

Approximations to the Stochastic Burgers equation

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Abstract

This article is devoted to the numerical study of various finite difference approximations to the stochastic Burgers equation. Of particular interest in the one-dimensional case is the situation where the driving noise is white both in space and in time. We demonstrate that in this case, different finite difference schemes converge to different limiting processes as the mesh size tends to zero. A theoretical explanation of this phenomenon is given and we formulate a number of conjectures for more general classes of equations, supported by numerical evidence.

1 Introduction

This article is devoted to the numerical study of several finite difference schemes for the viscous stochastic Burgers equation:

$$\partial_t u(x, t) = \nu \partial_x^2 u - g(u) \partial_x u + \sigma \xi(x, t), \quad x \in [0, 2\pi], \quad t \geq 0. \quad (1)$$

In this equation, ξ denotes space-time white noise, that is the centred, distribution-valued Gaussian random variable such that $\mathbb{E}(\xi(x, t)\xi(y, s)) = \delta(t - s)\delta(x - y)$.

Motivations for studying the stochastic Burgers equation are manifold. Just to name a few, it is used to model vortex lines in high-temperature superconductors [BFG⁺94], dislocations in disordered solids and kinetic roughening of interfaces in epitaxial growth [Bar96], formation of large-scale structures in the universe [GSS85, SZ89], constructive quantum field theory

[BCJL94], *etc.* Since in the case $\sigma = 0$ and $g(u) \propto u$ this equation is furthermore explicitly solvable via the Hopf-Cole transform ($u = v'/v$, where v solves the heat equation), it comes as no surprise that a wealth of numerical and analytical results are available. From a purely mathematical point of view, let us mention for example the well-posedness results from [BCF91, BCJL94, DPDT94, Gyö98, Kim06] and the ergodicity results obtained in [TZ06, GM05]. One remarkable achievement was the construction of a stationary solution in the inviscid limit with non-vanishing noise [EKMS00] (dissipation then occurs purely through shocks). From a more quantitative perspective, the scaling exponents of the solutions in the small viscosity limit have attracted considerable interest, both in the physics and the applied mathematics literature [YC96, BM96, Kra99, EVE00a, EVE00b].

Because of the presence of the white noise term ξ in (1), its solutions u will in general be very “rough”, and in particular will not be differentiable in the classical sense in the spatial variable x . As a consequence, it transpires that the solutions to (1) are extremely unstable under natural approximations of the nonlinearity. For example, it is natural to consider for any $a, b \geq 0$ with $a + b > 0$ the approximating equation

$$\partial_t u^\varepsilon(x, t) = \nu \partial_x^2 u^\varepsilon(x, t) - g(u^\varepsilon(x, t)) D_\varepsilon u^\varepsilon(x, t) + \sigma \xi(x, t), \quad (2)$$

where we defined the approximate derivative D_ε by

$$D_\varepsilon u(x, t) = \frac{u(x + a\varepsilon, t) - u(x - b\varepsilon, t)}{(a + b)\varepsilon}. \quad (3)$$

In the absence of the noise term ξ , it would be a standard exercise in numerical analysis to show that the solution to (2) does converge as $\varepsilon \rightarrow 0$ to the solution to (1). This is just an example

of the widely accepted ‘folklore’ fact that if an equation is well-posed, then any ‘reasonable’ approximation will converge to the exact solution.¹

In this article, we will argue that if ξ is taken to be space-time white noise, the limit of (2) as $\varepsilon \rightarrow 0$ depends on the values a and b and is equal to (1) if and only if either g is constant or $a = b$! Furthermore, it will follow from the argument that, if one considers driving noise that is slightly rougher than space-time white noise (taking a noise term equal to $(1 - \partial_x^2)^\alpha dw(t)$ with $\alpha \in (0, 1/4)$ still yields a well-posed equation), then one does not expect solutions to the approximate equation (2) to converge to anything at all, unless $a = b$. Our methodology here is to first present a heuristic argument which allows to derive quantitative predictions for the effect of the finite differences discretisation on the solution. We will then use numerical experiments to verify these predictions.

At this point we would like to emphasise that the aim of this article is certainly not to advocate the use of a finite difference scheme of the type (2) to effectively simulate (1). Indeed, we will show in Section 2.2 below that approximations of the nonlinearity of the type $D_\varepsilon G(u)$, where G is the antiderivative of g already have much better stability properties. Instead, our aim is merely to give a striking illustration of the fact that caution should be exercised in the simulation of stochastic PDEs driven by spatially rough noise.

The text is structured as follows: We start in section 2 by presenting our argument for the case of the stochastic Burgers equation, *i.e.* for $g(u) = u$. In section 3 we will present the corresponding results for more general equations and in section 4 we study the limit of vanishing noise and viscosity. Finally, in appendix A we give some details about how the numerical simulations in this article were performed.

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2 Stochastic Burgers Equation

In this section we consider the stochastic Burgers equation

$$du = \nu \partial_x^2 u dt - u \partial_x u dt + \sigma dw, \quad (4)$$

as well as the approximation given by

$$du^\varepsilon = \nu \partial_x^2 u^\varepsilon dt - u^\varepsilon D_\varepsilon u^\varepsilon dt + \sigma dw. \quad (5)$$

Here u and u^ε take values in the space $L^2([0, 2\pi], \mathbb{R})$, on which the operator ∂_x^2 is endowed with periodic boundary conditions, and w is an L^2 -cylindrical Wiener process (see [DPZ92] for details). Equation (4) is well-posed, since we can rewrite $u \partial_x u$ as $\frac{1}{2} \partial_x (u^2)$ which is locally Lipschitz from the Sobolev space $H^{1/4}$ into H^{-1} , thus allowing us to apply general local existence theorems as in [DPZ92, Hai09]. For fixed time $t > 0$, the solutions to (4) have the regularity of Brownian motion, when viewed as a function of the spatial variable x . In particular, they are not differentiable in x . Figure 1 shows numerical solutions of equation (5) for different values of a and b . One can see that different choices for these parameters lead to an $\mathcal{O}(1)$ difference in the solutions.

Our aim in this article is to understand and quantify these differences. In particular, in Conjecture 1 below, we compute a correction term to (4) and we verify numerically that the solutions to (5) converge to this corrected equation as $\varepsilon \rightarrow 0$. This understanding will then allow us to conjecture the appearance of similar correction terms in more complicated situations and we will also verify these conjectures numerically.

