^{\pm 1}, t^{\pm 1}]/(s^l + t^l - 1) \\ u & \mapsto & s^l. \end{array}$$

Lemma 7.2 (Properties of certain extensions and étale coverings). *Let l be an odd prime number and $\zeta_l \in \overline{k}$ a primitive l-th root of unity. Then the following hold:*

(i) *k_l is a* **finite abelian extension** *of $k(\zeta_l)$ of degree a power of l; moreover, the extension k_l of k is* **unramified** *outside $\mathfrak{P}(k;l)$.*

(ii) *The closed subgroup of* $\pi_1(\mathbb{P}^1_k \setminus \{0,1,\infty\})$ *obtained as the kernel of the surjection* $\pi_1(\mathbb{P}^1_{k(\zeta_l)} \setminus \{0,1,\infty\}) \twoheadrightarrow Q_l$ *is* **open**.

(iii) *The finite étale covering* $Y_l \to \mathbb{P}^1_k \setminus \{0,1,\infty\}$ **factors through** *the finite étale covering* $X_l \to \mathbb{P}^1_k \setminus \{0,1,\infty\}$*, i.e., we have a sequence* $Y_l \to X_l \to \mathbb{P}^1_k \setminus \{0,1,\infty\}$.

Proof. First, we verify assertion (i). It follows from Lemma 2.5 (respectively, Lemma 4.3, (ii)) that the quotient

$$G_{k(\zeta_l)} \twoheadrightarrow \rho^{\{l\}}_{0,3}(G_{k(\zeta_l)})$$

is *topologically finitely generated* (respectively, *pro-l*). Moreover, it follows from [1], Theorems A, B, that the algebraic extension of $k(\zeta_l)$ corresponding to the above quotient is *unramified outside* $\mathfrak{P}(k(\zeta_l); l)$. Therefore, assertion (i) follows immediately from the fact that the extension $k(\zeta_l)$ of k is *unramified outside* $\mathfrak{P}(k; l)$. This completes the proof of assertion (i).

Next, we verify assertion (ii). It follows from Lemma 2.5 that the quotient

$$\pi_1(\mathbb{P}^1_{k(\zeta_l)} \setminus \{0,1,\infty\}) \simeq \pi_1(\mathcal{M}_{0,4} \otimes_k k(\zeta_l)) \twoheadrightarrow \rho^{\{l\}}_{0,4}(\pi_1(\mathcal{M}_{0,4} \otimes_k k(\zeta_l)))$$

— where the first arrow is the isomorphism obtained by an isomorphism $\mathbb{P}^1_k \setminus \{0,1,\infty\} \simeq \mathcal{M}_{0,4}$ over k (cf. Lemma 4.1, (i)) — is *topologically finitely generated*. Therefore, the quotient Q_l is *finite*. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). It follows immediately from the definition of the connected finite étale covering $Y_l \to \mathbb{P}^1_{k(\zeta_l)} \setminus \{0,1,\infty\}$, together with Lemma 4.2, (i), that this covering is *Galois*, and the quotient by the normal open subgroup $\pi_1(Y_l) \subseteq \pi_1(\mathbb{P}^1_{k(\zeta_l)} \setminus \{0,1,\infty\})$ fits into an *exact* sequence

$$1 \longrightarrow \mathrm{Im}(\rho^{\{l\}\text{-geom}}_{0,4})^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{F}_l \longrightarrow \pi_1(\mathbb{P}^1_{k(\zeta_l)} \setminus \{0,1,\infty\})/\pi_1(Y_l)$$

$$\longrightarrow \rho^{\{l\}}_{0,3}(\mathrm{Gal}(\overline{k}/k(\zeta_l)))^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{F}_l \, (= \mathrm{Gal}(k_l/k(\zeta_l))) \longrightarrow 1\,.$$

Therefore, it follows immediately from the definition of the connected finite étale covering $X_l \to \mathbb{P}^1_{k(\zeta_l)} \setminus \{0,1,\infty\}$ that the natural surjection $\pi_1(\mathbb{P}^1_{k(\zeta_l)} \setminus \{0,1,\infty\}) \twoheadrightarrow Q_l$ *factors through* $\pi_1(\mathbb{P}^1_{k(\zeta_l)} \setminus \{0,1,\infty\}) \twoheadrightarrow \pi_1(\mathbb{P}^1_{k(\zeta_l)} \setminus \{0,1,\infty\})/\pi_1(Y_l)$. This completes the proof of assertion (iii). □

Lemma 7.3 (Fibers and monodromic fullness). *Let l be an odd prime number and $\zeta_l \in \overline{k}$ a primitive l-th root of unity. Consider the following four conditions:*

(i) *The fiber of* $X_l \to \mathbb{P}^1_k \setminus \{0,1,\infty\}$ *at the* (*image of the*) *k-rational point of* $\mathbb{P}^1_k \setminus \{0,1,\infty\}$ *corresponding to the element* $\lambda \in k \setminus \{0,1\}$ *is* **connected**.

