PLAQUETTES, SPHERES, AND ENTANGLEMENT

GEOFFREY R. GRIMMETT AND ALEXANDER E. HOLROYD

ABSTRACT. The high-density plaquette percolation model in d dimensions contains a surface that is homeomorphic to the $(d-1)$-sphere and encloses the origin. This is proved by a path-counting argument in a dual model. When $d = 3$, this permits an improved lower bound on the critical point p_e of entanglement percolation, namely $p_e \geq \mu^{-2}$ where μ is the connective constant for self-avoiding walks on $\mathbb{Z}^3$. Furthermore, when the edge density p is below this bound, the radius of the entanglement cluster containing the origin has an exponentially decaying tail.

1. INTRODUCTION AND RESULTS

The **plaquette percolation** model is a natural dual to bond percolation in two and more dimensions. Let $\mathbb{Z}^d$ be the integer lattice; elements of $\mathbb{Z}^d$ are called **sites**. For any site z, let $Q(z) := [-\frac{1}{2}, \frac{1}{2}]^d + z$ be the topologically closed unit d-cube centred at z. A **plaquette** is any topologically closed unit $(d-1)$-cube in $\mathbb{R}^d$ that is a face of some $Q(z)$ for $z \in \mathbb{Z}^d$. Let Π_d be the set of all plaquettes. For a set of plaquettes $S \subseteq \Pi_d$, we write $[S] := \bigcup_{\pi \in S} \pi$ for the associated subset of $\mathbb{R}^d$. In the plaquette percolation model with parameter $p \in [0, 1]$, each plaquette of Π_d is declared **occupied** with probability p, otherwise **unoccupied**, with different plaquettes receiving independent states; the associated probability measure is denoted $\mathbb{P}_p$.

The ℓ^s-norm on $\mathbb{R}^d$ is denoted $\| \cdot \|_s$. A **sphere** of $\mathbb{R}^d$ is a simplicial complex, embedded in $\mathbb{R}^d$, that is homeomorphic to the unit sphere $\{x \in \mathbb{R}^d : \|x\|_2 = 1\}$. By the generalized Schönflies theorem, the complement in $\mathbb{R}^d$ of a sphere has a bounded and an unbounded path-component, which we call respectively its **inside** and **outside**. The (ℓ^1-)**radius** of a set $A \subseteq \mathbb{R}^d$ with respect to the origin $0 \in \mathbb{R}^d$ is

$$\operatorname{rad} A = \operatorname{rad}_0 A := \sup\{\|x\|_1 : x \in A\}.$$

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Let μ_d be the **connective constant** of $\mathbb{Z}^d$, given as

$$\mu_d := \lim_{k\to\infty} \sigma(k)^{1/k},$$

where $\sigma(k)$ is the number of (ℓ^1-)nearest-neighbour self-avoiding paths from the origin with length k in $\mathbb{Z}^d$; it is a straightforward observation that $\mu_d \in [d, 2d-1]$, and stronger bounds may be found, for example, in [19].

Theorem 1. *Let $d \geq 2$, and consider the plaquette percolation model. If $p < \mu_d^{-2}$ then almost surely there exists a finite set S of unoccupied plaquettes whose union $[S]$ is a sphere with 0 in its inside. Moreover S may be chosen so that*

$$\mathbb{P}_p\big(\mathrm{rad}\,[S] \geq r\big) \leq C\alpha^r, \qquad r > 0,$$

for any $\alpha \in (\mu_d p, 1)$, and some $C = C(p, d, \alpha) < \infty$.

When $d = 2$, the first assertion of Theorem 1 amounts to the well known fact that there exists a suitable circuit of unoccupied bonds of the dual lattice (see, e.g., [8]). The result is more subtle in higher dimensions.

When $d = 3$, Theorem 1 has an application to entanglement percolation, which we explain next. Define a **bond** to be the topologically closed line segment in $\mathbb{R}^d$ joining any two sites $x, y \in \mathbb{Z}^d$ with $\|x-y\|_1 = 1$. Let $\mathbb{L}_d$ be the set of all bonds. In the **bond percolation** model, each bond is declared **occupied** with probability p, otherwise **unoccupied**, with the states of different bonds being independent. For a set of bonds K, write $[K] := \bigcup_{e\in K} e$. We say that K contains a site x if $x \in [K]$.

We say that a sphere $Z \subset \mathbb{R}^d$ **separates** a set $A \subset \mathbb{R}^d$ if A intersects both the inside and the outside of Z, but not Z itself. (We write $A \subset B$ if $A \subseteq B$ and $A \neq B$.) Let $d = 3$. We say that a set of bonds $K \subseteq \mathbb{L}_3$ is 1-**entangled** if no sphere of $\mathbb{R}^3$ separates $[K]$. The idea of this definition is that a 1-entangled set of bonds, if made of string or elastic, cannot be continuously "pulled apart". Any connected set of bonds is evidently 1-entangled. The simplest disconnected set that is 1-entangled consists of two linked loops. The prefix "1" reflects the fact that other natural definitions of entanglement are possible; see [10] and the discussion in Section 2 for more details. Entanglement of sets of bonds is intrinsically a three-dimensional issue, and therefore we shall always take $d = 3$ when discussing it.

