Internal stabilization by noise of the Navier–Stokes equation

Viorel Barbu*
University Al. I. Cuza, 700506, Iasi, Romania

Giuseppe Da Prato
Scuola Normale Superiore, 56126, Pisa, Italy

Abstract

One shows that the Navier-Stokes equation in $\mathscr{O}\subset R^d$, $d = 2,3$, around an unstable equilibrium solution is exponentially stabilizable in probability by an internal noise controller $V(t,\xi) = \sum_{i=1}^N V_i(t)\psi_i(\xi)\dot\beta_i(t)$, $\xi \in \mathscr{O}$, where $\{\beta_i\}_{i=1}^N$ are independent Brownian motions and $\{\psi_i\}_{i=1}^N$ is a system of functions on $\mathscr{O}$ with support in an arbitrary open subset $\mathscr{O}_0 \subset \mathscr{O}$. The stochastic control input $\{V_i\}_{i=1}^N$ is found in feedback form. The corresponding result for the linearized Navier-Stokes equation was established in [2].

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1 Introduction

Consider the Navier-Stokes equation

$$\begin{cases} X_t - \nu_0\Delta X + (X\cdot\nabla)X = f_e + \nabla p, & \text{in } (0,\infty)\times\mathscr{O} \\ \nabla\cdot X = 0, & \text{in } (0,\infty)\times\mathscr{O} \\ X = 0, & \text{on } (0,\infty)\times\partial\mathscr{O} \\ X(0) = x_0, & \text{in } \mathscr{O}. \end{cases} \tag{1.1}$$

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where $\mathscr{O}$ is an open and bounded subset of $\mathbb{R}^d$, $d=2,3$, with smooth boundary $\partial\mathscr{O}$. Here $f_e \in (L^2(\mathscr{O}))^d$ is given.Let X_e be an equilibrium solution to (1.1), i.e.,

$$\begin{cases} -\nu_0 \Delta X_e + (X_e \cdot \nabla) X_e = f_e + \nabla p_e, \quad \text{in } \ \mathscr{O} \\ \\ \nabla \cdot X_e = 0 \quad \text{in } \mathscr{O}, \\ \\ X_e = 0 \quad \text{on } (0,\infty) \times \partial\mathscr{O}. \end{cases} \tag{1.2}$$

If we replace X by $X - X_e$ equation (1.1) reduces to

$$\begin{cases} X_t - \nu_0 \Delta X + (X \cdot \nabla) X_e + (X_e \cdot \nabla) X \\ \\ \qquad\qquad\qquad +(X \cdot \nabla) X = \nabla p \quad \text{in } (0,\infty) \times \mathscr{O}, \\ \\ \nabla \cdot X = 0 \ \text{ in } \mathscr{O}, \\ \\ X = 0 \quad \text{on } (0,\infty) \times \partial\mathscr{O}, \\ \\ X(0) = x \quad \text{in } \mathscr{O}, \end{cases} \tag{1.3}$$

where $x = x_0 - X_e$. If we set

$$H = \left\{ X \in (L^2(\mathscr{O}))^d : \ \nabla \cdot X = 0, \, X \cdot \nu\big|_{\partial\mathscr{O}} = 0 \right\},$$

where ν is the normal to $\partial\mathscr{O}$ and denote by $P : (L^2(\mathscr{O}))^d \to H$ the Leray projector on H, we can rewrite system (1.3) as

$$\begin{cases} \dot{X}(t) + \mathscr{A} X(t) + B(X(t)) = 0, \quad t \geq 0, \\ \\ X(0) = x, \end{cases} \tag{1.4}$$

where

$$A = -P\Delta, \quad D(A) = (H_0^1(\mathscr{O}) \cap H^2(\mathscr{O}))^d \cap H,$$

$$A_0 x = P((x \cdot \nabla) X_e + (X_e \cdot \nabla) x), \quad B(x) = P((x \cdot \nabla) x),$$

$$\mathscr{A} = \nu_0 A + A_0, \quad D(\mathscr{A}) = D(A)$$

We have

$$\langle B(x), y \rangle_H = b(x,x,y), \quad \forall \, x, y \in D(A),$$

where $\langle \cdot, \cdot \rangle_H$ is the scalar product induced by H as pivot space and

$$b(x, z, y) = \sum_{j,k=1}^{d} \int_{\mathscr{O}} x_j D_j z_k y_k, \quad \forall\, x, y, z \in D(A).$$

We recall that for large values of the Reynolds number $\frac{1}{\nu_0}$ the stationary solution X_e to (1.1) is unstable. i.e. the corresponding flow is turbulent. Our purpose here is to stabilize (1.4) or, equivalently, the stationary solution X_e to (1.1), using a stochastic controller with support in an arbitrary open subset $\mathscr{O}_0 \subset \mathscr{O}$. To this aim we associate with (1.4) the controlled stochastic system

$$\begin{cases} dX(t) + (\mathscr{A} X(t) + B(X(t))dt = \displaystyle\sum_{j=1}^{N} V_j(t)\psi_j d\beta_j(t), \\ \\ X(0) = x, \end{cases} \tag{1.5}$$

where $\{\beta_j\}_{j=1}^N$ is an independent system of real Brownian motions in a filtered probability space $(\Omega, \mathbb{P}, \mathscr{F}, \{\mathscr{F}_t\}_{t>0})$.

The main result, Theorems 2.2 below, amounts to saying that, in the complexified space $\widetilde{H}$ associated with H, under appropriate assumptions on $\mathscr{A}$ (and, implicitly, on X_e), for each $\gamma > 0$ there exist $N \in \mathbb{N}$, $\{\psi_j\}_{j=1}^N \subset \widetilde{H}$, and an N-dimensional adapted process $\{V_j = V_j(t, \omega)\}_{j=1}^N$, $\omega \in \Omega$, such that for all x in a sufficiently small neighbourhood of the origin, $t \to e^{\frac{\gamma t}{4}} X(t, \omega)$ is decaying to zero for $t \to \infty$ in a set Ω_x^* of positive probability which is precisely estimated. Moreover, it turns out that the stabilizable controller arising in the right hand side of (1.5) is a linear feedback controller of the form

$$V_j(t) = \eta \langle X(t), \varphi_j^* \rangle_{\widetilde{H}}, \quad \psi_j = P(m\phi_j), \quad j = 1, ..., N, \tag{1.6}$$

where $|\eta| > 0$ and φ_j^* are the eigenfunctions of the dual Stokes-Oseen operator $\mathscr{A}^*$ corresponding to eigenvalues $\overline{\lambda}_j$ with $\operatorname{Re}\lambda_j \le \gamma$, $\{\phi_j\}_{j=1}^N$ is a system of functions related to φ_j^* and $m = 1\mathrm{l}_{\mathscr{O}_0}$ is the characteristic function of $\mathscr{O}_0$ where $\mathscr{O}_0$ is a given arbitrary open subset of $\mathscr{O}$.

We may view (1.5) as the deterministic system (1.4) perturbed by the white noise controller $\sum_{j=1}^N V_j(t)\psi_j\dot{\beta}_j$ with the support in $\mathscr{O}_0$.

This work is a continuation of [2] where such a result is proved for the linearized Navier-Stokes equation associated with (1.3). The previous treatment of internal stabilization of Navier-Stokes equations ([1],[3]) is based on the stabilization by a linear feedback provided by the solution of an algebraic infinite dimensional Riccati equation associated with the Stokes-Oseen operator $\mathscr{A}$. (This approach was also used in [4],[5], [14],[15],[19],[20] for boundary stabilization of Navier-Stokes equations.)

The main advantage of this stochastic based stabilization technique with respect to the Riccati-feedback based approach in above mentioned works, is that it avoids the difficult computation problems related to infinite dimensional Riccati equations. Also a nice features of this feedback control which has a stabilizing influence with high probability if applied in a small neighbourhood of a stationary solution is that besides its simplicity it is robust in the class of finite dimensional Gaussian multiplicative perturbations.

It should be said also that stabilization by noise of the dynamic PDEs was already used in the literature and we refer to [6], [7], [8] [9], [10],[11] ,[13] for related results. However, there is not overlap with existing literature and methods used here are different and may be viewed as a combination of spectral stabilization techniques ([1], [3]) with that of noise stabilization. In particular in [9] is studied the stabilization of some classes of PDE using Stratonovich noise which has a special interest in construction of an approximating stabilizing controller.

