Pathwise uniqueness and continuous dependence for SDEs with nonregular drift

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April 20, 2010

Abstract

A new proof of a pathwise uniqueness result of Krylov and Röckner is given. It concerns SDEs with drift having only certain integrability properties. In spite of the poor regularity of the drift, pathwise continuous dependence on initial conditions may be obtained, by means of this new proof. The proof is formulated in such a way to show that the only major tool is a good regularity theory for the heat equation forced by a function with the same regularity of the drift.

1 Introduction

Consider the stochastic differential equation in $\mathbb{R}^d$

$$X_t = x + \int_0^t b(s, X_s)\, \mathrm{d}s + W_t, \qquad t \in [0,T] \tag{1}$$

where W is a d-dimensional Brownian motion on a filtered probability space (Ω, F_t, P), $x \in \mathbb{R}^d$ and $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a measurable vector field with components of class $L_p^q(T) := L^q\left(0,T; L^p\left(\mathbb{R}^d\right)\right)$ for some $p, q \in (1, \infty)$ satisfying the condition

$$\frac{d}{p} + \frac{2}{q} < 1 \tag{2}$$

(known in fluid dynamics, with $\leq$, as the Prodi-Serrin condition). A measurable function $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is in $L_p^q(T)$ if

$$\|f\|_{L_p^q(T)} := \left(\int_0^T \left(\int_{\mathbb{R}^d} |f(r,y)|^p \, \mathrm{d}y \right)^{q/p} \mathrm{d}r \right)^{1/q} < \infty.$$

A remarkable result of Krylov and Röckner [KR05], which elaborates previous results of many authors, including Zwonkin [Zv74], Veretennikov [Ve80], Portenko [Po82], states that this equation has a unique strong solution, in the class of continuous processes such that

$$P\left(\int_0^T |b(s, X_s)|^2 \, \mathrm{d}s < \infty \right) = 1. \tag{3}$$

They also remark that the solution depends continuously on x in probability. The result is extended in [KR05] to local L_p^q-integrability conditions plus growth conditions; and there are extensions to state-dependent diffusion coefficients and other regularity assumptions, see [Zh05] and references therein.

The aim of this note is to give a new proof of the same result, based on a different argument, essentially based only on regularity theory of the heat equation with forcing equal to the drift or of the same class of regularity. We hope this new proof will look more elementary. The new proof is somewhat more quantitative (see in particular proposition 9) and will allow us to show the α-Hölder continuous dependence on x, for every $\alpha < 1$, *pathwise*, in the spirit of stochastic flows. This result is new and somewhat surprising, being b so rough. Precisely, we prove:

Theorem 1 *Equation (1), with $b \in L_p^q(T)$ with $p, q \in (1, \infty)$ satisfying the condition (2), for every $x \in \mathbb{R}^d$ has a unique strong solution X_t^x such that (3) holds true. The random field $\left\{X_t^x, t \in [0,T], x \in \mathbb{R}^d\right\}$ has a continuous modification, α-Hölder continuous in x, for every $\alpha < 1$.*

As we said, the aim of this note is to show a new simple argument to deal with SDEs with nonregular drift. In this spirit, we prefer to keep the exposition as simple as possible and thus we limit ourselves to the two claims of the theorem (uniqueness and pathwise Hölder continuity in the initial conditions). However, with longer arguments, we have also checked that an α-Hölder continuous stochastic flow exists; and moreover the solution is differentiable in x in an average sense, but not pathwise (we cannot get a differentiable stochastic flow). These results, mostly included in [Fe09], will be published elsewhere. Moreover, we do not stress the generality beyond the (already challenging) class $L_p^q(T)$, but it is clear that one can accept some form of local integrability plus suitable control of the growth, at the expenses of several more details. And presumably the extension to other regularity classes different from $L_p^q(T)$ is possible, preserving at least the basic property that ∇u is bounded (see below).

It will be clear from the proof below that a sort of principle emerges. *If we have a good theory for the heat equation*

$$\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u = \varphi \text{ on } [0,T], \qquad u|_{t=T} = 0 \tag{4}$$

when φ has the same regularity as the drift b, then we have the main tools to prove strong uniqueness and possibly stochastic flows of Hölder maps. The good theory must include (at our present level of understanding) a *uniform bound on the gradient ∇u*. This is the main reason for the assumption $b \in L_p^q(T)$ with $p, q \in (1, \infty)$ satisfying condition (2). Other properties of u, of course, are used below but they look more flexible, not optimized. It seems that this principle extends to infinite dimensional situations (replacing the heat semigroup by the Ornstein-Uhlenbeck one) and other finite dimensional cases beyond the one treated here.

Of course, this principle is just a reformulation of a known fact, because the non-trivial results on Kolmogorov type equations needed in other proofs of pathwise uniqueness (like those in the references mentioned above, see also [FGP10], [DF10]), are ultimately based on a perturbative analysis of the heat equation, in spaces with regularity related to the one of the drift. See also remark 3. But the presentation here is very direct and easily prompt to generalizations.

The proof, indeed, becomes slightly shorter if we use a good regularity theory for the backward Kolmogorov equation

$$\frac{\partial U}{\partial t} + \frac{1}{2}\Delta U + (b \cdot \nabla) U = -b \text{ on } [0,T], \qquad U_\Phi|_{t=T} = 0.$$

This is the approach developed in [FGP10] for Hölder continuous drift (see also the infinite dimensional generalization [DF10]), and in [Fe09] for L_p^q-drift. The proof is shorter (and to some extent more far reaching, if one wants to prove further properties like differentiability in x), but at the price of a careful preliminary analysis of the Kolmogorov equation. Even if ultimately the two approaches are equivalent, we think it is conceptually interesting to realize that only heat equation estimates, with forcing of the same type as the drift, are needed. For this reason we give a self-contained proof based only on (4).

2 First step of the proof

First, let us clarify that we prove only the strong uniqueness and pathwise dependence part of the theorem. Indeed, we give for granted the weak existence proved in previous works by means of Girsanov theorem (see [KR05] and proposition 15 in the appendix) and thus the strong existence follows from weak existence and strong uniqueness by the classical Yamada-Watanabe theorem, or by the construction given by Gyongy and Krylov [GK96].

Consider the backward heat equation (4) with $\varphi \in L_p^q(T)$. Denote by $H_{2,p}^q(T)$ the space

$$H_{2,p}^q(T) := L^q\left(0,T; W^{2,p}\left(\mathbb{R}^d\right)\right) \cap W^{1,q}\left(0,T; L^p\left(\mathbb{R}^d\right)\right)$$

with norm $\|.\|_{H_{2,p}^q(T)}$ given by the sum of the natural norms of $L^q\left(0,T;W^{2,p}\left(\mathbb{R}^d\right)\right)$ and $W^{1,q}\left(0,T;L^p\left(\mathbb{R}^d\right)\right)$. Denote by $\|.\|_{L^\infty(T)}$ the norm in the space $L^\infty\left([0,T]\times\mathbb{R}^d\right)$. All our analysis will be based only on the following classical result (see Krylov [Kr01] and [KR05, lemma 10.2]). More is known (uniqueness, Hölder continuity of ∇u), but we insist that we use only the following properties.

Theorem 2 *For every $\varphi \in L_p^q(T)$, the backward heat equation (4) has at least a solution $u \in H_{2,p}^q(T)$, with*

$$\left\|D^2 u\right\|_{L_p^q(T)} \leq C\|\varphi\|_{L_p^q(T)}. \tag{5}$$

Moreover, $\nabla u \in L^\infty\left([0,T]\times\mathbb{R}^d\right)$ and

$$\|\nabla u\|_{L^\infty(T)} \leq C(T)\|\varphi\|_{L_p^q(T)} \tag{6}$$

with

$$\lim_{T\to 0} C(T) = 0. \tag{7}$$

Given a vector field $\Phi : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, we still write $\Phi \in L_p^q(T)$ when all components Φ^i are of class $L_p^q(T)$. Denote by U_Φ the $\mathbb{R}^d$-valued field such that U_Φ^i solves the heat equation above with $\varphi = -\Phi^i$; in vector notations

$$\frac{\partial U_\Phi}{\partial t} + \frac{1}{2}\Delta U_\Phi = -\Phi \text{ on } [0,T], \qquad U_\Phi|_{t=T} = 0.$$

We have $U_\Phi^i \in H_{2,p}^q(T)$, $i = 1, ..., d$. As above, we write $U_\Phi \in H_{2,p}^q(T)$, for simplicity of notations.