In the bond percolation model in $d = 3$, let $\eta^1(p)$ be the probability that there exists an infinite 1-entangled set of occupied bonds containing the origin 0, and define the 1-entanglement **critical probability**

$p_{\mathrm{e}}^1 := \sup\{p : \eta^1(p) = 0\}$. The maximal 1-entangled set of occupied bonds containing a site x is called the 1-entanglement **cluster** at x.

Corollary 2. *The* 1*-entanglement critical probability in three dimensions satisfies*

$$p_{\mathrm{e}}^1 \geq \mu_3^{-2}.$$

Moreover, if $p < \mu_3^{-2}$*, the* 1*-entanglement cluster* E *at the origin satisfies*

$$\mathbb{P}_p\big(\mathrm{rad}\,[E] \geq r\big) \leq C\alpha^r, \qquad r > 0,$$

for any $\alpha \in (\mu_d p, 1)$*, and some* $C = C(p, d, \alpha) < \infty$*.*

The connective constant of $\mathbb{Z}^3$ satisfies the rigorous bound $\mu_3 \leq 4.7387$ (see [19]). Therefore, Corollary 2 gives

$$p_{\mathrm{e}}^1 \geq 0.04453 \cdots > \frac{1}{23}.$$

This is a significant improvement on the previous best lower bound of [4], namely $p_{\mathrm{e}}^1 > 1/597$, which in turn substantially improved the first non-zero lower bound, $p_{\mathrm{e}}^1 \geq 1/15616$, proved in [13]. In each case, the improvement is by a factor of approximately 26.

In Section 2 we discuss some history and background to our results. In Section 3 we prove Theorem 1 and Corollary 2. If p satisfies the stronger bound $p < (2d-1)^{-2}$, we shall see that our methods yield versions of these results with explicit formulae for the constants C and α. In Section 4 we consider the critical value of p associated with the event in Theorem 1, and its relationship to certain other critical values.

2. Remarks

2.1. **Duality.** To each bond $e \in \mathbb{L}_d$ there corresponds a unique plaquette $\pi(e) \in \Pi_d$ that intersects e. It is therefore natural to couple the bond and plaquette percolation models with common parameter p in such a way that $\pi(e)$ is occupied if and only if e is occupied. If p is less than the critical probability p_{c} for standard bond percolation (see, e.g., [8]), the connected component of occupied bonds at the origin is almost surely finite, and it is a straightforward consequence that there exists a finite set of unoccupied plaquettes whose union encloses the origin (i.e., the origin lies in some bounded component of its complement). Indeed, such plaquettes may be chosen so as to form a 'surface' enclosing the origin (although precise definition of such an object requires care). However, such a surface might be homeomorphic to a torus, or some other topological space. It is a key point of Theorem 1 that the surface $[S]$ is a sphere.

2.2. **Entanglement.** Entanglement in three-dimensional percolation was first studied, in a partly non-rigorous way, in [16] (some interesting remarks on the subject appeared earlier in [2]). The rigorous theory was systematically developed in [10], and further rigorous results appear in [3, 12, 13, 14, 15]. A discussion of physical applications of entanglement percolation may be found in [4].

As mentioned in Section 1, there are several (non-equivalent) ways of defining the property of entanglement for infinite graphs. One of these, namely 1-entanglement, was presented in that section, and a second follows next. We say that a set of bonds $K \subseteq \mathbb{L}_3$ is 0-**entangled** if every finite subset of K is contained in some finite 1-entangled subset of K. It was shown in [10] that the notions of 0-entanglement and 1-entanglement are extremal members of a certain class of natural candidate definitions, called *entanglement systems*, and furthermore that these two entanglement systems correspond (respectively) in a natural way to free and wired boundary conditions.

By combining inequalities of [3, 10, 13], we find that

$$0 < p_{\mathrm{e}}^1 \leq p_{\mathrm{e}}^{\mathcal{E}} \leq p_{\mathrm{e}}^0 < p_{\mathrm{c}} < 1,$$

where p_{e}^0, p_{e}^1, and $p_{\mathrm{e}}^{\mathcal{E}}$ are the critical probabilities for 0-entanglement, 1-entanglement, and for an arbitrary entanglement system $\mathcal{E}$, respectively. The inequality $p_{\mathrm{e}}^0 \leq p_{\mathrm{c}}$ reflects the straightforward fact that every connected set of bonds is 0-entangled. It was strengthened to the strict inequality $p_{\mathrm{e}}^0 < p_{\mathrm{c}}$ in [3, 15]. In [16] it was argued on the basis of numerical evidence that $p_{\mathrm{c}} - p_{\mathrm{e}} \approx 1.8 \times 10^{-7}$, for a certain notion of 'entanglement critical probability' p_{e}. It is an open question to decide whether or not $p_{\mathrm{e}}^0 = p_{\mathrm{e}}^1$.