1.1 Notations

Throughout in the following β_j, $j = 1, ..., N$ are independent real Brownian motions in a filtered probability space $(\Omega, \mathbb{P}, \mathscr{F}, \{\mathscr{F}_t\}_{t>0})$ and we shall refer to [11, 13] for definition and basic results on stochastic analysis of differential systems and spaces of stochastic processes adapted to filtration $\{\mathscr{F}_t\}_{t>0}$. The scalar product of H is denoted $\langle \cdot, \cdot \rangle_H$ and the norm $|\cdot|_H$. We shall denote by $\widetilde{H}$ the complexified space $H + iH$ with scalar product denoted by $\langle \cdot, \cdot \rangle_{\widetilde{H}}$ and norm by $|\cdot|_{\widetilde{H}}$. $C_W([0,T]; L^2(\Omega, \widetilde{H}))$ is the space of all adapted square-mean $\widetilde{H}$-valued continuous processes on $[0,T]$.

2 The main result

To begin with, let us briefly recall a few elementary spectral properties of the Stokes-Oseen operator $\mathscr{A}$. Denote again by $\mathscr{A}$ the extension of $\mathscr{A}$ to the complex space $\widetilde{H}$. The operator $\mathscr{A}$ has a compact resolvent $(\lambda I - \mathscr{A})^{-1}$ and $-\mathscr{A}$ generates a C_0-analytic semigroup $e^{-\mathscr{A}t}$ in $\widetilde{H}$. Consequently, $\mathscr{A}$ has a countable number of eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ with corresponding eigenfunctions φ_j each with finite algebraic multiplicity m_j. Of course, certain eigenfunctions φ_j might be generalized and so, in general, $\mathscr{A}$ is not diagonalizable, i.e., the algebraic multiplicity of λ_j might not coincide with its geometric multiplicity. Also, each eigenvalue λ_j will be repeated according to its algebraic multiplicity m_j.

We shall denote by N the number of eigenvalues λ_j with $\operatorname{Re}\lambda_j \leq \gamma$, $j = 1, ..., N$, where γ is a fixed positive number.

Denote by P_N the projector on the finite dimensional subspace

$$\mathscr{X}_u = \operatorname{lin\,span}\{\varphi_j\}_{j=1}^N.$$

We have $\mathscr{X}_u = P_N \widetilde{H}$ and

$$P_N = -\frac{1}{2\pi i}\int_\Gamma (\lambda I - \mathscr{A})^{-1} d\lambda, \tag{2.1}$$

where Γ is a closed smooth curve in $\mathbb{C}$ which is the boundary of a domain containing in interior the eigenvalues $\{\lambda_j\}_{j=1}^N$.

Let $\mathscr{A}_u = P_N\mathscr{A}$, $\mathscr{A}_s = (I - P_N)\mathscr{A}$. Then $\mathscr{A}_u$, $\mathscr{A}_s$ leave invariant the spaces $\mathscr{X}_u$ and $\mathscr{X}_s = (I - P_N)\widetilde{H}$ and the spectra $\sigma(\mathscr{A}_u)$, $\sigma(\mathscr{A}_s)$ are given by (see [9])

$$\sigma(\mathscr{A}_u) = \{\lambda_j\}_{j=1}^N, \quad \sigma(\mathscr{A}_s) = \{\lambda_j\}_{j=N+1}^\infty.$$

Since $\sigma(\mathscr{A}_s) \subset \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda > \gamma\}$ and $\mathscr{A}_s$ generates an analytic C_0-semigroup on $\widetilde{H}$, we have

$$|e^{-\mathscr{A}_s t}x|_{\widetilde{H}} \leq Ce^{-\gamma t}|x|_{\widetilde{H}}, \quad \forall x \in \widetilde{H},\ t \geq 0. \tag{2.2}$$

The eigenvalue λ_j is said to be *semi-simple* if its algebraic and geometrical multiplicity coincides, or, equivalently, λ_j is a simple pole for $(\lambda I - \mathscr{A})^{-1}$. If all eigenvalues $\{\lambda_j\}_{j=1}^N$ of the matrix $\mathscr{A}_u$ are semi-simple, then $\mathscr{A}_u$ *is diagonalizable.*

Herein, we shall assume that the following hypothesis holds.

(H_1) *All eigenvalues* λ_j, $j = 1, ..., N$, *are semi-simple.*

At regards Hypothesis (H_1) it should be said that it follows by a standard argument involving the Sard-Smale theorem that the property of eigenvalues of the Stokes-Oseen operator to be simple (and, consequently, semi-simple) is generic in the class of coefficients X_e. So, one might say that "almost everywhere" (in the sense of a set of first category), hypothesis (H_1) holds.

Denote by $\mathscr{A}^*$ the adjoint operator and by P_N^* the adjoint of P_N. We have

$$P_N^* = -\frac{1}{2\pi i}\int_{\overline{\Gamma}} (\lambda I - \mathscr{A}^*)^{-1} d\lambda. \tag{2.3}$$

The eigenvalues of $\mathscr{A}^*$ are precisely the complex conjugates $\overline{\lambda}_j$ of eigenvalues λ_j of $\mathscr{A}$ and they have the same multiplicity. Denote by φ_j^* the eigenfunction of $\mathscr{A}^*$ corresponding to the eigenvalue $\overline{\lambda_j}$. We have, therefore,

$$\mathscr{A}\varphi_j = \lambda_j\varphi_j, \quad \mathscr{A}^*\varphi_j^* = \overline{\lambda}_j\varphi_j^*, \quad j \in \mathbb{N}. \tag{2.4}$$

Since the eigenvalues $\{\lambda_j\}_{j=1}^N$ *are semi-simple, it turns out that the system consisting of* $\{\varphi_j\}_{j=1}^N$, $\{\varphi_j^*\}_{j=1}^N$ *can be chosen to form a bi-orthonormal sequence in* $\widetilde{H}$, i.e,

$$\langle \varphi_j, \varphi_k^* \rangle_{\widetilde{H}} = \delta_{jk}, \quad j,k = 1,...,N, \tag{2.5}$$

where δ_{jk} is the Kronecker symbol (see, e.g., [3]). We notice also that the functions φ_j and φ_j^* have the unique continuation property, i.e.,

$$\varphi_j \not\equiv 0, \quad \varphi_j^* \not\equiv 0 \ \text{ on } \mathscr{O}_0 \text{ for all } j = 1,...,N, \tag{2.6}$$

(see, e.g., Lemma 3.7 in [3]).

We have also the following property which will be proven in Appendix.

Lemma 2.1 *The system* $\{\varphi_1^*, ..., \varphi_N^*\}$ *is linearly independent in* $(L^2(\mathscr{O}_0))^d$.

If the eigenvalues λ_j are the same then Lemma 2.1 follows by the unique continuation property (2.6).

Consider the following stochastic perturbation of the system (1.4) considered in the complex space

$$\begin{cases} dX + (\mathscr{A}X + B(X))dt = \eta \displaystyle\sum_{j=1}^N \langle X, \varphi_j^* \rangle_{\widetilde{H}} P(m\phi_j) d\beta_j, \\ \\ X(0) = x, \end{cases} \tag{2.7}$$

where $|\eta| > 0$ and $m = 1\!\!1_{\mathscr{O}_0}$ is the characteristic function of the open subset $\mathscr{O}_0 \subset \mathscr{O}$. Here $\{\phi_j\}_{j=1}^N \subset \widetilde{H}$ is a system of functions to be precised in (2.9). This is a closed loop system with a stochastic linear feedback controller associated with (1.4)

In two dimensions the stochastic differential equation (2.7) has a global solution $X \in C_W([0,T]; L^2(\Omega, \widetilde{H}))$ for all $T > 0$ (see e.g. [13])

The closed loop system (2.7) can be equivalently written as

$$\begin{cases} dX(t) - \nu_0 \Delta X(t)dt + (X(t)\cdot\nabla)X_e dt + (X_e\cdot\nabla)X(t)dt + (X(t)\cdot\nabla)X(t)dt \\ = \eta m \displaystyle\sum_{j=1}^{N} \langle X(t), \varphi_j^*\rangle_{\widetilde{H}} \phi_j d\beta_j(t) + \nabla p(t)dt \text{ in } (0,\infty)\times\mathscr{O},\ \mathbb{P}\text{-a.s.} \\ \nabla\cdot X(t) = 0 \text{ in } \mathscr{O},\quad X(t)\Big|_{\partial\mathscr{O}} = 0,\ \forall t\ge 0,\ \mathbb{P}\text{-a.s.} \\ X(0) = x \text{ in } \mathscr{O}. \end{cases} \tag{2.8}$$

Hence, in the space $(L^2(\mathscr{O}))^d$, the feedback controller $\{u_j = \eta m\langle X, \varphi_j^*\rangle_{\widetilde{H}}\phi_j\}_{j=1}^N$ has the support in $\mathscr{O}_0$.