Moreover, denote by $\mathcal{T} : L_p^q(T) \to L_p^q(T)$ the map defined as

$$\mathcal{T}(\Phi) := (b \cdot \nabla) U_\Phi.$$

Using a generalization of Itô formula to $H_{2,p}^q(T)$-functions (see [KR05, theorem 3.7]), if X is a solution to equation (1) we have

$$dU_\Phi(t, X_t) = -\Phi(t, X_t)\,\mathrm{d}t + (b(t, X_t) \cdot \nabla) U_\Phi(t, X_t) + \nabla U_\Phi(t, X_t)\,\mathrm{d}W_t$$

namely

$$\int_0^t \Phi(s, X_s)\,\mathrm{d}s = U_\Phi(0, x) - U_\Phi(t, X_t) + \int_0^t \mathcal{T}(\Phi)(s, X_s)\,\mathrm{d}s + \int_0^t \nabla U_\Phi(s, X_s)\,\mathrm{d}W_s.$$

Hence, taking $\Phi = b$, we can rewrite equation (1) in the form

$$X_t = x + U_b(0, x) - U_b(t, X_t) + \int_0^t \mathcal{T}(b)(s, X_s)\,\mathrm{d}s + \int_0^t \nabla U_b(s, X_s)\,\mathrm{d}W_s + W_t.$$

Let us make several comments on this reformulation of equation (1). The difficulty in (1) is the non-regular field b, only of class $L_p^q(T)$. The terms $U_b(0, x)$, $U_b(t, X_t)$ and $\int_0^t \nabla U_b(s, X_s)\,\mathrm{d}W_s$ of the new equation involve more regular fields: U_b has even Hölder continuous gradient, while ∇U_b has gradient in $L_p^q(T)$. On the contrary, the term $\int_0^t \mathcal{T}(b)(s, X_s)\,\mathrm{d}s$ is not better than then original one, $\int_0^t b(s, X_s)\,\mathrm{d}s$, from the regularity viewpoint. But, if we take small T, the $L_p^q(T)$-norm of $(b \cdot \nabla) U_b$ is small as we want (because of (7)). So we have replaced the non-regular term in (1) by more regular ones plus a term which has the same degree of regularity but is much smaller. Iterating this procedure, namely replacing $\int_0^t \mathcal{T}(b)(s, X_s)\,\mathrm{d}s$ by analogous terms, and so on n times, we may keep the time interval $[0,T]$ small but given, and decrease arbitrarily the size of the non-regular term. We shall see that the sum of the other term is under control.

To be more precise, we repeat what we have done above for $\int_0^t b(s, X_s)\,\mathrm{d}s$ and get

$$\int_0^t \mathcal{T}(b)(s, X_s)\,\mathrm{d}s = U_{\mathcal{T}(b)}(0, x) - U_{\mathcal{T}(b)}(t, X_t) + \int_0^t \mathcal{T}^2(b)(s, X_s)\,\mathrm{d}s + \int_0^t \nabla U_{\mathcal{T}(b)}(s, X_s)\,\mathrm{d}W_s.$$

We iterate this procedure, substitute in the original equation and get

$$X_t = x + \sum_{k=0}^n U_{\mathcal{T}^k(b)}(0, x) - \sum_{k=0}^n U_{\mathcal{T}^k(b)}(t, X_t) + \int_0^t \mathcal{T}^{n+1}(b)(s, X_s)\,\mathrm{d}s$$
$$+ \int_0^t \left(\sum_{k=0}^n \nabla U_{\mathcal{T}^k(b)}(s, X_s) \right) \mathrm{d}W_s + W_t.$$

where we have set $\mathcal{T}^0(b) = b$. We shall prove our results (uniqueness and pathwise continuous dependence on initial conditions) for this equation.

To simplify a little the notations, let us set

$$U^{(n)}(t,x) = \sum_{k=0}^{n} U_{\mathcal{T}^k(b)}(t,x), \qquad b^{(n)} = \mathcal{T}^{n+1}(b).$$

The equation reads

$$X_t = x + U^{(n)}(0,x) - U^{(n)}(t,X_t) + \int_0^t b^{(n)}(s,X_s)\,\mathrm{d}s + \int_0^t \left(\nabla U^{(n)}(s,X_s) + I\right) \cdot \mathrm{d}W_s.$$

We discuss first the case when b is Hölder continuous, both to see this equation at work in an easier case, and to show two different ways to handle such an equation, in the C_b^α and L_p^q cases.

Remark 3 *Intuitively speaking (it can be made rigorous), if we pass to the limit in the previous identity we get*

$$X_t = x + U(0,x) - U(t,X_t) + \int_0^t (\nabla U(s,X_s) + I) \cdot \mathrm{d}W_s$$

where U is the solution of the backward Kolmogorov equation

$$\frac{\partial U}{\partial t} + \frac{1}{2}\Delta U + (b \cdot \nabla) U = -b \text{ on } [0,T], \qquad U_\Phi|_{t=T} = 0$$

used in [FGP10] (for Hölder continuous drift). These two approaches are thus equivalent, in principle, but for conceptual reasons and possibly for future extensions we would like to give a proof explicitly based only on the heat equation.

3 The case when b is Hölder continuous

For $\alpha \in (0,1)$, denote by $C_b^\alpha(\mathbb{R}^d)$ the space of all continuous $f : \mathbb{R}^d \to \mathbb{R}$ such that

$$\|u\|_{C_b^\alpha(T)} := \sup_{x \in \mathbb{R}^d} |u(x)| + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < \infty.$$

In this case we use the following well known result. In fact maximal regularity $u \in C\left([0,T]; C_b^{2,\alpha}(\mathbb{R}^d)\right)$ is known, and uniqueness, but again we do not need it for our result and strategy of proof.

Theorem 4 *For all $\varphi \in C([0,T]; C_b^\alpha(\mathbb{R}^d))$ there exists at least one solution u to the backward heat equation (4) of class*

$$u \in C\left([0,T]; C_b^2\left(\mathbb{R}^d\right)\right) \cap C^1\left([0,T]; C_b^\alpha\left(\mathbb{R}^d\right)\right)$$

with

$$\left\|D^2 u\right\|_{L^\infty(T)} \leq C \|\varphi\|_{C_b^\alpha(T)} \tag{8}$$

and

$$\|\nabla u\|_{C_b^\alpha(T)} \leq C(T) \|\varphi\|_{C_b^\alpha(T)} \text{ with } \lim_{T \to 0} C(T) = 0. \tag{9}$$

Due to estimate (8), the proof of theorem 1 simplifies a lot. Let us remark that in this case theorem 1 is known, see [FGP10], where it is proved that the equation has a stochastic flow of diffeomorphisms.

Lemma 5 *Set*

$$C_n(T) := \sum_{k=0}^{n} \left\| \nabla U_{\mathcal{T}^k(b)} \right\|_{L^\infty(T)}, \qquad D_n(T) := \sum_{k=0}^{n} \left\| D^2 U_{\mathcal{T}^k(b)} \right\|_{L^\infty(T)}$$

Then there exists T_0 *and* $C > 0$ *such that for all* $T \in (0, T_0]$ *we have*

$$C_n(T) \le \frac{1}{2}, \qquad D_n(T) \le C, \qquad \|\mathcal{T}^n(b)\|_{C_b^\alpha(T)} \le C \frac{1}{2^n}$$

for every $n \in \mathbb{N}$.

