Conditional distributions, exchangeable particle systems, and stochastic partial differential equations

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27 May 2010

Abstract

Stochastic partial differential equations whose solutions are probability-measure-valued processes are considered. Measure-valued processes of this type arise naturally as de Finetti measures of infinite exchangeable systems of particles and as the solutions for filtering problems. In both these cases, the solution is the conditional distribution of the solution of a stochastic differential equation. The main result states that, under mild nondegeneracy conditions on the coefficients of the stochastic differential equation, the conditional distribution of its solution charges any open set. Under stronger conditions we show that it is absolutely continuous with respect to Lebesgue measure and its density is positive almost everywhere. As applications we show the existence of a solution of a system of interacting diffusions and study the properties of the solution of the nonlinear filtering equation within a framework that allows for the signal noise and the observation noise to be correlated. The work was motivated by a model of asset price determination in which the price is given as a quantile of the valuations of infinitely many individual investors.

MSC 2010 subject classifications: 60H15, 60G09, 60G35, 60J25

Keywords: conditional distributions, stochastic partial differential equations, quantile processes, filtering equations, measure-valued processes

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and (E, r) a complete separable metric space. Let B and W be d and d'-dimensional standard Brownian motions, and let V be a cadlag E-valued

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process. We assume that B is independent of (W, V) and that W is compatible with V in the sense that for each $t \geq 0$, $W_{t+\cdot} - W_t$ is independent of $\mathcal{F}_t^{W,V}$, where $\mathcal{F}_t^{W,V} = \sigma(W_s, V_s, s \leq t)$. Let X be a d-dimensional stochastic process satisfying the equation

$$X_t = X_0 + \int_0^t f\left(X_s, V_s\right) ds + \int_0^t \sigma\left(X_s, V_s\right) dW_s + \int_0^t \bar{\sigma}\left(X_s, V_s\right) dB_s. \tag{1.1}$$

We assume that, given V_0, X_0 is conditionally independent of W, V and B, that is,

$$E[f(X_0)|\mathcal{F}_\infty^{W,V,B}] = E[f(X_0)|V_0]. \tag{1.2}$$

For reasons that we will make clear below, we are interested in the $\mathcal{P}(\mathbb{R}^d)$-valued process $\pi = \{\pi_t,\ t \geq 0\}$, where π_t is the conditional distribution of X_t given $\mathcal{F}_t^{W,V}$,

$$\pi_t\left(\varphi\right) = E\left[\varphi\left(X_t\right)|\mathcal{F}_t^{W,V}\right],$$

for any $\varphi \in B(\mathbb{R}^d)$, the bounded, Borel-measurable functions on $\mathbb{R}^d$.

The first result of the paper states that, under very general nondegeneracy and regularity conditions (Assumption **A1** below), for $t > 0$, π_t charges any open set $A \subset \mathbb{R}^d$ almost surely (and the null set can be chosen independent of A). Further, under additional conditions on the coefficients of (1.1), π_t is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$ and, with probability one, its density is strictly positive.

Our primary interest in these results is to treat infinite systems of stochastic differential equations

$$X_t^i = X_0^i + \int_0^t f\left(X_s^i, V_s\right) ds + \int_0^t \sigma\left(X_s^i, V_s\right) dW_s + \int_0^t \bar{\sigma}\left(X_s^i, V_s\right) dB_s^i, \tag{1.3}$$

where the B^i are independent standard Brownian motions, $\{X_0^i\}$ is an exchangeable sequence that is independent of W and $\{B^i\}$, and

$$V_t = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \delta_{X_t^i}. \tag{1.4}$$

We require the solution $\{X^i\}$ to be exchangeable so that the limit in (1.4) exists by deFinetti's theorem. (See Theorem A.1.) In particular, if the solution of the system is weakly unique, then $\{X^i\}$ must be exchangeable, so (1.4) must exist. Under a Lipschitz condition on the first variable and a continuity condition on the second, weak existence of an exchangeable solution can be shown for which the B^i are independent of W and V. By exchangeability,

$$\pi_t(\varphi) = E[\varphi(X_t^1)|\mathcal{F}_t^{W,V}] = \frac{1}{n} E\left[\sum_{i=1}^n \varphi(X_t^i)|\mathcal{F}_t^{W,V}\right]$$

and since V_t is measurable with respect to $\mathcal{F}_t^{W,V}$ it follows that

$$V_t(\varphi) = E\left[\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\varphi(X_t^i)|\mathcal{F}_t^{W,V}\right] = \lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i=1}^{n}\varphi(X_t^i)|\mathcal{F}_t^{W,V}\right] = \pi_t(\varphi).$$

If strong uniqueness holds,

$$V_t(\varphi) = E\left[\varphi\left(X_t\right)|\mathcal{F}_t^{W} \vee \sigma(V_0)\right].$$

In (1.3), the process W is common to all diffusions, while the processes B^i, $i \geq 1$ are mutually independent Brownian motions. Systems of this type have been considered by Kurtz and Protter [5] and Kurtz and Xiong [6, 7] under the assumption that the coefficients are Lipschitz functions of V in the Wasserstein metric on $\mathcal{P}(\mathbb{R}^d)$. This assumption excludes a variety of interesting examples. In particular, for $d = 1$, we are interested in equations whose coefficients are functions of quantiles of V,

$$V_t^{\alpha} = \inf_{x\in\mathbb{R}}\left\{x \in \mathbb{R}|V_t\left(-\infty, x\right] \geq \alpha\right\},$$

and the results on π_t play a central role in proving existence of solutions of a system in which the coefficients are continuous functions of the quantiles.

A second application of the support results is to the solution of stochastic filtering problems. Let (X, Y) be the solution of

$$\begin{aligned} X_t &= X_0 + \int_0^t f\left(X_s, Y_s\right)ds + \int_0^t \sigma\left(X_s, Y_s\right)dW_s + \int_0^t \bar{\sigma}\left(X_s, Y_s\right)dB_s \\ Y_t &= \int_0^t h\left(X_s, Y_s\right)ds + \int_0^t k\left(Y_s\right)dW_s. \end{aligned}$$

Here Y plays the role of V, so B is not independent of (W, Y). Assuming that $k(y)$ is invertible and setting

$$\widetilde{W}_t = W_t + \int_0^t k(Y_s)^{-1}h(X_s, Y_s)ds,$$

we have

$$\begin{aligned} X_t &= X_0 + \int_0^t \left(f(X_s, Y_s) + \sigma(X_s, Y_s)k(Y_s)^{-1}h(X_s, Y_s)\right)ds \\ &\quad + \int_0^t \sigma(X_s, Y_s)d\widetilde{W}_s + \int_0^t \bar{\sigma}\left(X_s, Y_s\right)dB_s \\ Y_t &= \int_0^t k\left(Y_s\right)d\widetilde{W}_s, \end{aligned}$$

and under modest assumptions on $h(x, y)/k(y)$, a Girsanov change of measure gives an equivalent probability measure under which B is independent of $(\widetilde{W}, Y)$. In this framework we show that the conditional distribution of X_t given $\mathcal{F}_t^Y$ charges any open set. Moreover, under additional conditions, it is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$, and with probability one its density is strictly positive.

The main results are proved under the following conditions on the coefficients of (1.1).

A1 $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^{d'} \times \mathbb{R}^d$, $\bar{\sigma} : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d \times \mathbb{R}^d$ are continuous functions, uniformly Lipschitz in the first argument. That is, there exists a constant c_1 such that

$$|f(x_1, y) - f(x_1, y)| \leq c_1 |x_1 - x_2|$$

for all $x_1, x_2 \in \mathbb{R}^d$ and $y \in \mathbb{R}^m$ with a similar inequality holding for σ and $\bar{\sigma}$.

$\bar{\sigma}$ is positive definite, i.e.,

$$\xi^\top \bar{\sigma}(x, y)\, \xi > 0$$

for any $y \in \mathbb{R}^m$ and $\xi, x \in \mathbb{R}^d$ with $\xi \neq 0$.

For $d > 1$, almost surely, V has paths with finite left limits. In other words, for all $t > 0$,

$$V_{t-} \stackrel{\text{def}}{=} \lim_{s \to t, s < t} V_t$$

exists and is finite.

A2 f, σ and $\bar{\sigma}$ are continuously differentiable in the first component.

Theorem 1.1 *Under assumption **A1**, there exists a set $\widetilde{\Omega} \in \mathcal{F}$ of full measure such that for every $\omega \in \widetilde{\Omega}$, π_t^ω charges every open set, i.e., $\pi_t^\omega(A) > 0$ for every nonempty, open set A.*

Theorem 1.2 *Under assumptions **A1+A2**, there exists a set $\widetilde{\Omega} \in \mathcal{F}$ of full measure such that for every $\omega \in \widetilde{\Omega}$, π_t^ω is absolutely continuous with respect to Lebesgue measure. Moreover if $y \to \rho_t^\omega(y)$ is the density of π_t^ω with respect to Lebesgue measure, ρ_t^ω is strictly positive.*

The additional condition in **A1** required to treat the multi-dimensional case is not needed when $d = 1$ because we are able to exploit the order structure of $\mathbb{R}$. For $d > 1$, the integral of a nonsingular, matrix-valued function may be singular, while for $d = 1$, the integral of a non-zero real-valued function is always non-zero, *provided it does not change sign.*

Acknowledgements. This work was completed while the first two authors were visiting the Isaac Newton Institute in Cambridge, UK. The hospitality and support provided by the Institute is gratefully acknowledged. The research of the first author was partially supported by the EPSRC grant EP/H000550/1. The research of the second author was supported in part by NSF grant DMS 08-0579.

2 Proof of the properties of the conditional distributions

Let F be a function $F:[0,\infty)\times\mathbb{R}^d\to\mathbb{R}^d$ with the following properties:

- For each $z\in\mathbb{R}^d$, the function $t\to F(t,z)$ is a measurable, locally-bounded function.
- For each $t\in[0,\infty)$, the function $z\to F(t,z)$ is differentiable. $F'(t,z)$ will denote the matrix of partial derivatives

$$(F'(t,z))_{ij}=\partial_j F_i(t,z).$$

- For each $z\in\mathbb{R}^d$, the function $t\to F'(t,z)$ is a measurable, locally-bounded function.

Now consider a new probability measure P^z, absolutely continuous with respect to P, defined by

$$\left.\frac{dP^z}{dP}\right|_{\mathcal{F}_t}=\exp\left(-\int_0^t F(s,z)^\top dB_s-\frac{1}{2}\int_0^t |F(s,z)|^2 ds\right),$$

where $F(s,z)^\top$ is the row vector $(F(s,z)_1,F(s,z)_2,...,F(s,z)_d)$. Then, by Girsanov's theorem, the process $B^z=\{B^z_t,t\geq 0\}$

$$B^z_t=B_t+\int_0^t F(s,z)\,ds$$

is a Brownian motion under P^z, independent of W and V. Since (B^z,W,V) has the same law under P^z as (B,W,V) has under P, it follows that $X(z)$ given by

$$\begin{aligned} dX_t(z) &= f(X_t(z),V_t)\,dt+\sigma(X_t(z),V_t)\,dW_t+\bar\sigma(X_t(z),V_t)\,dB^z_t \qquad (2.1)\\ &= f(X_t(z),V_t)\,dt+\sigma(X_t(z),V_t)\,dW_t+\bar\sigma(X_t(z),V_t)\,dB_t\\ &\quad+\bar\sigma(X_t(z),V_t)\,F(t,z)\,dt. \end{aligned}$$

has the same law under P^z as X has under P, and for $\varphi\in B(\mathbb{R}^d)$,

$$\begin{aligned} E\left[\varphi(X_t)\,|\mathcal{F}^{W,V}_t\right] &= E^z\left[\varphi(X_t(z))|\,\mathcal{F}^{W,V}_t\right]\\ &= E\left[\varphi(X_t(z))\,M_t(z)\,|\mathcal{F}^{W,V}_t\right] \end{aligned}$$

where $M_t(z)$ is defined as

$$M_t(z)=\exp\left(-\int_0^t F(s,z)^\top dB_s-\frac{1}{2}\int_0^t |F(s,z)|^2 ds\right),\quad t\geq 0. \qquad (2.2)$$

