Matematica – Existence for semilinear parabolic stochastic equations

Nota di Viorel Barbu presentata[1] dal Socio Corrispondente G. Da Prato

Abstract

The boundary value problem for semilinear parabolic stochastic equations of the form $dX - \Delta X\, dt + \beta(X)dt \ni \sqrt{Q}\, dW_t$, where W_t is a Wiener process and β is a maximal monotone graph everywhere defined, is well posed.

Key words: Wiener process, mild solution, random differential equation.

Riassunto. Il problema ai limiti per l'equazione stocastica semilineare di forma $dX - \Delta X\, dt + \beta(X)dt \ni \sqrt{Q}\, dW_t$, dove W_t é un processo Wiener e β é un grafico massimale monotono definito ovunque, é ben posto.

1 Introduction

Consider the stochastic differential equation

$$\begin{aligned}
&dX - \Delta X\, dt + \beta(X)dt \ni \sqrt{Q}\, dW_t && \text{in } (0,T)\times\mathcal{O} = Q_T,\\
(1)\qquad &X(0) = x && \text{in } \mathcal{O},\\
&X = 0 && \text{on } (0,T)\times\partial\mathcal{O} = \Sigma_T.
\end{aligned}$$

Here, $\mathcal{O}$ is an open and bounded subset of R^d with smooth boundary $\partial\mathcal{O}$, $d \geq 1$, and W_t is a cylindrical Wiener process in $L^2(\mathcal{O}) = H$ defined by

$$W_t = \sum_{k=1}^{\infty} e_k(\xi)\beta_k(t), \quad \xi \in \mathcal{O},\ t \geq 0,$$

where $\{\beta_k\}_k$ are mutually independent Brownian motions on a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ and $\{e_k\}$ is an orthonormal basis in H. The operator $Q \in L(H,H)$ is self–adjoint, positive and of finite trace.

Finally, $\beta : R \to 2^R$ is a maximal monotone graph (see [1]) everywhere defined on R.

[1] Nella seduta del...

The main result of this note is that, under suitable assumptions on Q (see (H1) below), equation (1) has a unique strong(mild) solution (Theorem 2). A similar result was proven in [2] for the stochastic porous media equation.

Compared with standard existence theory for equation (1) (see [3], [4]), where the main assumption is that β is continuous, monotonically increasing, here β might be multivalued and, therefore, discontinuous. Also, as seen later on, β might be a time dependent function $\beta = \beta(t, \cdot)$ measurable in $t \in [0, T]$.

Moreover, our existence results apply to multivalued graphs β everywhere defined on R. Such a graph (multivalued) arises naturally when in equation (1) the function β is monotonically increasing and discontinuous in $\{r_j\}_{j=1}^{\infty}$. Then, one redefines β by

$$\widetilde{\beta}(r) = \beta(r) \text{ for } r \neq r_j, \ \widetilde{\beta}(r_j) = [\beta(r_j), \ \beta(r_{j+1} - 0)]$$

and get a maximal monotone graph $\widetilde{\beta}$. So, one might say that the existence result established here in Theorem 2 below applies as well to discontinuous monotonically increasing besides continuous functions β.

We shall denote by $C_W([0, T]; H)$ the space of all adapted processes $X \in C([0, T]; L^2(\Omega, \mathcal{F}, \mathbb{P}, H))$, $H = L^2\mathcal{O})$ and by $L^2_W(0, T; H^1_0(\mathcal{O}))$ the space of all adapted processes $X \in L^2(0, T; L^2(\Omega, \mathcal{F}, \mathbb{P}, H^1_0(\mathcal{O}))$ (see [3]). Here, $H^1_0(\mathcal{O})$ is the standard Sobolev space.

We denote also by W_A the stochastic convolution

$$W_A(t) = \int_0^t e^{-A(t-s)} \sqrt{Q}\, dW_s, \quad t \geq 0,$$

where $A = -\Delta$, $D(A) = H^1_0(\mathcal{O}) \cap H^2(\mathcal{O})$. We recall that $W_A(t)$ is a Gaussian process and $E(|W_A(t)|^2] < \infty$, $\forall t \geq 0$ (see [3], p. 21).

2 The main result

The following hypotheses will be assumed.

(H1) $W_A(\cdot, \cdot)$ *is continuous on* $[0, T] \times \overline{\mathcal{O}}$, $\mathbb{P}$-*a.s.*.

(H2) $\beta : R \to 2^R$ *is a maximal monotone graph such that* $D(\beta) = R$.

Here, $D(\beta) = \{r \in R; \ \beta(r) \neq \emptyset\}$.

In particular, hypotheses (H2) holds if β is a monotonically nondecreasing and continuous function.

As regards hypotheses (H1), we refer to [3], Theorem 2.13, for sufficient conditions on Q under which it holds.