We shall define now $\phi_j,\ j = 1, ..., N$, as follows.

$$\phi_j(\xi) = \sum_{l=1}^{N} \alpha_{lj}\varphi_l^*(\xi),\quad \xi\in\mathscr{O}, \tag{2.9}$$

where α_{lj} are chosen in such a way that

$$\sum_{l=1}^{N} \alpha_{lj}\langle\varphi_l^*, \varphi_k^*\rangle_0 = \delta_{jk},\quad j,k = 1, ..., N.$$

(Since, in virtue of Lemma 2.1 the Gram matrix $\{\langle\varphi_l^*, \varphi_k^*\rangle_0\}_{l,k=1}^N$ is not singular, this is possible.) With this choice, we have

$$\langle\phi_j, \varphi_k^*\rangle_0 = \delta_{kj},\quad k, j = 1, ..., N. \tag{2.10}$$

Here, we have used the notation $\langle u, v\rangle_0 = \int_{\mathscr{O}_0} u(\xi)\bar{v}(\xi)d\xi$.

In the following we shall denote by A^α, $\alpha\in(0,1)$, the fractional power of order α of A, by $D(A^\alpha)$ its domain and set $|x|_\alpha = |A^\alpha x|$ for all $x\in D(A^\alpha)$. Moreover, we shall denote by W the space $D(A^{\frac{1}{4}})$ if $d = 2$ and $D(A^{\frac{1}{4}+\epsilon})$ if $d = 3$ where $\epsilon > 0$ is small.

Theorem 2.2 below is the main result of the paper.

Theorem 2.2 *Let $d = 2, 3$, $X_e \in C^2(\overline{\mathscr{O}})$ and*

$$|\eta| \ge \max_{1\le j\le N} \sqrt{6\gamma - 2\text{Re}\,\lambda_j}. \tag{2.11}$$

Then there is $C^* > 0$, independent of ω such that for each $x \in W$, $|x|_W \leq (C^*)^2$ there is $\Omega_x^* \subset \Omega$ with

$$\mathbb{P}(\Omega_x^*) \geq 1 - 2\left(C^*|x|_W^{-\frac{1}{2}} - 1\right)^{-\frac{\gamma}{2(\eta N)^2}}, \tag{2.12}$$

the solution $X(t,x)$ of (2.7) *satisfies*

$$\lim_{t\to\infty}\left(e^{\frac{\gamma t}{4}}|X(t,x)|_{\widetilde{H}}\right) = 0, \quad \mathbb{P}\text{-a.s. in } \Omega_x^*. \tag{2.13}$$

In particular, Theorem 2.2 implies that if $|x|_W \leq \rho_0 < (C^*)^{-2}$ then $X = X(t,x)$ is exponentially decaying to 0 on a set Ω_x^* of probability greater than

$$1 - 2\left(C^*|x|_W^{-\frac{1}{2}} - 1\right)^{-\frac{\gamma}{2(\eta N)^2}}.$$

The constant C^* depends of X_e only. The optimal η for which $\mathbb{P}(\Omega_x^*)$ is maximal is of course that which follows by (2.11), i.e.,

$$|\eta| = \max_{1\leq j\leq N}\sqrt{6\gamma - 2\text{Re }\lambda_j},$$

and we see that $\mathbb{P}(\Omega_x^*) \to 1$ as $|x|_W \leq \rho_0 \to 0$.

For the linearized Navier–Stokes equation, that is if one takes $B = 0$, the exponential decay in (2.7) occurs with probability one. In fact, as seen from the proof of Theorem 2.2 the constant C^* comes out from estimates on the nonlinear inertial term B and so, it is zero if this term is absent from the equation.

Remark 2.3 As mentioned earlier, system (2.8) is written here in the complex space $\widetilde{H}$. If set $X_1(t) = \text{Re}\, X(t)$, $X_2(t) = \text{Im}\, X(t)$, it can be rewritten as a real system in (X_1, X_2). In this case, the feedback controller is an implicit stabilizable feedback controller with support in $\mathscr{O}_0$ for the real Navier–Stokes equation (1.3). Of course, if λ_j, $j = 1, ..., N$, are real, then we may view $X(t)$ as a real valued function and so, in (2.12), $|X|_{\widetilde{H}} = |X|_H$.

In particular, by Theorem 2.2 we have

Corollary 2.4 *Under the assumptions of Theorem 2.2 the feedback controller*

$$\eta m \sum_{j=1}^{N} \langle X - X_e, \varphi_j^* \rangle_{\widetilde{H}} \phi_j \tag{2.14}$$

stabilizes exponentially the stationary solution X_e, $\mathbb{P}$-a.e in Ω_x^*.

3 Proof of Theorem 2.2

The idea of the proof is to transform equation (2.7) in a deterministic equation with random coefficients via substitution

$$y(t)=\prod_{j=1}^{N} e^{-\beta_j(t)\Gamma_j}\,X(t),\quad t\geq 0, \tag{3.1}$$

where $\Gamma_j:\widetilde{H}\to\widetilde{H}$ is the linear operator

$$\Gamma_j x:=\eta\langle x,\varphi_j^*\rangle_{\widetilde{H}}P(m\phi_j),\quad x\in\widetilde{H},\ j=1,...,N \tag{3.2}$$

and $e^{s\Gamma_j}\in L(\widetilde{H},\widetilde{H})$ is the C_0-group generated by Γ_j i.e.,

$$\frac{d}{ds}\,e^{s\Gamma_j}x-\Gamma_j e^{s\Gamma_j}x=0,\quad \forall\, s\in\mathbb{R},\ x\in\widetilde{H}. \tag{3.3}$$

We have by (3.2) and by (2.9) that

$$\Gamma_j\Gamma_k x=\eta^2\langle x,\varphi_j^*\rangle_{\widetilde{H}}P(m\phi_j)\delta_{jk},\quad \forall\, j,k=1,...,N \tag{3.4}$$

and therefore the operators $\Gamma_1,...,.\Gamma_N$ commute, because the Leray operator P is self-adjoint.

Then by [12, Theorem 7.22] we have that equation (2.7) reduces to

$$\begin{cases}\dfrac{dy(t)}{dt} & + \quad \mathscr{A}y(t)+\dfrac{1}{2}\displaystyle\sum_{j=1}^{N}\Gamma_j^2 y(t)+F(t)y(t)\\[2ex] & \quad +e^{-\sum_{j=1}^{N}\beta_j(t)\Gamma_j}B\left(e^{\sum_{j=1}^{N}\beta_j(t)\Gamma_j}y(t)\right)=0,\quad \forall\, t\geq 0,\ \mathbb{P}\text{-a.s.}\\[2ex] y(0) & = \quad x,\end{cases} \tag{3.5}$$

where

$$F(t)y(t)=e^{-\sum_{j=1}^{N}\beta_j(t)\Gamma_j}\mathscr{A}\left(e^{\sum_{j=1}^{N}\beta_j(t)\Gamma_j}y(t)\right)-\mathscr{A}y(t).$$

By a solution of (3.5) we mean a function $y\in C([0,\infty);D(A^{\frac{1}{4}}))\cap L^2(0,\infty;D(A))$ which fulfills (3.5) $\mathbb{P}$-a. s. in the mild sense (see Lemma 3.3 below).

Conversely, if y is a solution to (3.5) then it is an adapted process and so

$$X(t)=\prod_{j=1}^{N} e^{\beta_j(t)\Gamma_j}\,y(t),\quad t\geq 0, \tag{3.6}$$

belongs to $C_W([0,T];L^2(\Omega,\mathbb{P};D(A^{\frac{1}{4}}))\cap L^2(\Omega,\mathbb{P},C[0,T];D(A^{\frac{3}{4}}))$ and satisfies equation (2.7).

Then we shall confine in the following to study existence and exponential convergence in probability to solutions y to equation to (3.5).