¹Remember that we are working here at fixed non-zero viscosity, so we do not require an upwind scheme in order to obtain convergence.

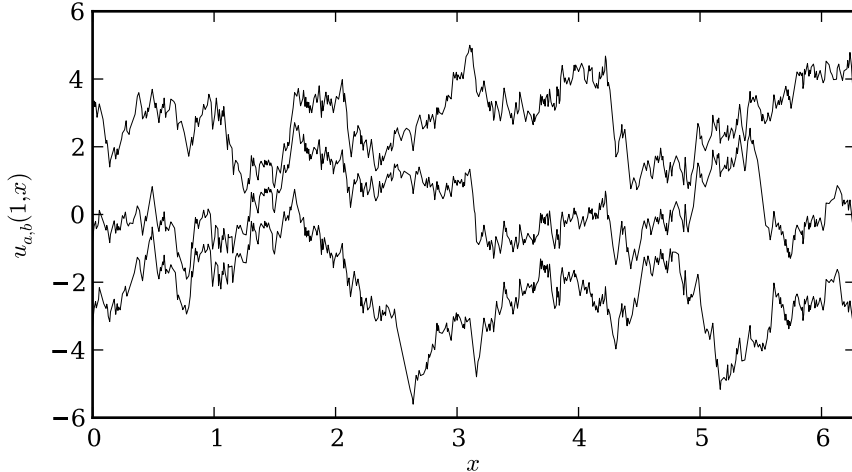


Figure 1: *Illustration of the discretisation error for the finite differences method (5). The three lines correspond to a right-handed discretisation ($a = 1, b = 0$; top line), a centred discretisation ($a = 1, b = 1$; middle line) and a left-handed discretisation ($a = 0, b = 1$; bottom line), all computed using the same instance of the noise. The picture clearly shows that there is an $\mathcal{O}(1)$ difference between the solutions obtained by the three different discretisation schemes. From the argument presented in the text, we assume that the exact solution of (4) will be closest to the middle of the three lines.*

2.1 Heuristic explanation

For simplicity, we consider the solution v to the stochastic heat equation

$$dv = \nu \partial_x^2 v dt + \sigma dw(t) \quad (6)$$

instead of u . Since the properties of the discretisation of differential operators only depend on local properties, and since v has the same spatial regularity as u , it will be sufficient to study how well $D_\varepsilon v$ approximates $v \partial_x v = \frac{1}{2} \partial_x v^2$.

By expressing v in the Fourier-basis $\{e^{inx}/\sqrt{2\pi}\}_{n \in \mathbb{Z}}$, it is easy to check that the stationary solution to (6) is

$$v(t, x) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\sigma}{in\sqrt{2\nu}} \xi_n(t) \frac{e^{inx}}{\sqrt{2\pi}} + \xi_0(t) \frac{1}{\sqrt{2\pi}},$$

where the ξ_0 is a (real-valued) standard Brownian motion and ξ_n for $n \neq 0$ are complex-valued Ornstein-Uhlenbeck processes with variance 1 (in the sense that $\mathbb{E}|\xi_n(t)|^2 = 1$) and time constant νn^2 which are independent, except for the condition that $\xi_{-n} = \bar{\xi}_n$. Therefore, the derivative of v is given (at least on a formal level) by

$$\partial_x v(x) = \sum_{n \neq 0} \frac{\sigma \xi_n(t) e^{inx}}{2\sqrt{\nu\pi}}. \quad (7)$$

The ε -approximation to the derivative given in (2) on the other hand is given by

$$D_\varepsilon v(x) = \sum_{n \neq 0} \frac{\sigma \xi_n(t) e^{inx}}{2\sqrt{\nu\pi}} \frac{e^{ina\varepsilon} - e^{-inb\varepsilon}}{(a+b)i\varepsilon n}. \quad (8)$$

It is clear that the terms in (8) are a good approximation to the terms in (7) only up to $n \approx \varepsilon^{-1}$. For larger n , the multiplier in (8) will decrease like n^{-1} .

For our analysis we restrict ourselves to the constant ($n = 0$) Fourier mode. Our numerical experiments, below, show that the contributions from this mode are already enough to explain the observed differences between the solutions of (5) and the exact solution. Since $\nu \partial_x v$ is a total

derivative, the 0-mode of this term vanishes. In contrast, the 0-mode of $vD_\varepsilon v$ does not vanish at all. We obtain instead for this term the sum

$$\begin{aligned}\langle 1, vD_\varepsilon v \rangle &= \sum_{n \neq 0} \frac{\sigma^2 \xi_{-n}(t) \xi_n(t)}{4\nu\pi(-in)} \frac{e^{ina\varepsilon} - e^{-inb\varepsilon}}{(a+b)\varepsilon in} \\ &= \frac{\sigma^2}{2\pi\nu} \sum_{n>0} |\xi_n(t)|^2 \frac{\cos a\varepsilon n - \cos b\varepsilon n}{(a+b)\varepsilon n^2}\end{aligned}\tag{9}$$

The expectation of this expression, as $\varepsilon \rightarrow 0$, converges to

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \langle 1, vD_\varepsilon v \rangle = \frac{\sigma^2}{2\pi\nu} \int_0^\infty \frac{\cos ax - \cos bx}{(a+b)x^2} dx = -\frac{\sigma^2}{4\nu} \frac{a-b}{a+b}.$$

As a consequence, one expects the following result.

Conjecture 1 *The solution of the approximating equation (5) converges, as $\varepsilon \rightarrow 0$, to the solution of*

$$du = \nu \partial_x^2 u dt - u \partial_x u dt + \frac{\sigma^2}{4\nu} \frac{a-b}{a+b} dt + \sigma dw.\tag{10}$$

Thus, the approximation converges to the stochastic Burgers equation (4) only for $a = b$.

Remark 2.1 The solution to the stochastic Burgers equation (or rather the integrated process which solves the corresponding KPZ equation) arising as the fluctuations process in the weakly asymmetric exclusion process [BG97, BQS09] is driven by the *derivative* of space-time white noise. As a consequence, it does *not* solve an SPDE that is well-posed in the classical sense and can currently only be defined via the Hopf-Cole transform. Such a process is even much rougher (by ‘‘one derivative’’) than the process considered here and one would expect the ‘wrong’ numerical approximation schemes to fail there in an even more spectacular way.