(ii) *The composite*

$$\mathrm{Gal}(\overline{k}/k(\zeta_l)) \longrightarrow \pi_1(\mathbb{P}^1_{k(\zeta_l)} \setminus \{0,1,\infty\}) \longrightarrow Q_l$$

— where the first arrow is the homomorphism (*which is determined up to* $\pi_1(\mathbb{P}^1_{\overline{k}} \setminus \{0,1,\infty\})$*-inner automorphism*) *induced by the* $k(\zeta_l)$*-rational point of* $\mathbb{P}^1_{k(\zeta_l)} \setminus \{0,1,\infty\}$ *corresponding to the element* $\lambda \in k \setminus \{0,1\} \subseteq k(\zeta_l) \setminus \{0,1\}$ *— is* **surjective**.

(iii) *The composite*

$$G_k \longrightarrow \pi_1(\mathbb{P}^1_k \setminus \{0,1,\infty\}) \xrightarrow{\sim} \pi_1(\mathcal{M}_{0,4}) \longrightarrow \mathrm{Im}(\rho^{\{l\}}_{0,4})$$

— where the first arrow is the homomorphism (*which is determined up to* $\pi_1(\mathbb{P}^1_{\overline{k}} \setminus \{0,1,\infty\})$*-inner automorphism*) *induced by the* k*-rational point of* $\mathbb{P}^1_k \setminus \{0,1,\infty\}$ *corresponding to the element* $\lambda \in k \setminus \{0,1\}$*, and the second arrow is the isomorphism over* G_k *obtained by an isomorphism* $\mathbb{P}^1_k \setminus \{0,1,\infty\}) \simeq \mathcal{M}_{0,4}$ *over* k (*cf. Lemma 4.1,* (i)) *— is* **surjective**.

(iv) *The hyperbolic curve* $(\mathbb{P}^1_k, \{0,1,\lambda,\infty\} \subseteq \mathbb{P}^1_k)$ *of type* $(0,4)$ *over* k *is* ***l*-monodromically full**.

Then the following implication and equivalences hold:

$$\text{(i)} \Longleftrightarrow \text{(ii)} \Longrightarrow \text{(iii)} \Longleftrightarrow \text{(iv)}$$

Proof. The equivalences

$$\text{(i)} \Longleftrightarrow \text{(ii)}\ ;\ \ \text{(iii)} \Longleftrightarrow \text{(iv)}$$

follow immediately from the various definitions involved. Thus, to verify Lemma 7.3, it suffices to show the *implication*

$$\text{(ii)} \Longrightarrow \text{(iii)}\,.$$

Suppose that condition (ii) is satisfied. It follows from the fact that $\mathrm{Ker}(\rho^{\{l\}}_{0,3}) \subseteq G_{k(\zeta_l)}$ (cf. Remark 5.2.1) that to verify condition (iii) — by replacing k by $k(\zeta_l)$ — we may assume without loss of generality that $\zeta_l \in k$. Then it follows from Lemma 2.5 (respectively, Lemma 4.3, (ii)) that $\mathrm{Im}(\rho^{\{l\}}_{0,4})$ is *topologically finitely generated* (respectively, *pro-l*). Therefore, it follows from [22], Lemma 2.8.7, (c); [22], Corollary 2.8.5, together with the definition of the quotient Q_l, that condition (iii) is satisfied. This completes the proof of the above *implication*. □

Definition 7.4.

(i) If $\mathfrak{p}$ is a nonarchimedean prime of k, then we shall write $v_{\mathfrak{p}}\colon k \to \mathbb{Z}$ for the $\mathfrak{p}$-adic valuation such that if p is the residue characteristic of $\mathfrak{p}$, then $v_{\mathfrak{p}}(p)$ coincides with the absolute ramification index of the completion of k at $\mathfrak{p}$.

(ii) If a is an element of k^*, then we shall write $[a]_{\pm}$ (respectively, $[a]_+$; $[a]_-$) for the (*necessarily finite*) set of nonarchimedean primes $\mathfrak{p}$ of k such that $v_{\mathfrak{p}}(a) \neq 0$ (respectively, $v_{\mathfrak{p}}(a) > 0$; $v_{\mathfrak{p}}(a) < 0$).

Lemma 7.5 (**Zeros and poles of certain divisors**).

(i) $[\lambda]_+ \cap [\lambda]_-$ *is* **empty**.
(ii) $[\lambda]_- = [1-\lambda]_-$.
(iii) $[\lambda]_+ \cap [1-\lambda]_+$ *is* **empty**.
(iv) $[\lambda]_+ \neq \emptyset$, $[1-\lambda]_+ \neq \emptyset$ *if and only if* $[\lambda]_\pm \not\subseteq [1-\lambda]_\pm$, $[1-\lambda]_\pm \not\subseteq [\lambda]_\pm$.
(v) *Suppose that*

$$\Big\{\lambda, 1-\lambda, \lambda/(\lambda-1)\Big\} \cap \mathfrak{o}_k^* = \emptyset\,.$$

Then there exists an element λ' *of*

$$\Big\{\lambda, 1/\lambda, 1-\lambda, 1/(1-\lambda), \lambda/(\lambda-1), (\lambda-1)/\lambda\Big\}$$

such that $[\lambda']_\pm \not\subseteq [1-\lambda']_\pm$ *and* $[1-\lambda']_\pm \not\subseteq [\lambda']_\pm$.