We notice first that, as easily follows by (3.2) and (3.4), we have

$$\begin{aligned} e^{s\Gamma_j}y &= \eta^{-1}\Gamma_j y(e^{\eta s}-1)+y \\ &= (e^{\eta s}-1)\langle y,\varphi_j^*\rangle_{\widetilde{H}}P(m\phi_j)+y, \quad \forall\, s>0,\ j=1,...,N,\ y\in H. \end{aligned} \tag{3.7}$$

respectively

$$\begin{aligned} e^{-s\Gamma_j}y &= \eta^{-1}\Gamma_j y(e^{-\eta s}-1)+y \\ &= (e^{-\eta s}-1)\langle y,\varphi_j^*\rangle_{\widetilde{H}}P(m\phi_j)+y, \quad \forall\, s>0,\ j=1,...,N,\ y\in H. \end{aligned}$$

This yields

$$F(t)y=\sum_{j=1}^N\left(e^{\beta_j(t)}-1\right)\langle y,\varphi_j^*\rangle_{\widetilde{H}}(\mathscr{A}P(m\phi_j)-\lambda_jP(m\phi_j)). \tag{3.8}$$

Next we consider the operator

$$\mathscr{A}_\Gamma y:=\mathscr{A}y+\frac{1}{2}\sum_{j=1}^N\Gamma_j^2y, \quad \forall\, y\in D(\mathscr{A}) \tag{3.9}$$

and notice that the C_0-semigroup $e^{-\mathscr{A}_\Gamma t}$ generated by $-\mathscr{A}_\Gamma$ on $\widetilde{H}$ is analytic. The operator $\mathscr{A}_\Gamma+F(t)$ generates an evolution operator $U(t,\tau)$ on $\widetilde{H}$, that is

$$\begin{cases} \dfrac{d}{dt}U(t,\tau)+(\mathscr{A}_\Gamma+F(t))U(t,\tau)=0, \quad 0\le\tau\le t \\ U(\tau,\tau)=I. \end{cases}$$

Lemma 3.1 *Let γ the number fixed at the beginning of Section 2. We have for $\eta\ge\max_{1\le j\le N}\sqrt{6\gamma-2\mathrm{Re}\,\lambda_j}$*

$$\|U(t,\tau)\|_{L(\widetilde{H},\widetilde{H})}\le Ce^{-\gamma(t-\tau)}\left(1+\int_\tau^t e^{-\gamma(t-s)}\zeta(s)ds\right), \quad \forall\, t\ge\tau, \quad \mathbb{P}\text{-a.s.}, \tag{3.10}$$

where C is independent of ω and $\zeta(t)=\sum_{j=1}^N e^{\beta_j(t)}$.

Proof. We shall use as in [2], [3] the spectral decomposition of the system

$$\begin{cases} \dfrac{dy}{dt} + \mathscr{A}_\Gamma y + F(t)y = 0, \quad t \geq \tau \\ \\ y(\tau) = x. \end{cases} \tag{3.11}$$

in the direct sum $\mathscr{X}_u \oplus \mathscr{X}_s$ of γ-unstable and γ-stable spaces of the operator $\mathscr{A}$. Namely we set

$$y_u = P_N y, \quad y_s = (I - P_N)y$$

and so, we rewrite system (3.11) as

$$\begin{cases} \dfrac{dy_u}{dt} + \mathscr{A}_u y_u + \dfrac{1}{2}\, P_N \displaystyle\sum_{j=1}^{N} \Gamma_j^2 y_u = 0, \quad t \geq \tau \\ \\ y_u(\tau) = P_N x. \end{cases} \tag{3.12}$$

and

$$\begin{cases} \dfrac{dy_s}{dt} + \mathscr{A}_s y_s + \dfrac{1}{2}\, (I - P_N) \displaystyle\sum_{j=1}^{N} \Gamma_j^2 y_u = 0, \quad t \geq \tau \\ \\ y_s(\tau) = (I - P_N)x. \end{cases} \tag{3.13}$$

We have $y = y_u + y_s$, $y_u = \sum_{j=1}^{N} y_j \varphi_j$ and by (2.4)

$$\mathscr{A}_u \varphi_j = \lambda_j \varphi_j, \quad j = 1, ..., N.$$

Recalling that in virtue of (3.4)

$$\Gamma_j^2 y = \eta \Gamma_j y = \eta^2 \langle y, \varphi_j^* \rangle_{\widetilde{H}} P(m\phi_j),$$

we may rewrite (3.12) as

$$\begin{cases} \dfrac{dy_j}{dt} + \lambda_j y_j + \dfrac{1}{2}\, \eta^2 y_j \langle P(m\phi_j), \varphi_j^* \rangle_{\widetilde{H}} = 0, \quad t \geq \tau,\ j = 1, ..., N, \\ \\ y_j(\tau) = \langle x, \varphi_j^* \rangle_{\widetilde{H}}. \end{cases}$$

Taking into account (2.10) it follows that

$$\begin{cases} \dfrac{dy_j}{dt} + \lambda_j y_j + \dfrac{1}{2}\, \eta^2 y_j = 0, \quad t \geq \tau,\ j = 1, ..., N, \\ \\ y_j(\tau) = \langle x, \varphi_j^* \rangle_{\widetilde{H}}. \end{cases}$$

This yields,

$$y_j(t) = e^{-(\lambda_j + \frac{1}{2}\,\eta^2)t} \langle x, \varphi_j^* \rangle_{\widetilde{H}}, \quad j = 1, ..., N,\ t \geq 0.$$

Hence for $\eta^2 \geq 6\gamma - 2\ \mathrm{Re}\ \lambda_j,\ j = 1, ..., N$, we have

$$|y_u(t)|_{\widetilde{H}} \leq Ce^{-3\gamma(t-\tau)} |x|_{\widetilde{H}}, \quad \forall\, t \geq \tau. \tag{3.14}$$

Now coming back to system (3.13) we shall rewrite it as

$$\begin{cases} \dfrac{dy_s}{dt} + \mathscr{A}_s y_s + \dfrac{1}{2}\,\eta^2 \displaystyle\sum_{j=1}^N y_j (I - P_N) P(m\phi_j) \\ \\ + \displaystyle\sum_{j=1}^N \left(e^{\beta_j(t)} - 1\right) y_j (I - P_N)(\mathscr{A} P(m\phi_j) - \lambda_j P(m\phi_j)) = 0, \quad t \geq \tau \\ \\ y_s(\tau) = (I - P_N)x. \end{cases} \tag{3.15}$$

Then by (3.14) and (2.2) we have that

$$\begin{aligned} &|y_s(t)|_{\widetilde{H}} \leq |e^{-\mathscr{A}_s(t-\tau)}(I - P_N)x|_{\widetilde{H}} \\ &+ \frac{1}{2}\,\eta^2 \int_\tau^t \sum_{j=1}^N \left(e^{\beta_j(s)} - 1\right) |e^{-\mathscr{A}_s(t-s)} y_j(s)(I - P_N) \\ &\times |P(m\phi_j) + \mathscr{A} P(m\phi_j) - \lambda_j P(m\phi_j)|_{\widetilde{H}} ds \\ &\leq Ce^{-\gamma(t-\tau)}|x| + C\,\frac{\eta^2}{2}\,|x|_H \int_0^t \sum_{j=1}^N |e^{-3\gamma s} e^{-\gamma(t-s)} \zeta(s) ds \\ &\leq Ce^{-\gamma(t-\tau)}(1 + \eta^2)|x|_H \int_\tau^t e^{-2\gamma(\tau+s)} \zeta(s) ds, \quad \forall\, t \geq 0,\ \mathbb{P}\text{-a.s.} \end{aligned}$$

for some constant C independent of x and $\omega \in \Omega$. This completes the proof of (3.10). □

Lemma 3.2 *We have*

$$\int_\tau^\infty e^{\gamma(t-\tau)} |U(t,\tau)x|^2_{\frac{3}{4}+\epsilon} e^{\gamma t} dt \leq C|x|^2_{\frac{1}{4}+\epsilon} \left(1 + \int_\tau^\infty e^{-2\gamma\tau} \zeta(t) dt\right)^2, \quad \forall\, x \in W, \tag{3.16}$$

where C *is independent of* $\omega \in \Omega,\ 0 \leq \epsilon < \frac{1}{2}$.

Proof. We set

$$z(t) := e^{\frac{\gamma}{2}(t-\tau)} U(t,\tau)x, \quad 0 < \tau < t.$$

Then by Lemma 3.1 we have

$$\int_\tau^\infty |z(t)|^2_{\widetilde{H}}\, dt \le C|x|^2 \left(1 + \int_\tau^\infty e^{-2\gamma\tau}\zeta(t)dt\right)^2, \quad \forall\, x \in H$$

while

$$\frac{dz}{dt} + \nu_0 Az + A_0 z + \frac{1}{2} \sum_{j=1}^N \Gamma_j^2 z + F(t)z = \frac{\gamma}{2}\, z, \quad t \ge \tau.$$

Multiplying the latter by z and $A^{\frac{1}{2}+\epsilon}z$ (scalarly in $\widetilde{H}$) we have the standard estimates for $d = 2, 3$

$$|\langle A_0 z, z\rangle| = |b(z, X_e, z)| \le C|z|_1\, |X_e|_{\frac{3}{4}}\, |z|_{\widetilde{H}} \le C|z|_{\frac{1}{4}}\, |z|_{\widetilde{H}}$$

and

$$|\langle A_0 z, A^{\frac{1}{2}} z\rangle| = |b(z, X_e, A^{\frac{1}{2}} z)| + |b(X_e, z, A^{\frac{1}{2}} z)|$$

$$\le C(|z|_{\frac{1}{4}}\, |X_e|_1\, |A^{\frac{1}{2}} z|_{\widetilde{H}} + |X_e|_1\, |z|_{\frac{1}{2}}\, |A^{\frac{1}{2}} z|_{\widetilde{H}}) \le C|z|^2_{\frac{1}{2}}.$$