Proof. Let

$$\varepsilon = \left(4 \|b\|_{C_b^\alpha(T)} \right)^{-1}$$

(unless $b = 0$, when there is nothing to prove). Due to (9), we may choose T_0 such that for all $T \in (0, T_0]$ we have

$$\|\nabla U_\Phi\|_{C_b^\alpha(T)} \le \varepsilon \|\Phi\|_{C_b^\alpha(T)}.$$

Thus

$$\|\nabla U_b\|_{C_b^\alpha(T)} \le \varepsilon \|b\|_{C_b^\alpha(T)}$$

$$\left\|\mathcal{T}^1(b)\right\|_{C_b^\alpha(T)} \le \|\nabla U_b\|_{C_b^\alpha(T)} \|b\|_{C_b^\alpha(T)} \le \varepsilon \|b\|^2_{C_b^\alpha(T)}$$

$$\left\|\nabla U_{\mathcal{T}^1(b)}\right\|_{C_b^\alpha(T)} \le \varepsilon^2 \|b\|^2_{C_b^\alpha(T)}$$

$$\left\|\mathcal{T}^2(b)\right\|_{C_b^\alpha(T)} \le \left\|\nabla U_{\mathcal{T}^1(b)}\right\|_{C_b^\alpha(T)} \|b\|_{C_b^\alpha(T)} \le \varepsilon^2 \|b\|^3_{C_b^\alpha(T)}$$

and so on; by induction, one can see that

$$\left\|\nabla U_{\mathcal{T}^k(b)}\right\|_{C_b^\alpha(T)} \le \varepsilon^{k+1} \|b\|^{k+1}_{C_b^\alpha(T)} \le 4^{-(k+1)}$$

$$\left\|\mathcal{T}^k(b)\right\|_{C_b^\alpha(T)} \le \varepsilon^k \|b\|^{k+1}_{C_b^\alpha(T)} \le 4^{-k} \|b\|_{C_b^\alpha(T)}.$$

Thus $C_n(T) \le \sum_{k=0}^{n} 4^{-(k+1)} \le 1/2$ and $\|\mathcal{T}^n(b)\|_{L_p^q(T)} \le C\frac{1}{2^n}$ for some constant $C > 0$. Moreover, for some constants $C', C'' > 0$

$$D_n(T) \le \sum_{k=0}^{n} C' \left\|\mathcal{T}^k(b)\right\|_{C_b^\alpha(T)} \le C'' \sum_{k=0}^{n} 2^{-k} \le 2C''.$$

The proof is complete. ■

Let $X_t^{(i)}$, $i = 1, 2$, be two solutions, with initial conditions $x^{(i)}$, $i = 1, 2$. Given any $p \ge 2$, let us estimate

$$E\left[\sup_{t\in[0,T]} \left|X_t^{(1)} - X_t^{(2)}\right|^p\right]. \tag{10}$$

We have

$$X_t^{(1)} - X_t^{(2)} = a_t + b_t + c_t + d_t$$

where

$$\begin{aligned}
a_t &= x^{(1)} - x^{(2)} + U^{(n)}\left(0, x^{(1)}\right) - U^{(n)}\left(0, x^{(2)}\right) \\
b_t &= U^{(n)}\left(t, X_t^{(1)}\right) - U^{(n)}\left(0, X_t^{(2)}\right) \\
c_t &= \int_0^t b^{(n)}\left(s, X_s^{(1)}\right) \mathrm{d}s - \int_0^t b^{(n)}\left(s, X_s^{(2)}\right) \mathrm{d}s \\
d_t &= \int_0^t \left(\nabla U^{(n)}\left(s, X_s^{(1)}\right) - \nabla U^{(n)}\left(s, X_s^{(2)}\right)\right) \cdot \mathrm{d}W_s.
\end{aligned}$$

We use the inequality

$$|a_t + b_t + c_t + d_t|^p \le \frac{3}{2}|b_t|^p + C_p|a_t|^p + C_p|c_t|^p + C_p|d_t|^p. \tag{11}$$

Let us take $T \le T_0$ given by the lemma. With new values of C_p when necessary, from the estimates of the lemma we have

$$E\left[\sup_{t\in[0,T]}|a_t|^p\right] \le C_p\left|x^{(1)} - x^{(2)}\right|^p + C_pC_n^p(T)\left|x^{(1)} - x^{(2)}\right|^p \le C_p\left|x^{(1)} - x^{(2)}\right|^p$$

because

$$\begin{aligned}
\left|\sum_{k=0}^n U_{\mathcal{T}^k(b)}\left(0, x^{(1)}\right) - \sum_{k=0}^n U_{\mathcal{T}^k(b)}\left(0, x^{(2)}\right)\right| &\le \sum_{k=0}^n \left|U_{\mathcal{T}^k(b)}\left(0, x^{(1)}\right) - U_{\mathcal{T}^k(b)}\left(0, x^{(2)}\right)\right| \\
&\le \sum_{k=0}^n \left\|\nabla U_{\mathcal{T}^k(b)}\right\|_{L^\infty(T)}\left|x^{(1)} - x^{(2)}\right| \\
&= C_n(T)\left|x^{(1)} - x^{(2)}\right|.
\end{aligned}$$

Similarly

$$E\left[\sup_{t\in[0,T]}|b_t|^p\right] \le \frac{1}{2^p}E\left[\sup_{t\in[0,T]}\left|X_t^{(1)} - X_t^{(2)}\right|^p\right].$$

These are the first two terms which contribute to estimate from above the quantity (10). The estimate of the third term $|c_t|^p$ is made by estimating the following two terms, $i = 1, 2$,

$$\begin{aligned}
E\left[\sup_{t\in[0,T]}\left|\int_0^t \mathcal{T}^{n+1}(b)\left(s, X_s^{(i)}\right)\mathrm{d}s\right|^p\right] &\le E\left[\left(\int_0^T \left|\mathcal{T}^{n+1}(b)\left(s, X_s^{(i)}\right)\right|\mathrm{d}s\right)^p\right] \\
&\le C^pT^p2^{-(n+1)p}.
\end{aligned}$$

Finally, the forth term is

$$E\left[\sup_{t\in[0,T]}|d_t|^p\right] \le C_pE\left[\left(\int_0^T\left\|\sum_{k=0}^n\left(\nabla U_{\mathcal{T}^k(b)}\left(s, X_s^{(1)}\right) - \nabla U_{\mathcal{T}^k(b)}\left(s, X_s^{(2)}\right)\right)\right\|^2\mathrm{d}s\right)^{p/2}\right].$$

We have

$$
\begin{aligned}
&\left\| \sum_{k=0}^{n} \left(\nabla U_{\mathcal{T}^k(b)} \left(s, X_s^{(1)} \right) - \nabla U_{\mathcal{T}^k(b)} \left(s, X_s^{(2)} \right) \right) \right\| \\
&\qquad\qquad \le \sum_{k=0}^{n} \left\| \nabla U_{\mathcal{T}^k(b)} \left(s, X_s^{(1)} \right) - \nabla U_{\mathcal{T}^k(b)} \left(s, X_s^{(2)} \right) \right\| \\
&\qquad\qquad \le \sum_{k=0}^{n} \left\| D^2 U_{\mathcal{T}^k(b)} \right\|_{L^\infty(T)} \left| X_s^{(1)} - X_s^{(2)} \right| = D_n(T) \left| X_s^{(1)} - X_s^{(2)} \right|
\end{aligned}
$$

hence

$$
\begin{aligned}
&E\left[\sup_{t\in[0,T]} \left| \int_0^t \left(\sum_{k=0}^{n} \nabla U_{\mathcal{T}^k(b)} \left(s, X_s^{(1)} \right) - \sum_{k=0}^{n} \nabla U_{\mathcal{T}^k(b)} \left(s, X_s^{(2)} \right) \right) \mathrm{d}W_s \right|^p \right] \\
&\qquad \le C_p E\left[\left(\int_0^T \left| X_s^{(1)} - X_s^{(2)} \right|^2 \mathrm{d}s \right)^{p/2} \right] \le C_p T^{p/2} E\left[\sup_{t\in[0,T]} \left| X_t^{(1)} - X_t^{(2)} \right|^p \right].
\end{aligned}
$$

Summarizing, using (11) we have proved

$$
\begin{aligned}
&E\left[\sup_{t\in[0,T]} \left| X_t^{(1)} - X_t^{(2)} \right|^p \right] \\
&\qquad \le C_p \left| x^{(1)} - x^{(2)} \right|^p + \left(\frac{3}{2}\frac{1}{2^p} + C_p T^{p/2} \right) E\left[\sup_{t\in[0,T]} \left| X_t^{(1)} - X_t^{(2)} \right|^p \right] + C^p T^p 2^{-(n+1)p}.
\end{aligned}
$$

Since this is true for every n, we have

$$
E\left[\sup_{t\in[0,T]} \left| X_t^{(1)} - X_t^{(2)} \right|^p \right] \le C_p \left| x^{(1)} - x^{(2)} \right|^p + \left(\frac{3}{2}\frac{1}{2^p} + C_p T^{p/2} \right) E\left[\sup_{t\in[0,T]} \left| X_t^{(1)} - X_t^{(2)} \right|^p \right].
$$

Then there exists $T_1 > 0$ such that

$$
E\left[\sup_{t\in[0,T_1]} \left| X_t^{(1)} - X_t^{(2)} \right|^p \right] \le C_p \left| x^{(1)} - x^{(2)} \right|^p.
$$

This implies uniqueness and the existence of the modification (by Kolmogorov regularity theorem for fields with values in $C([0,T];\mathbb{R})$ and the arbitrariness of $p \ge 2$), as claimed by the theorem, over the time interval $[0, T_1]$. By classical arguments one can iterate the result on successive intervals (their size does not change), so the result is true over $[0, T]$. The proof in the Hölder case is complete.