In the following, we will use a Fubini argument for the function ι, where

$$(z,\omega) \xrightarrow{\iota} \varphi\left(X_t\left(z\right)\right) M_t\left(z\right) \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{\frac{d}{2}}}$$

is defined on the product space $\mathbb{R}^d \times \Omega$. Consequently, we need to know that ι is $\mathcal{B}(\mathbb{R}^d) \times \mathcal{F}_t^{W,V}$-measurable. Measurability is not immediate as $X_t\left(z\right)$ is initially defined for each z individually. However, one can prove the existence of a process $\bar{X}_t\left(z\right)$ such that for each z, $\bar{X}\left(z\right)$ and $X\left(z\right)$ are indistinguishable and

$$(z,\omega) \xrightarrow{\bar{\iota}} \bar{X}_t\left(z\right)$$

is $\mathcal{B}(\mathbb{R}^d) \times \mathcal{F}_t^{W,V}$-measurable. More precisely, we can assume that $\bar{X}$ is optional, that is, the mapping

$$(t,z,\omega) \in [0,\infty) \times \mathbb{R}^d \times \Omega \to \bar{X}_t(z)$$

is measurable with respect to the σ-algebra generated by processes of the form

$$\sum \xi_i f_i(z) \mathbf{1}_{[t_i,t_{i+1})}(t),$$

where $0 = t_0 < t_1 < \cdots$, $f_i \in C(\mathbb{R}^d)$, and ξ_i is $\mathcal{F}_{t_i}^{W,V}$-measurable. To avoid further measurability complications, from now on, we will use this version of the solution of (2.1). Hence, if $\varphi : \mathbb{R}^d \times \Omega \to \mathbb{R}$ is a non-negative, $\mathcal{B}(\mathbb{R}^d) \times \mathcal{F}_t^{W,V}$-measurable function, the conditional version of Fubini's theorem (for nonnegative functions) gives

$$\begin{aligned} E\left[\varphi\left(X_t,\cdot\right)|\mathcal{F}_t^{W,V}\right] &= \int_{\mathbb{R}^d} E\left[\varphi\left(X_t\left(z\right),\cdot\right) M_t\left(z\right)|\mathcal{F}_t^{W,V}\right] \frac{e^{-\frac{1}{2}z^\top z}}{(2\pi)^{\frac{d}{2}}} dz \\ &= E\left[\int_{\mathbb{R}^d} \varphi\left(X_t\left(z\right),\cdot\right) M_t\left(z\right) \frac{e^{-\frac{1}{2}z^\top z}}{(2\pi)^{\frac{d}{2}}} dz|\mathcal{F}_t^{W,V}\right]. \end{aligned} \tag{2.3}$$

We treat the one-dimensional and the multi-dimensional cases separately.

2.1 The one-dimensional case

For $t_0 > 0$, let

$$F\left(t,z\right) = \begin{cases} z & \text{for } t \in [0,t_0] \\ 0 & \text{otherwise} \end{cases}, \tag{2.4}$$

where $a > 0$ is an arbitrary positive constant. In this case, (2.1) becomes

$$\begin{aligned} dX_t(z) = f\left(X_t\left(z\right),V_t\right)dt + \sigma\left(X_t\left(z\right),V_t\right)dW_t + \bar{\sigma}\left(X_t\left(z\right),V_t\right)dB_t \\ +\bar{\sigma}\left(X_t\left(z\right),V_t\right) z \mathbf{1}_{[0,t_0]}(t)dt. \end{aligned} \tag{2.5}$$

Since $\bar{\sigma}$ is positive, with probability 1, the function $z \to X_t(z)$ is a strictly increasing, continuous function and $\lim_{z\to-\infty} X_t(z) = -\infty$ and $\lim_{z\to\infty} X_t(z) = \infty$. In particular, $z \to$

$X_t(z)$ is a continuous bijection, so if $(\underline{\beta},\overline{\beta})$ is a (non-empty) open interval, then $X_t^{-1}(\underline{\beta},\overline{\beta})$ is a non-empty open interval. In particular, $X_t^{-1}(\underline{\beta},\overline{\beta})$ has positive Lebesgue measure. Hence, if we choose φ in (2.3) to be the indicator function of an open interval $(\underline{\beta},\overline{\beta})$, then

$$P\left[X_t \in (\underline{\beta},\overline{\beta})|\mathcal{F}_t^{W,V}\right] = \frac{1}{\sqrt{2\pi}}E\left[\int_{X_t^{-1}(\underline{\beta},\overline{\beta})} e^{-zB_t-\frac{z^2(t+1)}{2}}dz|\mathcal{F}_t^{W,V}\right].$$

Since $z \to e^{-zB_t-\frac{z^2(t+1)}{2}}$ is positive on $X_t^{-1}(\underline{\beta},\overline{\beta})$, it follows that $\int_{X_t^{-1}(\underline{\beta},\overline{\beta})} e^{-zB_t-\frac{z^2(t+1)}{2}}dz$ is positive (with probability 1) as is its conditional expectation. *This proves Theorem 1.1* in the case $d=1$.

Assuming that f,σ and $\bar{\sigma}$ are differentiable, $z \to X_t(z)$ is differentiable with probability 1. Its (positive) derivative is given by

$$J_t(z) \stackrel{\text{def}}{=} \frac{dX_t(z)}{dz} = \int_0^t \bar{\sigma}(X_s(z),V_s)\exp\left(i_s^t(z)\right)ds, \tag{2.6}$$

where

$$\begin{aligned} i_s^t(z) &= \int_s^t \left(f'(X_s(z),V_s) - \frac{1}{2}(\sigma'(X_s(z),V_s))^2 - \frac{1}{2}(\bar{\sigma}'(X_s(z),V_s))^2\right)ds \\ &\quad + \int_s^t \sigma'(X_s(z),V_s)\,dW_s + \int_s^t \bar{\sigma}'(X_s(z),V_s)\,dB_s + \int_s^t \bar{\sigma}'(X_s(z),V_s)\,azds. \end{aligned}$$

Now, since $z \to X_t(z)$ is a bijection, it is invertible, and we can define

$$\nu_t(y) = \frac{\exp\left\{-X_t^{-1}(y)\,B_t - \frac{\left(X_t^{-1}(y)\right)^2(t+1)}{2}\right\}}{J_t\left(X_t^{-1}(y)\right)}.$$

Taking $\varphi = \mathbf{1}_A$, $A \in \mathcal{B}(\mathbb{R})$, in (2.3) and using the change of variable $y = X_t(z)$,

$$P\left[X_t \in A|\mathcal{F}_t^{W,V}\right] = \frac{1}{\sqrt{2\pi}}E\left[\int_A \nu_t(y)dy\middle|\,\mathcal{F}_t^{W,V}\right] = \frac{1}{\sqrt{2\pi}}\int_A E\left[\nu_t(y)|\,\mathcal{F}_t^{W,V}\right]dy.$$

Hence, the conditional distribution of X_t given $\mathcal{F}_t^{W,V}$ is absolutely continuous with respect to Lebesgue measure with density

$$\rho_t(y) = \frac{1}{\sqrt{2\pi}}E\left[\nu_t(y)|\,\mathcal{F}_t^{W,V}\right].$$

Since $\nu_t(y)$ is strictly positive, by Lemma A.12, there exists a version of $\rho_t(y)$ such that with probability one, $\rho_t(y) > 0$ for all $y \in \mathbb{R}$ and $t \geq 0$. *This proves Theorem 1.2* in the case $d=1$.

Corollary 2.1 *Under assumptions **A1+A2**, there exists a random variable $c(s,t,k)$ positive almost surely such that*

$$\inf_{(r,y)\in[s,t]\times[-k,k]} \rho_r(y) \geq c(s,t,k). \tag{2.7}$$

In particular, the set $\widetilde{\Omega} \in \mathcal{F}$ of full measure appearing in the statement of Theorem 1.2 on which π_t^ω is absolutely continuous with respect to Lebesgue measure and the density of π_t^ω with respect to Lebesgue measure is strictly positive can be chosen independent of the time variable $t \in (0,\infty)$.

Proof. Using the independence properties of X_0, B, W, and V, we have

$$E[f(X_0,B)|\mathcal{F}_\infty^{W,V}] = E[f(X_0,B)|V_0],$$

for any reasonable function f, hence there exists h_f such that

$$E[f(X_0,B,W_{\cdot\wedge t},V_{\cdot\wedge t})|\mathcal{F}_\infty^{W,V}] = h_f(V_0,W_{\cdot\wedge t},V_{\cdot\wedge t})).$$

Since $\nu_t(y)$ is a function of X_0, B, $W_{\cdot\wedge t}$ and $V_{\cdot\wedge t}$, this implies that

$$\rho_t(y) = \frac{1}{\sqrt{2\pi}} E\left[\nu_t(y)|\,\mathcal{F}_t^{W,V}\right] = \frac{1}{\sqrt{2\pi}} E\left[\nu_t(y)|\,\mathcal{F}_\infty^{W,V}\right].$$

Choose m to be an arbitrary positive constant. Since the function $(t,x) \to \min(\nu_t(x),m)$ is bounded, positive and jointly continuous in (t,x) it follows that its conditional expectation

$$\rho_t^m(y) = \frac{1}{\sqrt{2\pi}} E\left[\min(\nu_t(x),m)|\,\mathcal{F}_\infty^{W,V}\right]$$

has a version which is bounded, positive and jointly continuous in (t,x). Hence, (2.7) holds true with $c(s,t,k) = \inf_{(r,y)\in[s,t]\times[-k,k]} \rho_r^m(y) > 0$. □

Lemma 2.2 *Under condition A1+A2, the density function $y \to \rho_t(y)$ is absolutely continuous. Moreover, it is differentiable almost everywhere and*

$$\frac{d\rho_t}{dy}(y) = \frac{1}{\sqrt{2\pi}} E\left[\left.\frac{d\nu_t}{dy}(y)\right|\mathcal{F}_t^{W,V}\right]. \tag{2.8}$$

More generally, if f, σ and $\bar{\sigma}$ are m-times continuously differentiable in the first component, then the density function $y \to \rho_t(y)$ is $(m-1)$-times continuously differentiable and m-times differentiable almost everywhere. A similar formula to (2.8) holds for higher derivative of ρ_t as well.

Proof. The function $y \to \nu_t(y)$ is continuously differentiable under condition A1+A2 and

$$\frac{d\nu_t(y)}{dy} = \iota_t^1(x) - \iota_t^2(x), \tag{2.9}$$

where

$$\begin{aligned}
\iota_t^1(x) &= \frac{\exp\left\{-X_t^{-1}(y)B_t - \frac{\left(X_t^{-1}(y)\right)^2(t+1)}{2}\right\}}{J_t\left(X_t^{-1}(y)\right)} \frac{B_t - X_t^{-1}(y)(t+1)}{J_t\left(X_t^{-1}(y)\right)} \\
\iota_t^2(x) &= \frac{\exp\left\{-X_t^{-1}(y)B_t - \frac{\left(X_t^{-1}(y)\right)^2(t+1)}{2}\right\}}{J_t\left(X_t^{-1}(y)\right)} \frac{\frac{dJ_t}{dx}(X_t^{-1}(y))}{J_t\left(X_t^{-1}(y)\right)^2}
\end{aligned}$$

We want to prove that

$$E\left[\int_{\mathbb{R}}\left|\frac{d\nu_t(y)}{dx}\right|dy\right] < \infty.$$

In order to do that, we show that the property holds for both functions on the right hand side of (2.9). We show how this is done for the first function. We have that

$$\begin{aligned}
E\left[\int_{\mathbb{R}}|\iota_t^1(y)|dy\right] &= E\left[\int_{\mathbb{R}}\frac{\exp\left\{-zB_t - \frac{z^2(t+1)}{2}\right\}}{J_t(z)}|B_t - z(t+1)|dz\right] && (2.10)\\
&= \int_{\mathbb{R}} e^{-\frac{z^2(t+1)}{2}} E\left[e^{-pzB_t}\right]^{\frac{1}{p}} E\left[J_t(z)^{-q}\right]^{\frac{1}{q}} E\left[|B_t - z(t+1)|^r\right]^{\frac{1}{r}} dz && (2.11)\\
&- \int_{\mathbb{R}} e^{-\frac{z^2((t+1-pt)}{2}} Q_r(|z|)^{\frac{1}{r}} E\left[J_t(z)^{-q}\right]^{\frac{1}{q}} dz, && (2.12)
\end{aligned}$$

where $p, q, r \in (1, \infty)$ are chosen so that $p < \frac{t+1}{t}$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ and Q_r is a suitably chosen polynomial so that $E\left[|B_t - z(t+1)|^r\right] \le Q_r(|z|)$ for any $z \in \mathbb{R}$. To get (2.10), we used the change of variable $z = X_t^{-1}(y)$ that and applied Hölder's inequality to obtain (2.11). From (2.6) it follows that

$$J_t \ge tac_{\bar\sigma} \exp\left(-tc_{f,\sigma,\bar\sigma} - tc'_{\bar\sigma}|z| - 2\sup_{s\in[0,t]}|C_s|\right), \tag{2.13}$$

where C is the martingale

$$C_s = \int_0^s \sigma'(X_s(z), V_s)\,dW_s + \int_0^s \bar\sigma'(X_s(z), V_s)\,dB_s, \quad s \in [0,t].$$