Proof. Assertion (i) follows from the definitions of "$[-]_+$" and "$[-]_-$". Assertions (ii) and (iii) follow immediately from a straightforward calculation. Assertion (iv) follows immediately from assertions (i), (ii), and (iii). Finally, we verify assertion (v). Suppose that any element of $\mathfrak{m}_\lambda \stackrel{\text{def}}{=} \{\lambda, 1/\lambda, 1-\lambda, 1/(1-\lambda), \lambda/(\lambda-1), (\lambda-1)/\lambda\}$ does *not* satisfy the desired condition. Now since $\mathfrak{m}_\lambda \cap \mathfrak{o}_k^* = \emptyset$, any element $\lambda' \in \mathfrak{m}_\lambda$ satisfies either $[\lambda']_+ \neq \emptyset$ or $[1/\lambda']_+ \neq \emptyset$; thus, it follows from assertion (iv) that — by replacing λ by an element of $\mathfrak{m}_\lambda$ — we may assume without loss of generality that

$$[\lambda]_+ = \emptyset\ ;\ [1/\lambda]_+ \neq \emptyset\ ;\ [(\lambda-1)/\lambda]_+ = \emptyset\ ;$$
$$[\lambda/(\lambda-1)]_+ \neq \emptyset\ ;\ [1/(1-\lambda)]_+ = \emptyset\ ;\ [1-\lambda]_+ \neq \emptyset\,.$$

Therefore, it follows that $1/\lambda$, $\lambda/(\lambda-1)$, $1-\lambda \in \mathfrak{o}_k$; in particular, we obtain that $\lambda/(\lambda-1) \in \mathfrak{o}_k^*$ — in contradiction to the assumption that $\mathfrak{m}_\lambda \cap \mathfrak{o}_k^* = \emptyset$. This completes the proof of assertion (v). □

Definition 7.6. Let l be an odd prime number. Then we shall say that l *satisfies the condition* $(\dagger_{\lambda\in k})$ if there exist nonarchimedean primes $\mathfrak{p}_0$ and $\mathfrak{q}_0$ of k satisfying the following conditions:

(i) $\mathfrak{p}_0 \notin \mathfrak{P}(k;l)$, $\mathfrak{p}_0 \in [\lambda]_\pm$, $\mathfrak{p}_0 \notin [1-\lambda]_\pm$, and l is *prime to* $v_{\mathfrak{p}_0}(\lambda)$.
(ii) $\mathfrak{q}_0 \notin \mathfrak{P}(k;l)$, $\mathfrak{q}_0 \notin [\lambda]_\pm$, $\mathfrak{q}_0 \in [1-\lambda]_\pm$, and l is *prime to* $v_{\mathfrak{q}_0}(1-\lambda)$.

Remark 7.6.1. It is easily verified that if $[\lambda]_\pm \not\subseteq [1-\lambda]_\pm$ and $[1-\lambda]_\pm \not\subseteq [\lambda]_\pm$, then there exists a *cofinite* set Σ of prime numbers — i.e., a (*necessarily infinite*) set of prime numbers obtained as the *complement of a finite set* of prime numbers in the set of all prime numbers — such that if $l \in \Sigma$, then l satisfies the condition $(\dagger_{\lambda\in k})$.

Lemma 7.7 (**Connectedness of a fiber**). *Let l be an odd prime number, $\zeta_l \in \overline{k}$ a primitive l-th root of unity, and $\alpha_l \in \overline{k}$ (respectively, $\beta_l \in \overline{k}$) a solution of $t^l - \lambda$ (respectively, $t^l - (1-\lambda)$) — where t is an indeterminate. Suppose that the prime number l* **satisfies** *the condition $(\dagger_{\lambda\in k})$. Then the following hold:*

(i) *The finite extension* $k(\zeta_l, \alpha_l)$ (*respectively,* $k(\zeta_l, \beta_l)$) *of* k *is* **ramified** *at* $\mathfrak{p}_0$ (*respectively,* $\mathfrak{q}_0$) — *cf. Definition 7.6 — and* **unramified** *at* $\mathfrak{q}_0$ (*respectively,* $\mathfrak{p}_0$).

(ii) *The extension* $k_l(\alpha_l, \beta_l)$ *of* k_l *is* **of degree** $\boldsymbol{l^2}$.

(iii) *The fiber of* $Y_l \to \mathbb{P}^1_k \setminus \{0, 1, \infty\}$ *at the* (*image of the*) k*-rational point of* $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ *corresponding to the element* $\lambda \in k \setminus \{0, 1\}$ *is* **connected**. *In particular, condition* (i) *of Lemma 7.3 is* **satisfied**.

Proof. Assertion (i) follows from the definition of the condition $(\dagger_{\lambda \in k})$, together with, for example, [19], Chapter V, Lemma 3.3. Assertion (ii) follows immediately from Lemma 7.2, (i), together with assertion (i). Assertion (iii) follows from assertion (ii), together with Lemma 7.2, (iii). □

Theorem 7.8 (**Monodromic fullness of certain split hyperbolic curves of type $(0, 4)$ over number fields**). *Let* l *be an odd prime number,* k *a* **number field** (*cf. the discussion entitled "Numbers" in* §*0*)*, and* $\lambda \in k \setminus \{0, 1\}$. *Suppose that* l **satisfies** *the condition* $(\dagger_{\lambda \in k})$ (*cf. Definition 7.6*). *Then the hyperbolic curve* $(\mathbb{P}^1_k, \{0, 1, \lambda, \infty\} \subseteq \mathbb{P}^1_k)$ *of type* $(0, 4)$ *over* k *is* $\boldsymbol{l}$**-monodromically full** (*cf. Definition 2.2,* (i)).