2.3. **Spheres, lower bounds, and exponential decay.** The inequality $p_{\mathrm{e}}^1 > 0$ expresses the fact that, for a sufficiently small density p of occupied bonds, there is no infinite entangled set of bonds. Prior to the current paper, proofs of this seemingly obvious statement have been very involved.

The proof in [13] employs topological arguments to show that, for $p < 1/15616$, almost surely the origin is enclosed by a sphere that intersects no occupied bond. The argument is specific to three dimensions, and does not resolve the question of the possible existence of a sphere of unoccupied plaquettes enclosing the origin. (See [10] for more on the distinction between spheres intersecting no occupied bond, and spheres of unoccupied plaquettes.) In [10], related arguments are used to show that, for sufficiently small p, the radius R of the 1-entanglement cluster at the origin has 'near-exponential' tail decay in that

$$\mathbb{P}(R > r) < \exp\bigl(-cr/\log\cdots\log r\bigr)$$

for an arbitrary iterate of the logarithm, and for some $c > 0$ depending on p and the number of logarithms.

In the recent paper [4], the above results are substantially improved in several respects. The lower bound on the critical point is improved to approximately $p_{\mathrm{e}}^1 > 1/597$, and it is proved also that the radius of the 1-entanglement cluster at the origin has exponential tail decay for p below the same value. The key innovation is a proof of an exponential upper bound on the number of possible 1-entangled sets of N bonds containing the origin, thereby answering a question posed in [10]. The method of proof is very different from that of [13].

In the current article, we improve on the proofs mentioned above in several regards. The lower bound on the critical point is further improved to approximately $p_{\mathrm{e}}^1 > 1/23$, and we establish exponential decay of the 1-entanglement cluster-radius at the origin for p below this value. We prove the existence of a sphere of unoccupied plaquettes enclosing the origin (rather than just a sphere intersecting no occupied bond), and we do so for all dimensions. Finally, our proofs are very simple. Our methods do not appear to imply the key result of [4] mentioned above, namely the exponential bound for entangled sets containing the origin.

3. Proofs

The geometric lemma below is the key to our construction of a sphere. For $x, y \in \mathbb{R}^d$, write $y \preceq x$ if for each $i = 1, 2, \ldots, d$ we have $|y_i| \le |x_i|$ and $x_i y_i \ge 0$ (equivalently, y lies in the closed cuboid with opposite corners at 0 and x). For a bond $e \in \mathbb{L}_d$, recall that $\pi(e) \in \Pi_d$ is the unique plaquette that intersects it.

Proposition 3. *Let $d \ge 2$. Suppose $K \subset \mathbb{Z}^d$ is a finite set of sites containing 0, with the property that, if $x \in K$, then every $y \in \mathbb{Z}^d$ with $y \preceq x$ lies in K. Let*

$$S := \bigl\{\pi(e) : e \text{ is a bond with exactly one endvertex in } K\bigr\}. \tag{1}$$

Then $[S]$ is a sphere with 0 in its inside.

Proof. Let $U := \bigcup_{x \in K} Q(x)$ be the union of the unit cubes corresponding to K. Note that $[S]$ is the topological boundary of U in $\mathbb{R}^d$. Let $\Sigma := \{z \in \mathbb{R}^d : \|z\|_2 = 1\}$ be the unit sphere; we will give an explicit homeomorphism between $[S]$ and Σ.

We claim first that U is strictly star-shaped, which is to say: if $x \in U$ then the line segment $\{\alpha x : \alpha \in [0,1)\}$ is a subset of the topological interior of U (i.e., of $U \setminus [S]$). To check this, suppose without loss of generality that x is in the non-negative orthant $[0,\infty)^d$. By the given properties of K, the open cuboid

$$H := \prod_{i=1}^{d} \big(-\tfrac{1}{2}\,,\, x_i \vee \tfrac{1}{2} \big)$$

is a subset of U (here it is important that the origin is at the centre of a cube, rather than on a boundary); now, H clearly contains the aforementioned line segment, and the claim is proved. In the above, $x \vee y$ denotes the maximum of x and y.

It follows that, for any point $z \in \Sigma$, the ray $\{\alpha z : \alpha \in [0,\infty)\}$ has exactly one point of intersection with $[S]$. Denote this point of intersection $f(z)$. Clearly f is a bijection from Σ to $[S]$; we must prove that it is a homeomorphism. Since Σ and $[S]$ are compact metric spaces, it suffices to express them as finite unions $\Sigma = \bigcup_{j=1}^{r} X_j$ and $[S] = \bigcup_{j=1}^{r} Y_j$, where the X_j and Y_j are compact, and such that f restricted to X_j is a homeomorphism from X_j to Y_j, for each j. This is achieved by taking $\{Y_1, \ldots, Y_r\}$ equal to the set of plaquettes S. Any plaquette in Π_d is a subset of some $(d-1)$-dimensional affine subspace (hyperplane) of $\mathbb{R}^d$ that does not pass through 0 (here the offset of $\frac{1}{2}$ is again important) and it is elementary to check that the projection through 0 from such a subspace to Σ is a homeomorphism to its image in Σ.