In (2.13) we used the fact that $c_{\bar\sigma} \stackrel{\text{def}}{=} \inf_{x,y}\bar\sigma(x,y) > 0$ and that

$$\begin{aligned}
c_{f,\sigma,\bar\sigma} &\stackrel{\text{def}}{=} \sup_{x,y}|f'(x,y) - \frac{1}{2}\sigma'(x,y)^2 - \frac{1}{2}\bar\sigma'(x,y)^2| \\
c'_{\bar\sigma} &\stackrel{\text{def}}{=} \sup_{x,y}|\bar\sigma'(x,y)|
\end{aligned}$$

are finite quantities. This follows from conditions A1+A2. Hence, immediately,

$$E\left[J_t(z)^{-q}\right] \leq k e^{qtc'_{\bar{\sigma}}|z|}, \tag{2.14}$$

where

$$k = (tac_{\bar{\sigma}})^{-q} \exp\left(qtc_{f,\sigma,\bar{\sigma}}\right) E\left[\exp\left(2q \sup_{s\in[0,t]} |C_s|\right)\right].$$

Note that k is finite as the running maximum of the martingale C has exponential moments of all orders. From (2.12) and (2.14) we deduce immediately the integrability of ι_t^1. The integrability of ι_t^2 follows in a similar manner as all the terms involved as similar to those appearing in ι_t^1. The only term that is different $\frac{dJ_t}{dz}$. Explicitly $\frac{dJ_t}{dz}$ is given by

$$\frac{dJ_t}{dz}(z) = \int_0^t \bar{\sigma}\left(X_s(z), V_s\right) \exp\left(i_s^t(z)\right)\left(\bar{\sigma}'\left(X_s(z), V_s\right) J_s(z) + \frac{di_s^t}{dz}(z)\right)\right) ds,$$

and one proves in a similar manner that

$$E\left[\left|\frac{dJ_t}{dz}\right|\right] \leq k' e^{k''|z|}, \tag{2.15}$$

where k' and k'' are some suitably chosen constants. It follows that

$$\begin{aligned}
\rho_t\left(y^1\right) - \rho_t\left(y^2\right) &= \frac{1}{\sqrt{2\pi}} E\left[\nu_t\left(y^1\right) - \nu_t\left(y^2\right) \middle| \mathcal{F}_t^{W,V}\right] \\
&= \frac{1}{\sqrt{2\pi}} E\left[\int_{y_2}^{y^1} \frac{d\nu_t}{dy}(y)\, dy \middle| \mathcal{F}_t^{W,V}\right] \\
&= \int_{y_2}^{y^1} \frac{1}{\sqrt{2\pi}} E\left[\frac{d\nu_t}{dy}(y) dy \middle| \mathcal{F}_t^{W,V}\right] dy \qquad (2.16)
\end{aligned}$$

and we deduce from the above the absolute continuity of ρ_t and, therefore, its differentiability almost everywhere. We note that the last identity follows by the (conditional) Fubini's theorem as we have proved the integrability of $\frac{d\nu_y}{dy}$ over the product space $\Omega \times \mathbb{R}$. The methodology to show that ρ_t has higher derivatives is similar. Observe first that

$$\begin{aligned}
\frac{d^m \nu_t(y)}{dy^m} = \quad & \frac{\exp\left\{-X_t^{-1}(y) B_t - \frac{\left(X_t^{-1}(y)\right)^2 (t+1)}{2}\right\}}{J_t\left(X_t^{-1}(y)\right)} \\
& \times T(t, B_t, X_t^{-1}(y), \frac{dX_t}{dx}\left(X_t^{-1}(y)\right), \ldots \frac{dX_t^m}{dx^m}\left(X_t^{-1}(y)\right)))
\end{aligned}$$

where $T(t, B_t, X_t^{-1}(y), \frac{dX_t}{dx}\left(X_t^{-1}(y)\right), \ldots \frac{dX_t^m}{dx^m}\left(X_t^{-1}(y)\right)))$ is a random variable which has moments of all order controlled by an upper bound of the type (2.15). One then shows the integrability of $\frac{d^m \nu_t(y)}{dy^m}$ over the product space $\Omega \times \mathbb{R}$ which implies the m-times differentiability of ρ_t. $\square$

2.2 The multidimensional case

For $\alpha \geq n^{-1}$, $n = 1, 2, \ldots$, define

$$\digamma^{\alpha,n}(s,z) = \mathbf{1}_{[\alpha-\frac{1}{n},\alpha]}(s)z. \tag{2.17}$$

Let $X^{\alpha,n}(z) = \{X_t^{\alpha,n}(z), t \geq 0\}$ be the solution of (2.1) with $\digamma$ replaced by $\digamma^{\alpha,n}$, and let $J_t^{\alpha,n}(z)$ be its Jacobian

$$(J_t^{\alpha,n}(z))_{ij} = \partial_j (X_t^{\alpha,n})_i (z).$$

Then $J^{\alpha,n}(z) = \{J_t^{\alpha,n}(z), t \geq 0\}$ is zero for $t \leq \alpha - n^{-1}$, and for $t \geq \alpha - n^{-1}$, $J^{\alpha,n}$ satisfies the following stochastic differential equation

$$\begin{aligned}
J_t^{\alpha,n}(z) &= \int_{\alpha-\frac{1}{n}}^t f'(X_s^{\alpha,n}(z), V_s) J_s^{\alpha,n}(z)\,ds + \sum_{i=1}^m \int_{\alpha-\frac{1}{n}}^t \sigma_i'(X_s^{\alpha,n}(z), V_s) J_s^{\alpha,n}(z)\,dW_s^i \\
&\quad + \sum_{i=1}^d \int_{\alpha-\frac{1}{n}}^t \bar{\sigma}_i'(X_s^{\alpha,n}(z), V_s) J_s^{\alpha,n}(z)\,dB_s^i \\
&\quad + \sum_{i=1}^d \int_{\alpha-\frac{1}{n}}^t \bar{\sigma}_i'(X_s^{\alpha,n}(z), V_s) \mathbf{1}_{[\alpha-\frac{1}{n},\alpha]}(s) z^i J_s^{\alpha,n}(z)\,ds \\
&\quad + \int_{\alpha-\frac{1}{n}}^t \bar{\sigma}(X_s^{\alpha,n}(z), V_s) \mathbf{1}_{[\alpha-\frac{1}{n},\alpha]}(s)ds,
\end{aligned} \tag{2.18}$$

where $f' : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^{d\times d}$ is the matrix-valued function defined as

$$(f'(x,v))_{ij} \stackrel{\text{def}}{=} \frac{\partial_j f(x,v)_i}{\partial x_j}$$

and σ_i', $i = 1, ..., d$ $\bar{\sigma}_i'$, $i = 1, ..., m$ are functions defined in the same manner (σ_i, $i = 1, ..., d$ $\bar{\sigma}_i$, $i = 1, ..., m$ are the column vectors of σ , respectively $\bar{\sigma}$, $\sigma = (\sigma_1, \sigma_2, ..., \sigma_d)$, $\bar{\sigma} = (\bar{\sigma}_1, \bar{\sigma}_2, ..., \bar{\sigma}_m)$). Let $\Phi^{\alpha,n}(z) = \{\Phi_t^{\alpha,n}(z), t \geq 0\}$ and $\Upsilon^{\alpha,n}(z) = \{\Upsilon_t^{\alpha,n}(z), t \geq 0\}$ be the solutions of the following matrix stochastic differential equations

$$\begin{aligned}
\Phi_t^{\alpha,n}(z) &= I + \int_{(\alpha-\frac{1}{n})\wedge t}^t \varpi(z, X_s^{\alpha,n}(z), V_s) \Phi_s^{\alpha,n}(z)\,ds \\
&\quad + \sum_{i=1}^m \int_{(\alpha-\frac{1}{n})\wedge t}^t \sigma_i'(X_s^{\alpha,n}(z), V_s) \Phi_s^{\alpha,n}(z)\,dW_s^i \\
&\quad + \sum_{i=1}^d \int_{(\alpha-\frac{1}{n})\wedge t}^t \bar{\sigma}_i'(X_s^{\alpha,n}(z), V_s) \Phi_s^{\alpha,n}(z)\,dB_s^i,
\end{aligned}$$

$$\Upsilon_t^{\alpha,n}(z) = I - \int_{(\alpha-\frac{1}{n})\wedge t}^t \Upsilon_s^{\alpha,n}(z) \kappa(z, X_s^{\alpha,n}(z), V_s)\,ds$$

$$
\begin{aligned}
&-\sum_{i=1}^{m}\int_{\left(\alpha-\frac{1}{n}\right)\wedge t}^{t}\Upsilon_s^{\alpha,n}(z)\,\sigma_i'\left(X_s^{\alpha,n}(z),V_s\right)dW_s^i\\
&-\sum_{i=1}^{d}\int_{\left(\alpha-\frac{1}{n}\right)\wedge t}^{t}\Upsilon_s^{\alpha,n}(z)\,\bar{\sigma}_i'\left(X_s^{\alpha,n}(z),V_s\right)dB_s^i,
\end{aligned}
$$

where

$$
\begin{aligned}
\varpi\left(z,X_s^{\alpha,n}(z),V_s\right) &= f'\left(X_s^{\alpha,n}(z),V_s\right)+\sum_{i=1}^{d}\bar{\sigma}_i'\left(X_s^{\alpha,n}(z),V_s\right)\mathbf{1}_{[\alpha-\frac{1}{n},\alpha]}(s)z^i\\
\kappa\left(z,X_s^{\alpha,n}(z),V_s\right) &= \varpi\left(z,X_s^{\alpha,n}(z),V_s\right)-\sum_{i=1}^{m}\sigma_i'\left(X_t^{\alpha,n}(z),V_t\right)^2-\sum_{i=1}^{d}\bar{\sigma}_i'\left(X_t^{\alpha,n}(z),V_t\right)^2.
\end{aligned}
$$

It is easy to check that

$$
d\left(\Upsilon_t^{\alpha,n}(z)\,\Phi_t^{\alpha,n}(z)\right)=0,
$$

and since $\Upsilon_0^{\alpha,n}(z)\,\Phi_0^{\alpha,n}(z)=I$, it follows that $\Upsilon_t^{\alpha,n}(z)\,\Phi_t^{\alpha,n}(z)=I$, for all $t\geq 0$, i.e., $\Phi_t^{\alpha,n}(z)$ and $\Upsilon_t^{\alpha,n}(z)$ are non-singular and inverse to each other. Then we can write the solution of (2.18) explicitly as

$$
J_t^{\alpha,n}(z)=\Phi_t^{\alpha,n}(z)\int_{\left(\alpha-\frac{1}{n}\right)\wedge t}^{\alpha\wedge t}\Upsilon_s^{\alpha,n}(z)\,\bar{\sigma}\left(X_s^{\alpha,n}(z),V_s\right)ds.
$$

Unlike the one-dimensional case, the Jacobian $J_t^{\alpha,n}(z)$ may be singular. However, since $\Phi_t^{\alpha,n}(z)$ is non singular, $J_t^{\alpha,n}(z)$ is nonsingular for $t\geq\alpha$ if and only if

$$
\Gamma^{\alpha,n}(z)=\int_{\left(\alpha-\frac{1}{n}\right)}^{\alpha}\Upsilon_s^{\alpha,n}(z)\,\bar{\sigma}\left(X_s^{\alpha,n}(z),V_s\right)ds
$$

is nonsingular.