Definition 1 By strong (or mild) solution to equation (1) we mean a process $X \in C([0,T];H)$ which satisfies

$$X(t) = e^{-At}x - \int_0^t e^{-A(t-s)}\eta(s)ds + W_A(t),\ \mathbb{P}\text{-a.s.},\ t \in [0,T], \tag{2}$$

where $\eta \in L^1((0,T) \times \mathcal{O} \times \Omega)$ is a process such that

$$\eta(t,\xi) \in \beta(X(t,\xi)),\quad \text{a.e. } (t,\xi) \in Q_T,\ \mathbb{P}\text{-a.s.} \tag{3}$$

Theorem 2 *Under hypotheses* (H1), (H2), *for each* $x \in H = L^2(\mathcal{O})$ *there is a unique strong solution* X *to equation* (1), *such that*

$$X \in L^2_W([0,T];H^1_0(\mathcal{O})), \tag{4}$$

$$j(X), j^*(\eta) \in L^1((Q,T) \times \mathcal{O} \times \Omega). \tag{5}$$

Here, j is the subpotential associated with β, i.e., $\partial j = \beta$ and j^* is the conjugate of j. (See the notation below.)

3 Proof of Theorem 2

Existence. By using a standard device, we shall reduce equation (1) to the random differential equation

$$\begin{aligned} & y_t - \Delta y + \beta(y + W_A) \ni 0, && (t,\xi) \in Q_T = (0,T) \times \mathcal{O}, \\ & y(0,\xi) = x(\xi), && \xi \in \mathcal{O}, \\ & y = 0 && \text{on } (0,T) \times \partial\mathcal{O} = \Sigma_T, \end{aligned} \tag{6}$$

where $y = X - W_A$.

We fix $\omega \in \Omega$ and approximate (6) by

$$\begin{aligned} & (y_\varepsilon)_t - \Delta y_\varepsilon + \beta_\varepsilon(y_\varepsilon + W_A) \ni 0, && (t,\xi) \in Q_T, \\ & y_\varepsilon(0,\xi) = x(\xi), && \text{in } \mathcal{O}, \\ & y = 0 && \text{on } \Sigma_T, \end{aligned} \tag{7}$$

where $\beta_\varepsilon = \frac{1}{\varepsilon}\left(1-(1+\varepsilon\beta)^{-1}\right)$ is the Yosida approximation of β (see, e.g., [1]). Since β_ε is Lipschitzian, equation (7) has a unique solution

$$\begin{aligned} y_\varepsilon &\in C([0,T];L^2(\mathcal{O}))\cap L^2(0,T;H_0^1(\mathcal{O})) \\ \sqrt{t}(y_\varepsilon)_t &\in L^2(0,T;L^2(\mathcal{O})),\ \sqrt{t}\,y_\varepsilon \in L^2(0,T);H^2(\mathcal{O})). \end{aligned}$$

Denote by $j : R \to R$ the subpotential function corresponding to β, that is $\partial j = \beta$, where ∂j is subdifferential of β (see, e.g., [1], p. 53). Let j^* be the conjugate of j, that is,

$$j^*(p) = \sup\{p\cdot r - j(r);\ r \in R\}$$

and recall that $p \in \partial\beta(r)$ if and only if

$$j(r) + j^*(p) = rp. \tag{8}$$

We have also $\beta_\varepsilon = \nabla j_\varepsilon$, where

$$\begin{aligned} j_\varepsilon(r) &= \inf\left\{\frac{|r-s|^2}{2\varepsilon} + j(s);\ s\in R\right\} \\ &= \frac{1}{2\varepsilon}\,|(1+\varepsilon\beta)^{-1}r - r|^2 + j((1+\varepsilon\beta)^{-1}r),\ \forall r\in R. \end{aligned} \tag{9}$$

Multiplying (7) by y_ε and integrating on $(0,T)\times\mathcal{O}$, we obtain that

$$\begin{aligned} \frac{1}{2}\,\|y_\varepsilon(t)\|^2_{L^2(\mathcal{O})} &+ \int_0^t \|y_\varepsilon(s)\|^2_{H_0^1(\mathcal{O})}ds + \int_0^t\int_{\mathcal{O}} j_\varepsilon(y_\varepsilon + W_A)ds\,d\xi \\ &\le \frac{1}{2}\,\|x\|^2_{L^2(\mathcal{O})} + \int_0^t\int_{\mathcal{O}} j_\varepsilon(W_A)ds\,d\xi \le C, \\ &\qquad \forall t\in[0,T]. \end{aligned} \tag{10}$$

Hence, on a subsequence $\varepsilon \to 0$, we have

(11) $y_\varepsilon \to y^*$ weakly in $L^2(0,T;H_0^1(\mathcal{O}))$ and weak-star in $L^\infty(0,T;L^2(\mathcal{O}))$.