Proof. This follows from Lemma 7.3, together with Lemma 7.7, (iii). □

Definition 7.9. Let X be a hyperbolic curve *of type* $(0, 4)$ over k. Then it follows immediately that there exists an element $\lambda_X \in \overline{k} \setminus \{0, 1\}$ such that the hyperbolic curve $X \otimes_k \overline{k}$ is *isomorphic over* $\overline{k}$ *to* the hyperbolic curve

$$(\mathbb{P}_{\overline{k}}, \{0, 1, \lambda_X, \infty\} \subseteq \mathbb{P}_{\overline{k}})$$

of type $(0, 4)$ over $\overline{k}$. Now we shall write

$$\mathfrak{m}_X \stackrel{\text{def}}{=} \{\lambda_X, 1/\lambda_X, 1 - \lambda_X, 1/(1 - \lambda_X), \lambda_X/(\lambda_X - 1), (\lambda_X - 1)/\lambda_X\} \subseteq \overline{k}\,.$$

Note that, as is well-known, $\mathfrak{m}_X$ *depends only on* (and *completely determines*!) the isomorphism class of the hyperbolic curve $X \otimes_k \overline{k}$ over $\overline{k}$.

Corollary 7.10 (**Monodromic fullness of certain hyperbolic curves of type $(0, 4)$ over number fields**). *Let* k *be a* **number field** (*cf. the discussion entitled "Numbers" in* §*0*), $\overline{k}$ *an algebraic closure of* k, $\mathfrak{o}_{\overline{k}}$ *the ring of integers of* $\overline{k}$, *and* X *a hyperbolic curve* (*cf. Definition 1.1,* (ii)) **of type $(0, 4)$** *over* k. *If* $\mathfrak{m}_X \cap \mathfrak{o}^*_{\overline{k}} = \emptyset$ (*cf. Definition 7.9*), *then there exists a* **cofinite** *set* Σ *of prime numbers — i.e., a* (**necessarily infinite**) *set of prime numbers obtained as the complement of a finite set of prime numbers in the set of all prime numbers — such that the* *hyperbolic curve* X *over* k *is* **Σ-monodromically full** (*cf. Definition 2.2,* (i)).

In particular, if $\mathfrak{o}_k \subseteq k$ *is the ring of integers of* k*, and* $\lambda \in k \setminus \{0,1\}$ *is an element of* $k \setminus \{0,1\}$ *such that*

$$\Big\{\lambda, 1-\lambda, \lambda/(\lambda-1)\Big\} \cap \mathfrak{o}_k^* = \emptyset\,,$$

then there exists a **cofinite** *set* Σ *of prime numbers such that the hyperbolic curve* $(\mathbb{P}^1_k, \{0,1,\lambda,\infty\} \subseteq \mathbb{P}^1_k)$ *of type* $(0,4)$ *over* k *is* **Σ-monodromically full**.

Proof. This follows from Theorem 7.8 together with Lemma 7.5, (v); Remark 7.6.1. □

Corollary 7.11 (**Monodromic fullness of split hyperbolic curves of type (0, 4) over the field of rational numbers or certain imaginary quadratic fields**)**.** *Let* d *be a square-free positive integer such that* $d \neq 1$, 3. *Write* k *for the* **field of rational numbers** $\mathbb{Q}$ *or the* **imaginary quadratic field** $\mathbb{Q}(\sqrt{-d})$. *If* $\lambda \in k$*, then the following conditions are* **equivalent***:*

- (i) *The hyperbolic curve* $(\mathbb{P}^1_k, \{0,1,\lambda,\infty\} \subseteq \mathbb{P}^1_k)$ *— of type* $(0,3)$ *or* $(0,4)$ *— over* k *is* **not isomorphic to** *the hyperbolic curve* $(\mathbb{P}^1_k, \{0,1,-1,\infty\} \subseteq \mathbb{P}^1_k)$ *of type* $(0,4)$ *over* k.
- (ii) λ *is* **not equal to** -1, 2, $1/2$.
- (iii) *There exists a* **cofinite** *set* Σ *of prime numbers — i.e., a* (**necessarily infinite**) *set of prime numbers obtained as the complement of a finite set of prime numbers in the set of all prime numbers — such that the hyperbolic curve* $(\mathbb{P}^1_k, \{0,1,\lambda,\infty\} \subseteq \mathbb{P}^1_k)$ *— of type* $(0,3)$ *or* $(0,4)$ *— over* k *is* **Σ-monodromically full** (*cf. Definition 2.2,* (i)).
- (iv) *There exists a prime number* l *such that the hyperbolic curve* $(\mathbb{P}^1_k, \{0,1,\lambda,\infty\} \subseteq \mathbb{P}^1_k)$ *— of type* $(0,3)$ *or* $(0,4)$ *— over* k *is* **l-monodromically full**.