Write

$$
\begin{aligned}
\Gamma^{\alpha,n}(z) &= \frac{1}{n}\bar{\sigma}\left(X_\alpha^{\alpha,n}(z),V_{\alpha-}\right)+\int_{\alpha-\frac{1}{n}}^{\alpha}\bar{\sigma}\left(X_s^{\alpha,n}(z),V_s\right)-\bar{\sigma}\left(X_\alpha^{\alpha,n}(z),V_{\alpha-}\right)ds\\
&\quad+\int_{\alpha-\frac{1}{n}}^{\alpha}\left(\Upsilon_s^{\alpha,n}(z)-I\right)\bar{\sigma}\left(X_s^{\alpha,n}(z),V_s\right)ds.
\end{aligned}
$$

Since $X_s^{\alpha,n}(z)$ and $\Upsilon_s^{\alpha,n}(z)$ are jointly continuous in s and z, $(x,v)\to\bar{\sigma}(x,v)$ is continuous, and $\lim_{s\to t}V_s=V_{t-}$, it follows that, for almost all $\omega\in\Omega$ and each compact $K\subset\mathbb{R}^d$,

$$
\begin{aligned}
\lim_{n\to\infty}\sup_{z\in K}\left|n\int_{\alpha-\frac{1}{n}}^{\alpha}\bar{\sigma}\left(X_s^{\alpha,n}(z),V_s\right)-\bar{\sigma}\left(X_t^{\alpha,n}(z),V_{t-}\right)ds\right| &= 0\\
\lim_{n\to\infty}\sup_{z\in K}\left|n\int_{\alpha-\frac{1}{n}}^{\alpha}\left(\Upsilon_s^{\alpha,n}(z)-I\right)\bar{\sigma}\left(X_s^{\alpha,n}(z),V_s\right)ds\right| &= 0.
\end{aligned}
$$

Hence,

$$\Omega^{\lim} = \left\{ \omega \in \Omega | \lim_{n\to\infty} \sup_{z\in K} \left| nJ_t^{\alpha,n}(z) - \bar{\sigma}(X_t, V_{t-}) \right| = 0 \text{ for each compact } K \right\}$$

has probability 1. Let $K_1 \subset K_2 \subset \cdots$ be compact subsets of $\mathbb{R}^d$ with $\mathbb{R}^d = \cup_k K_k$, and define

$$\Omega_{k,n} = \left\{ \omega \in \Omega | \ J_t^{\alpha,m}(z) \text{ is nonsingular for } m \geq n, z \in K_k \right\}.$$

Lemma 2.3

$$\Omega^{\lim} \subset \cap_k \cup_n \Omega_{k,n}$$

and, in particular, $P(\cap_k \cup_n \Omega_{k,n}) = 1$.

Proof. It is enough to prove that $\Omega^{\lim} \subset \cup_n \Omega_{k,n}$ for each k. Let $\omega \in \Omega^{\lim}$ but not in $\cup_n \Omega_{k,n}$. It follows that there exist (n_i, z_i), $n_i \to \infty$ and $z_i \in K_k$ such that, for this particular ω, the corresponding Jacobians $J_t^{n_i}(z_i) = J_t^{n_i}(z_i)(\omega)$ are singular. Hence there exist corresponding $\lambda_i \in \mathbb{R}^d$ with $|\lambda_i| = 1$ and

$$J_t^{n_i}(z_i)\lambda_i = 0.$$

If λ is a limit point of $\{\lambda_i\}$, the uniformity over compacts in the definition of $\Omega^{\lim}$ implies

$$\bar{\sigma}(X_t, V_{t-}) \cdot \lambda = 0.$$

Since $\bar{\sigma}(X_t, V_{t-})$ is nonsingular, we have a contradiction. $\square$

From (2.3) we get that, for any set A and any $k > \left[\frac{1}{t}\right]$

$$\begin{aligned}
\pi_t(A) &= P\left[X_t \in A | \mathcal{F}_t^{W,V}\right] \\
&= E\left[\int_{\{z | X_t^n(z) \in A\}} r^n(z)\, dz | \mathcal{F}_t^{W,V}\right] \\
&\geq E\left[\mathbf{1}_{\Omega_{k,n}} \int_{\left\{z \in \mathbb{R}^d | X_t^n(z) \in A\right\}} r^n(z)\, dz | \mathcal{F}_t^{W,V}\right],
\end{aligned}$$

where

$$r^n(z) \stackrel{\text{def}}{=} \exp\left(-z^\top \left(B_t - B_{t-\frac{1}{n}}\right) - \frac{z^\top z(n+1)}{2n}\right).$$

Now for $\omega \in \Omega_{k,n}$ the application $z \in K_k \to X_t(z)$ is a continuous bijection. It is injective since its Jacobian is always non-singular. The surjectivity follows by means of the Inverse Function Theorem: Since $\lim_{|z|\to\infty} X_t(z) = \infty$, the image of $z \to X_t(z)$ is a closed set. However, the image of $z \to X_t(z)$ is an open set, too. That is because any $z \in \mathbb{R}^d$ has the property that it has an open neighborhood U_z so that the function restricted to U_z is a (continuous) bijection from U_z to $V_{X_t(z)}$ where $V_{X_t(z)}$ is an open neighborhood of

$X_t(z)$. Hence for any $z \in \mathbb{R}^d$, the open set $V_{X_t(z)}$ is in the image of $z \to X_t(z)$. Since $\mathbb{R}^d$ has no proper subset which is both closed and open, we get the surjectivity of $z \to X_t(z)$.

So for any $\omega \in \Omega_k$ and A an open set, the set $\left\{z \in \mathbb{R}^d | \mathbb{X}_t(z) \in \mathbb{A}\right\}$ has positive Lebesgue measure, thus

$$\omega \to \int_{\left\{z \in \mathbb{R}^d | \mathbb{X}_t^{n_{k'}}(z) \in \mathbb{A}\right\}} r^{n_{k'}}(z)\, dz$$

is positive on Ω_k. Let now $\Omega^{\mathrm{p}} = \{\omega \in \Omega | \pi_t(A) > 0\}$. Then, for all $k > \left[\frac{1}{t}\right]$,

$$P(\Omega^{\mathrm{p}}) \geq P(\Omega_k).$$

That is because $P((\Omega \backslash \Omega^{\mathrm{p}}) \cap \Omega_k) = 0$. If not

$$0 = E\left[1_{\Omega \backslash \Omega^p} \pi_t(A)\right] \geq E\left[\mathbf{1}_{(\Omega \backslash \Omega^{\mathrm{p}}) \cap \Omega_k} \int_{\left\{z \in \mathbb{R}^d | \mathbb{X}_t^{n_{k'}}(z) \in \mathbb{A}\right\}} r^{n_{k'}}(z)\, dz | \mathcal{F}_t^{W,V}\right] > 0.$$

Hence from the previous lemma we deduce that π_t charges any open set A. Moreover the null set can be chosen to be independent of the set A, since the topology $\mathbb{R}^d$ has a countable base.

Now if A is a set of Lebesgue measure 0, then

$$\begin{aligned}
\pi_t(A) &= P\left[X_t \in A | \mathcal{F}_t^{W,V}\right] \\
&= E\left[\mathbf{1}_{\Omega_n} \int_{\left\{z \in \mathbb{R}^d | \mathbb{X}_t^n(z) \in \mathbb{A}\right\}} r^n(z)\, dz | \mathcal{F}_t^{W,V}\right] \\
&\quad + E\left[\mathbf{1}_{\Omega \backslash \Omega_n} \int_{\left\{z \in \mathbb{R}^d | \mathbb{X}_t^n(z) \in \mathbb{A}\right\}} r^n(z)\, dz | \mathcal{F}_t^{W,V}\right]. \qquad (2.19)
\end{aligned}$$

Since

$$\omega \to \int_{\mathbb{R}^d} r^n(z)\, dz$$

is uniformly integrable, it follows that the second term in (2.19) converges to 0. Hence

$$\begin{aligned}
\pi_t(A) &= \lim_{n \to \infty} E\left[\mathbf{1}_{\Omega_n} \int_{\left\{z \in \mathbb{R}^d | \mathbb{X}_t^n(z) \in \mathbb{A}\right\}} r^n(z)\, dz \right| \left. \mathcal{F}_t^{W,V}\right] \\
&= \lim_{n \to \infty} E\left[\mathbf{1}_{\Omega_n} \int_A r^n\left(\left(X_t^n\right)^{-1}(y)\right) \frac{1}{\det\left(J_t^n\left(\left(X_t^n\right)^{-1}(y)\right)\right)} dy \right| \left. \mathcal{F}_t^{W,V}\right]
\end{aligned}$$

Since

$$\int_A r^n\left(\left(X_t^n\right)^{-1}(y)\right) \frac{1}{\det\left(J_t^n\left(\left(X_t^n\right)^{-1}(y)\right)\right)} dy$$

is always 0 (an integral over a null set), it follows that the conditional distribution of X_t given $\mathcal{F}_t^{W,V}$ is absolutely continuous with respect to the Lebesgue measure. *This proves the first part Theorem 2*

For the second part of Theorem 2 let (as in the one-dimensional case) A_0 be the following random set

$$A_0 = \{x \in \mathbb{R} | \rho_t(y) = 0\}.$$

Then, from (2.19) we have that

$$\begin{aligned}
0 &= P\left[X_t \in A_0 | \mathcal{F}_t^{W,V}\right] \\
&\geq E\left[\mathbf{1}_{\Omega_n} \int_{\left\{z\in\mathbb{R}^d | \mathbb{X}_t^n(z)\in\mathbb{A}\right\}} r^n(z)\, dz | \mathcal{F}_t^{W,V}\right] \\
&= \int_{\mathbb{R}} \mathbf{1}_{A_0}(y) \frac{1}{\sqrt{2\pi}} E\left[\mathbf{1}_{\Omega_n}(y)\, r^n\left(\left(X_t^n\right)^{-1}(y)\right) \frac{1}{\det\left(J_t^n\left(\left(X_t^n\right)^{-1}(y)\right)\right)} \middle| \mathcal{F}_t^{W,V}\right] dy \\
&\geq 0
\end{aligned}$$

Hence

$$\int_{\mathbb{R}} \mathbf{1}_{A_0}(y) \frac{1}{\sqrt{2\pi}} E\left[\mathbf{1}_{\Omega_n}(y)\, r^n\left(\left(X_t^n\right)^{-1}(y)\right) \frac{1}{\det\left(J_t^n\left(\left(X_t^n\right)^{-1}(y)\right)\right)} \middle| \mathcal{F}_t^{W,V}\right] dy = 0$$

and therefore for any $\omega \in \widehat{\Omega}_n$, A_0 is a set of Lebesgue measure 0 where $\left(\widehat{\Omega}_n\right)_{n>\left[\frac{1}{t}\right]}$ is the following increasing sequence

$$\widehat{\Omega}_n = \left\{\omega \in \Omega \; E\left[\mathbf{1}_{\Omega_n}(y)\, r^n\left(\left(X_t^n\right)^{-1}(y)\right) \frac{1}{\det\left(J_t^n\left(\left(X_t^n\right)^{-1}(y)\right)\right)} \middle| \mathcal{F}_t^{W,V}\right] \middle| > 0, y\text{–a.e.}\right\}.$$

Since $P\left(\widehat{\Omega}_n\right) \geq P(\Omega_n)$ and $P(\lim \Omega_n) = 1$ it follows that A_0 is a set of Lebesgue measure 0 with probability 1. *This proves the second part of Theorem 2 for the multidimensional case.*

3 Weak existence for SPDEs with coefficients depending on quantiles

As described in the introduction, we now consider an infinite system of (one-dimensional) interacting diffusions

$$X_t^i = X_0^i + \int_0^t f\left(X_s^i, V_s^\alpha\right) ds + \int_0^t \sigma\left(X_s^i, V_s^\alpha\right) dW_s + \int_0^t \bar{\sigma}\left(X_s^i, V_s^\alpha\right) dB_s^i, \tag{3.1}$$

where

$$V_t^\alpha = \inf \{x \in \mathbb{R} | v_t (-\infty, x] \geq \alpha\}$$

and

$$v_t = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \delta_{X_t^i}. \tag{3.2}$$

We assume that $\{X_0^i\}$ is exchangeable and require the solution $\{X^i\}$ to be exchangeable so that the limit in (3.2) exists by deFinetti's theorem. (See Theorem A.1.)