Remark 2.2 One may think of two reasons why the correction term vanishes when $a = b$. On the one hand, this discretisation is more symmetric. On the other hand, it yields a second-order approximation to the derivative at x . The correct explanation is closer to the first one. Indeed, consider for example the discretisation

$$(\tilde{D}_\varepsilon u)(x) \approx \frac{cu(x+2\varepsilon) + (1-3c)u(x+\varepsilon) + 3cu(x) - (1+c)u(x-\varepsilon)}{2\varepsilon}.\tag{11}$$

This discretisation is of second order for every $c \in \mathbb{R}$. However, if we perform the same calculation as above with this discretisation, we obtain a correction term equal to

$$\frac{\sigma^2}{2\pi\nu} \int_0^\infty \frac{c \cos 2x - 4c \cos x + 3c}{2x^2} dx = -\frac{c\sigma^2}{8\nu},$$

which vanishes only if $c = 0$, thus reducing (11) to (3).

In the above calculation, both the limiting equation and the approximating equation live in the same space. It is possible to carry out a similar analysis in the case where the approximating equation takes values in a different space, typically \mathbb{R}^N for some large N . For example, we can consider the ‘finite differences’ approximation given by

$$\begin{aligned}\partial_x^2 u &\approx \frac{u(x+\delta) - 2u(x) + u(x-\delta)}{\delta^2} \stackrel{\text{def}}{=} (\Delta_N u)(x) \\ u \partial_x u &\approx u D_\delta u = u(x) \frac{u(x+\delta) - u(x)}{\delta} \stackrel{\text{def}}{=} F_N(u)(x),\end{aligned}\tag{12}$$

where we set $\delta = \frac{2\pi}{N}$ for the approximation with N gridpoints. In this setting, we approximate u by $u^N \in \mathbb{R}^N$ with $u_j^N \approx u(j\delta)$. The natural candidate for the approximation of space-time white noise is then given by $dW_j^N = \delta^{-1/2} dW_j$, where the W_j are independent, one-dimensional standard Brownian motions. This is the correct scaling, since it ensures for example that $\sum W_j^N \delta$

is a Wiener process with covariance 2π . With this notation, the approximating equation we consider is given by

$$du^N = \nu \Delta_N u^N dt - F_N(u^N) dt + \sigma dW^N(t). \quad (13)$$

Let us take N even for the sake of simplicity. It is then straightforward to check that the eigenvectors for Δ_N are given as before by e^{inx} with $n = -\frac{N}{2} + 1, \dots, \frac{N}{2}$, but the corresponding eigenvalues are

$$\lambda_n = \frac{2}{\delta^2} (\cos n\delta - 1) = -\left(\frac{2}{\delta} \sin\left(\frac{n\delta}{2}\right)\right)^2 \stackrel{\text{def}}{=} -\eta_n^2.$$

(Note that for fixed n and small δ , one has indeed $\lambda_n \approx -n^2$.) It then follows as previously that the solution to the linearised equation is given by

$$v(t, x) = \sum_{n \neq 0} \frac{\sigma}{2\sqrt{\nu\pi i\eta_n}} e^{inx} \xi_n(t) + \frac{1}{\sqrt{2\pi}} \xi_0(t),$$

and that its discrete derivative $D_\delta v$ is given by

$$D_\delta v(t, x) = \sum_{n \neq 0} \frac{\sigma e^{inx} \xi_n(t)}{2\sqrt{\nu\pi i\eta_n}} \frac{e^{in\delta} - 1}{\delta}.$$

Note that both sums in these expressions only run over all admissible values of n , that is $n = -\frac{N}{2} + 1, \dots, \frac{N}{2}$. Similarly to above, we obtain that the expectation of the zero mode of the product $v D_\delta v$ is given by

$$\mathbb{E}\langle 1, v D_\delta v \rangle = \frac{\sigma^2}{2\pi\nu} \sum_{n=1}^{N/2} \frac{\cos \delta n - 1}{\delta \eta_n^2} = -\frac{\sigma^2}{2\pi\nu} \sum_{n=1}^{N/2} \frac{\delta}{2} = -\frac{\sigma^2}{4\nu}. \quad (14)$$

One therefore expects the following result.

Conjecture 2 *The numerical approximation (13) converges, as $N \rightarrow \infty$, to the solution of*

$$du = \nu \partial_x^2 u dt - u \partial_x u dt + \frac{\sigma^2}{4\nu} dt + \sigma dw(t).$$

To test this conjecture, we use the following numerical experiment: We numerically solve both the ‘‘approximating’’ equation (13) and the ‘‘corrected’’ SPDE

$$du_\gamma = \nu \partial_x^2 u_\gamma dt - u_\gamma \partial_x u_\gamma dt + \gamma \frac{\sigma^2}{\nu} dt + \sigma dw(t), \quad (15)$$

until a fixed time T , using the same instance of the noise and $\gamma \in \mathbb{R}$. To discretise the term $u \partial_x u$ in (15) we use the approximation $(u(x)^2/2 - u(x-\delta)^2/2)/\delta$, which is known to converge to the exact solution. For increased accuracy we also use a finer grid for (15) than we did for (13). To compare the solutions, we consider $\|u^N(T, \cdot) - u_\gamma(T, \cdot)\|_2$ as a function of γ . If the conjecture is correct, we expect this function to take its minimum at $\gamma = 1/4$. The result of a simulation is given by the line labelled ‘‘finite differences’’ in figure 2. It can be seen that the minimum of the curve is indeed located close to $\gamma = 1/4$.