We get that

$$\frac{1}{2}\, \frac{d}{dt}\, |z(t)|^2_H + \nu_0 |z(t)|^2_{\frac{1}{2}} \le C(|z(t)|_{\frac{1}{2}} |z(t)|_{\widetilde{H}} + |z(t)|^2_{\widetilde{H}}) + \langle F(t)z, z(t)\rangle$$

(here $|\cdot| = |\cdot|_{\widetilde{H}}$) and

$$\frac{1}{2}\, \frac{d}{dt}\, |z(t)|^2_{\frac{1}{4}+\epsilon} + \nu_0 |z(t)|^2_{\frac{3}{4}+\epsilon} \le C(|z(t)||z(t)|_{\frac{1}{2}} + |z(t)|^2_{\frac{1}{2}}) + \langle F(t)z, A^{1/2} z(t)\rangle.$$

This yields, via interpolatory inequality

$$|z(t)|_\alpha \le |z(t)|^{\frac{4\alpha}{3}}_{\frac{3}{4}}\, |z(t)|^{1-\frac{4\alpha}{3}}, \quad \text{for } \alpha = \frac{1}{4}, \frac{1}{2}$$

and since by (3.8) $|\langle F(t)z, A^{1/2} z(t)\rangle| \le C|z|$, we get

$$\frac{d}{dt}\, |z(t)|^2_{\frac{1}{4}+\epsilon} + |z(t)|^2_{\frac{3}{4}+\epsilon} \le C|z(t)|^2, \quad t > \tau$$

which yields

$$\int_\tau^\infty |z(t)|^2_{\frac{3}{4}+\epsilon}\, dt \le C|x|^2_{\frac{1}{4}+\epsilon} \left(1 + \int_\tau^\infty e^{-2\gamma\tau}\zeta(t)dt\right)^2,$$

as claimed. (Here and everywhere in the following C is a positive constant independent of ω.) $\Box$

We come back to (3.5) and set

$$G(t,y) := e^{-\sum_{j=1}^N \beta_j(t)\Gamma_j} B(e^{\sum_{j=1}^N \beta_j(t)\Gamma_j} y), \quad \forall\, y \in \widetilde{H},\ t \geq 0.$$

Recalling (3.2) and (3.7) we see that

$$\begin{aligned} B(e^{\beta_j(t)\Gamma_j} y) &= B(y) + \langle y, \varphi_j^* \rangle_{\widetilde{H}}^2 (e^{\eta\beta_j} - 1)^2 B(P(m\phi_j)) \\ &\quad + (e^{\eta\beta_j(t)} - 1)\langle y, \varphi_j^* \rangle_{\widetilde{H}} [B_1(y, P(m\phi_j)) + B_2(y, P(m\phi_j))], \end{aligned} \tag{3.17}$$

where $B(y) = P((y\cdot\nabla)y)$ and

$$B_1(y,z) = P((y\cdot\nabla)z), \quad B_2(y,z) = P((z\cdot\nabla)y), \quad \forall\, y, z \in D(\mathscr{A}). \tag{3.18}$$

Then by (3.7),(3.8) and (3.17) we have for all $j,k = 1,...,N$

$$\begin{aligned} & e^{-\beta_k(t)\Gamma_k} B(e^{\beta_j(t)\Gamma_j} y) \\ & = e^{-\beta_k(t)\Gamma_k} [B(y) + \langle y, \varphi_j^* \rangle_{\widetilde{H}}^2 (e^{\eta\beta_j} - 1)^2 B(P(m\phi_j)) \\ & + (e^{\eta\beta_j(t)} - 1)\langle y, \varphi_j^* \rangle_{\widetilde{H}} [B_1(y, P(m\phi_j)) + B_2(y, P(m\phi_j))] \end{aligned}$$

But in virtue of (3.8) we have

$$e^{-\beta_k(t)\Gamma_k} y = (e^{-\eta\beta_k(t)} - 1)\langle y, \varphi_k^* \rangle_{\widetilde{H}} P(m\phi_k) + y$$

Therefore

$$\begin{aligned} e^{-\beta_k(t)\Gamma_k} B(e^{\beta_j(t)\Gamma_j} y) &= B(e^{\beta_j(t)\Gamma_j} y) + (e^{-\eta\beta_j(t)} - 1)\langle B(e^{\beta_j(t)\Gamma_j} y), \varphi_k^* \rangle_{\widetilde{H}} P(m\phi_k) \\ & = B(y) + \langle y, \varphi_j^* \rangle_{\widetilde{H}}^2 (e^{\eta\beta_j(t)} - 1)^2 B(P(m\phi_j)) \\ & + (e^{\eta\beta_j(t)} - 1)\langle y, \varphi_j^* \rangle_{\widetilde{H}} [B_1(y, P(m\phi_j)) + B_2(y, P(m\phi_j))] \\ & + (e^{-\eta\beta_j(t)} - 1)\langle B(e^{\beta_j(t)\Gamma_j} y), \varphi_k^* \rangle_{\widetilde{H}} P(m\phi_k). \end{aligned}$$

Taking into account that φ_j^*, φ_k^* are smooth we may write the previous relation as

$$e^{-\beta_k(t)\Gamma_k} B(e^{\beta_j(t)\Gamma_j} y) = B(y) + \Theta_{j,k}(t,y), \quad j,k = 1,...,N. \tag{3.19}$$

where

$$|\Theta_{j,k}(t,y)|_\alpha \le C(1+\delta(t))(|\langle y,\varphi_j^*\rangle_{\widetilde{H}}|^2 + |B_1(y,P(m\phi_j)|_\alpha^2$$

$$+|B_2(P(m\phi_j),y)|_\alpha^2 + |\langle B(y),\varphi_j^*\rangle_{\widetilde{H}}|), \quad \forall\, t\ge 0,\ y\in D(\mathscr{A}),\ j,k=1,...,N, \tag{3.20}$$

where $0<\alpha<1$ (recall that $|x|_\alpha = |A^\alpha x|$) and

$$\delta(t) = \sup_{1\le j\le N} \max\{e^{-4\eta\beta_j(t)}, e^{4\eta\beta_j(t)}\}. \tag{3.21}$$

To conclude, we have by (3.17)–(3.21) that

$$G(t,y) = B(y) + \Theta(t,y), \quad \forall\, t\ge 0,\ y\in D(\mathscr{A}). \tag{3.22}$$

Here for each $\alpha\in(0,1)$

$$\begin{aligned} &|\Theta(t,y)|_\alpha \le C(1+\delta^N(t)) \\ &\times(\max_{1\le j\le N}\{|B_1(y,P(m\phi_j)|_\alpha^2 + |B_2(P(m\phi_j),y)|_\alpha^2\} + |B(y)|_{\widetilde{H}}), \end{aligned} \tag{3.23}$$

where δ is given by (3.21) and C is independent of t,y and ω.

We write (3.5) as

$$\frac{dy(t)}{dt} + \mathscr{A}_\Gamma y(t) + G(t,y(t)) + F(t)y(t) = 0, \quad \forall\, t\ge 0,\ \mathbb{P}\text{-a.s..}$$

We set $z(t) = e^{\frac{1}{2}\gamma t}y(t)$ and rewrite it as

$$\begin{cases} \dfrac{dz(t)}{dt} + (\mathscr{A}_\Gamma - \dfrac{1}{2}\gamma)z(t) + e^{-\gamma t}G(t,z(t)) + F(t)y(t) = 0 \\ z(0) = x. \end{cases} \tag{3.24}$$

Equivalently

$$z(t) = S(t,0)x - \int_0^t S(t,s)e^{-\gamma s}G(s,z(s))ds, \quad \forall\, t\ge 0, \tag{3.25}$$

where

$$S(t,\tau) = U(t,\tau)e^{-\frac{1}{2}\gamma(t-\tau)}.$$

We have seen earlier in Lemma 3.1 that $S(t,\tau)$ is exponentially stable in H.

Lemma 3.3 *There is $\Omega_x \subset \Omega$, with*

$$\mathbb{P}(\Omega_x) \geq 1 - \left(C^* |x|_{\frac{1}{4}}^{-\frac{1}{2}} - 1\right)^{-\frac{\gamma}{8(\eta N)^2}},$$

with $C^* > 0$ independent of ω and x such that for each $x \in X$ with $|x|_W \leq C^*$ equation (3.25) *has a unique solution*

$$z \in C([0,\infty); W)) \cap L^2(0,\infty; Z).$$

Here $W = D(A^{\frac{1}{4}})$, $Z = D(A^{\frac{3}{4}})$ if $d = 2$ and $W = D(A^{\frac{1}{4}+\epsilon})$, $Z = D(A^{\frac{3}{4}+\epsilon})$ if $d = 3$.