As in Kurtz and Xiong [6], v_t will be a solution of the stochastic partial differential equation

$$\langle \phi, v(t) \rangle = \langle \phi, v(0) \rangle + \int_0^t \langle L(V^\alpha(s))\phi\, , v(s) \rangle ds + \int_0^t \langle \sigma(\cdot, V^\alpha(s))\phi'\, , v(s) \rangle dW_s, \tag{3.3}$$

where $\langle \phi, v(t) \rangle$ denotes

$$\langle \phi, v(t) \rangle = \int_{\mathbb{R}} \phi(x) v(t, dx)$$

and

$$L(V^\alpha)\phi = \frac{1}{2} \left[\sigma(x, V^\alpha)^2 + \bar{\sigma}(x, V^\alpha)^2\right] \frac{d^2\phi}{dx^2} + f(x, V^\alpha)\frac{d\phi}{dx}.$$

In (3.1), the process W is common to all diffusions, while the processes B^i, $i \geq 1$ are mutually independent Brownian motions. We will assume the following on the coefficients of the equations.

Q $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\bar{\sigma} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are *bounded*, continuous functions, uniformly Lipschitz in both arguments and continuously differentiable in the first component and $\bar{\sigma}$ is positive definite.

Then we have the following:

Theorem 3.1 *There exists a weak solution for the system (3.1)+(3.2) and, hence, for the stochastic partial differential equation (3.3).*

Proof. Consider the Euler-type approximation of (3.1)+(3.2) defined as follows:

$$X_t^{i,n} = X_0^{i,n} + \int_0^t f\left(X_s^{i,n}, V_s^{\alpha,n}\right) ds + \int_0^t \sigma\left(X_s^{i,n}, V_s^{\alpha,n}\right) dW_s + \int_0^t \bar{\sigma}\left(X_s^{i,n}, V_s^{\alpha}\right) dB_s^i, \tag{3.4}$$

where

$$V_t^{\alpha,n} = \inf \left\{x \in \mathbb{R} | v^n_{\frac{[tn]}{n}}((-\infty, x]) \leq \alpha\right\}$$

and v^n is defined as in (3.2). The system (3.4) has a unique strong solution. The existence and uniqueness of the solution is obtained progressively on intervals $\left[\frac{k}{n}, \frac{k+1}{n}\right]$. We note that, on each such interval, the process $V^{\alpha,n}$ is constant and equal to the quantile of the empirical measure of the system at the beginning of the interval. Existence and uniqueness of the solution follows from the assumption that f, σ, and $\bar{\sigma}$ are Lipschitz in the first component.

We also have

$$v^n(\varphi) = E\left[\varphi\left(X_t^{i,n}\right)|\mathcal{F}_t^W\right] = E\left[\int_{\mathbb{R}} \varphi\left(X_t^{i,n}(z)\right) M_t^{i,n}(z) \frac{e^{-\frac{1}{2}z^\top z}}{(2\pi)^{\frac{d}{2}}} dz \middle| \mathcal{F}_t^W\right], \tag{3.5}$$

where $X^n(z)$ is defined as in (2.1), and it follows that v_t^n charges every open set and, hence, that

$$V_t^{\alpha,n} = \inf\{x \in \mathbb{R} | v_t^n(-\infty, x] \geq \alpha\} = \sup\{x \in \mathbb{R} | v_t^n(-\infty, x)] < \alpha\}.$$

For each i, the boundedness of the coefficients implies the sequence $\{X^{i,n}\}_{n>0}$ is relatively compact (in distribution) in $D_{\mathbb{R}}([0,\infty)$. This relative compactness together with the continuity of the processes ensures relative compactness of $\{X^n\}_{n>0}$ in $D_{\mathbb{R}^\infty}([0,\infty)$. Taking a subsequence, if necessary, we can assume that $\{X^n\}_{n>0}$ converges in distribution to a continuous process $X = (X^i)_{i\geq 0}$. By Lemma A.3 in the Appendix v^n converges in distribution to v defined by

$$v_t = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \delta_{X_t^i}.$$

To complete the proof, we need the following two lemmas. In the first one, we drop the assumption that V^n be a quantile, allowing it to take values in a complete, separable metric space E, and only require that the coefficients be bounded, Lipschitz continuous in the first variable, and continuous in the second.

Lemma 3.2 *Suppose $\{V^n\}$ are E-valued processes and $X^n(z)$ satisfies*

$$dX_t^n(z) = f\left(X_t^n(z), V_t^n\right)dt + \sigma\left(X_t^n(z), V_t^n\right)dW_t + \bar{\sigma}\left(X_t^n(z), V_t^n\right)dB_t + \bar{\sigma}\left(X_t(z), V_t^n\right)zdt.$$

Define

$$\Gamma^n(C \times [0,t]) = \int_0^t \mathbf{1}_C(V^n(s))ds, \quad C \in \mathcal{B}(E),$$

$$M_B^n(\varphi, t) = \int_0^t \varphi(V^n(s))dB_s, \quad \varphi \in C_b(E),$$

and

$$M_W^n(\varphi, t) = \int_0^t \varphi(V^n(s))dW_s, \quad C_b(E).$$

Suppose that $\Gamma^n \Rightarrow \Gamma$, in $\mathcal{L}_m(E)$. (See Appendix A.3.) Then for $\varphi_1^B, \ldots, \varphi_k^B, \varphi_1^W, \ldots, \varphi_l^W \in C_b(E)$, $\{(\Gamma^n, M_B^n(\varphi_1^B), \ldots, M_B^n(\varphi_k^B), M_W^n(\varphi_1^W), \ldots, M_W^n(\varphi_l^W)\}$ is relatively compact in $\mathcal{L}_m(E) \times D_{\mathbb{R}^{k+l}}[0,\infty)$, and a subsequence can be selected along which convergence holds for all choices *of $\varphi_1^B,\ldots,\varphi_k^B,\varphi_1^W,\ldots,\varphi_l^W \in C_b(E)$. For any limit point, M_B and M_W are orthogonal martingale random measures satisfying*

$$\begin{aligned}
[M_B(\varphi_1),M_B(\varphi_2)]_t &= \int_E \varphi_1(y)\varphi_2(y)\Gamma(dy\times[0,t]) \\
[M_W(\varphi_1),M_W(\varphi_2)]_t &= \int_E \varphi_1(y)\varphi_2(y)\Gamma(dy\times[0,t]) \\
[M_B(\varphi_1),M_W(\varphi_2)]_t &= 0,
\end{aligned}$$

and $X^n(z) \Rightarrow X(z)$ satisfying

$$\begin{aligned}
X_t(z) &= X_0(z) + \int_{E\times[0,t]} f\left(X_s\left(z\right),v\right)\Gamma(dv\times ds) + \int_{E\times[0,t]} \sigma\left(X_s\left(z\right),v\right)M_W(dv\times ds) \\
&\quad + \int_{E\times[0,t]} \bar{\sigma}\left(X_s\left(z\right),v\right)M_B(dv\times ds) + \int_{E\times[0,t]} \bar{\sigma}\left(X_t\left(z\right),v\right)z\Gamma(dv\times ds), \qquad (3.6)
\end{aligned}$$

where the stochastic integrals are defined as in [5].

Proof. Relative compactness follows from the fact that

$$E[(M_B^n(\varphi,t+h)-M_B^n(\varphi,t))^2|\mathcal{F}_t^n] = E[\int_E \varphi(y)^2\Gamma^n(dy\times(t,t+h])|\mathcal{F}_t^n] \le \|\varphi\|^2 h$$

for each $\varphi \in C_b(E)$ and similarly for $\{M_W^n\}$. Along any convergent subsequence, $\{\Gamma^n, M_B^n, M_W^n\}$ satisfies the convergence conditions in Theorem 4.2 of Kurtz and Protter [5]. (See Example 12.1 of [5].) Under the boundedness and Lipschitz conditions on f, σ, and $\bar{\sigma}$, $X^n(z)$ converges to the solution of (3.6) by Theorem 7.4 of Kurtz and Protter [5]. □

Lemma 3.3 *Let $\{X_n\}$be a sequence of* uniformly integrable *random variables converging in distribution to a random variable X and $\{\mathcal{D}_n\}$ be a sequence of σ-fields defined on the probability spaces where $\{X_n\}$ reside. Let $\{Y_n\}$ be a sequence of S-valued random variables such that*

$$E\left[X_n|\mathcal{D}_n\right] = G(Y_n),$$

where $G : S \to \mathbb{R}$ is continuous. Suppose $(X_n,Y_n) \Rightarrow (X,Y)$. Then $E\left[X|Y\right] = G(Y)$.

Proof. Since $\{X_n\}$ is uniformly integrable, it follows by Jensen's inequality that $\{G(Y_n)\}$ is uniformly integrable. Then, employing the convergence in distribution and the uniform integrability,

$$E\left[G(Y)g(Y)\right] = \lim_{n\to\infty} E\left[G(Y_n)g(Y_n)\right] = \lim_{n\to\infty} E\left[X_n g(Y_n)\right] = E\left[Xg(Y)\right],$$

for every $g \in C_b(S)$, and the lemma follows. □

We return now to the proof of Theorem 3.1. Let

$$\rho_t^n(\varphi) = \int_{\mathbb{R}} \varphi\left(X_t^{i,n}(z)\right) M_t^{i,n}(z) \frac{e^{-\frac{1}{2}z^\top z}}{(2\pi)^{\frac{d}{2}}} dz.$$

From (3.5) and the definition of Γ^n and M_W^n, for any test function φ

$$E\left[\rho_t^n(\varphi) | \mathcal{F}_t^W\right] = E\left[\rho_t^n(\varphi) | \mathcal{F}_t^{\Gamma^n, M_W^n}\right] = v_t^n(\varphi).$$

Hence, letting $\Gamma^{n,t}$, Γ^t and W^t denote the restrictions of Γ^n, Γ, and W to the time interval [0,t], $(\rho_t^n, v_t^n, \Gamma^{n,t}, W^t) \Rightarrow (\rho_t, v_t, \Gamma^t, W^t)$, where

$$\rho_t(\varphi) = \int_{\mathbb{R}^d} \varphi\left(X_t^i(z)\right) M_t^i(z) \frac{e^{-\frac{1}{2}z^\top z}}{(2\pi)^{\frac{d}{2}}} dz.$$

By Lemma 3.3

$$E\left[\rho_t(\varphi) | \mathcal{F}_t^{\Gamma, M_W}\right] = v_t(\varphi).$$

As in the proof of Theorem 1.1, v_t charges any open set, and by Lemma A.8, $V^{\alpha,n}$ converges in distribution to V^α, where

$$V_t^\alpha = \inf\{x \in \mathbb{R} | v_t((-\infty, x]) \geq \alpha\}.$$

In turn, it follows that M_W and M_B satisfy

$$M_B(\varphi, t) = \int_0^t \varphi(V^\alpha(s)) dB_s, \quad \varphi \in C_b(E),$$

and

$$M_W^\alpha(\varphi, t) = \int_0^t \varphi(V^\alpha(s)) dW_s, \quad C_b(E).$$

Applying Theorem 7.4 of Kurtz and Protter [5], it follows that $(X^n, V^{\alpha,n}, v^n)$ converges in distribution to (X, V^α, v) which is a weak solution of (3.1). □

4 Quantile Process

Next, we find an equation for the quantile process

$$V_t^\alpha = \inf\{x \in \mathbb{R},\ v_t((-\infty, x]) \geq \alpha\}.$$

Recall that we considered an infinite system of (one-dimensional) interacting diffusions

$$X_t^i = X_0^i + \int_0^t f\left(X_s^i, V_s^\alpha\right) ds + \int_0^t \sigma\left(X_s^i, V_s^\alpha\right) dW_s + \int_0^t \bar{\sigma}\left(X_s^i, V_s^\alpha\right) dB_s^i, \qquad (4.1)$$

where

$$V_t^\alpha = \inf\left\{x \in \mathbb{R} | v_t\left(-\infty, x\right] \geq \alpha\right\}$$

and

$$v_t = \lim_{n\to\infty} v_t^n \quad \text{ where } \quad v_t^n = \frac{1}{n}\sum_{i=1}^n \delta_{X_t^i}. \tag{4.2}$$