Finally, we must check that 0 lies in the inside of the sphere $[S]$; this is clear because $0 \in U \setminus [S]$, and any unbounded path in $\mathbb{R}^d$ starting from 0 must leave U at some point, and thus must intersect $[S]$. $\square$

The next lemma is closely related to a recent result on random surfaces in [7]. Consider the bond percolation model with parameter p on $\mathbb{L}_d$. By a **path** we mean a self-avoiding path comprising sites in $\mathbb{Z}^d$ and bonds in $\mathbb{L}_d$. Recall that $\sigma(k)$ is the number of paths starting at the origin and having k edges, and

$$\mu_d := \lim_{k\to\infty} \sigma(k)^{1/k}$$

is the connective constant. Let $0 = v_0, v_1, \ldots, v_k$ be the sites (in order) of such a path. We call the path **good** if, for each i satisfying $\|v_{i-1}\|_1 < \|v_i\|_1$, the bond with endpoints v_{i-1}, v_i is occupied.

Lemma 4. *Let K be the set of sites $u \in \mathbb{Z}^d$ for which there is a good path from 0 to u. If $p < \mu_d^{-2}$ then K is a.s. finite, and moreover,*

$$\mathbb{P}_p(\operatorname{rad} K \geq r) \leq C'\alpha^r, \qquad r \geq 0, \tag{2}$$

for any $\alpha \in (\mu_d p, 1)$, and some $C' = C'(p, d, \alpha) < \infty$. If $p < (2d-1)^{-2}$ then (2) holds with $\alpha = p(2d-1)$ and $C' = 2/[1 - p(2d-1)^2]$.

Proof. Let $N(r)$ be the number of good paths that start at 0 and end on the ℓ^1-sphere $\{x \in \mathbb{Z}^d : \|x\|_1 = r\}$. Then,

$$\mathbb{P}_p(\operatorname{rad} K \geq r) \leq \mathbb{P}_p(N(r) > 0) \leq \mathbb{E}_p N(r).$$

For any path π with vertices $0 = v_0, v_1, \ldots, v_k = u$ with $\|u\|_1 = r$, let

$$A := \#\{i : \|v_i\|_1 > \|v_{i-1}\|_1\}; \qquad B := \#\{i : \|v_i\|_1 < \|v_{i-1}\|_1\}$$

be respectively the number of steps Away from, and Back towards, the origin 0. Note that $k = A + B$, and $\|u\|_1 = A - B$. Thus, the probability that π is good is p^A, while the number of possible paths having given values of A and B is at most $\sigma(A+B)$. Hence,

$$\mathbb{E}_p N(r) \leq \sum_{\substack{A,B\geq 0:\\ A-B=r}} \sigma(A+B)p^A = \sum_{B\geq 0} \sigma(2B+r)p^{B+r}. \tag{3}$$

For any $\epsilon > 0$, we have $\sigma(k) \leq (\mu+\epsilon)^k$ for k sufficiently large, where $\mu = \mu_d$. Therefore, (3) is at most

$$\sum_{B\geq 0} (\mu+\epsilon)^{2B+r} p^{B+r} = \frac{[(\mu+\epsilon)p]^r}{1-(\mu+\epsilon)^2 p}$$

provided $(\mu+\epsilon)^2 p < 1$ and r is sufficiently large; thus we can choose C' so that the required bound (2) holds for all $r \geq 0$.

The claimed explicit bound in the case $p < (2d-1)^{-2}$ follows similarly from (3) using $\sigma(k) \leq (2d)(2d-1)^{k-1} \leq 2(2d-1)^k$. ∎

Proof of Theorem 1. Couple the bond and plaquette percolation models by considering e to be occupied if and only $\pi(e)$ is occupied. Let $p < \mu_d^{-2}$ and let K be the random set of sites u for which there exists a good path from 0 to u. By Lemma 4, K satisfies (2) with the given constants α, C'.