The correction terms in conjectures 2 and 1 (with $a = 1$ and $b = 0$) coincide, even though the constants arise in a completely different ways. This might lead one to speculate that the value of this constant is an intrinsic property of the limiting equation, rather than of the particular way of approximating it. This is not the case, as one can also perform the same calculation with a ‘spectral Galerkin’ approximation of the linear part of the equation, but retaining a ‘finite difference’ approximation to the nonlinearity. In other words, we consider (13) as before, but we take for Δ_N the self-adjoint matrix with eigenvectors e^{inx} and eigenvalues $-n^2$. (This can be achieved by first applying the discrete Fourier transform, then multiplying the n th component by $-n^2$, and finally applying the inverse Fourier transform.) In this case one has $\eta_n = n$, and it transpires that the correction term is given by

$$\sum_{n=1}^{N/2} \frac{\sigma^2}{2\pi\nu\eta_n^2} \frac{1 - \cos n\delta}{\delta} \approx \frac{\sigma^2}{2\pi\nu} \int_0^\pi \frac{1 - \cos x}{x^2} dx \approx \frac{0.193 \sigma^2}{\nu}.$$

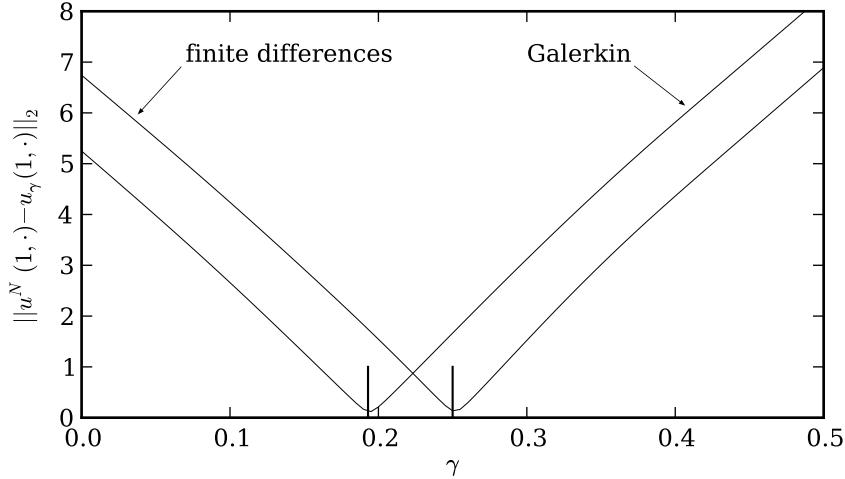


Figure 2: This figure compares the solution $u^N = u_{1,0}$ of (13) (using right-handed discretisation) to the solution of the “corrected” SPDE (15) which includes an additional drift term $\gamma\sigma^2/\nu$. The curves show the L^2 -norm difference between the solutions (for the same instance of the noise) as a function of γ , once using finite difference discretisation (12) of the linear part (corresponding to the top-most curve in figure 1) and once using the spectral Galerkin discretisation. The two vertical line segments give the predicted locations for the minima of the two curves. It can be seen that predictions and simulations are in good agreement.

which is clearly different from (14).

To verify that the spectral Galerkin discretisation of the linear part indeed leads to this changed correction term, we modify the code which we used to validate conjecture 2 above. The result of this simulation is given by the line labelled “Galerkin” in figure 2. Again, there is good agreement between our conjecture and the simulation results.

2.2 The case of more regular noise

To conclude this section, let us argue that the situation considered in this article is truly a borderline case in terms of regularity and that if we drive (5) by noise that gives rise to slightly more regular solutions, one would expect its solutions to converge to those of (4) without any correction term. Indeed, consider a general semilinear stochastic PDE driven by additive noise:

$$du = -Au dt + F(u) dt + Q dw(t), \quad (16)$$

where A is a strictly positive-definite selfadjoint operator on some Hilbert space \mathcal{H} , F is a (possibly unbounded) nonlinear operator from \mathcal{H} to \mathcal{H} , W is a standard cylindrical Wiener process on \mathcal{H} , and $Q: \mathcal{H} \rightarrow \mathcal{H}$ is a bounded operator.

Denote furthermore $\mathcal{H}^\alpha = \mathcal{D}(A^\alpha)$ for $\alpha > 0$ and let $\mathcal{H}^{-\alpha}$ be the dual space to \mathcal{H}^α under the dual pairing given by the Hilbert space structure of \mathcal{H} . (So that $\mathcal{H}^{-\alpha}$ is a superspace of \mathcal{H} for $\alpha > 0$.) Finally, we denote as before by v the solution to the linearised system

$$dv = -Av dt + Q dw(t),$$

which we assume to be an \mathcal{H} -valued Gaussian process with almost surely continuous sample paths. One then has the following convergence result:

Theorem 2.3 *Assume that there exists $\gamma \geq 0$ and $a \in [0, 1)$ such that $F: \mathcal{H}^\gamma \rightarrow \mathcal{H}^{\gamma-a}$ is locally Lipschitz continuous and such that the process v has continuous sample paths with values in \mathcal{H}^γ . Assume furthermore that $F_\varepsilon: \mathcal{H}^\gamma \rightarrow \mathcal{H}^{\gamma-a}$ is such that F_ε is locally Lipschitz and such that the convergence $F_\varepsilon(u) \rightarrow F(u)$ takes place in $\mathcal{H}^{\gamma-a}$, locally uniformly in \mathcal{H}^γ . Then, the solutions u_ε to*

$$du_\varepsilon = -Au_\varepsilon dt + F(u_\varepsilon) dt + Q dw(t), \quad (17)$$

converge to the solutions to (16) as $\varepsilon \rightarrow 0$.

Proof. The proof is straightforward and we only sketch it. We assume without loss of generality that the parameter ε is chosen in such a way that for every $R > 0$ there exists a constant C_R such that

$$\sup_{\|x\|_\gamma \leq R} \|F_\varepsilon(u) - F(u)\|_{\gamma-a} \leq C_R \varepsilon. \quad (18)$$

Denote now by u the solution to (16) with initial condition u_0 and by u_ε the solution to (17) with initial condition u_0^ε . Let $t > 0$ be small enough so that $\|u(s)\|_\gamma \leq R$ and $\|u(s) - u_\varepsilon(s)\|_\gamma \leq R$ for $s \leq t$. We then have

$$\begin{aligned} \|u(t) - u_\varepsilon(t)\|_\gamma &\leq \|u_0 - u_0^\varepsilon\|_\gamma + C \int_0^t (t-s)^{-a} \|F_\varepsilon(u_\varepsilon(s)) - F(u(s))\|_{\gamma-a} ds \\ &\leq \|u_0 - u_0^\varepsilon\|_\gamma + C \int_0^t (t-s)^{-a} \|F_\varepsilon(u_\varepsilon(s)) - F(u_\varepsilon(s))\|_{\gamma-a} ds \\ &\quad + C \int_0^t (t-s)^{-a} \|F(u_\varepsilon(s)) - F(u(s))\|_{\gamma-a} ds \\ &\leq \|u_0 - u_0^\varepsilon\|_\gamma + C_R \varepsilon + C_R t^{1-a} \sup_{s \leq t} \|u(s) - u_\varepsilon(s)\|_\gamma. \end{aligned}$$