Remark 6 *One can work on the full initial interval $[0, T]$ from the beginning, by means of the following trick, developed in [FGP10]: one takes the heat equation with damping*

$$
\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u = \lambda u + \varphi \text{ on } [0,T], \qquad u|_{t=T} = 0
$$

and use the fact that for large λ the gradient of u is uniformly small (like $\lambda^{-1/2}$).

4 General case

Let us now go back to the general case when $b \in L_p^q(T)$. The main novelty is that d_t cannot be estimated as above, since D^2U is not bounded. We use two tricks to overcome this apparently very serious difficulty, used before in other works on uniqueness for certain nonlinear equations: introduce a suitable increasing process A_t related to D^2U (see [Ve80]) and pre-multiply by e^{-A_t} (see [Sc97], [DD03]).

Let again $X_t^{(i)}$, $i = 1, 2$, be two solutions with initial conditions $x^{(i)}$, $i = 1, 2$. Given any $p \geq 2$, we want to estimate (10). We follow a different route with respect to the previous section. Set, for $i = 1, 2$ and $n \in \mathbb{N}$,

$$\begin{aligned} Y_t^{(i,n)} &:= X_t^{(i)} + U^{(n)}\left(t, X_t^{(i)}\right), \\ b_t^{(i,n)} &:= b^{(n)}\left(t, X_t^{(i)}\right), \qquad \sigma_t^{(i,n)} := \nabla U^{(n)}\left(t, X_t^{(i)}\right). \end{aligned}$$

We drop the index n in intermediate computations, when it is not essential to emphasize the dependence on n. The equation reads now

$$Y_t^{(i)} = Y_0^{(i)} + \int_0^t b_s^{(i)} \mathrm{d}s + \int_0^t \left(\sigma_s^{(i)} + I\right) \cdot \mathrm{d}W_s.$$

Controlling $\left|X_t^{(1)} - X_t^{(2)}\right|$ is the same as $\left|Y_t^{(1,n)} - Y_t^{(2,n)}\right|$, for each n, for small T (this reminds [KR05, lemma 10.6], although the approach is different)

Lemma 7 *Recall the notation $C_n(T)$ from lemma 5. There exists T_0 such that for all $T \in (0,T_0]$ we have*

$$C_n(T) \leq \frac{1}{2}, \qquad \|\mathcal{T}^n(b)\|_{L_p^q(T)} \leq C\frac{1}{2^n}. \tag{12}$$

It follows also

$$\begin{aligned} \left|Y_t^{(1,n)} - Y_t^{(2,n)}\right| &\leq \frac{3}{2}\left|X_t^{(1)} - X_t^{(2)}\right| \\ \left|X_t^{(1)} - X_t^{(2)}\right| &\leq 2\left|Y_t^{(1,n)} - Y_t^{(2,n)}\right| \end{aligned}$$

for $t \in [0, T]$ and

$$\left\|\nabla U^{(n)}\right\|_{L^\infty(T)} \leq \frac{1}{2}, \qquad \left\|D^2 U^{(n)}\right\|_{L_p^q(T)} \leq C$$

for some constant $C > 0$.

Proof. Using (6) instead of (9) one can prove (12) as in the case of lemma 5. The modifications are that we use

$$\varepsilon = \left(4\|b\|_{L_p^q(T)}\right)^{-1}$$

and we get the inequalities

$$\|\nabla U_b\|_{L^\infty(T)} \leq \varepsilon\|b\|_{L_p^q(T)}$$

$$\left\|\mathcal{T}^1(b)\right\|_{L_p^q(T)} \leq \|\nabla U_b\|_{L^\infty(T)} \|b\|_{L_p^q(T)} \leq \varepsilon \|b\|^2_{L_p^q(T)}$$

and so on by iteration. We do not rewrite all the details. Having proved (12), we have (using a simple approximation argument to write the estimate with $\|\nabla U_{\mathcal{T}^k(b)}\|_{L^\infty(T)}$)

$$\begin{aligned}\left|Y_t^{(1)} - Y_t^{(2)}\right| &\leq \left|X_t^{(1)} - X_t^{(2)}\right| + \sum_{k=0}^{n} \left\|\nabla U_{\mathcal{T}^k(b)}\right\|_{L^\infty(T)} \left|X_t^{(1)} - X_t^{(2)}\right| \\ &= \left|X_t^{(1)} - X_t^{(2)}\right| + C_n(T)\left|X_t^{(1)} - X_t^{(2)}\right| \leq \frac{3}{2}\left|X_t^{(1)} - X_t^{(2)}\right|\end{aligned}$$

and

$$\begin{aligned}\left|X_t^{(1)} - X_t^{(2)}\right| &\leq \left|Y_t^{(1)} - Y_t^{(2)}\right| + \left|\sum_{k=0}^{n} U_{\mathcal{T}^k(b)}\left(t, X_t^{(1)}\right) - U_{\mathcal{T}^k(b)}\left(t, X_t^{(2)}\right)\right| \\ &\leq \left|Y_t^{(1)} - Y_t^{(2)}\right| + \frac{1}{2}\left|X_t^{(1)} - X_t^{(2)}\right|\end{aligned}$$

and thus $\left|X_t^{(1)} - X_t^{(2)}\right| \leq 2\left|Y_t^{(1)} - Y_t^{(2)}\right|$. Finally, $\left\|\nabla U^{(n)}\right\|_{L^\infty(T)} \leq C_n(T)$ and

$$\left\|D^2 U^{(n)}\right\|_{L_p^q(T)} \leq \sum_{k=0}^{n} \left\|D^2 U_{\mathcal{T}^k(b)}\right\|_{L_p^q(T)} \leq C \sum_{k=0}^{n} \left\|\mathcal{T}^k(b)\right\|_{L_p^q(T)}$$

by (5), and the series converges by (12). The proof is complete. ∎

By Itô formula we have

$$\begin{aligned}d\left|Y_t^{(1)} - Y_t^{(2)}\right|^p &\leq p\left|Y_t^{(1)} - Y_t^{(2)}\right|^{p-1}\left|b_t^{(1)} - b_t^{(2)}\right| \mathrm{d}t \\ &+ p\left|Y_t^{(1)} - Y_t^{(2)}\right|^{p-2}\left\langle Y_t^{(1)} - Y_t^{(2)}, \left(\sigma_t^{(1)} - \sigma_t^{(2)}\right) \cdot \mathrm{d}W_t\right\rangle \\ &+ C_p^*\left|Y_t^{(1)} - Y_t^{(2)}\right|^{p-2}\left\|\sigma_t^{(1)} - \sigma_t^{(2)}\right\|^2 \mathrm{d}t\end{aligned}$$

for a suitable constant C_p^*. Following Veretennikov [Ve80], denote by $\frac{\left\|\sigma_t^{(1)} - \sigma_t^{(2)}\right\|^2}{\left|Y_t^{(1)} - Y_t^{(2)}\right|^2} 1_{\left\{Y_t^{(1)} \neq Y_t^{(2)}\right\}}$ the non negative function equal to $\frac{\left\|\sigma_t^{(1)} - \sigma_t^{(2)}\right\|^2}{\left|Y_t^{(1)} - Y_t^{(2)}\right|^2}$ when $Y_t^{(1)} \neq Y_t^{(2)}$ and equal to zero otherwise. Set

$$A_t^{(n)} := \int_0^t \frac{\left\|\sigma_s^{(1,n)} - \sigma_s^{(2,n)}\right\|^2}{\left|Y_s^{(1,n)} - Y_s^{(2,n)}\right|^2} 1_{\left\{Y_s^{(1,n)} \neq Y_s^{(2,n)}\right\}} \mathrm{d}s$$

(we write A_t when n is not the main concern) which a priori may be infinite. We shall prove below, lemma 10, that this is a finite, even exponentially integrable uniformly in n, increasing non negative process. Then

$$d\left(e^{-C_p^* A_t}\left|Y_t^{(1)}-Y_t^{(2)}\right|^p\right) \leq e^{-C_p^* A_t} p\left|Y_t^{(1)}-Y_t^{(2)}\right|^{p-1}\left|b_t^{(1)}-b_t^{(2)}\right| \mathrm{d}t$$
$$+e^{-C_p^* A_t} p\left|Y_t^{(1)}-Y_t^{(2)}\right|^{p-2}\left\langle Y_t^{(1)}-Y_t^{(2)},\left(\sigma_t^{(1)}-\sigma_t^{(2)}\right) \cdot \mathrm{d}W_t\right\rangle$$
$$+e^{-C_p^* A_t} C_p^*\left|Y_t^{(1)}-Y_t^{(2)}\right|^{p-2}\left\|\sigma_t^{(1)}-\sigma_t^{(2)}\right\|^2 \mathrm{d}t$$
$$-C_p^* e^{-C_p^* A_t}\left|Y_t^{(1)}-Y_t^{(2)}\right|^p dA_t.$$

