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ON THE EXCURSIONS OF REFLECTED LOCAL TIME PROCESSES AND STOCHASTIC FLUID QUEUES

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Abstract

This paper extends previous work by the authors. We consider the local time process of a strong Markov process, add negative drift, and reflect it à la Skorokhod. The resulting process is used to model a fluid queue. We derive an expression for the joint law of the duration of an excursion, the maximum value of the process on it, and the time distance between successive excursions. We work with a properly constructed stationary version of the process. Examples are also given in the paper.

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1. Introduction

Consider a stationary strong Markov process $X = (X_t, t \in \mathbb{R})$, defined on some filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t, t \in \mathbb{R}))$, with values in \mathbb{R}_+ , a.s. càdlàg paths, and adapted to (\mathcal{F}_t) . In this paper, the local time L of the process X at $x = 0$ is considered an (\mathcal{F}_t) -adapted stationary random measure that regenerates jointly with X at every (stopping) time that X hits 0. More precisely:

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(A1) L assigns a nonnegative random variable $L(B, \omega)$ to each $B \in \mathcal{B}(\mathbb{R})$ such that $L(\cdot, \omega)$ is a Radon measure for each $\omega \in \Omega$.

(A2) For any a.s. finite (\mathcal{F}_t) -stopping time T at which $X_T = 0$, the process $((X_{T+t}, L(T, T+t)), t \geq 0)$ is independent of \mathcal{F}_T .

We take the broader perspective with regard to the process L and we allow for the case that it is a local time of an irregular point (in which case L has discontinuous paths) as well as the case that 0 is a sticky point (in which case L is absolutely continuous with respect to the Lebesgue measure with density $c\mathbf{1}(X_t = 0)$ for some $c > 0$). We refer to [2, Chap. IV] (in particular Corollary 6), [17, Chap. 6], and [5, §V.3] for further discussion. For each $s \in \mathbb{R}$ define the inverse local time process with respect to t by

$$L_{s;u}^{-1} := \inf\{t > 0 : L[s, s+t] > u\}, \quad u \geq 0. \quad (1)$$

What is important is that, owing to this definition, the inverse of the cumulative local time is a Lévy process in the following sense:

Lemma 1. *If L is continuous then for every a.s. finite (\mathcal{F}_t) -stopping time T such that $X_T = 0$, the process $(L_{T;u}^{-1}, u \geq 0)$ is a subordinator with $L_{T;0}^{-1} = 0$.*

If L is not continuous, that is to say if 0 is an irregular point for X , then this Lemma is taken as an additional requirement to the definition of L . This is easily arranged by choosing L to be a modification of the counting process on \mathcal{Z} , the discrete set of times that X visits 0, so that the inverse is a subordinator. To do this, we assign, to each element of \mathcal{Z} , an i.i.d., unit-mean exponentially distributed weight. Then let the local time on an interval I to be the sum of all the weights of the points of \mathcal{Z} in I .

We summarise this as an assumption, in addition to (A1)-(A2) above:

(A3) *If L is discontinuous then we require that for every a.s. finite (\mathcal{F}_t) -stopping time T such that $X_T = 0$, the process $(L_{T;u}^{-1}, u \geq 0)$ is a subordinator.*

We will also need the following assumption:

(A4) The stationary random measure L has finite rate not exceeding 1, i.e.

$$EL(0, t) = \mu t,$$

where $0 < \mu < 1$.

Then, as in [19], [16], [22], and [15], we define a stationary process $Q = (Q_t, t \in \mathbb{R})$ by

$$Q_t = \sup_{-\infty < s \leq t} \{L(s, t] - (t - s)\}, \quad t \in \mathbb{R}. \quad (2)$$

Furthermore, Q is ergodic (its invariant σ -field is trivial.) Notice that Q also satisfies, pathwise,

$$\begin{aligned} Q_t &= Q_s + L(s, t] - (t - s) - \inf_{s \leq r \leq t} (Q_s + L(s, r] - (r - s)) \wedge 0, \\ &= \sup_{s \leq r \leq t} (L(r, t] - (t - r)) \vee (Q_s + L(s, t] - (t - s)) \end{aligned} \quad (3)$$

for all $-\infty < s < t < \infty$. It is worth recalling [15] that if we consider (3) as a fixed point equation for Q then process defined by (2) is the unique stationary and ergodic solution of (3). A typical sample path of Q is depicted in Figure 2 below. It consists of isolated excursions away from zero (also called “busy periods”), followed by intervals of time at which Q stays at zero (called “idle periods”). In this respect, the process Q is thought of as the workload in a stochastic fluid queue. Amongst other things in [16], [22], and [15], expressions are derived for the marginal distribution of Q and the Laplace transform of the duration of a typical idle and busy periods.

In this paper, we shall derive an expression for the joint law of three random variables: the duration of a busy period, the duration of an idle period, and the maximum of Q over a busy period.

It is assumed, throughout, that (Ω, \mathcal{F}, P) is endowed with a P -preserving measurable flow $\theta_t : \Omega \rightarrow \Omega$, $t \in \mathbb{R}$, with a measurable inverse $\theta_t^{-1} = \theta_{-t}$. All stationary random processes and measures can be constructed on Ω in such a way that the flow commutes with the natural shift, e.g., $Q_t(\theta_s \circ \omega) = Q_{t+s}(\omega)$,

and $L(B, \theta_s \omega) = L(B + s, \omega)$, for all $s, t \in \mathbb{R}$, Borel sets $B \subset \mathbb{R}$, and $\omega \in \Omega$. The flow will be explicitly used in Section 5 to obtain distributions conditional on observing a positive (or a zero) value of Q_0 .

2. A closer look at the reflected process

Consider now any a.s. finite (\mathcal{F}_t) -stopping time T , such that $X_T = 0$. Then $(L_{T;t}^{-1}, t \geq 0)$ is a subordinator starting from zero (owing to Lemma 1 or assumption (A3)) with law that does not depend on T . It turns out that the process of interest is $\Lambda_T = \{\Lambda_{T,t} : t \geq 0\}$, where

$$\Lambda_{T;t} = t - L_{T;t}^{-1}, \quad t \geq 0. \quad (4)$$

Note that, irrespective of T , the process Λ_T has the law of the same bounded variation, spectrally negative Lévy process which is issued from the origin at time zero. By (A4), L has rate $\mu < 1$; hence $E\Lambda_{T;1} = 1 - \frac{1}{\mu} < 0$. Since $L_{T;t}^{-1}$ is a subordinator, it has a well-defined, possibly nonzero, drift. If this drift is larger than or equal to unity then $-\Lambda_T$ is a subordinator and, as it will turn out, this is a trivial case.

We therefore assume in the sequel that the drift of L_T^{-1} is less than unity or, equivalently, that

(A5) *The drift δ_Λ of the process Λ defined by (4) is strictly positive.*

Under this assumption, the point 0 is irregular for $(-\infty, 0)$ for Λ_T (this follows as a standard results for bounded variation spectrally negative Lévy processes, see Bertoin [2, Chap. VII].)