The claim then follows by taking t sufficiently small and performing a simple iteration. \square

Despite its simplicity, this criterion is surprisingly sharp. Indeed, we argue that if we consider (4) but with the space-time white noise dw replaced by $(1 - \partial_x^2)^{-\delta} dw$ for $\delta > 0$, then the assumptions of Theorem 2.3 can be satisfied with some choice of exponent γ for the approximation

$$F_\varepsilon(u)(x) = u(x) \frac{u(x + \varepsilon) - u(x)}{\varepsilon} \stackrel{\text{def}}{=} u(x) (D_\varepsilon u)(x).$$

Obviously, this cannot be the case when $\delta = 0$, since we then observe the convergence to solutions to (10).

Indeed, we first note that since the linear operator appearing in (4) is of *second* order, we have the correspondence

$$\mathcal{H}^\gamma = H^{2\gamma},$$

between interpolation spaces and fractional Sobolev spaces. In order to keep our notation coherent throughout this section, we still denote by $\|\cdot\|_\gamma$ the norm of \mathcal{H}^γ , i.e. $\|u\|_\gamma = \|(1 - \partial_x^2)^\gamma u\|_{L^2}$, where we implicitly endow ∂_x^2 with periodic boundary conditions. With this notation, we have

Lemma 2.4 *For $\gamma \geq 0$ and $a \in [\frac{1}{2}, 1]$, we have $\|D_\varepsilon u - \partial_x u\|_{\gamma-a} \leq C \varepsilon^{2a-1} \|u\|_\gamma$.*

Proof. The operator $D_\varepsilon - \partial_x$ is given by the Fourier multiplier $M_\varepsilon(k) = \varepsilon^{-1}(e^{ik\varepsilon} - 1 - ik\varepsilon)$. We immediately obtain the bound

$$|M_\varepsilon(k)| \leq k(\varepsilon k \wedge 1) \leq k(\varepsilon k)^{2a-1},$$

from which the claim follows at once. \square

Since furthermore \mathcal{H}^γ is an algebra for $\gamma > \frac{1}{4}$, we conclude that the bound (18) holds (for some different power of ε), provided that we choose $\gamma \in (\frac{1}{4}, \frac{1}{2}]$ and $a = 2\gamma$. On the other hand, the solution to the linearised equation

$$dv = \partial_x^2 v dt + (1 - \partial_x^2)^{-\delta} dw$$

belongs to \mathcal{H}^γ if and only if $\gamma < \frac{1}{4} + \delta$, thus supporting our claim that the case $\delta = 0$ is precisely borderline for the applicability of Theorem 2.3.

If on the other hand we make the more natural choice

$$F_\varepsilon(u) = \frac{1}{2} D_\varepsilon(u^2), \quad (19)$$

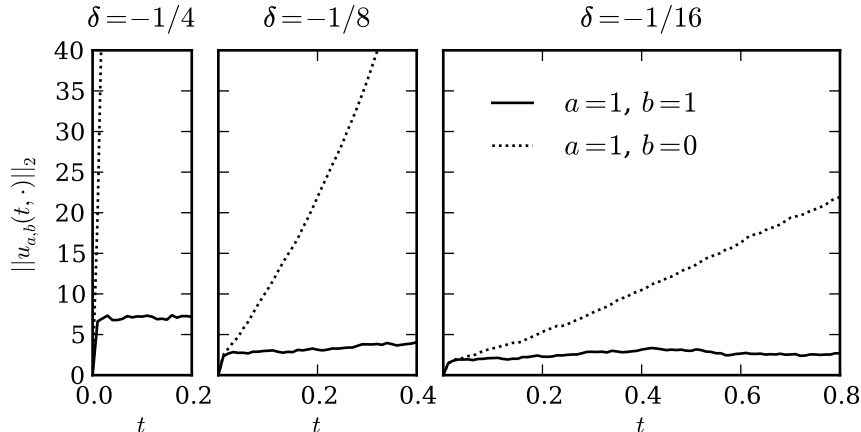


Figure 3: Illustration of the divergence of the right-hand discretisation for noise rougher than space-time white noise. The three panels show, for different values of δ , the L^2 -norm of numerical solutions of equation (20). The full lines correspond to a centred discretisation ($a = 1, b = 1$) whereas the dotted lines correspond to the right-hand discretisation ($a = 1, b = 0$). The figures show that only the centred discretisation seems to be stable for $\delta < 0$.

then it turns out that we can apply Theorem 2.3 even in the case $\delta = 0$. Indeed, it follows from standard Sobolev theory (see for example [Hai09]) that if $\gamma \in (\frac{1}{8}, \frac{1}{4})$, the map $u \mapsto u^2$ is locally Lipschitz from \mathcal{H}^γ into \mathcal{H}^β provided that $\beta < 2\gamma - \frac{1}{4}$. As a consequence, it follows from Lemma 2.4 that the approximation F_ε given by (19) converges to $u\partial_x u$ in the sense of (18), provided that $\gamma \in (\frac{1}{8}, \frac{1}{4})$ and $a \in (\frac{3}{4} - \gamma, 1)$.

Remark 2.5 Some ad hoc numerical scheme was shown to converge to the exact solution in [AG06]. Some other schemes are shown to converge in [GKN02], but only in the case $\delta > 0$ of more regular noise. A particle approximation to a specific modification was constructed in [GD04].

2.3 Numerical verification of the borderline case

We have performed numerical simulations that corroborate the argument presented in the previous section and show that $\delta = 0$ truly is the borderline case for the limiting equation to be independent on the type of discretisation performed on the nonlinearity. In the case $\delta < 0$ (i.e. the case where the driving noise is *rougher* than space-time white noise), our preceding discussion suggests that the centred discretisation should converge to the correct solution for $|\delta|$ sufficiently small, but that the solutions to both the left-handed and the right-handed discretisations should diverge as $\varepsilon \rightarrow 0$.