To prove the following result we choose a bounded, smooth, strictly positive function $q : \mathbb{R} \to \mathbb{R}$ with bounded first and second derivative such that $\int_{\mathbb{R}} q\left(x\right) dx = 1$ and

$$\sup_{x\in\mathbb{R}} \frac{q'(x)}{q(x)} < \infty^1. \tag{4.3}$$

Define the functions, $v_t^{n,\epsilon}, v_t^\epsilon, F_t^{n,\epsilon}, F_t^\epsilon : \mathbb{R} \to \mathbb{R}$ as follows

$$\begin{array}{cc} v_t^{n,\epsilon}\left(x\right) = \frac{1}{n}\sum_{i=1}^n q_\epsilon\left(x - X_t^i\right) & F_t^{n,\epsilon}\left(x\right) = \int_{-\infty}^x v_t^{n,\epsilon}\left(y\right) dy \\ v_t^\epsilon\left(x\right) = \int_{\mathbb{R}} q_\epsilon\left(x - y\right) v_t\left(dy\right) & F_t^\epsilon\left(x\right) = \int_{-\infty}^x v_t^\epsilon\left(y\right) dy \end{array},$$

where $q_\epsilon : \mathbb{R} \to \mathbb{R},\ q_\epsilon\left(x\right) = \frac{1}{\epsilon} q\left(\frac{x}{\epsilon}\right), x \in \mathbb{R}$. Then, the functions $v_t^{n,\epsilon}$ are uniformly bounded smooth functions and, since $\lim_{n\to\infty} v_t^n = v_t$, it follows that $v_t^{n,\epsilon}$ converges pointwise to v_t^ϵ. Hence the quantiles $V_t^{\alpha,n,\epsilon}$ of the probability measures with densities $v_t^{n,\epsilon}$ with respect to the Lebesgue measure uniquely defined by the formula

$$F^{n,\epsilon}\left(t, V_t^{\alpha,n,\epsilon}\right) = \alpha$$

converge to the quantiles $V_t^{\alpha,\epsilon}$ of the measure with density v_t^ϵ with respect to the Lebesgue measure, $\lim_{n\to\infty} V_t^{\alpha,n,\epsilon} = V_t^{\alpha,\epsilon}$. Moreover, since also the derivatives of the functions $v_t^{n,\epsilon}$ converge to the derivatives of the functions v_t^ϵ and are uniformly bounded, it follows that $v_t^{n,\epsilon}$ converges to v_t^ϵ *uniformly on compacts*. In particular this implies that $\lim_{n\to\infty} v_t^{n,\epsilon}(V_t^{\alpha,n,\epsilon}) = v_t^\epsilon(V_t^{\alpha,\epsilon})$. Similarly, $\lim_{n\to\infty} \frac{dv_t^{n,\epsilon}(x)}{dx}|_{x=V_t^{\alpha,n,\epsilon}} = \frac{dv_t^\epsilon(x)}{dx}|_{x=V_t^{\alpha,\epsilon}}$ This two facts will be used in the following proposition.

Proposition 4.1 *Assuming that* **A1+A2** *hold true and that f, σ and $\bar\sigma$ are twice continuously differentiable in the first component, then the quantiles V_t^α satisfy the following evolution equation*

$$\begin{aligned} V_t^\alpha &= V_s^\alpha + \int_s^t f(V_r^\alpha, V_r^\alpha) dr + \int_s^t \sigma(V_r^\alpha, V_r^\alpha) dW_r \\ &\qquad - \int_s^t \frac{1}{2v(r, V_r^\alpha)} \frac{\partial}{\partial x}\left[\sigma(x, V_r^\alpha) v_r(x)\right]\bigg|_{x=V_t^\alpha} dt. \end{aligned} \tag{4.4}$$

for any $t > s > 0$.

[1]It suffices to choose q such that $q(x) = c_q \exp(-|x|)$ for $|x| \geq 1$, where c_q is the normalization constant.

Proof. First, note that, by the definition of the quantiles,

$$\Upsilon^{\alpha,n,\epsilon}\left(V_t^{\alpha,n,\epsilon},X_t^1,...,X_t^n\right)=0,$$

where $\Upsilon^{\alpha,n,\epsilon}:\mathbb{R}^{n+1}\to\mathbb{R}$ is the smooth function

$$\Upsilon^{\alpha,n,\epsilon}\left(v,x_1,...,x_n\right)=\frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{v}q_\epsilon\left(y-x_i\right)dy-\alpha.$$

Since $\frac{\partial\Upsilon^{\alpha,n,\epsilon}}{\partial v}\left(v,x_1,...,x_n\right)=\frac{1}{n}\sum_{i=1}^{n}q_\epsilon\left(v-x_i\right)>0$, by the implicit function theorem there exists a countable set of balls $B\left(x_j,r_j\right)\in\mathbb{R}^n$ $j\geq1$ such that $\bigcup_{n\geq1}B\left(x_j,r_j\right)=\mathbb{R}^n$ and a countable set of *smooth* functions $Q^{\alpha,n,\epsilon,j}:B\left(x_j,r_j\right)\to\mathbb{R}$ such that

$$V_t^{\alpha,n,\epsilon}=Q^{\alpha,n,\epsilon,j}\left(X_t^1,...,X_t^n\right),\text{ if }\left(X_t^1,...,X_t^n\right)\in B\left(x_j,r_j\right).$$

In particular it follows that $V_t^{\alpha,n,\epsilon}$ is a semi-martingale. This fact allows us to deduce the evolution equation for the semimartingales $V_t^{\alpha,n,\epsilon}$. By applying the generalized Itô formula (see, for example, Kunita [3]) we have

$$\begin{aligned}0 &= d\Upsilon^{\alpha,n,\epsilon}\left(V_t^{\alpha,n,\epsilon},X_t^1,...,X_t^n\right)\\ &= \frac{\partial\Upsilon^{\alpha,n,\epsilon}}{\partial v}\left(V_t^{\alpha,n,\epsilon},X_t^1,...,X_t^n\right)dV_t^{\alpha,n,\epsilon}+\sum_{j=1}^{n}\frac{\partial\Upsilon^{\alpha,n,\epsilon}}{\partial x_j}\left(V_t^{\alpha,n,\epsilon},X_t^1,...,X_t^n\right)dX_t^j\\ &\quad+\frac{1}{2}\frac{\partial^2\Upsilon^{\alpha,n,\epsilon}}{\partial v^2}\left(V_t^{\alpha,n,\epsilon},X_t^1,...,X_t^n\right)d\langle V^{\alpha,n,\epsilon}\rangle_t+\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2\Upsilon^{\alpha,n,\epsilon}}{\partial x_j^2}\left(V_t^{\alpha,n,\epsilon},X_t^1,...,X_t^n\right)d\langle X^j\rangle_t\\ &\quad+\sum_{j=1}^{n}\frac{\partial\Upsilon^{\alpha,n,\epsilon}}{\partial x_j\partial v}\left(V_t^{\alpha,n,\epsilon},X_t^1,...,X_t^n\right)d\left\langle V^{\alpha,n,\epsilon},X^j\right\rangle_t.\end{aligned}$$

which implies that

$$\begin{aligned}0 &= v_t^{n,\epsilon}\left(V_t^{\alpha,n,\epsilon}\right)dV_t^{\alpha,n,\epsilon}-\frac{1}{n}\sum_{j=1}^{n}f\left(X_t^j,V_t^\alpha\right)q_\epsilon\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)dt\\ &\quad-\frac{1}{n}\sum_{j=1}^{n}\sigma\left(X_t^j,V_t^\alpha\right)q_\epsilon\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)dW_t-\frac{1}{n}\sum_{j=1}^{n}\bar\sigma\left(X_t^j,V_t^\alpha\right)q_\epsilon\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)dB_t^j\\ &\quad+\frac{1}{2n}\sum_{j=1}^{n}\bar\sigma^2\left(X_t^j,V_t^\alpha\right)q_\epsilon'\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)dt+\frac{1}{2n}\sum_{j=1}^{n}\sigma^2\left(X_t^j,V_t^\alpha\right)q_\epsilon'\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)dt\\ &\quad+\frac{1}{2n}\sum_{j=1}^{n}q_\epsilon'\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)d\langle V^{\alpha,n,\epsilon}\rangle_t-\frac{1}{n}\sum_{j=1}^{n}q_\epsilon'\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)\sigma\left(X_s^i,V_s^\alpha\right)d\langle W,V^{\alpha,n,\epsilon}\rangle_t\\ &\quad-\frac{1}{n}\sum_{j=1}^{n}q_\epsilon'\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)\bar\sigma\left(X_s^i,V_s^\alpha\right)d\left\langle B^j,V^{\alpha,n,\epsilon}\right\rangle_t\end{aligned}$$

From this identity it follows that

$$\begin{aligned}
\langle V^{\alpha,n,\epsilon}\rangle_t &= \int_0^t \frac{1}{v_s^{n,\epsilon}\left(V_s^{\alpha,n,\epsilon}\right)^2}\left(\frac{1}{n}\sum_{j=1}^n \sigma\left(X_s^j,V_s^\alpha\right)q_\epsilon\left(V_s^{\alpha,n,\epsilon}-X_s^j\right)\right)^2 ds \\
&\quad + \int_0^t \frac{1}{v_s^{n,\epsilon}\left(V_s^{\alpha,n,\epsilon}\right)^2}\left(\frac{1}{n^2}\sum_{j=1}^n \bar{\sigma}\left(X_s^j,V_s^\alpha\right)^2 q_\epsilon\left(V_s^{\alpha,n,\epsilon}-X_s^j\right)^2\right) ds \\
\langle W,V^{\alpha,n,\epsilon}\rangle_t &= \int_0^t \frac{1}{v_s^{n,\epsilon}\left(V_s^{\alpha,n,\epsilon}\right)}\left(\frac{1}{n}\sum_{j=1}^n \sigma\left(X_s^j,V_s^\alpha\right)q_\epsilon\left(V_s^{\alpha,n,\epsilon}-X_s^j\right)\right) ds \\
\left\langle B^i,V^{\alpha,n,\epsilon}\right\rangle_t &= \int_0^t \frac{1}{v_s^{n,\epsilon}\left(V_s^{\alpha,n,\epsilon}\right)}\frac{1}{n}\bar{\sigma}\left(X_s^j,V_s^\alpha\right)q_\epsilon\left(V_s^{\alpha,n,\epsilon}-X_s^j\right) ds
\end{aligned}$$

Therefore

$$\begin{aligned}
dV_t^{\alpha,n,\epsilon} &= \frac{1}{nv_t^{n,\epsilon}\left(V_t^{\alpha,n,\epsilon}\right)}\sum_{j=1}^n f\left(X_t^j,V_t^\alpha\right)q_\epsilon\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)dt \\
&\quad + \frac{1}{nv_t^{n,\epsilon}\left(V_t^{\alpha,n,\epsilon}\right)}\sum_{j=1}^n \sigma\left(X_t^j,V_t^\alpha\right)q_\epsilon\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)dW_t \\
&\quad + \frac{1}{nv_t^{n,\epsilon}\left(V_t^{\alpha,n,\epsilon}\right)}\sum_{j=1}^n \bar{\sigma}\left(X_t^j,V_t^\alpha\right)q_\epsilon\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)dB_t^j \\
&\quad - \frac{1}{2nv_t^{n,\epsilon}\left(V_t^{\alpha,n,\epsilon}\right)}\sum_{j=1}^n (\bar{\sigma}^2\left(X_t^j,V_t^\alpha\right)+\sigma^2\left(X_t^j,V_t^\alpha\right))q'_\epsilon\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)dt \\
&\quad - \frac{1}{2nv_t^{n,\epsilon}\left(V_t^{\alpha,n,\epsilon}\right)}\sum_{j=1}^n q'_\epsilon\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)d\langle V^{\alpha,n,\epsilon}\rangle_t \\
&\quad + \frac{1}{nv_t^{n,\epsilon}\left(V_t^{\alpha,n,\epsilon}\right)}\sum_{j=1}^n q'_\epsilon\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)\sigma\left(X_s^i,V_s^\alpha\right)d\langle W,V^{\alpha,n,\epsilon}\rangle_t \\
&\quad + \frac{1}{n}\sum_{j=1}^n q'_\epsilon\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)\bar{\sigma}\left(X_s^i,V_s^\alpha\right)d\left\langle B^j,V^{\alpha,n,\epsilon}\right\rangle_t
\end{aligned} \tag{4.5}$$