In addition, under (A5), it is clear that the time taken for Λ_T to first enter $(-\infty, 0)$ is almost surely strictly positive. It will be shown below (Lemma 3) that this implies that the excursions of the process Q , i.e. the busy periods, have strictly positive Lebesgue length with probability one. It can be intuitively seen, via a geometric argument involving the reflection of the space-time path

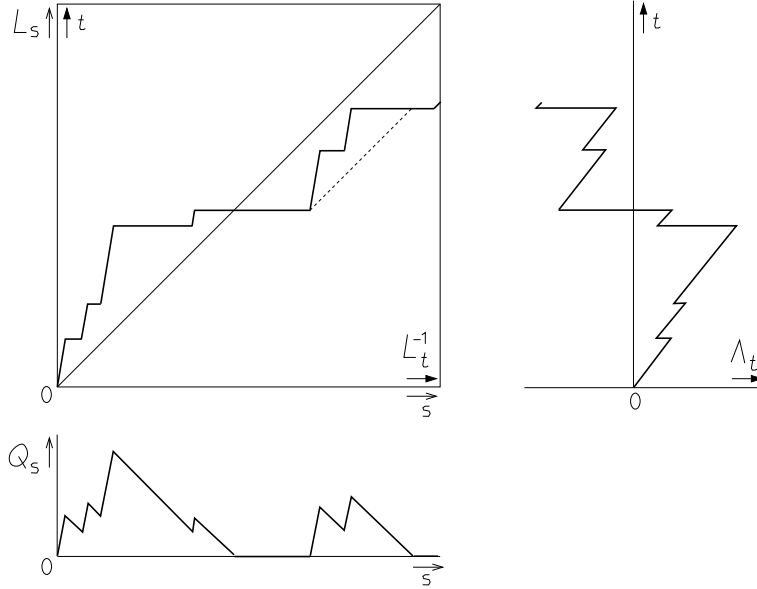


FIGURE 1: The construction of the process $(\Lambda_{T,t}, t \geq 0)$ and related processes, assuming that $T = 0$. Note that Λ may have countably many jumps on finite intervals.

of Λ_T about the diagonal (see Figure 1), that the time taken for Λ_T to first enter $(-\infty, 0)$ is almost surely equal to the length of the excursion of Q started at time T .

In this light, note also that Λ_T cannot creep downwards because it is spectrally negative with paths of bounded variation (cf. Bertoin [2, Chap. VII]). Hence the overshoot at first passage of Λ_T into $(-\infty, 0)$ is almost surely strictly positive. It will turn out (Lemma 1) that this overshoot agrees with the idle period following the aforementioned excursion of Q .

The above analysis implies that, on finite intervals of time, Q has finitely many excursions (busy periods) separated by positive-length idle periods. Denote by

$$\dots < g(-1) < g(0) < g(1) < g(2) < \dots$$

the beginnings of the idle periods and by

$$\dots < d(-1) < d(0) < d(1) < d(2) < \dots$$

their ends, see Figure 2. We choose the indexing so that $g(0) \leq 0 < g(1)$. Let N_g (respectively, N_d) be the point process with points $\{g(n) : n \in \mathbb{Z}\}$ (respectively, $\{d(n) : n \in \mathbb{Z}\}$). As Q is a stationary process, N_g and N_d are jointly stationary with finite, nonzero, intensity [15] denoted by λ (an expression for which is given by (26) and is derived in §4.3 below). Corresponding to point processes

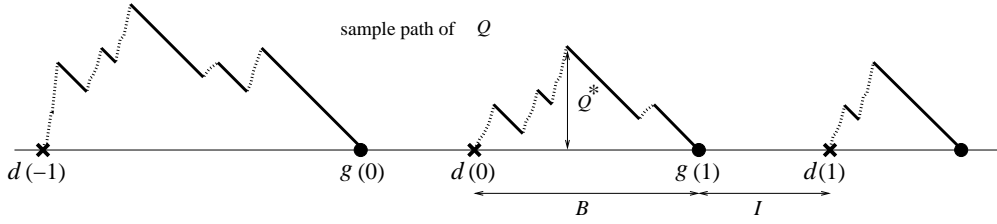


FIGURE 2: The definition of $g(n)$ and $d(n)$. By convention, the origin of time is between $g(0)$ and $g(1)$, under the original measure P . Under P_d , the origin of time is at $d(0)$. Under P_g , the origin of time is at $g(0)$. The random variable Q^* is the maximum deviation from 0 of Q within the typical busy period.

N_g, N_d we have the Palm probabilities P_g, P_d , respectively. Let us consider Q under the measure P_d . Then $P_d(d(0) = 0) = 1$, i.e. the origin of time is placed at the beginning of a busy period. By the strong Markov property, the “cycles”

$$\mathcal{C}_n := \{Q_t : d(n) \leq t < d(n+1)\}, \quad n \in \mathbb{Z},$$

are i.i.d. under measure P_d . In particular, the pairs of random variables

$$(g(n+1) - d(n), d(n+1) - g(n+1)), \quad n \in \mathbb{Z},$$

are i.i.d. under P_d . Consider the triple

$$(B, I, Q^*) := \left(g(1) - d(0), d(1) - g(1), \sup_{d(0) < t < g(1)} Q_t \right), \quad (5)$$

which is a function of \mathcal{C}_0 . We are primarily interested in the P_d -law of (B, I, Q^*) . Since, under P_d , the origin of time is placed at $d(0)$, we interpret B, I, Q^* as the typical busy period, the typical idle period, and the maximum value of Q over a typical busy period, respectively.

The next lemma is proved in [15]:

Lemma 2. *Let $D = \inf\{t > 0 : X_t = 0\}$ and $d = \inf\{t > 0 : Q_t > 0\}$. Then $d = D$ a.s. on $\{Q = 0\}$.*

We now obtain an alternative expression for $B = g(1) - d(0)$ and $I = d(1) - g(1)$ in terms of the inverse local time.

Lemma 3. *We have that*

$$B = g(1) - d(0) = \inf\{u > 0 : L_{d(0);u}^{-1} > u\}, \quad (6)$$

$$B + I = d(1) - d(0) = L_{d(0);g(1)-d(0)}^{-1}. \quad (7)$$

Proof. Since $d(0)$ is the end of an idle period (and the beginning of a busy period), we have $Q_{d(0)-} = 0$. Using then expression (3) we obtain

$$Q_t = L[d(0), t] - (t - d(0)), \quad d(0) \leq t < g(1),$$

which gives

$$B = g(1) - d(0) = \inf\{t > 0 : L[d(0), d(0) + t] = t\}.$$

Consider now $L_{d(0);u}^{-1}$, defined by (1). By Lemma 2, $d(0)$ is a point of increase of the function $t \mapsto L[d(0), d(0) + t]$. Hence $g(1) > d(0)$. Also, when $L[d(0), d(0) + t] - t$ decreases, it does so continuously. Therefore,

$$B = \inf\{t > 0 : L[d(0), d(0) + t] < t\}.$$

Notice also that, for all $t, x > 0$,

$$L[d(0), d(0) + t] < x \iff t < L_{d(0);x-}^{-1},$$

where $L_{d(0);x-}^{-1} = \lim_{\varepsilon \downarrow 0} L_{d(0);x-\varepsilon}^{-1}$. It follows that,

$$\begin{aligned} B &= \inf\{t > 0 : t < L_{d(0);t-}^{-1}\} \\ &= \inf\{t > 0 : L_{d(0);t}^{-1} > t\}, \end{aligned}$$

by the right continuity of $t \mapsto L_{d(0);t}^{-1}$. To prove the expression for $B + I$, notice that L does not charge the interval $[g(1), d(1))$ because, by definition, Q is zero for all t in this interval. \square

Henceforth it will be convenient to work with the process $\Lambda = (\Lambda_t, t \geq 0)$ where

$$\Lambda_t := t - L_{d(0);t}^{-1}, \quad t \geq 0.$$

Note also that $d(0)$ is an (\mathcal{F}_t) -stopping time at which X takes the value 0 and hence in our earlier notation $\Lambda_t = \Lambda_{d(0);t}$.

From the expression (6), and as discussed in the introduction of Section 2, we see that B is simply the first time at which Λ enters $(-\infty, 0)$,

$$B = \inf\{t > 0 : \Lambda_t < 0\}, \quad (8)$$

which is necessarily strictly positive thanks to the irregularity of 0 for $(-\infty, 0)$ for Λ . From (6) and (7) we see that

$$I = L_{d(0);g(1)}^{-1} - B = L_{d(0);B}^{-1} - B = -\Lambda_B, \quad (9)$$

i.e. I is, in absolute value, equal to the value of Λ at the first time it becomes negative. Again we recall from the discussion at the beginning of Section 2 that Λ cannot creep downwards and hence $I > 0$ almost surely.