In order to verify this effect, we numerically solve the SPDE

$$du = \nu \partial_x^2 u dt - u \partial_x u dt + (1 - \partial_x^2)^{-\delta} dw, \quad (20)$$

using different discretisation schemes and different values for δ . The results are shown in figure 3. The simulations are in good agreement with the argument outlined above.

3 Possible Generalisations of the Argument

In this section, we discuss a number of possible extensions of these results to more general Burgers-type equations. We restrict ourselves in this discussion to the case $a = 1, b = 0$, i.e. to right-handed discretisations. This is purely for notational convenience, and one would expect similar correction terms to appear for arbitrary values of a and b , just as before.

3.1 More General Nonlinearities

Consider the equation

$$du_i = \nu \partial_x^2 u_i dt + \sum_{j=1}^d \partial_j h_i(u) \partial_x u_j dt + \sigma dw_i, \quad (21)$$

for an \mathbb{R}^d -valued process u and a smooth function $h: \mathbb{R}^d \rightarrow \mathbb{R}^d$ with bounded second and third derivatives. Rewriting the nonlinearity as $\partial_x(h_i(u))$, we see that this equation is globally well-posed. As before, we consider the approximating equation

$$du_i^\varepsilon(x, t) = \nu \partial_x^2 u_i^\varepsilon(x, t) dt + \sum_j \partial_j h_i(u^\varepsilon(x, t)) D_\varepsilon u_j^\varepsilon(x, t) dt + \sigma dw_i(t). \quad (22)$$

The idea now is to introduce a cut-off frequency N and to write $u^\varepsilon = \bar{u}^\varepsilon + \tilde{u}^\varepsilon$, where \bar{u}^ε is the projection of u^ε onto Fourier modes with $|k| \leq N$. Since the linear part of the equation dominates the nonlinearity at high frequencies, one expects \tilde{u}^ε to be well approximated by \tilde{v} , the projection of v onto the high frequencies. This on the other hand is small in the L^∞ norm (it decreases like N^{-s} for every $s < \frac{1}{2}$), so that

$$\partial_j h_i(u^\varepsilon) \approx \partial_j h_i(\bar{u}^\varepsilon) + \sum_k \partial_{jk}^2 h_i(\bar{u}^\varepsilon) \tilde{v}_k.$$

It now follows from the same argument as before that the term $\partial_{jk}^2 h_i(\bar{u}^\varepsilon) \tilde{v}_k D_\varepsilon v_j$ is expected to yield a non-vanishing contribution for $k = j$ in the limit $\varepsilon \rightarrow 0$ and $N \rightarrow \infty$. Provided that we keep $N \ll \frac{1}{\varepsilon}$, this contribution will again be described by (9), so that we expect the following behaviour.

Conjecture 3 *The solution of (22) converges, as $\varepsilon \rightarrow 0$, to solutions of the equation*

$$du_i = \nu \partial_x^2 u_i dt + \sum_j \left(\partial_j h_i(u) \partial_x u_j - \frac{\sigma^2}{4\nu} \partial_{jj}^2 h_i(u) \right) dt + \sigma dw_i. \quad (23)$$

In the one-dimensional case we can recover conjecture 1 from conjecture 3 by choosing $h(u) = -u^2/2$.

We perform the following numerical experiment to validate the functional form of the correction term given in conjecture 3: We numerically solve both the ‘‘approximating’’ equation (22) (for $d = 1$; employing the same discretisation as for conjecture 2) and the ‘‘corrected’’ SPDE

$$d\hat{u} = \nu \partial_x^2 \hat{u} dt + h'(\hat{u}) \partial_x \hat{u} dt - \frac{\sigma^2}{4\nu} p(\hat{u}) dt + \sigma dw. \quad (24)$$

until a fixed time T , using the same instance of the noise and some function $p: \mathbb{R} \rightarrow \mathbb{R}$. To discretise the term $u \partial_x u$ in (15) we use again the approximation $(h(x) - h(x - \delta))/\delta$ and again we solve (15) on a finer grid. Then we numerically optimise p (using some parametric form) in order to minimise the distance $\|u^N(T, \cdot) - \hat{u}(T, \cdot)\|_2$. If the conjecture is correct, we expect the minimum to be attained for a function p which is close to the predicted correction term h'' . The result of a simulation is shown in figure 4, where we used fifth-order polynomials for p . The figure shows that there is indeed a good fit between the conjectured and numerically determined correction terms.

3.2 Classically Ill-Posed Equations

Pushing further the class of equations considered in the previous subsection, one may want to consider equations of the form

$$du_i = \nu \partial_x^2 u_i dt + \sum_{j=1}^d G_{ij}(u) \partial_x u_j dt + f(u) dt + \sigma dw_i \quad (25)$$