Also, by (9)∼(10), we see that, for $\varepsilon \to 0$,

(12) $(1+\varepsilon\beta)^{-1}(y_\varepsilon + W_A) \to y^* + W_A$ weak-star in $L^\infty(0,T;L^2(\mathcal{O}))$.

By (8), we have

$$j^*(\beta_\varepsilon(y_\varepsilon+W_A))+j((1+\varepsilon\beta)^{-1}(y_\varepsilon+W_A))$$
$$=(\beta_\varepsilon(y_\varepsilon+W_A))(1+\varepsilon\beta)^{-1}(y_\varepsilon+W_A)\le\beta_\varepsilon(y_\varepsilon+W_A)(y_\varepsilon+W_A).$$

This yields

$$\begin{aligned}&\int_{Q_T} j^*(\beta_\varepsilon(y_\varepsilon+W_A))d\xi\,dt\le\int_{Q_T}\beta_\varepsilon(y_\varepsilon+W_A)y_\varepsilon d\xi\,dt\\ &-\int_{Q_T}\beta_\varepsilon(y_\varepsilon+W_A)W_A d\xi\,dt=-\frac{1}{2}\|y_\varepsilon(T)\|^2_{L^2(\mathcal{O})}+\frac{1}{2}\|x\|^2_{L^2(\mathcal{O})}\\ &-\|y_\varepsilon\|^2_{L^2(0,T;H^1_0(\mathcal{O}))}-\int\int_{Q_T}\beta_\varepsilon(y_\varepsilon+W_A)W_A d\xi\,dt.\end{aligned}\tag{13}$$

Since $D(\beta)=R$, we have that

$$\lim_{|r|\to\infty}\frac{j^*(r)}{|r|}=+\infty.\tag{14}$$

Then, by (14) we obtain that for each n there is $C_n>0$ such that

$$\begin{aligned}&j^*(\beta_\varepsilon(y_\varepsilon+W_A))\ge n|\beta_\varepsilon(y_\varepsilon+W_A)|\\ &\qquad\text{a.e. on }\{(\xi,t);\ |\beta_\varepsilon(y_\varepsilon+W_A)(\xi,t)|\ge C_n\}.\end{aligned}\tag{15}$$

We shall use this to prove that $\{\beta_\varepsilon(y_\varepsilon+W_A)\}_{\varepsilon>0}$ is weakly compact in $L^1(Q_T)$. To this purpose, it suffices to show that

$$\int_{Q_T}|\beta_\varepsilon(t_\varepsilon+W_A)|d\xi\,dt\le C,\quad\forall\varepsilon>0,\tag{16}$$

and that, for each $\delta>0$, there is $C(\delta)$ such that for any measurable subset $Q^*\subset Q_T$ with the Lebesgue measure $m(Q^*)\le C_\delta$, we have

$$\int_{Q^*}|\beta_\varepsilon(y_\varepsilon+W_A)|d\xi\,dt\le\delta,\quad\forall\varepsilon>0,\tag{17}$$

(C_δ independent of ε).

Estimate (16) follows by (13) and (15). As regards (17), we start from the inequality

$$\int_{Q^*} |\beta_\varepsilon(y_\varepsilon + W_A)| d\xi\, dt \le \int_{Q^* \cap [|\beta_\varepsilon(y_\varepsilon+W_A)|\ge n]} |\beta_\varepsilon(y_\varepsilon + W_A)| d\xi\, dt$$

$$+nm(Q^*) \le \frac{1}{n} \int_{Q^*} j_\varepsilon^*(\beta_\varepsilon(y_\varepsilon + W_A)) d\xi\, dt + nm(Q^*)$$

$$\le \frac{1}{n} \|W_A\|_{L^\infty(Q_T)} \|\beta_\varepsilon(y_\varepsilon + W_A)\|_{L^1(Q_T)} \le \frac{C}{n} + nm(Q^*).$$

(Here, we have used (13), (15), (16) and (H1).)

Hence, for $n \ge \dfrac{\delta}{2C}$ and $m(Q^*) \le \dfrac{\delta}{2n}$, we obtain (17), as claimed.