Since a good path may always be extended by a step towards the origin (provided the new site is not already in the path), K satisfies the condition that $x \in K$ and $y \preceq x$ imply $y \in K$. Therefore, Proposition 3 applies. If e is a bond with exactly one endvertex in K, then by the definition of a good path, it is the end closer to 0 that is in K, and e must be unoccupied. Therefore, all plaquettes in the set S in (1) are unoccupied. Finally, the tail bound in (2) implies the bound in Theorem 1 because $\mathrm{rad}\,[S] \le \mathrm{rad}\,K + d/2$. □

Proof of Corollary 2. Couple the bond and plaquette models as usual. Let $p < \mu_3^{-2}$, and let S be the set of plaquettes from Theorem 1. The sphere $[S]$ intersects no occupied bond and has 0 in its inside, hence it has $[E]$ in its inside. □

4. Critical values

We consider next the critical value of p for the event of Theorem 1, namely the event that there exists a finite set S of unoccupied plaquettes whose union $[S]$ is a sphere with 0 in its inside. In so doing, we shall make use of the definition of a good path from Section 3. We shall frequently regard $\mathbb{Z}^d$ as a graph with bond-set $\mathbb{L}_d$. A directed path of $\mathbb{Z}^d$ is called **oriented** if every step is in the direction of increasing coordinate-value.

Let $d \ge 2$, and (as after Theorem 1) declare a bond of $\mathbb{L}_d$ to be **occupied** with probability p. Let $\theta_{\mathrm{g}}(p)$ be the probability that there exists an infinite good path beginning at the origin. Since θ_{g} is a non-decreasing function, we define a critical value

$$p_{\mathrm{g}} := \sup\{p : \theta_{\mathrm{g}}(p) = 0\}.$$

Note that

$$\mu_d^{-2} \le p_{\mathrm{g}} \le \vec{p}_{\mathrm{c}}, \tag{4}$$

where $\vec{p}_{\mathrm{c}} = \vec{p}_{\mathrm{c}}(d)$ denotes the critical probability of oriented percolation on $\mathbb{Z}^d$. That $\mu_d^{-2} \le p_{\mathrm{g}}$ follows by Theorem 1; the second inequality $p_{\mathrm{g}} \le \vec{p}_{\mathrm{c}}$ holds since every occupied oriented path from 0 is necessarily good.

For $x = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d$, let

$$s(x) := \sum_{i=1}^{d} x_i,$$

and let

$$H_n := \{x \in \mathbb{Z}^d : s(x) = n\};\ H := \{x \in \mathbb{Z}^d : s(x) \ge 0\};\ H_+ := H \setminus H_0.$$

A finite or infinite path $v_0, v_1, \ldots$ is called **admissible** if, for each i satisfying $s(v_{i-1}) < s(v_i)$, the bond with endpoints v_{i-1}, v_i is occupied. If there exists an admissible path from x to y, we write $x \to_{\mathrm{a}} y$; if such a path exists using only sites in some set S, we write $x \to_{\mathrm{a}} y$ in S.

Let $e := (1, 1, \ldots, 1)$ and

$$R := \sup\{n \ge 0 : 0 \to_{\mathrm{a}} ne\}.$$

Let $\theta_{\mathrm{a}}(p) := \mathbb{P}_p(R = \infty)$, with associated critical value

$$p_{\mathrm{a}} := \sup\{p : \theta_{\mathrm{a}}(p) = 0\}.$$

By the definition of admissibility,

$$\theta_{\mathrm{a}}(p) = \mathbb{P}_p(\forall x \in \mathbb{Z}^d,\ 0 \rightarrow_{\mathrm{a}} x), \tag{5}$$

and indeed the associated events are equal.

If $x \rightarrow_{\mathrm{a}} y$ by an admissible path using only sites of H_+ except possibly for the first site x, we write $x \rightarrow_{\mathrm{a}}^{H} y$. We write $x \rightarrow_{\mathrm{a}}^{H} \infty$ if x is the endvertex of some infinite admissible path, all of whose vertices except possibly x lie in H_+. Let $\theta_{\mathrm{a}}^{H}(p) = \mathbb{P}_p(0 \rightarrow_{\mathrm{a}}^{H} \infty)$, with associated critical value

$$p_{\mathrm{a}}^{H} := \sup\{p : \theta_{\mathrm{a}}^{H}(p) = 0\}.$$

Also define the orthant

$$K := \{x \in \mathbb{Z}^d : x_i \geq 0 \text{ for all } i\}.$$

Let $\theta_{\mathrm{a}}^{K}(p)$ be the probability of an infinite admissible path in K starting at 0, and let p_{a}^{K} be the associated critical value. Since $K \subseteq H_+ \cup \{0\}$, we have $\theta_{\mathrm{a}}^{H} \geq \theta_{\mathrm{a}}^{K}$ and $p_{\mathrm{a}}^{H} \leq p_{\mathrm{a}}^{K}$.

Theorem 5. *For $d \geq 2$ we have $p_{\mathrm{a}} \leq p_{\mathrm{a}}^{H} = p_{\mathrm{a}}^{K}$.*

Since every admissible path in the orthant K is good, we have that $p_{\mathrm{g}} \leq p_{\mathrm{a}}^{K}$, and therefore $p_{\mathrm{g}} \leq p_{\mathrm{a}}^{H}$ by Theorem 5. We pose two questions.