Consider now the random variable $Q^* = \sup_{d(0) < t < g(1)} Q_t$. If we define

$$\tau_x := \inf\{t > 0 : \Lambda_t > x\} = \inf\{t > 0 : \Lambda_t = x\} \quad (10)$$

we immediately see that

$$\{Q^* < x\} = \{B < \tau_x\}. \quad (11)$$

3. The triple law

Recall that P_d is the Palm probability with respect to the point process $\{d(n), n \in \mathbb{Z}\}$. The function

$$H(\alpha, \beta, x) = E_d \left[e^{-\alpha B - \beta I} \mathbf{1}(Q^* \leq x) \right]$$

characterizes the joint law of the triple (B, I, Q^*) under P_d . Since $P_d(d(0) = 0) = 1$, we have that

$$\Lambda_t = t - L_{0;t}^{-1}, \text{ with } \Lambda_0 = 0, P_d\text{-a.s.} \quad (12)$$

Recalling the expressions (8), (9) and (11) for B , I and Q^* , respectively, we write

$$H(\alpha, \beta, x) = E_d \left[e^{-\alpha B + \beta \Lambda_B} \mathbf{1}(B \leq \tau_x) \right]. \quad (13)$$

Since our primary object is the process Λ defined in (12), and in view of (4) and (13), it makes sense to consider the process on its canonical probability space and denote its law by \mathbb{P}_0 . Then

$$H(\alpha, \beta, x) = \mathbb{E}_0 \left[e^{-\alpha B + \beta \Lambda_B} \mathbf{1}(B < \tau_x) \right]. \quad (14)$$

The latter function may now be expressed in terms of so-called scale functions for spectrally negative Lévy processes. To define the latter, let

$$\psi_\Lambda(\theta) := \log \mathbb{E}_0 e^{\theta \Lambda_1}, \theta \geq 0,$$

be the Laplace exponent of Λ under \mathbb{P}_0 . Then the, so-called, q -scale function for (Λ, \mathbb{P}_0) , denoted by $W^{(q)}(x)$, satisfies $W^{(q)}(x) = 0$ for $x < 0$ and on $[0, \infty)$ it is the unique continuous (right continuous at the origin) monotone increasing function whose Laplace transform is given by

$$\int_0^\infty e^{-\theta x} W^{(q)}(x) dx = \frac{1}{\psi_\Lambda(\theta) - q}, \quad \text{for } \beta > \Phi_\Lambda(q), \quad (15)$$

where

$$\Phi_\Lambda(q) = \sup\{\theta \geq 0 : \psi_\Lambda(\theta) = q\}$$

is the right inverse of ψ_Λ . (See for example the discussion in Chapter 9 of [17]).

Theorem 1. *Let Λ be the process defined by (12), B its first entry time to $(-\infty, 0)$ as in (8), and τ_x the first hitting time of $\{x\}$ as in (10). For $\alpha, \beta, x \geq 0$*

we have

$$\begin{aligned} H(\alpha, \beta, x) &= \mathbb{E}_0 \left[e^{-\alpha B + \beta \Lambda_B} \mathbf{1}(B < \tau_x) \right] \\ &= 1 - \frac{1}{\delta_\Lambda} \frac{1 + (\alpha - \psi_\Lambda(\beta)) \int_0^x e^{-\beta y} W^{(\alpha)}(y) dy}{e^{-\beta x} W^{(\alpha)}(x)}. \end{aligned} \quad (16)$$

Proof. Let $\mathcal{G}_t := \sigma(\Lambda_s, s \leq t)$ and define, for all $\beta \geq 0$, the exponential (\mathcal{G}_t) -martingale

$$M_t^\beta := e^{\beta \Lambda_t - \psi_\Lambda(\beta)t}, \quad t \geq 0.$$

Let, on the canonical space of Λ , \mathbb{P}_0^β be a probability measure, absolutely continuous with respect to \mathbb{P}_0 on \mathcal{G}_t for each t , with Radon-Nikodým derivative

$$\left. \frac{d\mathbb{P}_0^\beta}{d\mathbb{P}_0} \right|_{\mathcal{G}_t} := M_t^\beta.$$

Notice that Λ is still a Lévy process under \mathbb{P}_0^β with Laplace exponent

$$\psi_\Lambda^\beta(\theta) = \log \mathbb{E}_0^\beta e^{\theta \Lambda_1} = \psi_\Lambda(\beta + \theta) - \psi_\Lambda(\beta). \quad (17)$$

It is straightforward to check from the above formula that, under \mathbb{P}_0^β , Λ is spectrally negative, with bounded variation paths and drift coefficient equal to δ_Λ . Since on the stopped σ -field \mathcal{G}_B we have $(d\mathbb{P}_0^\beta/d\mathbb{P}_0)|_{\mathcal{G}_B} = M_B^\beta$, we may substitute

$$e^{\beta \Lambda_B} = M_B^\beta e^{\psi_\Lambda(\beta)B}$$

in the equation (14) for H to obtain

$$H(\alpha, \beta, x) = \mathbb{E}_0 \left[M_B^\beta e^{\psi_\Lambda(\beta)B} e^{-\alpha B} \mathbf{1}(B < \tau_x) \right] = \mathbb{E}_0^\beta \left[e^{-(\alpha - \psi_\Lambda(\beta))B} \mathbf{1}(B < \tau_x) \right].$$

Let

$$q := \alpha - \psi_\Lambda(\beta),$$

and assume that $q \geq 0$. It follows from [17, Thm, 8.1(iii)] that

$$H(\alpha, \beta, x) = \mathbb{E}_0^\beta \left[e^{-qB} \mathbf{1}(B < \tau_x) \right] = Z_\beta^{(q)}(0) - Z_\beta^{(q)}(x) \frac{W_\beta^{(q)}(0)}{W_\beta^{(q)}(x)}, \quad (18)$$

where $W_\beta^{(q)}$ is the q -scale function for $(\Lambda, \mathbb{P}_0^\beta)$ and $Z_\beta^{(q)}$ is given by

$$Z_\beta^{(q)}(x) = 1 + q \int_0^x W_\beta^{(q)}(t) dt.$$

It is easy to see [17, Lemma 8.4] that the Laplace transform of $W_\beta^{(q)}(\cdot)$ is the Laplace transform of $W^{(q)}(\cdot)$ shifted by β and this ensures that

$$W_\beta^{(q)}(x) = e^{-\beta x} W^{(q)}(x). \quad (19)$$

Moreover, since Λ still has drift coefficient δ_Λ under \mathbb{P}_0^β , [17, Lemma 8.6] tells us that, irrespective of the value of q and β , $W_\beta^{(q)}(0) = 1/\delta_\Lambda$. Putting the pieces together, this gives us the desired expression for $\alpha \geq \psi_\Lambda(\beta)$. However [17, Lemma 8.3], since $W^{(q)}(x)$ is analytic in q , the condition on α can be relaxed to $\alpha \geq 0$ by using a straightforward analytic extension argument. \square

In view of (8), (9), (11), and (13), we get the following corollary.

Corollary 1. (Joint law of typical B , I and Q^* .) *Assume that (A1)–(A5) hold. Then the joint law of the length B of a typical busy period, the length I of a typical idle period, and the maximum Q^* of Q over the typical busy period is expressed by the formula*

$$E_d[e^{-\alpha B - \beta I} \mathbf{1}(Q^* \leq x)] = 1 - \frac{1}{\delta_\Lambda} \frac{1 + (\alpha - \psi_\Lambda(\beta)) \int_0^x e^{-\beta y} W^{(\alpha)}(y) dy}{e^{-\beta x} W^{(\alpha)}(x)} \quad (20)$$

where $\alpha, \beta, x \geq 0$.