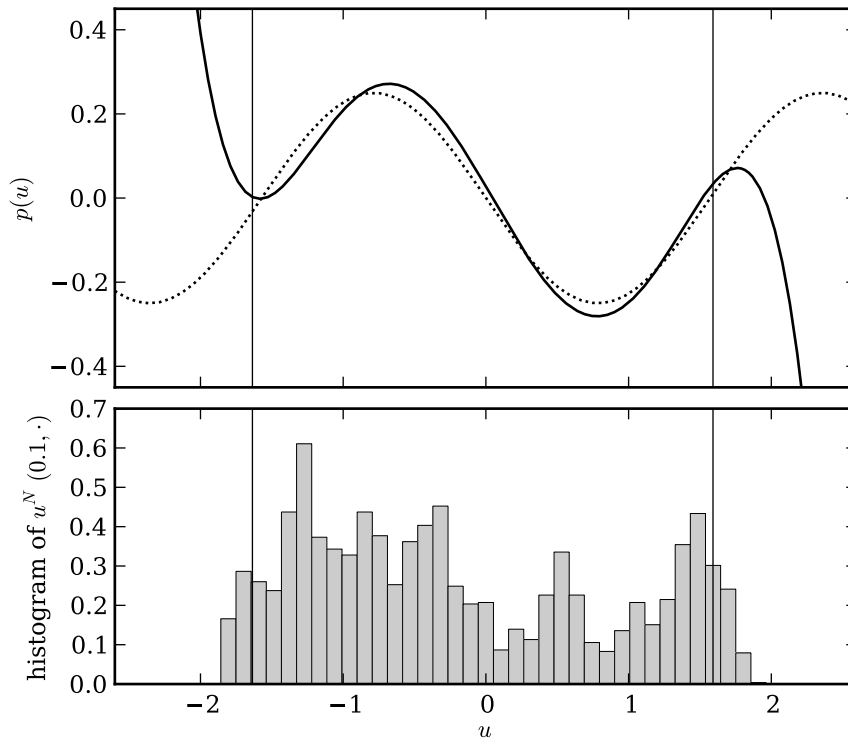


Figure 4: *Illustration of the convergence of (22) to (23) for the one-dimensional case $h'(x) = \sin(x)^2$. For the figure we numerically compute the finite differences solution u^N to (22). We then compute solutions \hat{u} to (24), with the same noise and using a fifth-order polynomial for the correction term p . This polynomial is then numerically fitted to minimise $\|u^N(1, \cdot) - \hat{u}(1, \cdot)\|_2$. The top panel shows the resulting fitted correction term $-\sigma^2 p / 4\nu$ (full line) together with the correction term $-\sigma^2 h''(u) / 4\nu$ from conjecture 3 (dotted line). To give an idea which range of the correction term is actually used in the computation, the lower panel shows the histogram of the values of u^N (the vertical bars indicate the 5% and 95% quantiles). The graphs shows a good fit between the numerically determined and conjectured correction terms.*

for some functions $G: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ and $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$. If we do *not* assume that G has an antiderivative and since solutions are only expected to be α -Hölder continuous in space for $\alpha < \frac{1}{2}$, it is no longer even clear what it means to be a solution to this equation. So, at least classically, (25) is ill-posed and the mere *concept* of solutions to such an evolution equation is difficult to establish.

However, the discretised equation does of course still make sense for any fixed value of ε . Furthermore, we observed numerically that there seems to be no instability as $\varepsilon \rightarrow 0$; indeed one observes pathwise convergence to a limiting process. By analogy with the behaviour observed for the situations where (25) is classically well-posed (i.e. when G has an antiderivative), one would then be tempted to define solutions to (25) to be those processes that can be obtained as limits as $\varepsilon \rightarrow 0$ of the solutions to the equation where $\partial_x u_j$ is replaced by its symmetric discretisation. Formally, we would then expect solutions of the discretised equation with right-handed discretisation to converge, as $\varepsilon \rightarrow 0$, to solutions of the corrected equation

$$du_i = \nu \partial_x^2 u_i dt + \sum_j (G_{ij}(u) \partial_x u_j - \frac{\sigma^2}{4\nu} \partial_j G_{ij}(u)) dt + f(u) dt + \sigma dw_i.$$

Conjecture 4 *As $\varepsilon \rightarrow 0$, the equations*

$$du_i^\varepsilon = \nu \partial_x^2 u_i^\varepsilon dt + \sum_{j=1}^d G_{ij}(u^\varepsilon) D_\varepsilon^{1,0} u_j^\varepsilon dt + f(u^\varepsilon) dt + \sigma dw_i, \quad (26)$$

where $D_\varepsilon^{1,0}$ denotes the right-handed discretisation, and

$$d\tilde{u}_i^\varepsilon = \nu \partial_x^2 \tilde{u}_i^\varepsilon dt + \sum_{j=1}^d (G_{ij}(\tilde{u}^\varepsilon) D_\varepsilon^{1,1} \tilde{u}_j^\varepsilon - \frac{\sigma^2}{4\nu} \partial_j G_{ij}(\tilde{u}^\varepsilon)) dt + f(\tilde{u}^\varepsilon) dt + \sigma dw_i, \quad (27)$$

where $D_\varepsilon^{1,1}$ denotes centred discretisation, converge to the same limit.

To test conjecture 4, we use the following numerical experiment: we consider the SPDE

$$\begin{aligned} \partial_t u = \frac{1}{\sigma^2} \partial_x^2 u + \frac{2}{\sigma^2} \begin{pmatrix} 0 & \cos(u_2) - \sin(u_1) \\ \sin(u_1) - \cos(u_2) & 0 \end{pmatrix} \partial_x u \\ + \frac{4}{\sigma^2} \begin{pmatrix} \sin(u_1) \cos(u_1) \\ -\cos(u_2) \sin(u_2) \end{pmatrix} + \sqrt{2} \partial_t w. \end{aligned} \quad (28)$$

This SPDE is of the form (25) where G has no antiderivative. SPDEs like (28) occur in the problem described in [HSV07, section 9] where we conjecture that the stationary distribution of this SPDE on $L^2([0, 2\pi], \mathbb{R}^2)$ (when equipped with appropriate boundary conditions) coincides with the distribution of the SDE

$$dU(t) = 2 \begin{pmatrix} -\sin(U_2(t)) \\ \cos(U_1(t)) \end{pmatrix} dt + \sigma dB(t).$$

For our experiment we numerically solve the SPDEs (26) and (27) and compare the solutions. The result is displayed in figure 5. As can be seen from the figure, the simulation results are in good agreement with conjecture 4.

3.3 Multiplicative noise

Consider the equation

$$du = \nu \partial_x^2 u dt + g(u) \partial_x u dt + f(u) dw,$$

where g is as before and f is a smooth bounded function with bounded derivatives of all orders. Such an equation is well-posed if the stochastic integral is interpreted in the Itô sense [GN99]. (Note that it is *not* well-posed if the stochastic integral is interpreted in the Stratonowich sense. This follows from the fact that, at least formally, the Itô correction term is infinite when f is not constant.)