Observe that the term $\frac{1}{nv_t^{n,\epsilon}\left(V_t^{\alpha,n,\epsilon}\right)}\sum_{j=1}^n f\left(X_t^j,V_t^\alpha\right)q_\epsilon\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)$ is bounded by $\|f\|_\infty$, the supremum norm of f, with similar bounds holding for the second and the third term in (4.5) and for the terms appearing in the expression for $\langle V^{\alpha,n,\epsilon}\rangle_t$, $\langle W,V^{\alpha,n,\epsilon}\rangle_t$, $\langle B^i,V^{\alpha,n,\epsilon}\rangle_t$. The term

$$x \to \frac{1}{2nv_t^{n,\epsilon}(x)}\sum_{j=1}^n (\bar{\sigma}^2\left(x,V_t^\alpha\right)+\sigma^2\left(x,V_t^\alpha\right))q'_\epsilon\left(x-X_t^j\right)$$

is uniformly bounded by $\frac{1}{\epsilon}(\|\bar{\sigma}\|^2+\|\sigma\|^2)$ following property (4.3) of the function q. A similar bound can be proved for all the remaining terms in (4.5) are uniformly bounded on compacts as $\inf_n \inf_{r\in[s,t]} v_s^{n,\epsilon}(x)$ is strictly positive on compacts (using the tightness of the sequence v^n) and $\bar{\sigma}$, σ, and q'_ϵ are bounded. Using these bounds, we take the limit in (4.5) as n tends to infinity to obtain that

$$
\begin{aligned}
dV_t^{\alpha,\epsilon} = {} & \frac{1}{v_t^\epsilon\left(V_t^{\alpha,\epsilon}\right)}\left(\int_{\mathbb{R}} f\left(x, V_t^\alpha\right) q_\epsilon\left(V_t^{\alpha,\epsilon}-x\right) v_t(dx)\right) dt \\
& +\frac{1}{v_t^\epsilon\left(V_t^{\alpha,\epsilon}\right)}\left(\int_{\mathbb{R}} \sigma\left(x, V_t^\alpha\right) q_\epsilon\left(V_t^{\alpha,\epsilon}-x\right) v_t(dx)\right) dW_t \\
& -\frac{1}{2v_t^\epsilon\left(V_t^{\alpha,\epsilon}\right)}\left(\int_{\mathbb{R}} (\bar{\sigma}^2\left(x, V_t^\alpha\right)+\sigma^2\left(x, V_t^\alpha\right)) q'_\epsilon\left(V_t^{\alpha,\epsilon}-x\right) v_t(dx)\right) dt \\
& -\frac{1}{2v_t^\epsilon\left(V_t^{\alpha,\epsilon}\right)}\left(\int_{\mathbb{R}} q'_\epsilon\left(V_t^{\alpha,\epsilon}-x\right) v_t(dx)\right) \\
& \qquad \times\frac{1}{v_t^\epsilon\left(V_t^{\alpha,\epsilon}\right)^2}\left(\int_{\mathbb{R}} \sigma\left(x, V_t^\alpha\right) q_\epsilon\left(V_t^{\alpha,\epsilon}-x\right) v_t(dx)\right)^2 dt \\
& +\frac{1}{v_t^\epsilon\left(V_t^{\alpha,\epsilon}\right)}\left(\int_{\mathbb{R}} q'_\epsilon\left(V_t^{\alpha,\epsilon}-x\right) \sigma\left(x, V_s^\alpha\right) v_t(dx)\right) \\
& \qquad \times\frac{1}{v_t^\epsilon\left(V_t^{\alpha,\epsilon}\right)}\left(\int_{\mathbb{R}} \sigma\left(x, V_t^\alpha\right) q_\epsilon\left(V_t^{\alpha,\epsilon}-x\right) v_t(dx)\right) dt
\end{aligned} \tag{4.6}
$$

Next since $v_t(x) = \lim_{\epsilon\to 0} v_t^\epsilon(x)$ as ϵ tends to 0, it follows that $V_t^\alpha = \lim_{\epsilon\to 0} V_t^{\alpha,\epsilon}$. Following from Corollary 2.1 and the boundedness of both $v_r(x)$ and $\frac{\partial}{\partial x}[v_r(x)]$ on sets of the form $[s,t]\times[-k,k]$, we can take the limit in (4.6) as ϵ tends to 0 to obtain that

$$
\begin{aligned}
dV_t^\alpha = {} & f(V_t^\alpha, V_t^\alpha)dt + \sigma(V_t^\alpha, V_t^\alpha)dW_t - \frac{1}{2v(t,V_t^\alpha)}\frac{\partial}{\partial x}\left[(\bar{\sigma}^2(x,V_t^\alpha)+\sigma^2(x,V_t^\alpha))v_t(x)\right]\Big|_{x=V_t^\alpha} dt \\
& -\frac{1}{2v(t,V_t^\alpha)}\sigma^2(V_t^\alpha,V_t^\alpha)\frac{\partial}{\partial x}\left[v_t(x)\right]\Big|_{x=V_t^\alpha} dt + \frac{\sigma(V_t^\alpha,V_t^\alpha)}{v(t,V_t^\alpha)}\frac{\partial}{\partial x}\left[\sigma(x,V_t^\alpha)v_t(x)\right]\Big|_{x=V_t^\alpha} dt
\end{aligned}
$$

which gives (4.4). □

Remark 4.2 *See also [8] for the equation (4.4).*

Remark 4.3 *Under additional assumptions on the initial distribution of X (for example if the distribution of X_0 is absolutely continuous with respect to the Lebesgue measure and its density is twice continuously differentiable) one can show that (4.4) holds true also for $s = 0$.*

5 Application to nonlinear filtering

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which we have defined two independent d-dimensional, respectively m -dimensional standard Brownian motions $B = \{(B_t^i)_{i=1}^d, t \geq 0\}$ and $W =$

$\{(W_t^i)_{i=1}^m, t \geq 0\}$ Let (X, Y) be the solution of the following stochastic system

$$\begin{aligned} X_t &= X_0 + \int_0^t f(X_s, Y_s)\, ds + \int_0^t \bar{\sigma}(X_s, Y_s)\, dW_s + \int_0^t \bar{\sigma}(X_s, Y_s)\, dB_s \\ Y_t &= \int_0^t h(X_s, Y_s)\, ds + \int_0^t k(Y_s)\, dW_s. \end{aligned}$$

Let $\mathcal{F}_t^Y$ be σ-field generated by the process Y and π_t be the conditional distribution of X_t given the σ-field generated by the process Y. We show that π_t charges any open set. Moreover, under additional conditions, we show that it is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ and has a positive density. Here are the required conditions:

F1 $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$, $h : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^m$, $\sigma : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^d$, and $\bar{\sigma} : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d \times \mathbb{R}^d$ are continuous functions, uniformly Lipschitz in the first argument. We assume that $\bar{\sigma}$ is positive definite, $k : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$ is invertible, k^{-1} is bounded and $\sigma k^{-1} h : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ is a continuous functions, uniformly Lipschitz in the first argument. The random variable X_0 has finite second moment.

F2 f, $\sigma k^{-1} h$, σ and $\bar{\sigma}$ are continuously differentiable in the first component.

We have the following

Corollary 5.1 *Under assumption **F1**, there exists a set $\widetilde{\Omega} \in \mathcal{F}$ of full measure such that for every $\omega \in \widetilde{\Omega}$, π_t^ω charges any open set. Moreover under assumptions **A1+A2**, there exists a set $\widetilde{\Omega} \in \mathcal{F}$ of full measure such that for every $\omega \in \widetilde{\Omega}$, π_t^ω is absolutely continuous with respect to the Lebesgue measure and the density of π_t^ω with respect to the Lebesgue measure is positive almost everywhere.*

Proof. Let $Z = \{Z_t, t \geq 0\}$ be defined as

$$\begin{aligned} Z_t = \exp\Bigg(&- \int_0^t \left(k^{-1}(Y_s)\, h(X_s, Y_s)\right)^\top dW_s \\ &- \frac{1}{2} \int_0^t \left(k^{-1}(Y_s)\, h(X_s, Y_s)\right)^\top \left(k^{-1}(Y_s)\, h(X_s, Y_s)\right) ds \Bigg). \end{aligned}$$

Under condition **F1,** Z is a martingale. Consider the probability measure $\widetilde{P}$ absolutely continuous with respect to P defined as

$$\left.\frac{d\widetilde{P}}{dP}\right|_{\mathcal{F}_t} = Z_t.$$

Then, by Girsanov's theorem, the process $\widetilde{W} = \left\{\widetilde{W}_t, t \geq 0\right\}$ defined by

$$\widetilde{W}_t = W_t - \int_0^t k^{-1}(Y_s)\, h(X_s, Y_s)\, ds$$

for $t \geq 0$ is a Brownian motion under $\widetilde{P}$ independent of B and, by Kallianpur-Striebel's formula,

$$E\left[\varphi\left(X_t\right)|\mathcal{F}_t^Y\right] = \widetilde{E}\left[\varphi\left(X_t\right)\zeta_t|\,\mathcal{F}_t^Y\right], \tag{5.1}$$

where $\zeta_t = \frac{Z_t^{-1}}{\widetilde{E}\left[Z_t^{-1}|\mathcal{F}_t^Y\right]}$ and

$$X_t = X_0 + \int_0^t \left(f + \sigma k^{-1}h\right)\left(X_s, Y_s\right)ds + \int_0^t \sigma\left(X_s, Y_s\right)d\widetilde{W}_s + \int_0^t \bar{\sigma}\left(X_s, Y_s\right)dB_s.$$

We note that, under $\widetilde{P}$, Y satisfies the SDE

$$Y_t = \int_0^t k\left(Y_s\right)d\widetilde{W}_s,$$

hence

$$\widetilde{W}_t = \int_0^t k^{-1}\left(Y_s\right)dY_s$$

and in particular $\mathcal{F}_t^Y = \mathcal{F}_t^{\widetilde{W},Y}$ for all $t \geq 0$. From (5.1) we obtain that as in (2.3) that

$$\pi_t\left(\varphi\right) = \int_{\mathbb{R}^d} E\left[\varphi\left(X_t\left(z\right)\right)M_t\left(z\right)\zeta_t\frac{e^{-\frac{1}{2}z^\top z}}{\left(2\pi\right)^{\frac{d}{2}}}|\mathcal{F}_t^Y\right]dz$$

where $M_t(z)$ is the martingale defined in (2.2). The analysis then proceeds in an identical fashion to that in the proofs of Theorems 1.1 and 1.2. □

Remark 5.2 *Note that we cannot apply the results of the Theorems 1.1 and 1.2 under the original measure P as the Brownian motion B is not independent of Y under P.*

A Appendix

A.1 Convergence of sequences of exchangeable families.

Let S be a complete, separable metric space. A family of S-valued random variables $\{\xi_1, \ldots, \xi_m\}$ is exchangeable if for every permutaion $(\sigma_1, \ldots, \sigma_m)$ of $(1, \ldots, m)$, $\{\xi_{\sigma_1}, \ldots, \xi_{\sigma_m}\}$ has the same distribution as $\{\xi_1, \ldots, \xi_m\}$. A sequence $\xi_1, \xi_2, \ldots$ is exchangeable if every finite subfamily $\xi_1, \ldots, \xi_m$ is exchangeable.

Theorem A.1 *(*deFinetti*) Let $\xi_1, \xi_2, \ldots$ be an exchangeable sequence of S-valued random variables. Then there is a $\mathcal{P}(S)$-valued random variable Ξ such that*

$$\Xi = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \delta_{\xi_i}$$

and, conditioned on Ξ, $\xi_1, \xi_2, \ldots$ are iid with distribution Z, that is, for each $f \in B(S^m)$, $m = 1, 2, \ldots$,

$$E[f(\xi_1, \ldots, \xi_m)|\Xi] = \langle f, \Xi^m \rangle.$$

We will refer to Ξ as the deFinetti measure for $\xi_1, \xi_2, \ldots$.

Proofs of the following lemmas can be found in the Appendix of [2].

Lemma A.2 *For $n = 1, 2, \ldots$, let $\{\xi_1^n, \ldots, \xi_{N_n}^n\}$ be exchangeable, S-valued random variables. (We allow $N_n = \infty$.) Let Ξ^n be the corresponding empirical measure,*

$$\Xi^n = \frac{1}{N_n} \sum_{i=1}^{N_n} \delta_{\xi_i^n},$$

where if $N_n = \infty$, we mean

$$\Xi^n = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \delta_{\xi_i^n}.$$

Assume that $N_n \to \infty$ and that for each $m = 1, 2, \ldots$, $\{\xi_1^n, \ldots, \xi_m^n\} \Rightarrow \{\xi_1, \ldots, \xi_m\}$ in S^m. Then $\{\xi_i\}$ is exchangeable and setting $\xi_i^n = s_0 \in S$ for $i > N_n$, $\{\Xi^n, \xi_1^n, \xi_2^n \ldots\} \Rightarrow \{\Xi, \xi_1, \xi_2, \ldots\}$ in $\mathcal{P}(S) \times S^\infty$, where Ξ is the deFinetti measure for $\{\xi_i\}$. If for each m, $\{\xi_1^n, \ldots, \xi_m^n\} \to \{\xi_1, \ldots, \xi_m\}$ in probability in S^m, then $\Xi^n \to \Xi$ in probability in $\mathcal{P}(S)$.