Question 1. *For $d \geq 3$, is it the case that $p_{\mathrm{a}} = p_{\mathrm{a}}^{H}$?*

Question 2. *For $d \geq 3$, is it the case that $p_{\mathrm{g}} = p_{\mathrm{a}}^{H}$?*

These matters are resolved as follows when $d = 2$.

Theorem 6. *For $d = 2$ we have $p_{\mathrm{g}} = p_{\mathrm{a}}^{H} = p_{\mathrm{a}} = 1 - \vec{p}_{\mathrm{c}}$, where $\vec{p}_{\mathrm{c}} = \vec{p}_{\mathrm{c}}(2)$ is the critical probability of oriented percolation on $\mathbb{Z}^2$.*

In advance of the proofs, we present a brief discussion of Question 1 above. By Lemma 7 below, one has that

$$\lim_{n\to\infty} \mathbb{P}_p(0 \rightarrow_{\mathrm{a}}^{H} H_n) \begin{cases} = 0 & \text{if } p < p_{\mathrm{a}}, \\ > 0 & \text{if } p > p_{\mathrm{a}}. \end{cases}$$

Now,

$$\begin{aligned}
\mathbb{P}_p(0 \rightarrow_{\mathrm{a}} ne) &\leq \sum_{x \in H_0} \mathbb{P}_p(x \rightarrow_{\mathrm{a}}^{H} ne) \\
&= \sum_{x \in H_0} \mathbb{P}_p(0 \rightarrow_{\mathrm{a}}^{H} ne - x) = \sum_{x \in H_n} \mathbb{P}_p(0 \rightarrow_{\mathrm{a}}^{H} x).
\end{aligned}$$

If one could prove that

$$\sum_{x\in H_n} \mathbb{P}_p(0 \to_{\mathrm{a}}^{H} x) \to 0 \quad \text{as} \quad n \to \infty,$$

whenever $p < p_{\mathrm{a}}^H$, it would follow by Theorem 5 that $p_{\mathrm{a}} = p_{\mathrm{a}}^H$. This is similar to the percolation problem solved by Aizenman–Barsky and Menshikov, [1, 17, 18] (see also [8, Chap. 5]). It seems possible to adapt Menshikov's proof to prove an exponential-decay theorem for *admissible* paths, but perhaps not for admissible connections *restricted to* H.

The proofs of the two theorems above will make heavy use of the next lemma. Let $d \geq 2$ and $0 \leq a < b \leq \infty$. Define the cone $K_{a,b}$ to be the set of sites $x = (x_1, x_2, \dots, x_d) \in \mathbb{Z}^d$ satisfying:

$$x_1 \geq 0, \quad \text{and } ax_1 \leq x_j \leq bx_1 \text{ for } j = 2, 3, \dots, d. \tag{6}$$

It is easy to see that $K_{a,b}$ comprises a unique infinite component, denoted $I(K_{a,b})$, together with a finite number of finite components.

Lemma 7. *For $d \geq 2$, let $p > p_{\mathrm{a}}^H$ and $0 \leq a < b \leq \infty$. Then*

$$\mathbb{P}_p(K_{a,b} \text{ contains some infinite admissible path}) = 1,$$

and for all $v \in I(K_{a,b})$,

$$\mathbb{P}_p(v \to_{\mathrm{a}} \infty \text{ in } K_{a,b}) > 0.$$

The remainder of this section is set out as follows. First, we deduce Theorems 5 and 6 from Lemma 7. The proof of Lemma 7 is not presented in this paper, since it would be long and would repeat many constructions found elsewhere. Instead, this section ends with some comments concerning that proof.

Proof of Theorem 5. As noted before the statement of Theorem 5, $K \subseteq H_+ \cup \{0\}$, whence $p_{\mathrm{a}}^H \leq p_{\mathrm{a}}^K$. Let $p > p_{\mathrm{a}}^H$. By Lemma 7 with $a = 0$ and $b = \infty$, we have $\theta_{\mathrm{a}}^K(p) > 0$, so that $p \geq p_{\mathrm{a}}^K$. Therefore, $p_{\mathrm{a}}^H = p_{\mathrm{a}}^K$.

There is more than one way of showing $p_{\mathrm{a}} \leq p_{\mathrm{a}}^H$, of which the following is one. Let $d \geq 3$; the proof is similar when $d = 2$. Let $p > p_{\mathrm{a}}^H$ and fix $0 < a < b < \infty$ arbitrarily. By Lemma 7, $K_{a,b}$ contains a.s. some infinite admissible path π. Any infinite path π in $K_{a,b}$ has the property that, for all $x \in \mathbb{Z}^d$, there exists $z \in \pi$ with $x \leq z$ (in that $x_i \leq z_i$ for every coordinate i). By (5), $\theta_{\mathrm{a}}(p) > 0$, so that $p \geq p_{\mathrm{a}}$ and $p_{\mathrm{a}}^H \geq p_{\mathrm{a}}$ as claimed. □

Proof of Theorem 6. We shall make extensive use of two-dimensional duality. We call the graph $\mathbb{Z}^2$ the **primal lattice**, and we call the shifted graph $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ the **dual lattice**. Thus, the dual bonds are precisely the plaquettes of Π_2. Recall that a dual bond is declared occupied if and only if the primal bond that crosses it is occupied. For consistency with standard terminology, we now call a dual bond **open** if and only if it is *unoccupied* (so a dual bond is open with probability $q := 1-p$). We assign directions to dual bonds as follows: a horizontal dual bond is directed from left to right, and a vertical bond from top to bottom.