Proof. We shall proceed as in the proof of [5, Theorem 5.1]. Namely, we rewrite (3.25) as

$$z(t) = S(t,0)x + \mathscr{N} z(t) := \Lambda z(t), \quad t \geq 0,$$

where $\mathscr{N} : L^2(0,\infty; Z)$ is the integral operator

$$\mathscr{N} z(t) = -\int_0^t S(t,s) e^{-\gamma s} G(s, z(s)) ds.$$

We shall prove first the following estimate

$$|\mathscr{N} z|_{L^2(0,\infty;Z)} \leq C \int_0^\infty e^{-\gamma t} |G(t, z(t))|_{\frac{1}{4}} dt. \tag{3.26}$$

Indeed for any $\zeta \in L^2(0,\infty; Z')$ (Z' is the dual of Z) we have via Fubini's theorem

$$\int_0^\infty \langle \mathscr{N} z(t), \zeta(t) \rangle dt = \int_0^\infty dt \left\langle \int_0^t S(t,s) e^{-\gamma s} G(s, z(s)) ds, \zeta(t) \right\rangle$$

$$\leq \int_0^\infty dt \int_0^t |S(t,s) e^{-\gamma s} G(s, z(s))|_Z ds \, |\zeta(t)|_{Z'}$$

$$= \int_0^\infty d\tau \int_\tau^\infty |S(t,\tau) e^{-\gamma \tau} G(\tau, z(\tau))|_Z \, |\zeta(t)|_{Z'} dt$$

$$\leq \int_0^\infty d\tau \left(\int_\tau^\infty |S(t,\tau) e^{-\gamma \tau} G(\tau, z(\tau))|_Z^2 \, dt \right)^{\frac{1}{2}} \, |\zeta|_{L^2(0,\infty;Z')}.$$

Now we set

$$I := \int_0^\infty d\tau \left(\int_\tau^\infty |S(t-\tau) e^{-\gamma \tau} G(\tau, z(\tau))|_Z^2 \, dt \right)^{\frac{1}{2}}.$$

By Lemma 3.2 we have

$$\int_\tau^\infty |S(t,\tau)x|^2_{\frac{3}{4}}dt \le C|x|^2_W \left(1+\int_\tau^\infty e^{-2\gamma\tau}\zeta(t)dt\right)^2, \quad \forall\, x\in W.$$

Next we apply this for $x=e^{-\gamma\tau}G(\tau,z(\tau))$ and get

$$\int_\tau^\infty |S(t-\tau)e^{-\gamma\tau}G(\tau,z(\tau))|^2_Z dt$$

$$\le C|G(\tau,z(\tau))|^2_W e^{-2\gamma\tau}\left(1+\int_\tau^\infty e^{-2\gamma\tau}\zeta(t)dt\right)^2, \quad \forall\, x\in W$$

and therefore

$$I\le C\int_0^\infty |G(\tau,z(\tau))|_W e^{-\gamma\tau}d\tau\left(1+\int_0^\infty e^{-2\gamma s}\zeta(s)ds\right),$$

as claimed.

Next by (3.26) and Lemma 3.2 we have

$$|\Lambda z|_{L^2(0,\infty;Z)}\le C\left(|x|_W+\left(1+\int_0^\infty e^{-2\gamma s}\zeta(s)ds\right)\int_0^\infty e^{-\gamma\tau}|G(\tau,z(\tau))|_W\,d\tau\right). \tag{3.27}$$

On the other hand by (3.22), (3.23) we have

$$|G(t,y)|_W\le |By|_W+|\Theta(t,y)|_W.$$

By [5, Lemma 5.4] we deduce also that

$$|By|_W\le C|y|^2_Z, \quad \forall\, y\in Z$$

and similarly by (3.20) we have

$$|\Theta(t,y)|_W\le C(1+\delta^N(t))|y|^2_W, \quad \forall\, y\in Z.$$

Then (3.27) yields

$$|\Lambda z|_{L^2(0,\infty;Z)}\le C_1^*\left(|x|_W+\int_0^\infty (1+\delta^N(t))e^{-\gamma t}|z(t)|^2_Z dt\right), \quad \mathbb{P}\text{-a.s.}, \tag{3.28}$$

where C_1^* is a positive constant independent of ω. By (3.21) we have

$$\sup_{t\ge0}(1+\delta^N(t)(\omega))e^{-\gamma t}=1+\sup_{t\ge0}\max_{0\le j\le N}\{e^{4\eta N\beta_j(t)-\gamma t}\}=1+\mu(\omega), \quad \omega\in\Omega.$$

$$(3.29)$$

Similarly, we have

$$\int_0^\infty e^{-\gamma t}\zeta(t)dt \le \frac{1}{\gamma}\sup_{1\le j\le N}\sup_{t\ge 0} e^{\beta_j(t)-\gamma t} \le \frac{1}{\gamma}\,\mu(\omega).$$

So (3.28) yields

$$|\Lambda z|_{L^2(0,\infty;Z)} \le C_1^*\left(|x|_W + (1+\mu(\omega)^2)|z|^2_{L^2(0,\infty;Z)}\right), \quad \mathbb{P}\text{-a.s.} \tag{3.30}$$

In order to estimate the right hand side of (3.30) we need the following lemma.

Lemma 3.4 *Let $\beta(t),\ t\ge 0$ be a real Brownian motion in some probability space $(\Omega,\mathscr{F},\mathbb{P})$. Then for each $\lambda>0$ we have*

$$\begin{aligned}&\mathbb{P}\big(\sup_{t>0} e^{\beta(t)-\lambda t}\ge r\big) = \mathbb{P}\big(e^{\sup_{t>0}(\beta(t)-\lambda t)}\ge r\big)\\ &= \mathbb{P}\big(\sup_{s>0}(\beta(s)-\lambda s)\ge \log r\big) = r^{-2\lambda}.\end{aligned} \tag{3.31}$$

Proof. Fix $T>0$. By Girsanov's theorem, $\widetilde{\beta}(t) := \beta(t)-\lambda t,\ t\le T$ is a Brownian motion in $(\Omega,\mathscr{F},\widetilde{\mathbb{P}})$ where

$$d\widetilde{\mathbb{P}} = e^{\lambda\beta(T)-\frac{1}{2}\lambda^2 T}d\mathbb{P}.$$

We have

$$\mathbb{P}\left(\sup_{0\le t\le T} e^{\beta(t)-\lambda t}\ge r\right) = \mathbb{P}\left(\sup_{0\le t\le T} e^{\widetilde{\beta}(t)}\ge r\right).$$

Setting $M_T = \sup_{0\le t\le T} e^{\widetilde{\beta}(t)}$ we have

$$\mathbb{P}(M_T\ge r) = \int_\Omega \mathbb{1}_{[r,+\infty)}(M_T)d\mathbb{P} = \int_\Omega \mathbb{1}_{[r,+\infty)}(M_T)e^{-\lambda\beta(T)+\frac{1}{2}\lambda^2 T}d\widetilde{\mathbb{P}}.$$

Replacing in the latter identity $\beta(t)$ by $\widetilde{\beta}(t)+\lambda t$ yields

$$\mathbb{P}(M_T\ge r) = \int_\Omega \mathbb{1}_{[r,+\infty)}(M_T)e^{-\lambda\widetilde{\beta}(T)-\frac{1}{2}\lambda^2 T}d\widetilde{\mathbb{P}}.$$

Because $\widetilde{\beta}$ is a Brownian motion with respect to $\widetilde{\mathbb{P}}$ we can compute the integral above by using the well known expression of the law of $(M_t,\widetilde{\beta}(t))$, see e.g. [17, (8.2) page 9]. We obtain that

$$\mathbb{P}\big(M_T\ge r\big) = \frac{2}{\sqrt{2\pi T^3}}\int_r^\infty db\int_{-\infty}^b (b-a)e^{-\lambda a-\frac{1}{2}\lambda^2 T}\ e^{-\frac{(2b-a)^2}{2T}}da.$$

It follows that

$$\mathbb{P}(M_T \geq r) = \frac{1}{2}\, e^{-2\lambda r}\text{Erfc}\left(\frac{r-\lambda T}{\sqrt{2T}}\right) + \frac{1}{2}\, e^{2\lambda r}\text{Erfc}\left(\frac{r+\lambda T}{\sqrt{2T}}\right),$$

where

$$\text{Erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2} dt.$$

For $T \to \infty$ we obtain (3.31). □

Proof of Lemma 3.3 (continued). By (3.31) it follows that

$$\mathbb{P}\left(\sup_{t\geq 0} e^{4\eta N\beta_j(t)-\gamma t} \leq r\right) \geq 1 - r^{-\frac{\gamma}{8(N\eta)^2}}, \quad j = 1, ..., N, \tag{3.32}$$

and therefore by (3.29)

$$\mathbb{P}(1+\mu \leq r) \geq 1 - (r-1)^{-\frac{\gamma}{8(N\eta)^2}}, \quad \forall\, r \geq 1. \tag{3.33}$$

We set

$$\mathscr{U}(\omega) := \{z \in L^2(0,\infty;Z) : \ |z|_{L^2(0,\infty;Z)} \leq R(\omega)\},$$

where $R : \Omega \to \mathbb{R}^+$ is a random variable such that

$$\frac{2C_1^*|x|_{\frac14}}{1+\sqrt{1-4(C_1^*)^2|x|_{\frac14}(1+\mu)^2}} \leq R(\omega) \leq \frac{2C_1^*|x|_{\frac14}}{1-\sqrt{1-4(C_1^*)^2|x|_{\frac14}(1+\mu)^2}}, \quad \omega \in \Omega. \tag{3.34}$$

Then, as easily follows from (3.30) and (3.34) for

$$|x|_W \leq \rho_1(\omega) := [8(1+\mu(\omega)^2)(C_1^*)^2]^{-1}, \tag{3.35}$$

we have

$$\Lambda\mathscr{U}(\omega) \subset \mathscr{U}(\omega).$$

Now we shall apply the Banach fixed point theorem to $\mathscr{N}$ on the set $\mathscr{U}(\omega)$.