Since

$$e^{-C_p^* A_t} C_p^*\left|Y_t^{(1)}-Y_t^{(2)}\right|^{p-2}\left\|\sigma_t^{(1)}-\sigma_t^{(2)}\right\|^2 \mathrm{d}t-C_p^* e^{-C_p^* A_t}\left|Y_t^{(1)}-Y_t^{(2)}\right|^p dA_t=0$$

The inequality simplifies to

$$d\left(e^{-C_p^* A_t}\left|Y_t^{(1)}-Y_t^{(2)}\right|^p\right) \leq e^{-C_p^* A_t} p\left|Y_t^{(1)}-Y_t^{(2)}\right|^{p-1}\left|b_t^{(1)}-b_t^{(2)}\right| \mathrm{d}t$$
$$+e^{-C_p^* A_t} p\left|Y_t^{(1)}-Y_t^{(2)}\right|^{p-2}\left\langle Y_t^{(1)}-Y_t^{(2)},\left(\sigma_t^{(1)}-\sigma_t^{(2)}\right) \cdot \mathrm{d}W_t\right\rangle.$$

The last term is a martingale: the processes $\sigma_t^{(i)}$ are bounded (recall that ∇U is bounded), $e^{-C_p^* A_t}$ is bounded by 1, and $\left|Y_t^{(1)}-Y_t^{(2)}\right|$ is integrable at any power, since it is smaller than $\frac{3}{2}\left|X_t^{(1)}-X_t^{(2)}\right|$ (lemma 7) and we know that solutions of equation (1) are integrable to any power, see proposition 17 in the Appendix. Therefore, using also $\left|Y_0^{(1)}-Y_0^{(2)}\right| \leq \frac{3}{2}\left|x^{(1)}-x^{(2)}\right|$ (lemma 7)

$$E\left[e^{-C_p^* A_t}\left|Y_t^{(1)}-Y_t^{(2)}\right|^p\right] \leq C_p\left|x^{(1)}-x^{(2)}\right|^p+p \int_0^t E\left[\left|Y_s^{(1)}-Y_s^{(2)}\right|^{p-1}\left|b_s^{(1,n)}-b_s^{(2,n)}\right|\right] \mathrm{d}s.$$

Using again lemma 7, both in the first and last term, we get

$$E\left[e^{-C_p^* A_t^{(n)}}\left|X_t^{(1)}-X_t^{(2)}\right|^p\right] \leq C_p\left|x^{(1)}-x^{(2)}\right|^p+C_p \int_0^T E\left[\left|X_s^{(1)}-X_s^{(2)}\right|^{p-1}\left|b_s^{(1,n)}-b_s^{(2,n)}\right|\right] \mathrm{d}s. \tag{13}$$

Lemma 8 *For every* $\alpha, \beta \geq 1$,

$$\lim_{n \rightarrow \infty} E\left[\left(\int_0^T\left|X_s^{(1)}-X_s^{(2)}\right|^\alpha\left|b_s^{(1,n)}-b_s^{(2,n)}\right| \mathrm{d}s\right)^\beta\right]=0.$$

Proof. We have

$$E\left[\left(\int_0^T\left|X_s^{(1)}-X_s^{(2)}\right|^\alpha\left|b_s^{(1,n)}-b_s^{(2,n)}\right| \mathrm{d}s\right)^\beta\right]$$
$$\leq E\left[\left(\int_0^T\left|X_s^{(1)}-X_s^{(2)}\right|^{2\alpha} \mathrm{d}s\right)^\beta\right]^{1/2} E\left[\left(\int_0^T\left|b_s^{(1,n)}-b_s^{(2,n)}\right|^2 \mathrm{d}s\right)^\beta\right]^{1/2}.$$

The first term is bounded since $E\left[\int_0^T \left|X_s^{(i)}\right|^N \mathrm{d}s\right] < \infty$ for each $N > 0$, $i = 1, 2$, see proposition 17 in the Appendix. Let us prove that the second term converges to zero. For each $i = 1, 2$, we have

$$E\left[\left(\int_0^T \left|2^{n+1} b_s^{(i,n)}\right|^2 \mathrm{d}s\right)^\beta\right] = E\left[\left(\int_0^T \left|f_n\left(s, X_s^{(i)}\right)\right|^2 \mathrm{d}s\right)^\beta\right]$$

where

$$f_n := 2^{n+1} \mathcal{T}^{n+1}(b)$$

are equibounded in $L_p^q(T)$ by lemma 7. From Girsanov formula (24) of the Appendix we have

$$\begin{aligned} &E\left[\left(\int_0^T \left|f_n\left(s, X_s^{(i)}\right)\right|^2 \mathrm{d}s\right)^\beta\right] \\ &= E\left[\left(\int_0^T \left|f_n\left(s, x^{(i)} + W_s\right)\right|^2 \mathrm{d}s\right)^\beta e^{\int_0^T b(s,x^{(i)}+W_s)\,\mathrm{d}W_s - 1/2\int_0^T \left|b(s,x^{(i)}+W_s)\right|^2 \mathrm{d}s}\right]. \end{aligned}$$

This is equal to

$$\begin{aligned} &= E\left[\left(\int_0^T \left|f_n(s, x + W_s)\right|^2 \mathrm{d}s\right)^\beta e^{\int_0^T b(s,x+W_s)\,\mathrm{d}W_s - \frac{2}{2}\int_0^T |b(s,x+W_s)|^2 \mathrm{d}s} e^{\frac{(2-1)}{2}\int_0^T |b(s,x+W_s)|^2 \mathrm{d}s}\right] \\ &\le E\left[\left(\int_0^T \left|f_n(s, x + W_s)\right|^2 \mathrm{d}s\right)^{2\beta} e^{(2-1)\int_0^T |b(s,x+W_s)|^2 \mathrm{d}s}\right]^{1/2} \\ &\le E\left[\left(\int_0^T \left|f_n(s, x + W_s)\right|^2 \mathrm{d}s\right)^{4\beta}\right]^{1/4} E\left[e^{2\int_0^T |b(s,x+W_s)|^2 \mathrm{d}s}\right]^{1/4} \end{aligned}$$

where we have used

$$E\left[e^{\int_0^T 2b(s,x+W_s)\,\mathrm{d}W_s - \frac{1}{2}\int_0^T |2b(s,x+W_s)|^2 \mathrm{d}s}\right] = 1.$$

Both factors of the last inequality are bounded, by the exponential moment estimates of corollary 14. Therefore we can find a constant K_β independent of n, such that

$$E\left[\left(\int_0^T \left|f_n\left(s, X_s^{(i)}\right)\right|^2 \mathrm{d}s\right)^\beta\right] \le K_\beta \tag{14}$$

which implies $E\left[\left(\int_0^T \left|b_s^{(i,n)}\right|^2 \mathrm{d}s\right)^\beta\right] \le K_\beta \frac{1}{(2^{n+1})^{2\beta}}$. The proof is complete. ■

From (13) and lemma 8 we get

$$\limsup_{n\to\infty} \sup_{t\in[0,T]} E\left[e^{-C_p^* A_t^{(n)}} \left|X_t^{(1)} - X_t^{(2)}\right|^p\right] \le C_p \left|x^{(1)} - x^{(2)}\right|^p.$$

But we have

$$E\left[\left|X_t^{(1)}-X_t^{(2)}\right|^{p/2}\right]=E\left[e^{C_p^* A_t^{(n)}/2}e^{-C_p^* A_t^{(n)}/2}\left|X_t^{(1)}-X_t^{(2)}\right|^{p/2}\right]$$
$$\leq E\left[e^{C_p^* A_t^{(n)}}\right]^{1/2} E\left[e^{-C_p^* A_t^{(n)}}\left|X_t^{(1)}-X_t^{(2)}\right|^{p}\right]^{1/2}$$
$$\leq E\left[e^{C_p^* A_t^{(n)}}\right]^{1/2} C_p\left|x^{(1)}-x^{(2)}\right|^{p/2}.$$

From lemma 10, $E\left[e^{C_p^* A_t^{(n)}}\right]$ is uniformly bounded, so we include it into the constant and get (renaming p)

$$\limsup_{n\to\infty}\sup_{t\in[0,T]} E\left[\left|X_t^{(1)}-X_t^{(2)}\right|^{p/2}\right]\leq C_p\left|x^{(1)}-x^{(2)}\right|^{p/2}.$$

But now the left-hand-side is independent of n. We have proved the following result, of independent interest. It is proved here for small T, but by iteration or by the trick described in remark 6, it holds true on the original time interval $[0,T]$.