Proof. The implication

$$\text{(i)} \Longrightarrow \text{(ii)}$$

is immediate. The implication

$$\text{(ii)} \Longrightarrow \text{(iii)}$$

follows from Theorem 7.10, together with the fact that $\mathfrak{o}_k^* = \{\pm 1\}$. The implication

$$\text{(iii)} \Longrightarrow \text{(iv)}$$

is immediate. Finally, we verify the *implication*

$$\text{(iv)} \Longrightarrow \text{(i)}\,.$$

It is easily verified that $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, -1, \infty\}$ *has some special symmetry* — i.e., $\mathrm{Aut}_{\overline{\mathbb{Q}}}(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, -1, \infty\})$ is *not* isomorphic to $G_{0,4}$ (cf. Definition 3.1). Therefore, the above *implication* follows from Proposition 3.4. $\square$

Corollary 7.12 (**Galois-theoretic characterization of isomorphism classes of certain hyperbolic curves of type $(0, 4)$ over number fields**). *Let k be a* **number field** (*cf. the discussion entitled "Numbers" in* §*0*)*; $\overline{k}$ an algebraic closure of k; $\mathfrak{o}_{\overline{k}}$ the ring of integers of $\overline{k}$; $G_k \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$; $X_1 = (C_1, D_1 \subseteq C_1)$, $X_2 = (C_2, D_2 \subseteq C_2)$ hyperbolic curves* (*cf. Definition 1.1,* (ii)) **of type $(0, 4)$** *over k. Suppose that $\mathfrak{m}_{X_1} \cap \mathfrak{o}^*_{\overline{k}} = \mathfrak{m}_{X_2} \cap \mathfrak{o}^*_{\overline{k}} = \emptyset$* (*cf. Definition 7.9*)*. Then the following conditions are* **equivalent***:*

(i) *X_1 is* **isomorphic to** *X_2 over k.*

(ii) *There exists an* **infinite** *set Σ of prime numbers such that, for any $l \in \Sigma$, if we write*

$$\rho^{\{l\}}_{X_i/k} \colon G_k \longrightarrow \mathrm{Out}\Big(\pi_1((C_i \setminus D_i) \otimes_k \overline{k})^{(l)}\Big)$$

for the pro-l outer Galois representation associated to X_i, then $\mathrm{Ker}(\rho^{\{l\}}_{X_1/k}) = \mathrm{Ker}(\rho^{\{l\}}_{X_2/k})$.

Proof. The implication

$$\text{(i)} \Longrightarrow \text{(ii)}$$

is immediate; on the other hand, the implication

$$\text{(ii)} \Longrightarrow \text{(i)}$$

follows immediately from Theorem 6.1, together with Corollary 7.10. $\square$

8. Example II: Nonisotrivial hyperbolic curves of type (0,4)

In the present §, we consider the monodromic fullness of *nonisotrivial hyperbolic curves of type* $(0, 4)$.

Definition 8.1. Let X be a hyperbolic curve over k. Then we shall say that X is *NF-isotrivial* (where the "NF" stands for "number field") if there exist a finite extension $k' \subseteq \overline{k}$ of k, a number field $k_0 \subseteq k'$ (cf. the discussion entitled "Numbers" in §0) contained in k', and a hyperbolic curve X_0 over k_0 such that $X \otimes_k k'$ is isomorphic to $X_0 \otimes_{k_0} k'$ over k'. (cf. [26], Proposition 1.2, (i)).

Corollary 8.2 (**Monodromic fullness of nonisotrivial hyperbolic curves of type $(0, 4)$**). *Let k be a* **finitely generated field of characteristic zero** (*cf. the discussion entitled "Numbers" in* §*0*) *and X a hyperbolic curve* (*cf. Definition 1.1,* (ii)) **of type $(0, 4)$** *over k which is* **not NF-isotrivial** (*cf. Definition 8.1*)*. Then there exists a* **cofinite**

set Σ *of prime numbers — i.e., a* (**necessarily infinite**) *set of prime numbers obtained as the complement of a finite set of prime numbers in the set of all prime numbers — such that the hyperbolic curve* X *over* k *is* **Σ-monodromically full** (*cf. Definition 2.2,* (i)).

Proof. It is immediate that to verify Corollary 8.2 — by replacing k by a suitable finite extension of k — we may assume without loss of generality that X is *split*. Now since k is *finitely generated field of characteristic zero*, there exist a subfield $k_0 \subseteq k$ of k and a scheme V_0 over k_0 satisfying the following conditions:

(i) k_0 is a *number field* (cf. the discussion entitled "Numbers" in §0).
(ii) V_0 is regular, separated, geometrically connected, and of finite type over k_0.
(iii) The function field of V_0 is isomorphic to k.
(iv) The *split* hyperbolic curve X over k extends to a *split* hyperbolic curve X_0 over V_0.

Now since the natural homomorphism $\pi_1(\operatorname{Spec} k) \to \pi_1(V_0)$ (cf. (iii)) is *surjective* (cf. (ii)), and the pro-Σ outer monodromy representation $\rho^{\Sigma}_{X/k}$ factors through $\rho^{\Sigma}_{X_0/V_0}$ (cf. (iv)), it follows from the definition of the term "monodromically full" that, to verify Corollary 8.2, it suffices to show that there exists a closed point $v \in V_0$ of V_0 such that the hyperbolic curve $(X_0)_v$ over the residue field at $v \in V_0$ obtained as the fiber of the hyperbolic curve X_0 over V_0 at $v \in V_0$ is Σ-*monodromically full* for some *cofinite* set Σ of prime numbers.