Consider the sets

$$\begin{aligned} D^+ &:= \{(-u,u) + (-\tfrac{1}{2}, \tfrac{1}{2}) : u \geq 0\}, \\ D^- &:= \{(u,-u) + (\tfrac{1}{2}, -\tfrac{1}{2}) : u \geq 0\}, \end{aligned}$$

of dual sites. The primal origin 0 lies in some infinite admissible path of H^+ if and only if no site of D^+ is connected by an directed open dual path of the north–east half-plane $\{(x,y) : x+y \geq 0\}$ to some site of D^-. If $1-p < \vec{p}_{\mathrm{c}}$ (respectively, $1-p > \vec{p}_{\mathrm{c}}$) the latter occurs with strictly positive probability (respectively, probability 0). Therefore, $p_{\mathrm{a}}^H = 1 - \vec{p}_{\mathrm{c}}$.

We show next that $p_{\mathrm{g}} = p_{\mathrm{a}}^H$. Since $p_{\mathrm{g}} \leq p_{\mathrm{a}}^H$, it suffices to show that $p_{\mathrm{g}} \geq 1 - \vec{p}_{\mathrm{c}}$. Let $p < 1 - \vec{p}_{\mathrm{c}}$, so that $1-p > \vec{p}_{\mathrm{c}}$. We shall prove the required inequality $p \leq p_{\mathrm{g}}$. Define the set of dual sites

$$Q := \{(x,y) + (\tfrac{1}{2}, \tfrac{1}{2}) : x, y \geq 0\}.$$

Let $B(k) \subseteq Q$ be given by $B(k) := [0,k]^2 + (\frac{1}{2}, \frac{1}{2})$. For $n \geq 0$, let C_n be the event that there exists a directed open dual path from $v_n := (n,0) + (\frac{1}{2}, \frac{1}{2})$ to $w_n := (0,n) + (\frac{1}{2}, \frac{1}{2})$ lying entirely within the region $B(n) \setminus B(\frac{1}{3}n)$. We claim that there exists $\beta > 0$ such that

$$\mathbb{P}_p(C_n) \geq \beta, \qquad n \geq 1, \tag{7}$$

and the proof of this follows.

Let V_n be the event that there exists a directed open dual path from v_n to the line $\{(n,k) + (\frac{1}{2}, \frac{1}{2}) : 0 \leq k \leq n\}$ lying entirely within the cone $\{(x,y) + (\frac{1}{2}, \frac{1}{2}) : 0 \leq n-y \leq x,\ x \geq 0\}$; let W_n be the event that such a path exists to w_n from some site on the line $\{(k,n) + (\frac{1}{2}, \frac{1}{2}) : 0 \leq k \leq n\}$, this path lying entirely within $\{(x,y) + (\frac{1}{2}, \frac{1}{2}) : n-x \leq y < \infty,\ x \leq n\}$. By Lemma 7, $\mathbb{P}_p(V_n) > 0$. By reversing the directions of all dual bonds, we see that $\mathbb{P}_p(W_n) = \mathbb{P}_p(V_n)$. On the event $V_n \cap W_n$, there exists a directed open dual path of $B(n) \setminus B(\frac{1}{3}n)$ from v_n to w_n, and hence, by the Harris–FKG inequality,

$$\mathbb{P}_p(C_n) \geq \mathbb{P}_p(V_n)\mathbb{P}_p(W_n) > 0,$$

as required for (7).

By considering corresponding events in the other three quadrants of $\mathbb{Z}^2$ (with appropriately chosen bond-orientations), we conclude that each annulus of $\mathbb{R}^2$ with inner (respectively, outer) ℓ^∞-radius $\frac{1}{3}n$ (respectively, $n+\frac{1}{2}$) contains, with probability at least $(1-p)^4\beta^4$, a dual cycle blocking good paths from the origin. It follows that $p \le p_{\mathrm{g}}$ as required.