Proposition 9

$$\sup_{t\in[0,T]} E\left[\left|X_t^{(1)}-X_t^{(2)}\right|^{p}\right]\leq C_p\left|x^{(1)}-x^{(2)}\right|^{p}. \tag{15}$$

Let us stress that, in our opinion, this proposition is a remarkable step forward with respect to what was known before for equation (1) under L_p^q-drift. In a sense, the rest are more or less classical details.

Let us improve the proposition to an estimate for $E\left[\sup_{t\in[0,T]}\left|X_t^{(1)}-X_t^{(2)}\right|^{p}\right]$. We may use the inequality proved above

$$e^{-C_p^* A_t}\left|Y_t^{(1)}-Y_t^{(2)}\right|^{p}\leq\frac{3}{2}\left|x^{(1)}-x^{(2)}\right|^{p}+p\int_0^T\left|Y_s^{(1)}-Y_s^{(2)}\right|^{p-1}\left|b_s^{(1)}-b_s^{(2)}\right|\mathrm{d}t$$
$$+p\int_0^t\left|Y_s^{(1)}-Y_s^{(2)}\right|^{p-2}\left\langle Y_s^{(1)}-Y_s^{(2)},\left(\sigma_s^{(1)}-\sigma_s^{(2)}\right)\cdot\mathrm{d}W_t\right\rangle$$

square it

$$e^{-2C_p^* A_t}\left|Y_t^{(1)}-Y_t^{(2)}\right|^{2p}\leq C\left|x^{(1)}-x^{(2)}\right|^{2p}+C_p\left(\int_0^T\left|Y_s^{(1)}-Y_s^{(2)}\right|^{p-1}\left|b_s^{(1)}-b_s^{(2)}\right|\mathrm{d}t\right)^2$$
$$+C_p\left(\int_0^t\left|Y_s^{(1)}-Y_s^{(2)}\right|^{p-2}\left\langle Y_s^{(1)}-Y_s^{(2)},\left(\sigma_s^{(1)}-\sigma_s^{(2)}\right)\cdot\mathrm{d}W_t\right\rangle\right)^2$$

and apply Doob's inequality, and lemma 7, to get

$$\begin{aligned} E\left[\sup_{t\in[0,T]}\left(e^{-2C_p^* A_t^{(n)}}\left|X_t^{(1)}-X_t^{(2)}\right|^{2p}\right)\right] &\le 2^{2p}E\left[\sup_{t\in[0,T]}\left(e^{-2C_p^* A_t^{(n)}}\left|Y_t^{(1)}-Y_t^{(2)}\right|^{2p}\right)\right] \\ &\le C_p\left|x^{(1)}-x^{(2)}\right|^{2p}+C_pE\left[\left(\int_0^T\left|Y_s^{(1)}-Y_s^{(2)}\right|^{p-1}\left|b_s^{(1)}-b_s^{(2)}\right|\mathrm{d}t\right)^2\right] \\ &\quad + C_pE\left[\int_0^T\left|Y_s^{(1)}-Y_s^{(2)}\right|^{2(p-1)}\left\|\sigma_s^{(1)}-\sigma_s^{(2)}\right\|^2\mathrm{d}s\right]. \end{aligned}$$

One one side we have

$$\begin{aligned} E\left[\left(\int_0^T\left|Y_s^{(1,n)}-Y_s^{(2,n)}\right|^{p-1}\left|b_s^{(1,n)}-b_s^{(2,n)}\right|\mathrm{d}t\right)^2\right] &\le C_pE\left[\left(\int_0^T\left|X_s^{(1)}-X_s^{(2)}\right|^{p-1}\left|b_s^{(1,n)}-b_s^{(2,n)}\right|\mathrm{d}t\right)^2\right] \end{aligned}$$

which converges to zero as $n\to\infty$, by lemma 8. On the other side, since by definition of $\sigma_t^{(i,n)}$ and inequality (12) we have $\left|\sigma_s^{(i)}\right|\le\frac{1}{2}$, one has the estimate

$$\begin{aligned} E\left[\int_0^T\left|Y_s^{(1)}-Y_s^{(2)}\right|^{2(p-1)}\left\|\sigma_s^{(1)}-\sigma_s^{(2)}\right\|^2\mathrm{d}s\right] &\le CE\left[\int_0^T\left|Y_s^{(1)}-Y_s^{(2)}\right|^{2(p-1)}\mathrm{d}s\right] \\ &\le C'E\left[\int_0^T\left|X_s^{(1)}-X_s^{(2)}\right|^{2(p-1)}\mathrm{d}s\right] \\ &\le C_p\left|x^{(1)}-x^{(2)}\right|^{2(p-1)} \end{aligned}$$

by means of (15). Summarizing and taking the limit as $n\to\infty$ we have

$$\limsup_{n\to\infty}E\left[\sup_{t\in[0,T]}\left(e^{-2C_p^* A_t^{(n)}}\left|X_t^{(1)}-X_t^{(2)}\right|^{2p}\right)\right]\le C\left|x^{(1)}-x^{(2)}\right|^{2p}+C_p\left|x^{(1)}-x^{(2)}\right|^{2(p-1)}.$$

Moreover,

$$E\left[e^{-2C_p^* A_T^{(n)}}\sup_{t\in[0,T]}\left(\left|X_t^{(1)}-X_t^{(2)}\right|^{2p}\right)\right]\le E\left[\sup_{t\in[0,T]}\left(e^{-2C_p^* A_t^{(n)}}\left|X_t^{(1)}-X_t^{(2)}\right|^{2p}\right)\right]$$

and finally

$$\begin{aligned} E\left[\sup_{t\in[0,T]}\left(\left|X_t^{(1)}-X_t^{(2)}\right|^{p}\right)\right] &= E\left[e^{C_p^* A_T^{(n)}}e^{-C_p^* A_T^{(n)}}\sup_{t\in[0,T]}\left(\left|X_t^{(1)}-X_t^{(2)}\right|^{p}\right)\right] \\ &\le E\left[e^{2C_p^* A_T^{(n)}}\right]^{1/2}E\left[e^{-2C_p^* A_T^{(n)}}\sup_{t\in[0,T]}\left(\left|X_t^{(1)}-X_t^{(2)}\right|^{2p}\right)\right]^{1/2}. \end{aligned}$$

By lemma 10 below, the previous inequalities give us

$$E\left[\sup_{t\in[0,T]}\left(\left|X_t^{(1)}-X_t^{(2)}\right|^p\right)\right]\leq C_p\left(\left|x^{(1)}-x^{(2)}\right|^{2p}+\left|x^{(1)}-x^{(2)}\right|^{2(p-1)}\right)^{1/2}.$$

By Kolmogorov theorem, we deduce the pathwise properties of our main theorem. To complete the proof we need the following exponential estimate. The L^1-integrability of an expression very similar to $A_T^{(n)}$ has been proved in [KR05].

Lemma 10 *For any $k\in\mathbb{R}$ there is a constant $C_k>0$ such that*

$$E\left[e^{kA_T^{(n)}}\right]\leq C_k \tag{16}$$

uniformly in $n\in\mathbb{N}$.