4. Marginal distributions

Clearly, formula (20) can be used to extract more detailed information about typical behaviour of Q . Let us first derive the distribution (Laplace transform) of the pair (B, I) under the measure P_d . We have

$$E_d[e^{-\alpha B} e^{-\beta I}] = \mathbb{E}_0[e^{-\alpha B} e^{\beta \Lambda_B}] = \lim_{x \rightarrow \infty} H(\alpha, \beta, x).$$

To derive the limit, let us temporarily assume that $q = \alpha - \psi_\Lambda(\beta) > 0$ and $\beta \geq 0$. Consider (16) in the form (18) and use the limiting result

$$\lim_{x \rightarrow \infty} \frac{Z_\beta^{(q)}(x)}{W_\beta^{(q)}(x)} = \frac{q}{\Phi_\Lambda^\beta(q)},$$

from [17, Exercise 8.5] where the function Φ_Λ^β is the right inverse of ψ_Λ^β . That is to say

$$\begin{aligned} \Phi_\Lambda^\beta(q) &= \sup\{\theta \geq 0 : \psi_\Lambda^\beta(\theta) = q\} \\ &= \sup\{\theta \geq 0 : \psi_\Lambda^\beta(\theta) = \alpha - \psi_\Lambda(\beta)\} \\ &= \sup\{\theta \geq 0 : \psi_\Lambda(\theta + \beta) = \alpha\} \\ &= \Phi_\Lambda(\alpha) - \beta. \end{aligned}$$

This gives

$$E_d[e^{-\alpha B} e^{-\beta I}] = 1 - \frac{1}{\delta_\Lambda} \frac{\alpha - \psi_\Lambda(\beta)}{\Phi_\Lambda(\alpha) - \beta}. \quad (21)$$

To remove the restriction that $\alpha > \psi_\Lambda(\beta)$ in (21) and replace it instead by just $\alpha \geq 0$, one may again proceed with an argument involving analytical extension taking care to note for the case that $\alpha = \psi_\Lambda(\beta)$,

$$\lim_{|\alpha - \psi_\Lambda(\beta)| \rightarrow 0} \frac{\alpha - \psi_\Lambda(\beta)}{\Phi_\Lambda(\alpha) - \beta} = \lim_{|\alpha - \psi_\Lambda(\beta)| \rightarrow 0} \frac{\psi_\Lambda^\beta(\Phi_\Lambda(\alpha) - \beta)}{\Phi_\Lambda(\alpha) - \beta} = \psi_\Lambda^{\beta'}(0+) = \psi_\Lambda'(\beta).$$

4.1. Busy period

Letting $\beta = 0$ in (21), we find the P_d -law of B . That is to say,

$$E_d[e^{-\alpha B}] = 1 - \frac{1}{\delta_\Lambda} \frac{\alpha}{\Phi_\Lambda(\alpha)}.$$

This formula is consistent with the result of [15, Prop. 8] and, moreover, we see that the mean duration of the busy period is given by

$$E_d[B] = \frac{1}{\delta_\Lambda \Phi_\Lambda(0)}. \quad (22)$$

4.2. Idle period

To find the P_d -law of I we need to set $\alpha = 0$. Recall however from the beginning of Section 2 that $E_d(\Lambda_1) < 0$. This implies that $\Phi_\Lambda(0) > 0$ and hence we have

$$E_d[e^{-\beta I}] = 1 - \frac{1}{\delta_\Lambda} \frac{\psi_\Lambda(\beta)}{\beta - \Phi_\Lambda(0)}. \quad (23)$$

It follows that the mean idle period is thus equal to

$$E_d[I] = \frac{-\psi'_\Lambda(0+)}{\delta_\Lambda \Phi_\Lambda(0)}, \quad (24)$$

where $\psi'_\Lambda(0+) = E_d(\Lambda_1) < 0$.

4.3. Rates

A cycle of the process Q is defined as the interval from the beginning of a busy period until the beginning of the next busy period. We therefore have

$$\text{mean cycle length} = E_d[B + I] = \frac{1 - \psi'_\Lambda(0+)}{\delta_\Lambda \Phi_\Lambda(0)}. \quad (25)$$

We can express the common rate, λ , of N_g and N_d as the inverse of the mean cycle length:

$$\lambda := EN_d(0, 1) = EN_g(0, 1) = \frac{1}{E_d[B + I]} = \frac{\delta_\Lambda \Phi_\Lambda(0)}{1 - \psi'_\Lambda(0+)}. \quad (26)$$

4.4. The maximum over a busy period

We now derive the P_d -distribution of Q^* . Letting $\alpha = \beta = 0$ in (16) we obtain

$$P_d(Q^* \leq x) = 1 - \frac{1}{\delta_\Lambda W(x)},$$

where $W(x) \equiv W^{(0)}(x)$ is defined through its Laplace transform

$$\int_0^\infty e^{-\theta x} W(x) dx = \frac{1}{\psi_\Lambda(\theta)}, \quad \text{for } \theta > \Phi_\Lambda(0). \quad (27)$$

An immediate observation is that $\lim_{x \rightarrow 0} P_d(Q^* \leq x) = 0$, since $W(0) = \lim_{\theta \rightarrow 0} \theta / \psi_\Lambda(\theta) = 1 / \delta_\Lambda$. So under P_d , the random variable Q^* has no atom at zero—which is, of course, expected.

We now show that Q^* has exponential tail under P_d and derive the precise asymptotics. To do this, let

$$\beta^* := \Phi_\Lambda(0).$$

Then (27) gives that the Laplace transform of $x \mapsto e^{-\beta^*x}W(x)$ is $\theta \mapsto 1/\psi_\Lambda(\beta^* + \theta)$. From the final value theorem for Laplace transforms,

$$\lim_{x \rightarrow \infty} e^{-\beta^*x}W(x) = \lim_{\theta \rightarrow 0} \frac{\theta}{\psi_\Lambda(\beta^* + \theta)} = \frac{1}{\psi_\Lambda'(\beta^*)},$$

where we used the fact that $\psi_\Lambda(\beta^*) = 0$. It follows that

$$P_d(Q^* > x) \sim \frac{\psi_\Lambda'(\Phi_\Lambda(0))}{\delta_\Lambda} e^{-\Phi_\Lambda(0)x}$$

as $x \rightarrow \infty$.

5. Cycle formulae

We now show how the use of cycle formulae of Palm calculus enable us to find (Proposition 1 below) the joint law of the endpoints of an idle period conditional on the event that the idle period contains the origin of time. Also (Proposition 2 below) we characterise the joint law of the endpoints of a busy period, together with the maximum of Q over this busy period, conditional on the event that the busy period contains the origin of time.

It is well-known that if (Ω, \mathcal{F}, P) is endowed with a P -preserving flow $(\theta_t, t \geq 0)$ (see end of Section 1) then for any random measure M with finite intensity λ_M , and any point process N with finite intensity λ_N such that $M(B, \theta_t\omega) = M(B + t, \omega)$, $N(B, \theta_t\omega) = N(B + t, \omega)$, for $t \in \mathbb{R}$, B Borel subset of \mathbb{R} , and $\omega \in \Omega$, and any nonnegative measurable $Z : \Omega \rightarrow \mathbb{R}$, we have

$$\lambda_M E_M[Z] = \lambda_N E_N \int_{T_k}^{T_{k+1}} Z \circ \theta_t M(dt), \quad (28)$$

where P_M, E_M (respectively, P_N, E_N) denotes Palm probability and expectation with respect to M (respectively, N), T_0 is the first atom of N which is ≤ 0 , and T_k, T_{k+1} are any two successive atoms of N .

The next result can be found for some special cases in [16] (diffusions), [15] (Lévy processes), and in [22] the general expression is derived. Here we offer a new proof in the general case based on (28).

Proposition 1. (Joint law of endpoints of idle period.) *Assume that (A1)–(A5) hold. Then, conditional on $Q_0 = 0$, the left end-point, $g(0)$, and right end-point, $d(0)$, of the idle period containing $t = 0$ have joint Laplace transform given by*

$$E[e^{-\alpha d(0) + \beta g(0)} \mid Q_0 = 0] = \frac{\Phi_\Lambda(0)}{-\psi'_\Lambda(0+)} \cdot \frac{1}{\alpha - \beta} \left(\frac{\psi_\Lambda(\alpha)}{\alpha - \Phi_\Lambda(0)} - \frac{\psi_\Lambda(\beta)}{\beta - \Phi_\Lambda(0)} \right),$$

for non-negative α and β ($\alpha \neq \beta$).