Then, by the Pettis theorem, $\{\beta_\varepsilon(y_\varepsilon + W_A)\}_{\varepsilon>0}$ is weakly compact in $L^1(Q_T)$ and so, on a subsequence, again denoted ε, we have

$$\beta_\varepsilon(y_\varepsilon + W_A) \to \eta \text{ weakly in } L^1(Q_T). \tag{18}$$

Inasmuch as $\{\beta_\varepsilon(y_\varepsilon + W_A)\}$ is bounded in $L^1(Q_T)$, it follows by (7) that $\{y_\varepsilon\}$ is compact in $C([0,T]; L^1(\mathcal{O}))$ and, therefore, for $\varepsilon \to 0$,

$$y_\varepsilon \to y^* \quad \text{strongly in } C([0,T]; L^1(\mathcal{O})) \tag{19}$$

and

$$\begin{aligned} & y_t^* - \Delta y^* + \eta = 0 && \text{in } Q_T, \\ & y^*(0) = x, \quad y^*(t) \in H_0^1(\mathcal{O}), && \text{a.e. } t \in [0,T]. \end{aligned} \tag{20}$$

In order to conclude the proof of existence for equation (6), it remains to be proven that

$$\eta(t,\xi) \in \beta(y^*(t,\xi)) + W_A(t,\xi)), \text{ a.e. } (t,\xi) \in Q_T. \tag{21}$$

To this end, we start from the inequality

$$\begin{aligned} & \int_{Q_0} \beta_\varepsilon(y_\varepsilon + W_A)(y_\varepsilon + W_A - z) d\xi\, dt \\ & \quad \ge \int_{Q_0} j_\varepsilon(y_\varepsilon + W_A) d\xi\, dt - \int_{Q_0} j_\varepsilon(z) d\xi\, dt, \ \forall z \in L^\infty(Q_0), \end{aligned} \tag{22}$$

for any measurable subset $Q_0 \subset Q_T$.

On the other hand, by (19), by Egorov Theorem, it follows that for each $\delta > 0$ there is $Q_\delta \subset Q_T$ such that $m(Q_T \setminus Q_\delta) \leq \delta$ and $y_\varepsilon \to y^*$ uniformly on Q_δ as $\varepsilon \to 0$. Taking $Q_0 = Q_T$ in (22), we obtain

$$\int_{Q_\delta} \eta(y^* + W_A - z) d\xi\, dt \geq \int_{Q_\delta} (j(y^* + W_A) - j(z)) d\xi\, dt, \quad \forall z \in L^\infty(Q_\delta).$$

The latter implies by a standard device the pointwise inequality

$$\eta(y^* + W_A - z) \geq j(y^* + W_A) - j(z), \text{ a.e. in } Q_\delta, \ \forall z \in R,$$

and, therefore, $\eta \in \partial j(y^* + W_A) = \beta(y^* + W_A)$, a.e. in Q_δ, and since δ is arbitrary, we obtain (21), as claimed.

Now, it is clearly seen that $X(t) = y(t) + W_A$ is a solution to (1) in the sense precised in Definition 1. (The fact that the process $X(t) = \lim\limits_{\varepsilon \to 0} y_\varepsilon(t) + W_A(t)$ is adapted is obvious because so is $X_\varepsilon(t) = y_\varepsilon(t) + W_A(t)$.)

By (10) and (13), it is also easily seen that $j(X), j^*(\eta \in L^1((0,T) \times \mathcal{O} \times \Omega)$. This completes the proof of the existence.

Uniqueness. It is immediate, because if X_i, $i = 1, 2$, are solutions to (1) in the above sense, then $y_i = X_i - W_A$, $i = 1, 2$, are $\mathbb{P}$-a.s. solutions to equation (6), which clearly has a unique solution by monotonicity of β.

Remark 3 Theorem 2 remains true for time dependent maximal monotone graphs $\beta = \beta(t, \cdot)$ which satisfy the following assumptions.

(H2)$'$ *For almost all* $t \in (0, T)$, $\beta(t, \cdot) : R \to 2^R$ *is maximal monotone, measurable in* t *and for each* $M > 0$ *there is* C_M *independent of* t *such that*

(23) $$|\beta(t, r)| \leq C_M \quad a.e.\ t \in (0, T), \ \forall r \in [-M, M].$$

If β is independent of t, (H2)$'$ is implied by (H2). The proof is exactly the same as that of Theorem 2.

Acknowledgements. This paper was written while the author was visiting the Isaac Newton Institute for Mathematical Sciences in Cambridge (UK).

References

[1] V. Barbu, *Nonlinear Differential Equations of Monotone Type in Banach Spaces*, Springer, Berlin. New York, 2010.

[2] V. Barbu, G. Da Prato, M. Roeckner, Existence of strong solutions for stochastic porous media equation under general monotonicity conditions, *The Annals of Probability*, vol. 37, no. 2 (2009), 428–452.

[3] G. Da Prato, *Kolmogorov Equations for Stochastic PDEs*, Birkhäuser Berlag, Basel, 2004.

[4] G. Da Prato, J. Zabzyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, UK, 1992.

V. Barbu

Al.I. Cuza University and
Octav Mayer Institute of Mathematics
Iaşi (Romania)