Proof: For $Y_s^{(1)}\neq Y_s^{(2)}$, we also have $X_s^{(1)}\neq X_s^{(2)}$, by lemma 7 (and vice versa, so the functions $1_{\left\{Y_s^{(1)}\neq Y_s^{(2)}\right\}}$ and $1_{\left\{X_s^{(1)}\neq X_s^{(2)}\right\}}$ coincide), so we may also write

$$\begin{aligned}\frac{\left\|\sigma_s^{(1)}-\sigma_s^{(2)}\right\|^2}{\left|Y_s^{(1)}-Y_s^{(2)}\right|^2}&=\frac{\left\|\nabla U^{(n)}\left(s,X_s^{(1)}\right)-\nabla U^{(n)}\left(s,X_s^{(2)}\right)\right\|^2}{\left|X_s^{(1)}-X_s^{(2)}\right|^2}\frac{\left|X_s^{(1)}-X_s^{(2)}\right|^2}{\left|Y_s^{(1)}-Y_s^{(2)}\right|^2}\\&\leq 2\frac{\left\|\nabla U^{(n)}\left(s,X_s^{(1)}\right)-\nabla U^{(n)}\left(s,X_s^{(2)}\right)\right\|^2}{\left|X_s^{(1)}-X_s^{(2)}\right|^2}\end{aligned}$$

where we have used again lemma 7. Thus it is sufficient to prove that

$$E\left[\exp\left(k\int_0^T\frac{\left\|\nabla U^{(n)}\left(s,X_s^{(1)}\right)-\nabla U^{(n)}\left(s,X_s^{(2)}\right)\right\|^2}{\left|X_s^{(1)}-X_s^{(2)}\right|^2}1_{\left\{X_s^{(1)}\neq X_s^{(2)}\right\}}\mathrm{d}s\right)\right]\leq C_k$$

where C_k is a constant independent of n. Notice that $\nabla U^{(n)}$ are equibounded in $L^q\left(0,T;W^{1,p}(\mathbb{R}^d)\right)$ by the last assertion of lemma 7. Thus, by the density of $C_c^\infty\left([0,T]\times\mathbb{R}^d\right)$ in $L^q\left(0,T;W^{1,p}(\mathbb{R}^d)\right)$, it is sufficient to prove the following claim: for all smooth functions $f\in C_c^\infty\left([0,T]\times\mathbb{R}^d\right)$ with $\|f\|_{L^q\left(0,T;W^{1,p}(\mathbb{R}^d)\right)}\leq R$ we have

$$E\left[\exp\left(k\int_0^T\frac{\left|f\left(s,X_s^{(1)}\right)-f\left(s,X_s^{(2)}\right)\right|^2}{\left|X_s^{(1)}-X_s^{(2)}\right|^2}1_{\left\{X_s^{(1)}\neq X_s^{(2)}\right\}}\mathrm{d}s\right)\right]\leq C_{k,R} \tag{17}$$

where $C_{k,R}$ depends only on k and R.

For smooth functions f we have

$$\frac{\left|f\left(s,X_s^{(1)}\right)-f\left(s,X_s^{(2)}\right)\right|^2}{\left|X_s^{(1)}-X_s^{(2)}\right|^2}1_{\left\{X_s^{(1)}\neq X_s^{(2)}\right\}}\leq\int_0^1\left\|\nabla f\left(s,rX_s^{(1)}+(1-r)X_s^{(2)}\right)\right\|^2\mathrm{d}r.$$

Using the convexity of the exponential function, we obtain that the left–hand side of (17) is less than a constant times

$$\int_0^1 E\left[\exp\left(k\int_0^T \left\|\nabla f\left(s, rX_s^{(1)} + (1-r)X_s^{(2)}\right)\right\|^2 \mathrm{d}s\right)\right] \mathrm{d}r. \tag{18}$$

With the notations

$$X_s^{(r)} = rX_s^{(1)} + (1-r)X_s^{(2)}, \qquad x^{(r)} = rx^{(1)} + (1-r)x^{(2)}$$
$$b_s^{(r)} = rb\left(s, X_s^{(1)}\right) + (1-r)b\left(s, X_s^{(2)}\right)$$

the process $X_t^{(r)}$ is given by

$$X_t^{(r)} = x^{(r)} + \int_0^t b_s^{(r)} \,\mathrm{d}s + W_t.$$

We have

$$E\left[e^{\lambda\int_0^T \left|b_t^{(r)}\right|^2 \mathrm{d}t}\right] \le E\left[e^{2\lambda r^2\int_0^T \left|b\left(t,X_t^{(1)}\right)\right|^2 \mathrm{d}t} e^{2\lambda(1-r)^2\int_0^T \left|b\left(t,X_t^{(2)}\right)\right|^2 \mathrm{d}t}\right]$$

which is finite (by Hölder inequality) using the exponential estimates on solutions of equation (1) proved in the Appendix, see (25). Hence Novikov condition is fulfilled; by Girsanov theorem, $X_t^{(r)}$ is a Brownian motion from $x^{(r)}$, on $\left(\Omega, F_t, Q^{(r)}\right)$ with

$$\left.\frac{dQ^{(r)}}{dP}\right|_{F_T} = \rho_T^{(r)} := \exp\left(-\int_0^T b_t^{(r)} \cdot \mathrm{d}W_t - \frac{1}{2}\int_0^T \left|b_t^{(r)}\right|^2 \mathrm{d}t\right).$$

Therefore we obtain (we indicate by superscripts the measure used in the expected values)

$$\begin{aligned}
E^P\left[\exp\left(k\int_0^T \left\|\nabla f\left(s, X_s^{(r)}\right)\right\|^2 \mathrm{d}s\right)\right] &= E^P\left[\left(\rho_T^{(r)}\right)^{-1/2}\left(\rho_T^{(r)}\right)^{1/2}\exp\left(k\int_0^T \left\|\nabla f\left(s, X_s^{(r)}\right)\right\|^2 \mathrm{d}s\right)\right] \\
&\le CE^P\left[\rho_T^{(r)}\exp\left(2k\int_0^T \left\|\nabla f\left(s, X_s^{(r)}\right)\right\|^2 \mathrm{d}s\right)\right]^{1/2} \\
&= E^Q\left[\exp\left(2k\int_0^T \left\|\nabla f\left(s, x^{(r)} + W_s\right)\right\|^2 \mathrm{d}s\right)\right]^{1/2}.
\end{aligned}$$

This is bounded by a constant depending only on the L_p^q norm of ∇f, and on k, see corollary 14 in the Appendix. The proof is complete.

5 Appendix

We collect here known results, taken from the paper [KR05] and previous works, see for instance [Po82], [Ve80]. They include weak existence of a solution X by Girsanov theorem, a formula for the density of the law of the solution with respect to Wiener measure, weak uniqueness and the exponential integrability of the process $|f(t, X_t)|^2$ when $f \in L_p^q(T)$ with $\frac{d}{p} + \frac{2}{q} < 1$.

Lemma 11 *Given p', $q' \in [1,\infty]$ such that*

$$\frac{d}{p'} + \frac{2}{q'} < 2 \tag{19}$$

there exist two positive constants C and β (it is $2\beta = 2 - 2/q' - d/p'$) with the following property: for every $f \in L^{q'}_{p'}(T)$ and every $t > s$, $t, s \in [0,T]$,

$$\sup_{x\in\mathbb{R}^d} E\left[\int_s^t f(r, x + W_{r-s})\,\mathrm{d}r\right] \le C(t-s)^\beta \|f\|_{L^{q'}_{p'}(T)}. \tag{20}$$

The proof is elementary (we write it only for p', $q' \in (1,\infty)$): with $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$, since

$$\int_{\mathbb{R}^d} (2\pi(r-s))^{-pd/2}\, e^{\frac{-p|y|^2}{2(r-s)}}\,\mathrm{d}y = C\,(r-s)^{(1-p)d/2}$$

(we denote by C a generic constant) and $\frac{q(1-p)d+2p}{2pq} = -\frac{d}{2p'} + 1 - \frac{1}{q'}$ we have

$$\begin{aligned}
E\left[\int_s^t f(r, x + W_{r-s})\,\mathrm{d}r\right] &\le \int_s^t \left(\int_{\mathbb{R}^d} f^{p'}(r,y)\,\mathrm{d}y\right)^{1/p'} \left(\int_{\mathbb{R}^d} (2\pi(r-s))^{-pd/2}\, e^{\frac{-p|y|^2}{2(r-s)}}\,\mathrm{d}y\right)^{1/p} \mathrm{d}r \\
&\le C\|f\|_{L^{q'}_{p'}(T)} \left(\int_s^t (r-s)^{q(1-p)d/2p}\,\mathrm{d}r\right)^{1/q} \\
&= C\|f\|_{L^{q'}_{p'}(T)} (t-s)^{1-1/q'-d/2p'}.
\end{aligned}$$

Remark 12 *As a consequence, if $f \in L^q_p(T)$ with $\frac{d}{p} + \frac{2}{q} < 1$ (condition (2)), then $f^2 \in L^{q'}_{p'}(T)$ with $q' = q/2$, $p' = p/2$ satisfying $\frac{d}{p'} + \frac{2}{q'} < 2$, and $\|f^2\|_{L^{q'}_{p'}(T)} \le \|f\|^2_{L^q_p(T)}$. Therefore*

$$\sup_{x\in\mathbb{R}^d} E\left[\int_s^t f^2(r, x + W_{r-s})\,\mathrm{d}r\right] \le C(t-s)^\beta \|f\|^2_{L^q_p(T)}.$$

Lemma 13 (Khas'minskii) *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a positive Borel function such that*

$$\alpha := \sup_{x\in\mathbb{R}^d} E\left[\int_0^T f(s, x + W_s)\,\mathrm{d}s\right] < 1. \tag{21}$$

Then

$$\sup_{x\in\mathbb{R}^d} E\left[e^{\int_0^T f(s,x+W_s)\,\mathrm{d}s}\right] \le \frac{1}{1-\alpha}.$$

See [Kh59] or [Sz98, Chapter 1, lemma 2.1].