Finally we show that $p_{\mathrm{a}} = 1-\vec{p}_{\mathrm{c}}$. Since $p_{\mathrm{a}} \le p_{\mathrm{a}}^H$ by Theorem 5, we have only to show that $p_{\mathrm{a}} \ge 1-\vec{p}_{\mathrm{c}}$. This follows by (5) and the fact that, when $q = 1-p > \vec{p}_{\mathrm{c}}$, there exists $\mathbb{P}_p$-a.s. a doubly-infinite directed open dual path intersecting the positive y-axis. Here is a proof of the latter assertion. Let $\psi(q)$ be the probability that there exists an infinite oriented path from the origin in oriented percolation with density q. By reversing the arrows in the fourth quadrant, the probability that 0 lies in a doubly-infinite directed open path of $\mathbb{Z}^2$ is ψ^2. The event

$$J := \{\text{there exists a doubly-infinite open dual path}\}$$

is a zero–one event and $\mathbb{P}_p(J) \ge \psi^2$, so that $\mathbb{P}_p(J) = 1$. Let J^+ (respectively, J^-) be the event that such a path exists and intersects the positive (respectively, the non-positive) y-axis. By reversing the directions of bonds, we have that $\mathbb{P}_p(J^+) = \mathbb{P}_p(J^-)$. By the Harris–FKG inequality,

$$0 = \mathbb{P}_p(\overline{J}) = \mathbb{P}_p(\overline{J^+} \cap \overline{J^-}) \ge \mathbb{P}_p(\overline{J^+})\mathbb{P}_p(\overline{J^-}) = \mathbb{P}_p(\overline{J^+})^2,$$

so that $\mathbb{P}_p(J^+) = 1$. $\square$

We finish with some comments on Lemma 7. This may be proved by the dynamic-renormalization arguments developed for percolation in [5, 11], for the contact model in [6], and elaborated for directed percolation in [9]. An account of dynamic renormalization for percolation may be found in [8]. The proof of Lemma 7 is omitted, since it requires no novelty beyond the above works, but extensive duplication of material therein. The reader is directed mainly at [9], since the present lemma involves a model in which the edge-orientations are important. The method yields substantially more than the statement of the lemma, but this is not developed here.

Three aspects of the proof are highlighted, since they involve minor variations on the method of [9]. First, the box $B_{L,K}$ of [9, Sect. 4] is replaced by

$$B_{l,k} := \{x \in H : s(x) \le k,\ \|x\|_1 \le l\},$$

with an amended version of [9, Lemma 4.1]. Secondly, the current proof uses the technique known as 'sprinkling', as at the corresponding point of the proofs presented in [11] and [8, Sect. 7.2].

Finally, since Lemma 7 is concerned with admissible paths in subcones of the orthant K, we require a straightforward fact about oriented percolation on $\mathbb{Z}^2$, namely that the associated critical probability is strictly less than 1. This weak statement leads via renormalization to the stronger Lemma 7. Consider oriented bond percolation on $\mathbb{Z}^2$ with edge-probability p. We write $u \to v$ if there exists an open oriented path from u to v, and $u \to \infty$ if u is the first site of some infinite oriented open path.

Lemma 8. *Let $0 \le a < b \le \infty$, and let $K_{a,b}$ be the cone of $\mathbb{Z}^2$ containing all sites (x, y) with $ax \le y \le bx$ and $x \ge 0$. There exists $\epsilon = \epsilon_{a,b} > 0$ such that: if $p > 1 - \epsilon$,*

$$\mathbb{P}_p(K_{a,b} \text{ contains some infinite open oriented path}) = 1,$$

and for all $v \in I(K_{a,b})$,

$$\mathbb{P}_p(v \to \infty \text{ in } K_{a,b}) > 0.$$

Proof. Let $r = r_2/r_1$, $s = s_2/s_1$ be rationals satisfying $a < r < s < b$, and write $R = (r_1, r_2)$, $S = (s_1, s_2)$. Let $v \in I(K_{a,b})$, and consider the set $v_{i,j} := v + iR + jS$, $i, j \ge 0$, of sites of $K_{a,b}$. Choose an oriented path π_1 (respectively, π_2) of $K_{a,b}$ from v to $v + R$ (respectively, $v + S$) such that the last bond is horizontal (respectively, vertical). The paths π_1 and π_2 may have bonds in common. Let $\alpha = p^N$ where N is the number of bonds in either π_1 or π_2. We declare $v_{i,j}$ **black** if all bonds in both $v_{i,j} + \pi_1$ and $v_{i,j} + \pi_2$ are open. Note that the states of different sites $v_{i,j}$ are independent.

If $1-\alpha < (1-\vec{p}_c)^2$, the set of black vertices dominates (stochastically) the set of sites w of a supercritical oriented percolation model with the property that both bonds directed away from w are open. The claims of the lemma follow by standard properties of oriented percolation. □

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(G. R. Grimmett) Statistical Laboratory, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 0WB, UK

E-mail address: g.r.grimmett@statslab.cam.ac.uk

URL: http://www.statslab.cam.ac.uk/∼grg/

(A. E. Holroyd) Microsoft Research, 1 Microsoft Way, Redmond WA 98052, USA; and Department of Mathematics, University of British Columbia, 121–1984 Mathematics Road, Vancouver, BC V6T 1Z2, Canada

E-mail address: holroyd at math.ubc.ca

URL: http://math.ubc.ca/∼holroyd/