Proof. Let M_I be the restriction of the Lebesgue measure on the idle periods:

$$M_I(A) = \int_A \mathbf{1}(Q_t = 0) dt, \quad A \in \mathcal{B}(\mathbb{R}).$$

Then $E_{M_I}[Z] = E[Z \mid Q_0 = 0]$ for all nonnegative random variables Z . Apply (28) with $M = M_I$, $N = N_d$, and $Z = e^{-\alpha d(0) + \beta g(0)}$:

$$\lambda_{M_I} E_{M_I}[e^{-\alpha d(0) + \beta g(0)}] = \lambda E_d \int_{d(-1)}^{d(0)} e^{-\alpha d(0) \circ \theta_t + \beta g(0) \circ \theta_t} M_I(dt).$$

Here λ is the rate of N_d and is given by (26). The rate λ_{M_I} is given by

$$\lambda_{M_I} = \frac{E_d[I]}{E_d[B + I]}.$$

Hence

$$\frac{\lambda}{\lambda_{M_I}} = \frac{1}{E_d[I]} = \frac{\delta_\Lambda \Phi_\Lambda(0)}{-\psi'_\Lambda(0+)},$$

where we used (24) and (25). Now, $P_d(d(0) = 0) = 1$. To compute the integral above, note that M_I is zero on the interval $(d(-1), g(0))$, and that, for $g(0) \leq t \leq 0$, we have $d(0) \circ \theta_t = -t$, and $g(0) \circ \theta_t = g(0) - t$. So the integral above equals

$$\int_{g(0)}^0 e^{(\alpha - \beta)t - \alpha g(0)} dt = \frac{e^{\beta g(0)} - e^{\alpha g(0)}}{\alpha - \beta}.$$

Combining the above we obtain

$$E[e^{-\alpha d(0)+\beta g(0)} \mid Q_0 = 0] = \frac{\Phi_\Lambda(0)}{-\psi'_\Lambda(0+)} \cdot \frac{E_d[e^{\beta g(0)}] - E_d[e^{\alpha g(0)}]}{\alpha - \beta}.$$

Since $E_d[e^{\beta g(0)}] = E_d[e^{-\beta I}]$, the result is obtained by using (23). \square

Proposition 2. (Joint law of endpoints of busy period and maximum over it.)
Assume that (A1)–(A5) hold. Then, conditional on $Q_0 > 0$, the left end-point, $d(0)$, and right end-point, $g(1)$, of the busy period containing $t = 0$, together with the maximum of Q_s for s ranging over this busy period have a joint law which is characterised by

$$\begin{aligned} & E[e^{-\alpha g(1)+\beta d(0)} \mathbf{1}(Q^* \leq x) \mid Q_0 > 0] \\ &= \frac{\Phi_\Lambda(0)}{\alpha - \beta} \left(\frac{1 + \alpha \int_0^x W^{(\alpha)}(y) dy}{W^{(\alpha)}(x)} - \frac{1 + \beta \int_0^x W^{(\beta)}(y) dy}{W^{(\beta)}(x)} \right), \end{aligned} \quad (29)$$

for non-negative α and β ($\alpha \neq \beta$).

Proof. Let M_B be the restriction of the Lebesgue measure on the busy periods:

$$M_B(A) = \int_A \mathbf{1}(Q_t > 0) dt, \quad A \in \mathcal{B}(\mathbb{R}).$$

Then $E_{M_B}[Z] = E[Z \mid Q_0 > 0]$ for all random variables $Z \geq 0$. Apply (28):

$$\begin{aligned} & \lambda_{M_B} E_{M_B}[e^{-\alpha g(1)+\beta d(0)} \mathbf{1}(Q^* \leq x)] \\ &= \lambda E_d \int_{d(0)}^{d(1)} e^{-\alpha g(1) \circ \theta_t + \beta d(0) \circ \theta_t} \mathbf{1}(Q^* \circ \theta_t \leq x) M_B(dt) \\ &= \lambda E_d \int_0^{g(1)} e^{-\alpha(g(1)-t) - \beta t} \mathbf{1}(Q^* \leq x) dt \\ &= \lambda E_d \left[\mathbf{1}(Q^* \leq x) e^{-\alpha g(1)} \frac{e^{(\alpha-\beta)g(1)} - 1}{\alpha - \beta} \right] \\ &= \frac{\lambda}{\alpha - \beta} \left(E_d[e^{-\beta g(1)} \mathbf{1}(Q^* \leq x)] - E_d[e^{-\alpha g(1)} \mathbf{1}(Q^* \leq x)] \right) \\ &= \frac{\lambda}{\alpha - \beta} (H(\beta, 0, x) - H(\alpha, 0, x)), \end{aligned} \quad (30)$$

where $H(\alpha, \beta, x)$ is the right-hand side of (20). Using (26), (25) and (22), we have

$$\frac{\lambda}{\lambda_{M_B}} = \frac{1}{E_d[B]} = \delta_\Lambda \Phi_\Lambda(0).$$

Combining the above we obtain the announced formula. \square

Proposition 2 yields the next corollary which recovers a result obtained in [22] using different methods (for special cases, see [16] and [15]). Clearly, Corollary 2 could also be proved analogously as Proposition 1.

Corollary 2. *Assume that (A1)–(A5) hold. Then, conditional on $Q_0 > 0$, the left end-point, $d(0)$, and right end-point, $g(1)$, of the busy period containing $t = 0$ have joint Laplace transform given by*

$$E[e^{-\alpha g(1) + \beta d(0)} \mid Q_0 > 0] = \frac{\Phi_\Lambda(0)}{\alpha - \beta} \cdot \left(\frac{\alpha}{\Phi_\Lambda(\alpha)} - \frac{\beta}{\Phi_\Lambda(\beta)} \right),$$

for non-negative α and β ($\alpha \neq \beta$).

Proof. The argument proceeds as in the proof Proposition 2 by omitting the factor $\mathbf{1}(Q^* \leq x)$, i.e. by formally replacing x with $+\infty$. The last line of (30) will give $\frac{\lambda}{\alpha - \beta}(H(\beta, 0, \infty) - H(\alpha, 0, \infty))$, where $H(\alpha, \beta, \infty)$ is given by the right-hand side of (21). \square

Corollary 3. *Assume that (A1)–(A5) hold. Then, conditional on $Q_0 > 0$, the maximum of Q over the busy period containing $t = 0$ has distribution*

$$P(Q^* \leq x \mid Q_0 > 0) = \Phi_\Lambda(0) \frac{W(x) \int_0^x W(y) dy - \int_0^x W(x-y)W(y) dy}{W(x)^2} \quad (31)$$

for $x \geq 0$.

Proof. Letting $\alpha, \beta \rightarrow 0$ in (29) yields

$$P(Q^* \leq x \mid Q_0 > 0) = \Phi_\Lambda(0) \lim_{\alpha \rightarrow 0} \frac{\partial \hat{H}}{\partial \alpha}(\alpha, 0, x),$$

where

$$\hat{H}(\alpha, 0, x) = 1 - \frac{1 + \alpha \int_0^x W^{(\alpha)}(y) dy}{W^{(\alpha)}(x)}.$$

Next recall that for each $x > 0$, $W^{(\alpha)}(x)$ is an entire function in the variable α and in particular

$$W^{(\alpha)}(x) = \sum_{k \geq 0} \alpha^k W^{*(k+1)}(x)$$

where $W^{*(k+1)}(x)$ is the $(k+1)$ -th convolution of W (cf. Bertoin [3]). From this one easily deduces that

$$\frac{\partial}{\partial \alpha} W^{(\alpha)}(x)|_{\alpha=0} = \int_0^x W(y)W(x-y)dy.$$

The result now follows from straightforward differentiation. \square

6. Example: Local time storage from reflected Brownian motion with negative drift

Let $X = \{X_t, t \in \mathbb{R}\}$ be a reflected Brownian motion with drift $-c < 0$ in stationary state living on $I = [0, \infty)$, and let P_0 denote the probability measure associated with X when initiated from 0 at time 0. Its local time (at 0) for $s < t$ is given by

$$L(s, t] := \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_s^t \mathbf{1}_{[0, \varepsilon)}(X_u) du. \quad (32)$$

Let Q be the stationary process defined as in (2):

$$Q_t := \sup_{s \leq t} \{L(s, t] - (t - s)\}.$$

This particular example of fluid queues was introduced and analysed in [19] and further studied in [16] and [15].