Write $\widetilde{s}_{X_0/V_0} \colon V_0 \to \mathbb{P}^1_{k_0} \setminus \{0, 1, \infty\}$ for the classifying morphism of the *split* hyperbolic curve X_0 over V_0 (cf. (iv), together with Lemma 4.1, (i)). Then since X is *not NF-isotrivial*, and $\mathbb{P}^1_{k_0} \setminus \{0, 1, \infty\}$ is *of dimension one*, it follows that the image of the morphism $\widetilde{s}_{X_0/V_0}$ is *open*; in particular, there exists a closed point $\overline{v}$ of $\mathbb{P}^1_{k_0} \setminus \{0, 1, \infty\}$ contained in the image of $\widetilde{s}_{X_0/V_0}$ such that if $\lambda \in \overline{k}_0 \setminus \{0, 1\}$ is an element of $\overline{k}_0 \setminus \{0, 1\}$ naturally corresponding to $\overline{v} \in \mathbb{P}^1_{k_0} \setminus \{0, 1, \infty\}$, then $\{\lambda, 1-\lambda, \lambda/(\lambda-1)\} \cap \mathfrak{o}^*_{\overline{k}_0} = \emptyset$ — where $\overline{k}_0$ is an algebraic closure of k_0 and $\mathfrak{o}_{\overline{k}_0}$ is the ring of integers of $\overline{k}_0$ (cf. (i)). Let $v \in V_0$ be a closed point of V_0 whose image via $\widetilde{s}_{X_0/V_0}$ is $\overline{v}$. Then it follows immediately from Corollary 7.10 that the hyperbolic curve $(X_0)_v$ over the residue field at $v \in V_0$ obtained as the fiber of the hyperbolic curve X_0 over V_0 at $v \in V_0$ is Σ-*monodromically full* for some *cofinite* set Σ of prime numbers. This completes the proof of Corollary 8.2. ☐

Remark 8.2.1. It is immediate that Corollary 8.2 implies the following *assertion*:

> *Let* k *be a* **finitely generated field of characteristic zero** (*cf. the discussion entitled "Numbers" in* §*0*) *and* X *a hyperbolic curve* (*cf. Definition 1.1,* (ii)) **of type**

$(0,4)$ *over k. Suppose that there exists an* **infinite** *set Σ of prime numbers such that if $l \in \Sigma$, then X is* **not l-monodromically full** (*cf. Definition 2.2,* (i)). *Then X is* **NF-isotrivial** (*cf. Definition 8.1*).

On the other hand, if, in the above *assertion*, one replaces "$(0,4)$" by "$(0,r)$" for some $r \geq 5$, then the conclusion no longer holds in general. A counter-example is as follows: Let k_0 be a number field, $S \stackrel{\text{def}}{=} \mathbb{P}^1_{k_0} \setminus \{0,1,-1,\infty\}$, and k the function field of S. Then the natural open immersion

$$S \hookrightarrow \mathbb{P}^1_{k_0} \setminus \{0,1,\infty\}$$

and the composite

$$S \to \operatorname{Spec} k_0 \hookrightarrow \mathbb{P}^1_{k_0} \setminus \{0,1,\infty\}$$

— where the first arrow is the structure morphism of S, and the second arrow is the k_0-rational point corresponding to $-1 \in k_0 \setminus \{0,1\}$ — determine a morphism over k from S to the second configuration space of $\mathbb{P}^1_{k_0} \setminus \{0,1,\infty\}$; in particular, it follows immediately from Lemma 4.1, (i), that we obtain a split hyperbolic curve X over k of type $(0,5)$. Now since X *may be embedded as an open subscheme of* $\mathbb{P}^1_k \setminus \{0,1,-1,\infty\}$, it follows immediately from Proposition 3.4 (cf. also the argument used in the proof of the implication (iv) $\Rightarrow$ (i) in the proof of Corollary 7.11), together with Remark 2.2.5, that, for any prime number l, the hyperbolic curve X over k is *not l-monodromically full*. On the other hand, it follows immediately from the definition of X that X is *not NF-isotrivial*.

Corollary 8.3 (**Galois-theoretic characterization of isomorphism classes of nonisotrivial hyperbolic curves of type $(0,4)$**). *Let k be a* **finitely generated field of characteristic zero** (*cf. the discussion entitled "Numbers" in* §*0*)*; $\overline{k}$ an algebraic closure of k; $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$; $X_1 = (C_1, D_1 \subseteq C_1)$, $X_2 = (C_2, D_2 \subseteq C_2)$ hyperbolic curves* (*cf. Definition 1.1,* (ii)) **of type $(0,4)$** *over k which are* **not NF-isotrivial** (*cf. Definition 8.1*). *Then the following conditions are* **equivalent***:*

(i) *X_1 is* **isomorphic to** *X_2 over k.*

(ii) *There exists an* **infinite** *set Σ of prime numbers such that, for any $l \in \Sigma$, if we write*

$$\rho^{\{l\}}_{X_i/k} \colon G_k \longrightarrow \mathrm{Out}\Big(\pi_1((C_i \setminus D_i) \otimes_k \overline{k})^{(l)}\Big)$$

for the pro-l outer Galois representation associated to X_i, then $\mathrm{Ker}(\rho^{\{l\}}_{X_1/k}) = \mathrm{Ker}(\rho^{\{l\}}_{X_2/k})$.