The converse also holds in the sense that $\Xi^n \Rightarrow \Xi$ implies $\{\xi_1^n, \ldots, \xi_m^n\} \Rightarrow \{\xi_1, \ldots, \xi_m\}$.

We are interested in applying the above lemma in the case $S = D_E[0, \infty)$. In that setting, in addition to the $\mathcal{P}(D_E[0, \infty))$-valued random variables Ξ_n, it is natural to consider the $\mathcal{P}(E)$-valued processes

$$Z_n(t) = \frac{1}{N_n} \sum_{i=1}^{N_n} \delta_{X_i^n(t)},$$

(where N_n may be infinite) which will have sample paths in $D_{\mathcal{P}(E)}[0, \infty)$. Unlike Ξ_n, convergence of Z_n is not always assured.

Lemma A.3 *For $n = 1, 2, \ldots$, let $X^n = (X_1^n, \ldots, X_{N_n}^n)$ be exchangeable families of $D_E[0,\infty)$-valued random variables such that $N_n \Rightarrow \infty$ and $X^n \Rightarrow X$ in $D_E[0,\infty)^\infty$. Define $\Xi_n = \frac{1}{N_n}\sum_{i=1}^{N_n} \delta_{X_i^n} \in \mathcal{P}(D_E[0,\infty))$, $\Xi = \lim_{m\to\infty} \frac{1}{m}\sum_{i=}^{m} \delta_{X_i}$, $Z_n(t) = \frac{1}{N_n}\sum_{i=1}^{N_n} \delta_{X_i^n(t)} \in \mathcal{P}(E)$, and $Z(t) = \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} \delta_{X_i(t)}$.*

a) Let $D_\Xi = \{t : E[\Xi\{x : x(t) \neq x(t-)\}] > 0\}$. Then for $t_1, \ldots, t_l \notin D_\Xi$,

$$(\Xi_n, Z_n(t_1), \ldots, Z_n(t_l)) \Rightarrow (\Xi, Z(t_1), \ldots, Z(t_l)).$$

b) If $X^n \Rightarrow X$ in $D_{E^\infty}[0,\infty)$, then $(X^n, Z_n) \Rightarrow (X, Z)$ in $D_{E^\infty\times\mathcal{P}(E)}[0,\infty)$. If $X^n \to X$ in probability in $D_{E^\infty}[0,\infty)$, then $(X^n, Z_n) \to (X, Z)$ in probability in $D_{E^\infty\times\mathcal{P}(E)}[0,\infty)$.

Remark A.4 *a) The set D_Ξ is at most countable.*

b) If for $i \neq j$, with probability one, X_i and X_j have no simultaneous discontinuities, then $D_\Xi = \emptyset$ and convergence of X^n to X in $D_E[0,\infty)^\infty$ implies convergence in $D_{E^\infty}[0,\infty)$. In particular, this conclusion holds if the X_i are continuous.

c) If $\{X^n\}$ is relatively compact in $D_{E^\infty}[0,\infty)$, then $\{(X^n, Z_n)\}$ is relatively compact in $D_{E^\infty\times\mathcal{P}(E)}[0,\infty)$.

Lemma A.5 *If $X = (X_1, X_2, \ldots)$ is an exchangeable sequence in $D_E[0,\infty)$, then Z is continuous if and only if for $i \neq j$, with probability one X_i and X_j have no simultaneous discontinuity.*

A.2 Convergence of quantiles

For $0 < \alpha < 1$, and for $\mu \in \mathcal{P}(\mathbb{R})$, define $q_\alpha(\mu) = \inf\{x : \mu(-\infty, x] \geq \alpha\}$. Note that μ is a point of continuity for q_α if and only if $\mu(q_\alpha(\mu), q_\alpha(\mu)+\epsilon) > 0$ and $\mu(q_\alpha(\mu)-\epsilon), q_\alpha(\mu)) > 0$ for every $\epsilon > 0$.

Lemma A.6 *Let $\{Y_n\}$ be a sequence of $\mathcal{P}(\mathbb{R})$-valued random variables such that $Y_n \Rightarrow Y$. Suppose that with probability 1, the measure Y charges every open set. Then $q_\alpha(Y_n) \Rightarrow q_\alpha(Y)$ for each $0 < \alpha < 1$.*

Proof. The lemma follows by the continuous mapping theorem. □

Lemma A.7 *Suppose $z \in D_{\mathcal{P}(\mathbb{R})}[0,\infty)$ and for each $t \geq 0$, $z(t)$ and $z(t-)$ charge every open set. Then if $0 < \alpha < 1$ and $z_n \to z$ in $D_{\mathcal{P}(\mathbb{R})}[0,\infty)$, $q_\alpha(z_n) \to q_\alpha(z)$ in $D_\mathbb{R}[0,\infty)$.*

Proof. The lemma follows by Proposition 3.6.5 of Ethier and Kurtz [1] and the continuity properties of q_α. □

The continuous mapping theorem gives the following.

Lemma A.8 *Suppose $\{Z_n\}$ is a sequence of processes in $D_{\mathcal{P}(\mathbb{R})}[0,\infty)$ such that $Z_n \Rightarrow Z$. If, with probability 1, $Z(t)$ and $Z(t-)$ charge every open set for all t, then for $0 < \alpha < 1$, $q_\alpha(Z_n) \Rightarrow q_\alpha(Z)$.*

A.3 Convergence of random measures

The following results are from Kurtz [4]. Let $\mathcal{L}(S)$ be the space of measures μ on $[0,\infty)\times S$ such that $\mu([0,t]\times S)<\infty$ for each $t>0$, and let $\mathcal{L}_m(S)\subset\mathcal{L}(S)$ be the subspace on which $\mu([0,t]\times S)=t$. For $\mu\in\mathcal{L}(S)$, let μ^t denote the restriction of μ to $[0,t]\times S$. Let ρ_t denote the Prohorov metric on $\mathcal{M}([0,t]\times S)$, and define $\widehat{\rho}$ on $\mathcal{L}(S)$ by

$$\widehat{\rho}(\mu,\nu)=\int_0^\infty e^{-t}1\wedge\rho_t(\mu^t,\nu^t)dt,$$

that is, $\{\mu_n\}$ converges in $\widehat{\rho}$ if and only if $\{\mu_n^t\}$ converges weakly for almost every t.

Lemma A.9 *A sequence of $(\mathcal{L}_m(S),\widehat{\rho})$-valued random variables $\{\Gamma_n\}$ is relatively compact if and only if for each $\epsilon>0$ and each $t>0$, there exists a compact $K\subset S$ such that $\inf_n E[\Gamma_n([0,t]\times K)]\geq(1-\epsilon)t$.*

Lemma A.10 *Let $\{(x_n,\mu_n)\}\subset D_E[0,\infty)\times\mathcal{L}(S)$, and $(x_n,\mu_n)\to(x,\mu)$. Let $h\in\bar{C}(E\times S)$. Define*

$$u_n(t)=\int_{[0,t]\times S}h(x_n(s),y)\mu_n(ds\times dy),\quad u(t)=\int_{[0,t]\times S}h(x(s),y)\mu(ds\times dy)$$

$z_n(t)=\mu_n([0,t]\times S)$, and $z(t)=\mu([0,t]\times S)$.

a) If x is continuous on $[0,t]$ and $\lim_{n\to\infty}z_n(t)=z(t)$, then $\lim_{n\to\infty}u_n(t)=u(t)$.

b) If $(x_n,z_n,\mu_n)\to(x,z,\mu)$ in $D_{E\times\mathbb{R}}[0,\infty)\times\mathcal{L}(S)$, then $(x_n,z_n,u_n,\mu_n)\to(x,z,u,\mu)$ in $D_{E\times\mathbb{R}\times\mathbb{R}}[0,\infty)\times\mathcal{L}(S)$. In particular, $\lim_{n\to\infty}u_n(t)=u(t)$ at all points of continuity of z.

c) The continuity assumption on h can be replaced by the assumption that h is continuous a.e. ν_t for each t, where $\nu_t\in\mathcal{M}(E\times S)$ is the measure determined by $\nu_t(A\times B)=\mu\{(s,y):x(s)\in A,s\leq t,y\in B\}$.

d) In both (a) and (b), the boundedness assumption on h can be replaced by the assumption that there exists a nonnegative convex function ψ on $[0,\infty)$ satisfying $\lim_{r\to\infty}\psi(r)/r=\infty$ such that

$$\sup_n\int_{[0,t]\times S}\psi(|h(x_n(s),y)|)\mu_n(ds\times dy)<\infty\tag{A.1}$$

for each $t>0$.

A.4 Measurability and positivity of random functions given by conditional expectations

Lemma A.11 *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, E a complete, separable metric space, and $\{\mathcal{F}_x, x \in E\}$ a collection of complete sub-σ-algebras of $\mathcal{F}$. Suppose that for each $A \in \mathcal{F}$, there exists a $\mathcal{B}(E) \times \mathcal{F}$ measurable process X_A indexed by E such that for each x,*

$$P(A|\mathcal{F}_x) = X_A(x) \quad a.s.$$

Then for each bounded, $\mathcal{B}(E) \times \mathcal{F}$-measurable process Y there exists another $\mathcal{B}(E) \times \mathcal{F}$-measurable process $\widehat{Y}$ such that

$$E[Y(x)|\mathcal{F}_x] = \widehat{Y}(x) \quad a.s.$$

Proof. If $Y(x) = \mathbf{1}_B(x)\mathbf{1}_A$ for $B \in \mathcal{B}(E)$ and $A \in \mathcal{F}$, then $\widehat{Y}(x) = \mathbf{1}_B(x)X_A(x)$ satisfies the requirements of the lemma. Since $\{B \times A : B \in \mathcal{B}(E), A \in \mathcal{F}\}$ is closed under intersections and generates $\mathcal{B}(E) \times \mathcal{F}$ and the collection of Y for which the conclusion of the lemma holds is closed under bounded monotone increasing limits, the lemma follows by the monotone class theorem for functions. (See Theorem 4.3 in the Appendix of Ethier and Kurtz [1].) □

Lemma A.12 *Suppose that the conclusion of Lemma A.11 holds and that Y is $\mathcal{B}(E) \times \mathcal{F}$-measurable and strictly positive. Then $\widehat{Y}$ can be taken to be strictly positive.*

Proof. Let $A_0 = \{(x,\omega) : Y(x,\omega) \geq 1\}$ and $A_n = \{(x,\omega) : 2^{-n} \leq Y(x,\omega) < 2^{-(n-1)}\}$, $n = 1, 2, \ldots$. Then $\cup_{n=0}^{\infty} A_n = E \times \Omega$, and we can assume that $E[\mathbf{1}_{A_n}|\mathcal{F}_x] \geq 0$ for all (x,ω). Note that

$$1 = \lim_{n\to\infty} \sum_{k=0}^{n} E[\mathbf{1}_{A_k}|\mathcal{F}_x] \quad a.s.$$

for all x. If necessary, we can replace $E[\mathbf{1}_{A_n}|\mathcal{F}_x]$ by

$$1 \wedge \sum_{k=0}^{n} E[\mathbf{1}_{A_k}|\mathcal{F}_x] - 1 \wedge \sum_{k=0}^{n-1} E[\mathbf{1}_{A_k}|\mathcal{F}_x]$$

to ensure $\sum_{k=0}^{\infty} E[\mathbf{1}_{A_k}|\mathcal{F}_x] \leq 1$ and then replace $E[\mathbf{1}_{A_0}|\mathcal{F}_x]$ by

$$1 - \sum_{k=1}^{\infty} E[\mathbf{1}_{A_k}|\mathcal{F}_x]$$

to ensure $\sum_{k=0}^{\infty} E[\mathbf{1}_{A_k}|\mathcal{F}_x] = 1$ for all (x,ω). Then

$$\sum_{n=0}^{\infty} 2^{-n} E[\mathbf{1}_{A_n}|\mathcal{F}_x] \leq \widehat{Y}(x) \quad a.s.$$

and we can replace $\widehat{Y}(x)$ by $\widehat{Y}(x) \vee \sum_{n=0}^{\infty} 2^{-n} E[\mathbf{1}_{A_n}|\mathcal{F}_x]$ to be assured that $\widehat{Y}(x) > 0$ for all (x,ω). □

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