Recall that $E_0 L(0, 1] = c$, and, hence Q is well-defined if and only if $0 < c < 1$. Here we make this example more complete by finding the α -scale function associated with process $\Lambda_t := t - L_t^{-1}, t \geq 0$, where

$$L_t^{-1} := \inf\{s : L(0, s] > t\}, \quad t \geq 0.$$

is the inverse local time process. As seen from formulae (31) and (16), the α -scale function is the key ingredient needed for computing the distribution of the maximum of Q over a busy period and related random variables.

To begin with, we recall some basic formulae. When normalising as in (32), see [10, pg. 214], [6, pg. 22] it holds that

$$\begin{aligned} E_0(\exp\{-\theta L_t^{-1}\}) &= \exp\left\{-t \int_0^\infty (1 - e^{-\theta u}) \frac{1}{\sqrt{2\pi u^3}} e^{-c^2 u/2} du\right\} \\ &= \exp\left\{-\frac{t}{G_\theta(0,0)}\right\}, \end{aligned} \quad (33)$$

where

$$G_\theta(0,0) := \frac{1}{\sqrt{2\theta + c^2} - c}$$

is the resolvent kernel (Green kernel) of X at $(0,0)$; see [6, pg. 129]. Consequently, we have

$$E_0(\exp\{\theta \Lambda_t\}) = \exp\left\{t \left(\theta - \frac{1}{G_\theta(0,0)}\right)\right\} = \exp\{t\psi_\Lambda(\theta)\},$$

where

$$\psi_\Lambda(\theta) := \theta - \sqrt{2\theta + c^2} + c, \quad \theta \geq 0.$$

Recall (cf. (15)) that the α -scale function ($\alpha \geq 0$) associated with Λ is defined for $x \geq 0$ via

$$\int_0^\infty e^{-\theta x} W^{(\alpha)}(x) dx = \frac{1}{\psi_\Lambda(\theta) - \alpha}; \quad (34)$$

for $x < 0$ we set $W^{(\alpha)}(x) = 0$. The 0-scale function is called simply the scale function and denoted W . For the next proposition introduce

$$\text{Erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt,$$

and notice that $\text{Erfc}(0) = 1$, $\text{Erfc}(+\infty) = 0$, and $\text{Erfc}(-\infty) = 2$.

Proposition 3. *The α -scale function $W^{(\alpha)}$ of Λ is for $x \geq 0$ given by*

$$\begin{aligned} W^{(\alpha)}(x) &= \frac{e^{-c^2 x/2}}{\lambda_1 - \lambda_2} \left(\lambda_1 e^{\lambda_1^2 x/2} \text{Erfc}(-\lambda_1 \sqrt{x/2}) \right. \\ &\quad \left. - \lambda_2 e^{\lambda_2^2 x/2} \text{Erfc}(-\lambda_2 \sqrt{x/2}) \right), \end{aligned} \quad (35)$$

where

$$\lambda_1 := 1 + \sqrt{(1-c)^2 + 2\alpha}, \quad \lambda_2 := 1 - \sqrt{(1-c)^2 + 2\alpha} \quad (36)$$

In particular,

$$W(x) = \frac{e^{-c^2 x/2}}{2(1-c)} \left((2-c) e^{(2-c)^2 x/2} \operatorname{Erfc}(-(2-c)\sqrt{x/2}) - c e^{c^2 x/2} \operatorname{Erfc}(-c\sqrt{x/2}) \right), \quad (37)$$

and $W(0) = 1$.

Proof. From (34) we have

$$\int_0^\infty e^{-\theta x} W^{(\alpha)}(x) dx = \frac{1}{\theta - \sqrt{2\theta + c^2} + c - \alpha}. \quad (38)$$

To invert this Laplace transform, introduce $\lambda := 2\theta + c^2$. With this notation,

$$\begin{aligned} \frac{1}{\theta - \sqrt{2\theta + c^2} + c - \alpha} &= \frac{2}{\lambda - 2\sqrt{\lambda} + 2(c - \alpha) - c^2} \\ &= \frac{2}{(\sqrt{\lambda} - \lambda_1)(\sqrt{\lambda} - \lambda_2)} \\ &= \frac{2}{\lambda_1 - \lambda_2} \left(\frac{1}{\sqrt{\lambda} - \lambda_1} - \frac{1}{\sqrt{\lambda} - \lambda_2} \right), \end{aligned} \quad (39)$$

where $\lambda_{1,2}$ are the roots of the equation $z^2 - 2z + 2(c - \alpha) - c^2 = 0$, i.e., as in (36). Next, recall the following Laplace inversion formula (cf. Erdélyi [7, pg. 233])

$$\mathcal{L}^{-1} \left(\frac{1}{\sqrt{\lambda} + \beta} \right) = \frac{1}{\sqrt{\pi x}} - \beta e^{\beta^2 x} \operatorname{Erfc}(\beta\sqrt{x}) \quad (40)$$

valid for $\lambda - \beta^2 > 0$. Since

$$\int_0^\infty e^{-\theta x} W^{(\alpha)}(x) dx = \int_0^\infty e^{-\lambda y} e^{c^2 y} W^{(\alpha)}(2y) 2dy$$

we obtain using (40)

$$2e^{c^2 y} W^{(\alpha)}(2y) = \frac{2}{\lambda_1 - \lambda_2} \left(\lambda_1 e^{\lambda_1^2 y} \operatorname{Erfc}(-\lambda_1\sqrt{y}) - \lambda_2 e^{\lambda_2^2 y} \operatorname{Erfc}(-\lambda_2\sqrt{y}) \right),$$

which is formula (35). In particular, when $\alpha = 0$ it holds $\lambda_1 = 2 - c$ and $\lambda_2 = c$ yielding formula (37). \square

Using the scale function W and the fact

$$\Phi_\Lambda(0) = \sup\{\theta > 0 : \psi_\Lambda(\theta) = 0\} = 2(1 - c)$$

formula (31) yields the distribution of the maximum Q^* over an observed busy period (i.e. over a busy period containing the origin of time).

Proposition 4. *Let $0 < c < 1$. The distribution of the maximum Q^* over an observed busy period of a reflected Brownian motion (with drift $-c$) local time storage is given by*

$$P(Q^* \leq x \mid Q_0 > 0) = 2(1 - c) \frac{\int_0^x W(y) (W(x) - W(x - y)) dy}{W^2(x)}, \quad (41)$$

where the scale function W is given by (37).

We plot the derivative of (41) for $c = 1/2$ in Figure 3 below.

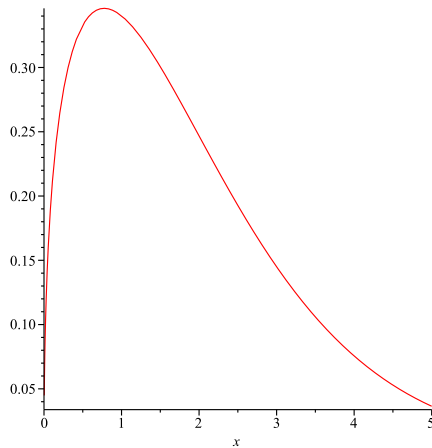


FIGURE 3: The density of Q^* conditional on $\{Q_0 > 0\}$ for the example corresponding to Brownian motion with drift $-c = -1/2$.