Proof. The implication

$$\text{(i)} \Longrightarrow \text{(ii)}$$

is immediate; on the other hand, the implication

$$\text{(ii)} \Longrightarrow \text{(i)}$$

follows immediately from Theorem 6.1, together with Corollary 8.2. □

Appendix. Ramification of outer Galois representations and isomorphism classes of hyperbolic curves

In the present §, we prove *finiteness results*, which are related to the main result of the present paper (cf. Theorem A.3, Corollary A.4 below). It seems to the author that the results appearing in the present § are likely to be well-known; since, however, the results could not be found in the literature, the author decided to give proofs of the results in the present §. In the present §, let l be a prime number, k a *number field* (cf. the discussion entitled "Numbers" in §0), and (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$.

Definition A.1. Let $N \subseteq G_k$ be a normal closed subgroup of G_k and $\mathfrak{P}$ a set of primes of k. Then we shall write

$$\mathcal{I}^{\mathrm{Gal}}(l, k, g, r, N) \ \ (\text{respectively, } \ \mathcal{I}^{\mathrm{unr}}(l, k, g, r, \mathfrak{P}))$$

for the set of the isomorphism classes over k of hyperbolic curves $X = (C, D \subseteq C)$ *of type* (g, r) *over* k satisfying the following condition: If we write

$$\rho^{\{l\}}_{X/k} \colon G_k \longrightarrow \mathrm{Out}\Big(\pi_1((C \setminus D) \otimes_k \overline{k})^{(l)}\Big)$$

for the pro-l outer Galois representation associated to X, then the kernel of $\rho^{\{l\}}_{X/k}$ *coincides with* $N \subseteq G_k$ (respectively, then $\rho^{\{l\}}_{X/k}$ is *unramified outside* $\mathfrak{P}$).

Remark A.2. If $N \subseteq G_k$ is a normal closed subgroup of G_k *obtained as the kernel of the pro-l outer Galois representation associated to a hyperbolic curve over k*, then it is easily verified that there exists a *finite set* $\mathfrak{P}$ of primes of k such that $\mathcal{I}^{\mathrm{Gal}}(l, k, g, r, N) \subseteq \mathcal{I}^{\mathrm{unr}}(l, k, g, r, \mathfrak{P})$.

The main purpose of the present § is to prove the following fact.

Theorem A.3. *Let l be a prime number, k a* **number field** (*cf. the discussion entitled "Numbers" in* §*0*)*, $\overline{k}$ an algebraic closure of k, $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$, (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$, and $\mathfrak{P}$ a* **finite** *set of primes of k. Then the set $\mathcal{I}^{\mathrm{unr}}(l, k, g, r, \mathfrak{P})$* (*cf. Definition A.1*) *is* **finite**.

By Theorem A.3, together with Remark A.2, we obtain the following corollary.

Corollary A. 4. *Let l be a prime number, k a* **number field** (*cf. the discussion entitled "Numbers" in* §*0*)*, $\overline{k}$ an algebraic closure of k, $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$, (g, r) a pair of nonnegative integers such that $2g -$*

$2+r>0$, and $N \subseteq G_k$ a normal closed subgroup of G_k. Then the set $\mathcal{I}^{\mathrm{Gal}}(l,k,g,r,N)$ (cf. Definition A.1) is **finite**.

The rest of the present § is devoted to prove Theorem A.3.

Lemma A.5. *Let $\mathfrak{P}$ be a* **finite** *set of primes of k and $X=(C, D \subseteq C)$ a hyperbolic curve over k whose isomorphism class over k is in $\mathcal{I}^{\mathrm{unr}}(l,k,g,r,\mathfrak{P})$. Then there exists a* **finite** *extension $k(l,k,r,\mathfrak{P}) \subseteq \overline{k}$ of k that* **depends only on** *l, k, r, and $\mathfrak{P}$ such that the hyperbolic curve $X \otimes_k k(l,k,r,\mathfrak{P})$ over $k(l,k,r,\mathfrak{P})$ is* **split** *(cf. Definition 1.5,* (i)*).*

Proof. To prove Lemma A.5 — by replacing $\mathfrak{P}$ by a finite set of primes of k containing $\mathfrak{P}$ and the set of the primes of k whose residue characteristic are l — we may assume without loss of generality that the set of primes of k whose residue characteristic are l is *contained in* $\mathfrak{P}$. Then it follows immediately from the criterion of Oda-Tamagawa for good reduction of hyperbolic curves (cf. [25], Theorem 0.8) that any irreducible component of D is isomorphic to the spectrum of a finite extension of k which is *unramified outside* $\mathfrak{P}$. On the other hand, it follows immediately from a *well-known theorem of Hermite-Minkowski* that there are *only finitely many* isomorphism classes of finite extensions of k of extension degree $\leq r$ which are unramified outside $\mathfrak{P}$. Therefore, if we write $k(l,k,r,\mathfrak{P})$ for the composite field of all extension fields (in $\overline{k}$) of extension degree $\leq r$ which are unramified outside $\mathfrak{P}$, then $k(l,k,r,\mathfrak{P})$ satisfies the desired condition. This completes the proof of Lemma A.5. □