Corollary 14 *If f is a vector field of class $L^q_p(T)$ for some p, $q \in [1,\infty]$ such that (2) holds, then there exists a constant K_f depending on $\|f\|_{L^q_p(T)}$ such that*

$$\sup_{x\in\mathbb{R}^d} E\left[e^{\int_0^T |f(s,x+W_s)|^2 \mathrm{d}s}\right] \le K_f\,.$$

Moreover, all (positive and negative) moments of

$$\rho_T := \exp\left(\int_0^T f(s, x + W_s) \cdot \mathrm{d}W_s - \frac{1}{2}\int_0^T |f(s, x + W_s)|^2 \,\mathrm{d}s\right) \tag{22}$$

are finite.

Proof. Since $f \in L_p^q(T)$ with p, q satisfying (2), $f^2 \in L_{p'}^{q'}(T)$ with $p' = p/2$, $q' = q/2$ satisfying (19). Since (19) is a strict inequality, we may choose $\delta > 0$ such that $|f|^{2+\delta} \in L_{p'}^{q'}(T)$ for some new p', q' satisfying (19). Then we have inequality (20) with f replaced by $|f|^{2+\delta}$. Choose $\varepsilon > 0$ such that

$$\sup_{x\in\mathbb{R}^d} E\left[\int_0^T \varepsilon\, |f|^{2+\delta}(s, x + W_s)\,\mathrm{d}s\right] < 1.$$

Then, by Khas'minskii lemma,

$$\sup_{x\in\mathbb{R}^d} E\left[e^{\int_0^T \varepsilon|f|^{2+\delta}(s,x+W_s)\,\mathrm{d}s}\right] < \infty.$$

From Young inequality, there exists a constant $C_{\varepsilon,\delta} > 0$ such that $f^2 \leq \varepsilon\, |f|^{2+\delta} + C_{\varepsilon,\delta}$. Then

$$\sup_{x\in\mathbb{R}^d} E\left[e^{\int_0^T f^2(s,x+W_s)\mathrm{d}s}\right] \leq \sup_{x\in\mathbb{R}^d} E\left[e^{\int_0^T \varepsilon|f|^{2+\delta}(s,x+W_s)\mathrm{d}s}\right] e^{C_{\varepsilon,\delta}} < \infty.$$

By inspection into the previous inequalities, we see that this bound depends only on $\|f\|_{L_p^q(T)}$.

For the last claim, notice that, by Novikov condition, the process $\rho_t = \exp\left(\int_0^t f(s, x + W_s) \cdot \mathrm{d}W_s - \frac{1}{2}\int_0^t |f(s, x + W_s)|^2\,\mathrm{d}s\right)$ is an exponential martingale, in particular with $E[\rho_T] = 1$. Take any $\alpha > 0$ and set $\overline{f} = 2\alpha f$. This is again an element of $L_p^q(T)$. Then we can define the corresponding exponential martingale $\bar{\rho}$ with $\bar{b}$ in place of b, with $E[\bar{\rho}_T] = 1$. Then, for β such that $\sqrt{2\alpha\beta} = 2\alpha$,

$$\begin{aligned}
E\left[\rho_T^\alpha\right] &= E\left[e^{\int_0^T \alpha f(s,x+W_s)\cdot \mathrm{d}W_s - \frac{\alpha\beta}{2}\int_0^T |f(s,x+W_s)|^2\mathrm{d}s} e^{\frac{\alpha(\beta-1)}{2}\int_0^T |f(s,x+W_s)|^2\mathrm{d}s}\right] \\
&\leq E\left[e^{\int_0^T 2\alpha f(s,x+W_s)\cdot \mathrm{d}W_s - \frac{1}{2}\int_0^T \left|\sqrt{2\alpha\beta} f(s,x+W_s)\right|^2\mathrm{d}s}\right]^{1/2} E\left[e^{\alpha(\beta-1)\int_0^T |f(s,x+W_s)|^2\mathrm{d}s}\right]^{1/2}
\end{aligned}$$

which is finite since the first factor is $E[\bar{\rho}_T]^{1/2} = 1$ and the second is finite by the first claim of the corollary applied to $\sqrt{|\alpha(\beta-1)|} f \in L_p^q(T)$. For $\alpha < 0$ the computations are similar. The proof is complete. ■

By a classical application of Girsanov theorem (see [KR05, lemma 3.2] for details) we have:

Proposition 15 *Given $b \in L_p^q(T)$ with p, q satisfying (2) and $x \in \mathbb{R}^d$, there exist processes X_t, W_t defined for $t \in [0, T]$ on a filtered space (Ω, F, F_t, P) such that W_t is a d–dimensional*

$\{F_t\}$–Wiener process and X_t is an $\{F_t\}$–adapted, continuous, d–dimensional process for which

$$P\left(\int_0^T |b(t, X_t)|^2 \ \mathrm{d}t < \infty\right) = 1 \tag{23}$$

and almost surely, for all $t \in [0, T]$

$$X_t = x + \int_0^t b(s, X_s)\, \mathrm{d}s + W_t.$$

When both a solution X of equation (1) and the Brownian motion itself satisfy condition (23), we may apply a result of absolutely continuous change of measures, see Liptser–Shiryaev [LS77, theorems 7.7 and 7.9]. We know that Brownian motion satisfies this condition, when $b \in L_p^q(T)$, by remark 12. We have to impose by assumption the condition (23) on solutions.

Corollary 16 *Take $b \in L_p^q(T)$ for p, q such that (2) holds. Let (X, W) be a (weak) solution of equation (1) in the sense of theorem 15, in particular with X satisfying condition (23). Then, for any non negative Borel function Φ defined on the space $C\left([0, T]; \mathbb{R}^d\right)$ we have*

$$E\left[\Phi(X)\right] = E\left[\Phi(x + W)\ e^{\int_0^T b(s, x+W_s)\cdot \mathrm{d}W_s - 1/2 \int_0^T |b(s, x+W_s)|^2 \mathrm{d}s}\right]. \tag{24}$$

In particular, weak uniqueness holds for the equation (1), in the class of solutions satisfying (23). Moreover, if $f \in L_{\tilde{p}}^{\tilde{q}}(T)$ where $\tilde{p}, \tilde{q}$ are such that $d/\tilde{p} + 2/\tilde{q} < 1$, then, for any $k \in \mathbb{R}$ there exists a constant C_f depending on $\|f\|_{L_{\tilde{p}}^{\tilde{q}}(T)}$ such that

$$E\left[e^{k \int_0^T |f(t, X_t)|^2 \, \mathrm{d}t}\right] \le C_f. \tag{25}$$

The first part of the corollary depends on the above mentioned results of [LS77, theorems 7.7 and 7.9]. To prove the exponential integrability of $|f(t, X_t)|^2$, notice that by (24) we have

$$E\left[e^{k \int_0^T |f(t, X_t)|^2 \mathrm{d}t}\right] = E\left[\ e^{\int_0^T b(s, x+W_s)\cdot \mathrm{d}W_s - 1/2 \int_0^T |b(s, x+W_s)|^2 \mathrm{d}s + k \int_0^T |f(t, x+W_t)|^2 \mathrm{d}t}\right]$$

and thus it is sufficient to repeat the estimates made above to prove that $E\left[\rho_T^\alpha\right]$ was finite.

With the same proof, namely

$$E\left[|X_t|^p\right] = E\left[\ |x + W_t|^p\, e^{\int_0^T b(s, x+W_s)\cdot \mathrm{d}W_s - 1/2 \int_0^T |b(s, x+W_s)|^2 \mathrm{d}s}\right]$$

followed by Hölder inequality as in the proof made above to prove that $E\left[\rho_T^\alpha\right]$ was finite, we also have:

Proposition 17 *Let (X, W) be a (weak) solution of equation (1). Then*

$$\sup_{t \in [0, T]} E\left[|X_t|^p\right] < \infty$$

for every $p \ge 1$.

Acknowledgements

Part of the work was done at the Newton institute for Mathematical Sciences in Cambridge (UK), whose support is gratefully acknowledged, during the program "Stochastic partial differential equations".

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