We recall some formulas from [16]. First

$$\begin{aligned} E \left(e^{\theta d(0) - \beta g(1)} \mid Q_0 > 0 \right) &= \frac{8(1 - c)}{\sqrt{2\theta + (1 - c)^2} + \sqrt{2\beta + (1 - c)^2}} \\ &\quad \times \frac{1}{(\sqrt{2\theta + (1 - c)^2} + 1 + c)(\sqrt{2\beta + (1 - c)^2} + 1 + c)} \\ &=: F(\theta, \beta; 1 - c), \end{aligned} \quad (42)$$

and

$$\mathbb{E} \left(e^{\theta g(0) - \beta d(0)} \mid Q_0 = 0 \right) = F(\theta, \beta; c). \quad (43)$$

Setting $\beta = \theta$ in the right-hand side of (42) and (43), respectively, we get

$$E\left(e^{-\theta(g(1)-d(0))} \mid Q_0 > 0\right) = \frac{4(1-c)}{\sqrt{2\theta + (1-c)^2}(\sqrt{2\theta + (1-c)^2} + 1 + c)^2}, \quad (44)$$

and

$$E\left(e^{-\theta(d(0)-g(0))} \mid Q_0 = 0\right) = \frac{4c}{\sqrt{2\theta + c^2}(\sqrt{2\theta + c^2} + 2 - c)^2}, \quad (45)$$

Taking the inverse Laplace transform of (44) (cf. Erdélyi [7, pg. 234]) we obtain the density of the length of the busy period $g(1) - d(0)$, given $Q_0 > 0$, as

$$f_{g-b}(v) = 2(1-c)e^{-(1-c)^2v/2} \left(\sqrt{2v/\pi} - (1+c)v e^{(1+c)^2v/2} \operatorname{Erfc}((1+c)\sqrt{v/2}) \right)$$

Note that the density of the length of the idle period $d(0) - g(0)$, given that $Q_0 = 0$, is obtained from $f_{g-b}(v)$ by substituting c for $1-c$. In Figure 4 we have plotted $f_{g-b}(v)$ for three different values of c . We notice also that the mean

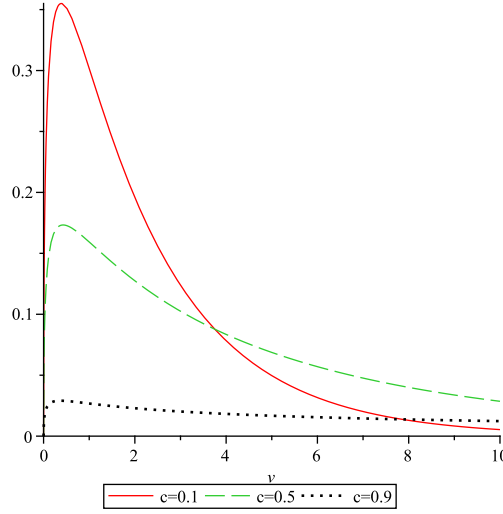


FIGURE 4: The density of length of the busy period, given that $Q_0 > 0$ for three different values of c .

busy period length has a simple expression:

$$E[g(1) - d(0) \mid Q_0 > 0] = \frac{2-c}{(1-c)^2}.$$

The joint density of $d(0)$ and $g(1)$ is given by

$$\begin{aligned} \mathbb{P}(-d(0) \in dx, g(1) \in dy \mid Q_0 > 0) &= 2(1-c)e^{-(1-c)^2(x+y)/2} \\ &\times \left(\sqrt{2/(\pi(x+y))} - (1+c)e^{(1+c)^2(x+y)/2} \operatorname{Erfc}((1+c)\sqrt{(x+y)/2}) \right) \end{aligned}$$

and, again, the density for $(g(0), d(0))$ is obtained by substituting c for $1-c$.

Next we find the density of $g(1)$ (recall that $-d(0)$ is identical in law with $g(1)$) by inverting the Laplace transform (obtained from (42) by choosing $\theta = 0$):

$$\begin{aligned} E\left(e^{-\beta g(1)} \mid Q_0 > 0\right) \\ = \frac{4(1-c)}{\left(\sqrt{2\beta + (1-c)^2} + 1 - c\right) \left(\sqrt{2\beta + (1-c)^2} + 1 + c\right)}. \end{aligned} \quad (46)$$

Letting $\lambda := 2\beta + (1-c)^2$ we rewrite (46) as

$$E\left(e^{-\beta g(1)} \mid Q_0 > 0\right) = \frac{2(1-c)}{c} \left(\frac{1}{\sqrt{\lambda} + 1 - c} - \frac{1}{\sqrt{\lambda} + 1 + c} \right).$$

From (40)

$$\begin{aligned} \mathcal{L}^{-1} \left(\frac{1}{\sqrt{\lambda} + 1 - c} - \frac{1}{\sqrt{\lambda} + 1 + c} \right) \\ = (1+c)e^{(1+c)^2 x} \operatorname{Erfc}((1+c)\sqrt{x}) - (1-c)e^{(1-c)^2 x} \operatorname{Erfc}((1-c)\sqrt{x}). \end{aligned} \quad (47)$$

Consequently,

$$\begin{aligned} 2e^{-(1-c)^2 x} f_{g(1)}(2x) &= \frac{2(1-c)}{c} \left((1+c)e^{(1+c)^2 x} \operatorname{Erfc}((1+c)\sqrt{x}) \right. \\ &\quad \left. - (1-c)e^{(1-c)^2 x} \operatorname{Erfc}((1-c)\sqrt{x}) \right), \end{aligned} \quad (48)$$

where $f_{g(1)}$ denotes the density of $g(1)$ conditioned on $\{Q_0 > 0\}$. From (48) we obtain

$$\begin{aligned} f_{g(1)}(x) &= \frac{(1-c)e^{-(1-c)^2 x/2}}{c} \left((1+c)e^{(1+c)^2 x/2} \operatorname{Erfc} \left((1+c)\sqrt{x/2} \right) \right. \\ &\quad \left. - (1-c)e^{(1-c)^2 x/2} \operatorname{Erfc} \left((1-c)\sqrt{x/2} \right) \right). \end{aligned} \quad (49)$$

Moreover, the density $f_{d(0)}$ of $d(0)$ conditional on $\{Q_0 = 0\}$ is obtained from (49) by substituting c instead of $1 - c$:

$$f_{d(0)}(x) = \frac{c e^{-c^2 x/2}}{1 - c} \left((2 - c) e^{(2-c)^2 x/2} \operatorname{Erfc} \left((2 - c) \sqrt{x/2} \right) - c e^{c^2 x/2} \operatorname{Erfc} \left(c \sqrt{x/2} \right) \right). \quad (50)$$

It is striking how similar formulae (37) and (50) are.

Remark 1. The scale function formulae (35) and (37) are clearly valid for all $c \geq 0$. In case $c = 0$ the process $\{L(0, t); t \geq 0\}$ is a version of the Brownian local time, and the α -scale function $W_0^{(\alpha)}$ of the corresponding process Λ is given by

$$W_0^{(\alpha)}(x) = \frac{e^{(1+\alpha)x}}{2\sqrt{1+2\alpha}} \left((1 + \sqrt{1+2\alpha}) e^{x\sqrt{1+2\alpha}} \operatorname{Erfc} \left(-(1 + \sqrt{1+2\alpha}) \sqrt{x/2} \right) - (1 - \sqrt{1+2\alpha}) e^{-x\sqrt{1+2\alpha}} \operatorname{Erfc} \left(-(1 - \sqrt{1+2\alpha}) \sqrt{x/2} \right) \right).$$

In particular,

$$W_0(x) = e^{2x} \operatorname{Erfc}(-\sqrt{2x}). \quad (51)$$

In case $c = 1$ it holds $\lambda_{1,2} = 1 \pm \sqrt{2\alpha}$ and for $\alpha \neq 0$ formula (37) can be used directly. For the 0-scale function we need to take the limit as $c \rightarrow 1$ in (37):

$$W_1(x) = (1 + x) \operatorname{Erfc}(-\sqrt{x/2}) + \sqrt{\frac{2x}{\pi}} e^{-x/2}. \quad (52)$$

Remark 2. Here we display some formulae for Laplace transforms apparent from above and point out a misprint in Erdélyi et al. [7].