Lemma A. 6. *Let k' be a finite extension of k and Y a hyperbolic curve over k'. Then there are* **only finitely many** *isomorphism classes over k of hyperbolic curves X over k satisfying the following condition: $X \otimes_k k'$ is* **isomorphic to** *Y over k'.*

Proof. To verify Lemma A.6 — by replacing k' by a finite extension of k' — we may assume without loss of generality that the extension k' of k is *Galois*. Write $\mathcal{D}$ for the set of the isomorphism classes $[X, \phi\colon X \otimes_k k' \xrightarrow{\sim} Y]$ of pairs $(X, \phi\colon X \otimes_k k' \xrightarrow{\sim} Y)$ of hyperbolic curves X over k and isomorphisms $\phi\colon X \otimes_k k' \xrightarrow{\sim} Y$ over k' — where we shall say that a pair $(X_1, \phi_1\colon X_1 \otimes_k k' \xrightarrow{\sim} Y)$ is *isomorphic to* a pair $(X_2, \phi_2\colon X_2 \otimes_k k' \xrightarrow{\sim} Y)$ if there exists an isomorphism $\psi\colon X_1 \xrightarrow{\sim} X_2$ over k such that $\phi_2 \circ \psi = \phi_1$. To verify Lemma A.6, it is immediate that it suffices to show that this set $\mathcal{D}$ is *finite*. Moreover, to verify the *finiteness* of $\mathcal{D}$, it is immediate that we may assume without loss of generality that $\mathcal{D}$ is *nonempty*. Let us fix an element $[X_0, \phi_0\colon X_0 \otimes_k k' \xrightarrow{\sim} Y] \in \mathcal{D}$ of $\mathcal{D}$. Then we obtain a map

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & Z^1(\mathrm{Gal}(k'/k), \mathrm{Aut}_{k'}(Y)) \\ [X, \phi\colon X \otimes_k k' \xrightarrow{\sim} Y] & \mapsto & (g \mapsto \phi \circ g^{-1} \circ \phi^{-1} \circ \phi_0 \circ g \circ \phi_0^{-1}) \end{array}$$

— where the action of $\mathrm{Gal}(k'/k)$ on $\mathrm{Aut}_{k'}(Y)$ is given by

$$\begin{array}{ccc} \mathrm{Gal}(k'/k) & \longrightarrow & \mathrm{Aut}\Big(\mathrm{Aut}_{k'}(Y)\Big) \\ g & \mapsto & (f \mapsto \phi_0 \circ g^{-1} \circ \phi_0^{-1} \circ f \circ \phi_0 \circ g \circ \phi_0^{-1}) \,. \end{array}$$

Moreover, by *Galois descent*, this map is *injective*. Therefore, the *finiteness* of $\mathcal{D}$ follows from the *finiteness* of $\mathrm{Gal}(k'/k)$ and $\mathrm{Aut}_{k'}(Y)$. □

Proof of Theorem A.3. To prove Theorem A.3 — by replacing $\mathfrak{P}$ by a finite set of primes of k containing $\mathfrak{P}$ and the set of the primes of k whose residue characteristic are l — we may assume without loss of generality that the set of primes of k whose residue characteristic are l is *contained in* $\mathfrak{P}$. Moreover, it follows from Lemma A.6 that to prove Theorem A.3, it suffices to verify that if we write $\mathcal{I}_1 \subseteq \mathcal{I}^{\mathrm{unr}}(l, k, g, r, \mathfrak{P})$ for

> the subset of $\mathcal{I}^{\mathrm{unr}}(l, k, g, r, \mathfrak{P})$ consisting of the isomorphism classes over k of hyperbolic curves which are *split*,

then $\mathcal{I}_1$ is *finite*. Now if $X = (C, D \subseteq C)$ is a hyperbolic curve over k whose isomorphism class over k is in $\mathcal{I}_1$, then it follows from the criterion of Oda-Tamagawa for good reduction of hyperbolic curves (cf. [25], Theorem 0.8) that the proper curve C *admits good reduction at all primes outside* $\mathfrak{P}$. Therefore, if $g \geq 1$ (respectively, if $g = 0$), then it follows from a *well-known theorem of Faltings-Shafarevich* (respectively, the fact that X is *split*) that

> the set consisting of the isomorphism classes over k of the proper curves "C" appearing in the elements of $\mathcal{I}_1$

is *finite*. Thus, to prove Theorem A.3, it suffices to verify that for a hyperbolic curve $X_0 = (C_0, D_0 \subseteq C_0)$ over k whose isomorphism class over k is in $\mathcal{I}_1$, if we write $\mathcal{I}_2 \subseteq \mathcal{I}_1$ for

> the subset of $\mathcal{I}_1$ consisting of the isomorphism classes over k of hyperbolic curves $X = (C, D \subseteq C)$ over k whose isomorphism classes over k are in $\mathcal{I}_1$ such that the proper curves C are *isomorphic to* the proper curve C_0 over k,

then $\mathcal{I}_2$ is *finite*. On the other hand, this follows immediately from two *well-known theorems of Mahler-Siegel* and *Faltings-Mordell*, together with the criterion of Oda-Tamagawa for good reduction of hyperbolic curves (cf. [25], Theorem 0.8). This completes the proof of Theorem A.3. □

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