Let $z_1, z_2 \in \mathscr{U}(\omega)$. Arguing as in the proof of (3.30) we find that

$$|\mathscr{N} z_1 - \mathscr{N} z_2|_{L^2(0,\infty;Z)} \leq C_1^* \int_0^\infty e^{-\gamma t}|G(t,z_1) - G(t,z_2)|_{\frac{1}{4}}\, dt$$

$$\left(1 + \int_0^\infty e^{-2\gamma s}\zeta(s)ds\right)$$

$$\leq C_1^* C_2^* \int_0^\infty (1+\delta(t))e^{-\gamma t}|z_1(t) - z_2(t)|_Z(|z_1(t)|_Z + |z_2(t)|_Z)dt$$

$$\left(1 + \int_0^\infty e^{-2\gamma s}\zeta(s)ds\right)$$

$$\leq C_1^* C_2^* \left(\int_0^\infty |z_1(t) - z_2(t)|_Z^2 dt\right)^{\frac{1}{2}} \left(\int_0^\infty e^{-\gamma t}(|z_1(t)|_Z^2 + |z_2(t)|_Z^2)dt\right)^{\frac{1}{2}}$$

$$(1+\mu(\omega))^2 \leq 2C_1^* C_2^* (1+\mu(\omega))^2 R(\omega)|z_1 - z_2|_{L^2(0,\infty;Z)},$$

where C_1^*, C_2^* are independent of ω.

Now if we choose x such that besides (3.35) to have also

$$|x|_{\frac{1}{4}} \leq \frac{\sqrt{2}+1}{2\sqrt{2}(C_1^*)^2 C_2^*(1+\mu)^2} =: \rho_2(\omega)$$

we see that there is $R = R(\omega)$ satisfying (3.34) and such that

$$2C_1^* C_2^*(1+\mu)^2 R < 1.$$

Now we take

$$|x|_{\frac{1}{4}} \leq \rho(\omega) := \min\{\rho_1(\omega), \rho_2(\omega)\} = ((C^*)^2(1+\mu)^2)^{-1}, \tag{3.36}$$

where C^* is a suitable chosen constant independent of ω. Then for x satisfying (3.36) $\mathscr{N}$ is a contraction on $\mathscr{U}(\omega)$ and maps $\mathscr{U}(\omega)$ on itself.

We set

$$\Omega_x = \{\omega \in \Omega : \ |x|_W \leq \rho(\omega)\}. \tag{3.37}$$

Hence for each $\omega \in \Omega_x$ the equation (3.25) has a unique solution z satisfying conditions in Lemma 3.3. On the other hand, by (3.33) and (3.37) we see that

$$\mathbb{P}(\Omega_x) \geq 1 - \left(C^*|x|_W^{-\frac{1}{2}} - 1\right)^{-\frac{\gamma}{8(\eta N)^2}},$$

as claimed. $\square$

Lemma 3.5 *Let z be the solution to* (3.24) *given by Lemma 3.3. Then*

$$\lim_{t\to\infty} |z(t)|_{\widetilde{H}} = 0, \quad \mathbb{P}\text{-a.s. in } \Omega_x. \tag{3.38}$$

Proof. By (3.24) it follows as in the proof of Lemma 3.2 that

$$\frac{1}{2}\,\frac{d}{dt}\,|z(t)|^2_{\widetilde{H}} + \frac{\nu_0}{2}\,|z(t)|^2_{\frac{1}{2}} \le C_1|z(t)|^2_{\widetilde{H}} + e^{-\gamma t}\langle G(t,z(t)),z(t)\rangle.$$

Taking into account that

$$|e^{-\gamma t}\langle G(t,z(t)),z(t)\rangle| = e^{-\gamma t}|\langle \Theta(t,z(t)),z(t)\rangle| \le C_2|z(t)|^2_Z$$

and that $z \in L^2(0,\infty;D(A^{\frac{3}{4}}))$ we infer that

$$\frac{d}{dt}\,|z(t)|^2_{\widetilde{H}} \in L^\infty(0,\infty),$$

and together with $z \in L^2(0,\infty;\widetilde{H}))$ this implies (3.38) as claimed. $\square$

Proof of Theorem 2.2 (continued)

By Lemma 3.5 we have that

$$\lim_{t\to\infty} |y(t)|_{\widetilde{H}} e^{\frac{1}{2}\gamma t} = 0, \quad \forall\, \omega \in \Omega_x. \tag{3.39}$$

Then as seen earlier

$$X(t) = \prod_{j=1}^N e^{\beta_j(t)\Gamma_j}\; y(t), \quad \mathbb{P}\text{-a.s.}$$

is the solution to (2.7). Then by (3.7) and (3.8) we see that

$$|X(t)|_{\widetilde{H}}\; e^{\frac{\gamma t}{4}} \le C_1^*\left(1 + \max_{1\le j\le N}\left\{e^{N\eta\beta_j(t)-\frac{\gamma t}{4}}, e^{-N\eta\beta_j(t)-\frac{\gamma t}{4}}\right\}\right)|y(t)|_{\widetilde{H}}\; e^{\frac{\gamma t}{2}}. \tag{3.40}$$

We set

$$\Omega_x^r = \left\{\omega \in \Omega:\ \sup_{t\ge 0}\max_{1\le j\le N}\left\{e^{N\eta\beta_j(t)-\frac{\gamma t}{4}}, e^{-N\eta\beta_j(t)-\frac{\gamma t}{4}}\right\} \le r\right\},$$

where $r > 0$. By Lemma 3.4 (see (3.32)) we have

$$\mathbb{P}(\Omega_x^r) \ge 1 - r^{-\frac{\gamma}{2(\eta N)^2}}. \tag{3.41}$$

This yields

$$\mathbb{P}(\Omega_x \cap \Omega_x^r) \ge 1 - \left(C^*|x|_W^{-\frac{1}{2}} - 1\right)^{-\frac{\gamma}{2(\eta N)^2}} - r^{-\frac{\gamma}{2(\eta N)^2}}, \tag{3.42}$$

for any $r > 0$. We set $\Omega_x^* = \Omega_x \cap \Omega_x^r$ where

$$r = \left(C^* |x|_W^{-\frac{1}{2}} - 1 \right)^{\frac{1}{4}}$$

and by (3.41), (3.42) we get (2.12) and

$$\lim_{t\to\infty} |X(t)|_{\widetilde{H}} e^{\frac{\gamma t}{4}} = 0 \quad \mathbb{P}\text{-a.s. in } \Omega_x^*.$$