First, from (38), (39), and (52) we have the following Laplace inversion formula valid for $\lambda > 1$:

$$\mathcal{L}^{-1} \left(\frac{1}{(\sqrt{\lambda} - 1)^2} \right) = (1 + 2x) e^x \operatorname{Erfc}(-\sqrt{x}) + \frac{2\sqrt{x}}{\sqrt{\pi}}, \quad (53)$$

and this can be “extended” (for $a > 0$) to

$$\mathcal{L}^{-1} \left(\frac{1}{(\sqrt{\lambda} - a)^2} \right) = (1 + 2a^2 x) e^{a^2 x} \operatorname{Erfc}(-a\sqrt{x}) + \frac{2a\sqrt{x}}{\sqrt{\pi}}. \quad (54)$$

Furthermore, it can be checked that (54) is valid for all $a < 0$ by evaluating the Laplace transform of the right-hand side. This can be done term by term by using, e.g., Erdélyi et al. [7, pp. 137, 177] (well-known formulae):

$$\begin{aligned}\mathcal{L}(\sqrt{x}) &= \frac{\sqrt{\pi}}{2} \lambda^{-3/2}, \\ \mathcal{L}\left(e^{a^2x} \operatorname{Erfc}(a\sqrt{x})\right) &= \lambda^{-1/2} (\lambda^{1/2} + a)^{-1},\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}\left(xe^{a^2x} \operatorname{Erfc}(a\sqrt{x})\right) &= -\frac{\partial}{\partial \lambda} \mathcal{L}\left(e^{a^2x} \operatorname{Erfc}(a\sqrt{x})\right) \\ &= -\frac{\partial}{\partial \lambda} \lambda^{-1/2} (\lambda^{1/2} + a)^{-1} \\ &= \frac{1}{2a} \left(\lambda^{-3/2} - \lambda^{-1/2} (\lambda^{1/2} + a)^{-2}\right).\end{aligned}$$

We remark that formula (10) in [7] p. 234:

$$\mathcal{L}^{-1}\left(\frac{1}{(\sqrt{\lambda} + \sqrt{b})^2}\right) = 1 - 2\sqrt{bx/\pi} + (1 - 2bx) e^{bx} \left(\operatorname{Erf}(\sqrt{bx}) - 1\right). \quad (55)$$

is **not** correct since it does not coincide with formula (54) (for $a < 0$). Indeed, because

$$\operatorname{Erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

the right-hand side of (55) is zero at zero but the right-hand side of (54) is 1 at zero.

7. Further examples

In the previous example we derived a local time process from a given Markov process. However, it is also possible to consider examples where just the local time process L , or equivalently the subordinator L^{-1} , is specified. Indeed the subordinator that will play the role of L^{-1} in this example has no drift and has Lévy measure given by

$$\Pi(x, \infty) = \frac{\gamma^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\gamma x} + \varphi \frac{\gamma^\nu}{\Gamma(\nu)} \int_x^\infty y^{\nu-1} e^{-\gamma y} dy,$$

where the constants $\varphi, \gamma > 0$ and $\nu \in (0, 1)$. Note in particular then that L^{-1} is the sum of two independent subordinators, one of which is a compound Poisson process with gamma distributed jumps, the other has infinite activity and is of the so called tempered-stable type. Clearly Π also describes the Lévy measure of $-\Lambda$ too.

According to [9], the process Λ belongs to the Gaussian Tempered Stable Convolution class and moreover,

$$\psi_\Lambda(\theta) = (\theta - \varphi) \left(1 - \left(\frac{\gamma}{\gamma + \theta} \right)^\nu \right)$$

for $\theta \geq 0$. In particular $\delta_\Lambda = 1$ and $\Phi_\Lambda(0) = \varphi$. It is a straightforward exercise to show that

$$\mathbb{E}(\Lambda_1) = \psi'_\Lambda(0+) = -\varphi \frac{\nu}{\gamma}$$

and this implies that

$$\mu = \frac{1}{1 + \varphi\nu/\gamma} < 1,$$

as required.

From [9] we also know that

$$W(x) = e^{\varphi x} + \gamma^\nu e^{\varphi x} \int^x e^{-(\gamma+\varphi)y} y^{\nu-1} \mathbf{E}_{\nu,\nu}(\gamma^\nu y^\nu) dy$$

where

$$\mathbf{E}_{\alpha,\beta}(x) := \sum_{n \geq 0} \frac{z^n}{\Gamma(\alpha n + \beta)}$$

is the two parameter Mittag-Leffler function.

We may now deduce from the theory presented earlier that, for example,

$$P_d(Q^* \leq x) = \frac{1 - e^{-\varphi x} + \gamma^\nu \int^x e^{-(\gamma+\varphi)y} y^{\nu-1} \mathbf{E}_{\nu,\nu}(\gamma^\nu y^\nu) dy}{1 + \gamma^\nu \int^x e^{-(\gamma+\varphi)y} y^{\nu-1} \mathbf{E}_{\nu,\nu}(\gamma^\nu y^\nu) dy}$$

and

$$P_d(Q^* > x) \sim \left(1 - \left(\frac{\gamma}{\gamma + \varphi} \right)^\nu \right) e^{-\varphi x}.$$

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References

- [1] Baccelli, F., and Brémaud, P. (2003) *Elements of Queueing Theory: Palm Martingale Calculus and Stochastic Recurrences*, 2nd edition, Springer.
- [2] Bertoin, J. (1996) *Lévy processes*. Cambridge University Press.
- [3] Bertoin, J. (1997) Exponential decay and ergodicity of completely asymmetric Lévy processes in a finite interval. *Ann. Appl. Probab.* **7**, 56-169.
- [4] Bingham, N.H. (1975) Fluctuation theory in continuous time. *Adv. Appl. Prob.* **7**, 705-766.
- [5] Blumenthal, R.M. and Gettoor, R.K. (1968) *Markov Processes and Potential Theory*. Academic Press, New York.
- [6] Borodin, A.N., Salminen, P. (2002) *Handbook of Brownian Motion - Facts and Formulae, 2nd ed.* Birkhäuser.
- [7] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G. (1954) *Tables of Integral Transforms*. McGraw-Hill.
- [8] Fristedt, B.E. (1974) Sample functions of stochastic processes with stationary independent increments. *Adv. Probab.* **3**, 241–396. Dekker, New York.
- [9] Hubalek, F. and Kyprianou A.E. (2010) Old and new examples of scale functions for spectrally negative Lévy processes. To appear in *Sixth Seminar on Stochastic Analysis, Random Fields and Applications*, eds R. Dalang, M. Dozzi, F. Russo. *Progress in Probability*, Birkhäuser.
- [10] Itô, K. and McKean, H.P. (1974) *Diffusion Processes and Their Sample Paths*. Springer Verlag.
- [11] Kallenberg, O. (1983) *Random Measures*. Academic Press.
- [12] Kallenberg, O. (2002) *Foundations of Modern Probability*. Springer-Verlag.

- [13] Konstantopoulos, T. and Last, G. (2000) On the dynamics and performance of stochastic fluid systems. *J. Appl. Prob.* **37**, 652-667.
- [14] Konstantopoulos, T., Zazanis, M. and de Veciana, G. (1997) Conservation laws and reflection mappings with an application to multiclass mean value analysis for stochastic fluid queues. *Stoch. Proc. Appl.* **65**, 139-146.
- [15] Konstantopoulos, T., Kyprianou, A. E. , Salminen, P. and Sirviö, M. (2008) Analysis of stochastic fluid queues driven by local time processes. *Adv. Appl. Prob.* **40(4)**, 1072-1103.
- [16] Kozlova, M. and Salminen, P. (2004) Diffusion local time storage. *Stoch. Proc. Appl.* **114**, 211-229.
- [17] Kyprianou, A. E. (2006) *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer.
- [18] Kyprianou, A. E. and Palmowski, Z. (2004) A martingale review of some fluctuation theory for spectrally negative Lévy processes. *Sem. Prob.* **28**, 16-29, Lecture Notes in Math., Springer.
- [19] Mannersalo, P., Norros, I. and Salminen, P. (2004) A storage process with local time input. *Queueing Syst.* **46**, 557-577.
- [20] Pitman, J. (1986) Stationary excursions. *Sem. Prob.* **XXI**, 289-302, Lecture Notes in Math. 1247, Springer.
- [21] Salminen, P. (1993) On the distribution of diffusion local time. *Stat. Prob. Letters* **18**, 219-225.
- [22] Sirviö, M. (2006) On an inverse subordinator storage. *Helsinki University of Technology, Inst. Math. Report series* **A501**.
- [23] Tsirelson, B. (2004) Non-classical stochastic flows and continuous products. *Probability Surveys* **1**, 173-298.

- [24] Zolotarev, V.M. (1964) The first-passage time of a level and the behaviour at infinity for a class of processes with independent increments. *Theory Prob. Appl.* **9**, 653-664.