This completes the proof of Theorem 2.2. □

4 Final remarks

4.1 Stochastic stabilization versus deterministic stabilization

By the same proofs that of Theorem 2.2 it follows that the deterministic feedback controller

$$u = -\eta \sum_{j=1}^{N} \langle X, \varphi_j^* \rangle_{\widetilde{H}} P(m\phi_j), \tag{4.1}$$

where η is sufficiently large, stabilizes exponentially system (1.4) in a neighbourhood $\{x \in H : |x|_{\frac{1}{2}} < \rho\}$. Here ϕ_j are chosen as in (2.10). Apparently the feedback controller (4.1) is simpler than its stochastic counterpart (1.6) above while the stabilization performances are comparable. It should be said, however, that the controller (4.1) though stabilizable is not robust while the stochastic one designed here is. In fact it is easily seen that (4.1) is very sensitive to structural perturbations in system (1.1) because small variations of the spectral system ϕ, $\{\varphi_j^*\}$ might break the orthogonality condition (2.10) from which ϕ_j are determined. In this way the deterministic linear closed loop equation

$$dX + AXdt = -\eta \sum_{j=1}^{N} \langle X, \varphi_j^* \rangle_{\widetilde{H}} P(m\phi_j)dt$$

might became unstable even for $\eta > 0$ and large. By contrary this does not happen for the stochastic system

$$dX + AXdt = -\eta \sum_{j=1}^{N} \langle X, \varphi_j^* \rangle_{\widetilde{H}} P(m\phi_j)d\beta_j, \tag{4.2}$$

because its unstable part that is $X=\sum_{j=1}^N X_j\phi_j$, where

$$dX_j+\lambda_jX_jdt=-\eta\sum_{j=1}^{N}X_j\langle\phi_j,\varphi_j^*\rangle_0P(m\phi_j)d\beta_j,\qquad \text{Re}\,\lambda_j\le\gamma,\ j=1,...,N, \tag{4.3}$$

still remains exponentially stable with probability one to small perturbations of $\{\varphi_j^*\}$. Indeed in this case instead of (2.10) we have

$$|\langle\phi_j,\varphi_k^*\rangle-\delta_{j,k}|\le\epsilon,\quad \forall\, j,k=1,...,N$$

and therefore

$$\sum_{j=1}^{N}\sum_{i=1}^{N}|\langle\phi_j,\varphi_j^*\rangle_0|^2\,|X_j|^2\ge\mu\sum_{j=1}^{N}|X_j|^2$$

which, as seen earlier in [2] implies stabilization of (4.3) for sufficiently large $|\eta|$.

As mentioned in introduction one might design starting from (4.1) a robuste stabilizable controller via infinite dimensional Riccati equations associated with the linear system but this involves, however, hard numerical computations.

4.2 Giving up to assumption (H1)

One might design a feedback stochastic feedback controller of the above form in absence of assumption (H1).

Indeed if we replace $\{\varphi_j\}_1^N$ by its Schmidt's orthogonalization $\{\widetilde{\varphi}_j\}_1^N$, we still have $\mathscr{X}_u=\text{lin span }\{\widetilde{\varphi}_j\}_1^N$ and $\mathscr{X}_s=\text{lin span }\{\widetilde{\varphi}_j\}_{N+1}^\infty$.

Consider the feedback controller

$$u=\eta\sum_{j=1}^{N}\langle X,\phi_j^*\rangle_{\widetilde{H}}P(m\Phi_j)\dot{\beta}_j \tag{4.4}$$

where $\{\widetilde{\Phi}_j\}$ are determined by

$$\langle\widetilde{\Phi}_j,\widetilde{\varphi}_k\rangle_0=\delta_{j,k},\quad j,k=1,...,N. \tag{4.5}$$

By Lemma 2.1 it follows that system $\{\widetilde{\varphi}_j\}_1^N$, is independent on $\mathscr{O}_0$ and so such a system $\{\widetilde{\Phi}_j\}_1^N$, exists. Then the proof of Theorem 2.2 applies with minor modifications to show that the controller u defined by (4.4) is exponentially stabilizable in the sense of Theorem 2.2.The details are omitted.

A Proof of Lemma 2.1

Consider the Stokes–Oseen operator

$$\mathscr{L}\varphi = -\nu_0\Delta\varphi + (y_e\cdot\nabla)\varphi + (\varphi\cdot\nabla)y_e, \quad \text{in } \mathscr{O}$$

and recall the following unique continuation result

Lemma A.1 *Assume $y_e \in C^2(\overline{\mathscr{O}})$ and let $\varphi \in C^2(\overline{\mathscr{O}})$ be the solution to the problem*

$$\begin{cases} \mathscr{L}\varphi = \lambda\varphi + \nabla p, \quad \text{in } \mathscr{O}, \\ \\ \nabla\cdot\varphi = 0, \quad \text{in } \mathscr{O}, \;\; \varphi = 0, \quad \text{on } \partial\mathscr{O}, \end{cases} \tag{A.1}$$

such that $\varphi \equiv 0 \quad \text{on } \mathscr{O}_0$ where $\mathscr{O}_0$ is an open subset of $\mathscr{O}$. Then $\varphi \equiv 0$.

This result is well known in literature and a proof can be found in [3, Lemma 3.7]. A simple proof of Lemma 2.1 in $2-d$ can be given to reducing A!, via vorticity transformation $\psi = \text{curl}\ \varphi$, to

$$-\nu_0\Delta\psi + y_e\cdot\nabla\psi + \nabla(\text{curl}\ y_e) = \lambda\psi, \quad \text{in } \mathscr{O}$$

and by stream function ϕ to

$$-\nu_0\Delta^2\phi + y_e\cdot\nabla\phi + \nabla\phi\cdot\Delta y_e - \lambda\Delta\phi = 0, \quad \text{in } \mathscr{O}. \tag{A.2}$$

(Here $\varphi = \nabla^\perp\phi = \{D_2\phi, -D_1\phi\}$.)

Then if $\varphi \equiv 0$, on $\mathscr{O}_0$ it follows that $\nabla\phi = 0$ in $\mathscr{O}_0$ and by the Carleman inequality combined with unique continuation arguments as in [16, Theorem 8.9.1] it follows by (A.2) that $\phi \equiv 0$ and therefore $\varphi \equiv 0$ as claimed.

Let $\{\varphi_j\}_{j=1}^N$ be eigenfunctions corresponding to eigenvalues λ_j, i.e.,

$$\begin{cases} \mathscr{L}\varphi_j = \lambda_j\varphi_j + \nabla p_j, \quad \text{in } \mathscr{O}, \\ \\ \nabla\cdot\varphi_j = 0, \quad \text{in } \mathscr{O} \\ \\ \varphi_j = 0, \quad \text{on } \partial\mathscr{O}. \end{cases} \tag{A.3}$$

One must prove that each system $\{\varphi_1, ..., \varphi_m\}$, $1 \le m \le N$, is linearly independent in $\mathscr{O}_0$. As mentioned earlier this is immediate if all φ_j are eigenfunctions corresponding to the same eigenvalue λ_j and so, it suffices to prove this for distinct eigenvalues λ_j. For $m = 1$ this follows by Lemma A.1.

Let $m = 2$ and let φ_1, φ_2 two eigenfunctions with corresponding eigenvalues λ_1, λ_2. Then we have

$$\mathscr{L}(\lambda_2\varphi_1 - \lambda_1\varphi_2) = \lambda_2\nabla p_1 - \lambda_1\nabla p_2 = \nabla p, \quad \text{in } \mathscr{O}. \tag{A.4}$$

Assume that $\alpha_1\varphi_1 + \alpha_2\varphi_2 \equiv 0$ on $\mathscr{O}_0$ for $\alpha_1, \alpha_2 \neq 0$ and argue from this to a contradiction. Indeed, in this case replacing φ_1 by $\frac{\lambda_2}{\alpha_1}\varphi_1$ and φ_2 by $-\frac{\lambda_1}{\alpha_2}\varphi_2$ we see that $\lambda_2\varphi_1 - \lambda_1\varphi_2 \equiv 0$ on $\mathscr{O}_0$ and so by (A.4) and Lemma A.1 we infer that $\lambda_2\varphi_1 - \lambda_1\varphi_2 \equiv 0$ on $\mathscr{O}$ which is of course absurd. We shall treat now the case $m = 3$. We have as above besides (A.4) that

$$\mathscr{L}(\lambda_3\varphi_1 - \lambda_1\varphi_3) = \nabla q, \quad \text{in } \mathscr{O}$$

and therefore

$$\mathscr{L}((\lambda_2 - \lambda_3)\varphi_1 - \lambda_1\varphi_2 + \lambda_1\varphi_3) = \nabla q, \quad \text{in } \mathscr{O}_0. \tag{A.5}$$

If $\alpha_1\varphi_1 + \alpha_2\varphi_2 + +\alpha_3\varphi_3 \equiv 0$ on $\mathscr{O}_0$, replacing $\varphi_1, \varphi_2, \varphi_3$ by $\frac{\lambda_2-\lambda_3}{\alpha_1}\varphi_1$, $-\frac{\lambda_1}{\alpha_2}\varphi_2$, $\frac{\lambda_1}{\alpha_3}\varphi_3$ respectively. we obtain that

$$(\lambda_2 - \lambda_3)\varphi_1 - \lambda_1\varphi_2 + \lambda_1\varphi_3 \equiv 0, \quad \text{in } \mathscr{O}_0,$$

which in virtue of (A.5) and Lemma A.1 implies

$$(\lambda_2 - \lambda_3)\varphi_1 - \lambda_1\varphi_2 + \lambda_1\varphi_3 \equiv 0, \quad \text{in } \mathscr{O},$$

which is again absurd.

The argument works for all $m \in \mathbb{N}$ and this concludes the proof of Lemma 2.1. □

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