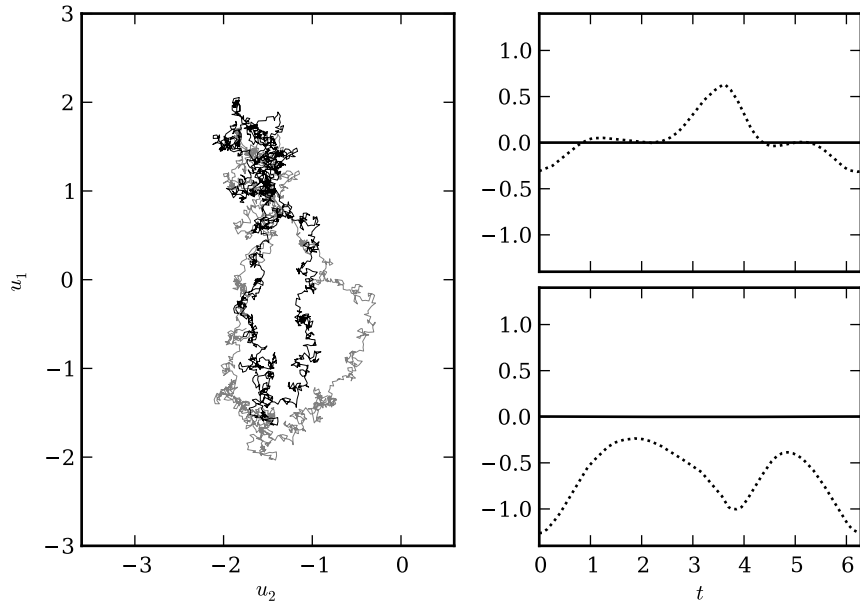


Figure 5: *Illustration of the convergence in conjecture 4. The left-hand panel shows a numerical solution of the SPDE (28) at a fixed time t . Since we use periodic boundary conditions, the plotted graph of $u(t, \cdot)$ forms a loop in \mathbb{R}^2 . The black line in this plot was obtained using a centred discretisation, whereas the gray line in the background was obtained using a right-handed discretisation. As in figure 1, there is an $O(1)$ difference between the two discretisation schemes. The two plots on the right-hand side show the differences $u_1^\varepsilon - \tilde{u}_1^\varepsilon$ (upper panel) and $u_2^\varepsilon - \tilde{u}_2^\varepsilon$ (lower panel) between the discretisation schemes (26) and (27) (full lines). For comparison, the plots also show the functions $u_1^\varepsilon - \bar{u}_1^\varepsilon$ and $u_2^\varepsilon - \bar{u}_2^\varepsilon$ where \bar{u}^ε is the solution of the SPDE (28) using a centred discretisation (dotted lines). The graphs show good agreement between the solutions of (26) and (27).*

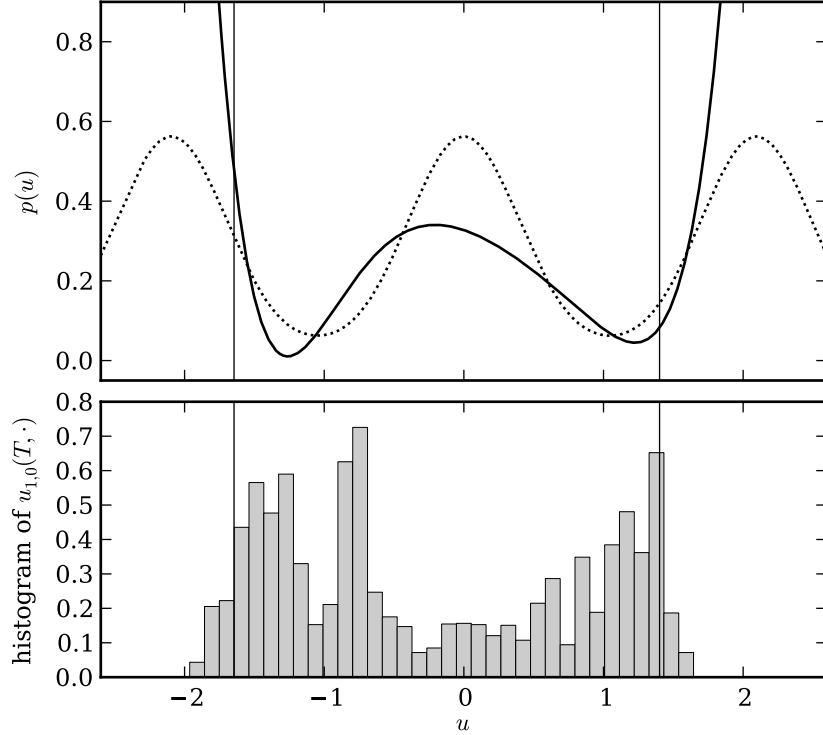


Figure 6: *Illustration of the convergence of (29) to (30) in conjecture 5, for the case $g(u) = u$ and $f(u) = 1 + \frac{1}{2} \cos(3u)$. See figure 4 for an explanation of the graphs. Here we use a sixth-order polynomial p to fit the correction term. The figure shows that the fit is significantly worse than in figure 4. See the text for a discussion of possible reasons for this effect.*

In such a case, the local quadratic variation of the solution is expected to be proportional to $f^2(u)$, so that one expects the right-handed discretisation to exhibit a correction term proportional to $g'(u)f^2(u)$. More precisely, in analogy to conjecture 3 one would expect the following conjecture to hold.

Conjecture 5 *The solutions of*

$$du = \nu \partial_x^2 u dt + g(u) D_\varepsilon u dt + f(u) dw, \quad (29)$$

converge, as $\varepsilon \rightarrow 0$, to solutions of the equation

$$du = \nu \partial_x^2 u dt + g(u) \partial_x u dt + \frac{1}{4\nu} g'(u) f^2(u) dt + f(u) dw. \quad (30)$$

To test this conjecture we perform a numerical experiment, similar to the one for conjecture 3; the result is shown in figure 6. The fit between predicted and numerically determined correction term in figure 6 is worse than in figure 4 and thus the numerical test is not entirely conclusive.

One possible reason is that the spatial resolution of our numerical simulations is not sufficient. Indeed, the argument of the previous sections is based on a spatial averaging of the small-scale fluctuations of the process. In the case of multiplicative noise, these small-scale fluctuations are themselves multiplied by the process $f(u)$, which is spatially quite rough. Therefore, this spatial averaging will hold only on extremely small scales, essentially sufficiently small so that $f(u)$ is constant for all practical purposes. In order to be seen by the numerical simulation, these scales still need to be resolved at sufficient precision to have some version of the law of large numbers.

4 Small noise / viscosity limit

One regime that is of particular interest is the small noise/small viscosity limit. If one takes $\nu \propto \sigma^2$ in conjectures 1 and 2, one obtains a non-vanishing correction even for arbitrarily small ν and σ ! It is therefore of interest to study approximations to

$$du = \varepsilon \partial_x^2 u dt - u \partial_x u dt + \sqrt{\varepsilon} dw \quad (31)$$

for $\varepsilon \ll 1$.

It is well-known that in the absence of viscosity, finite difference schemes for the Burgers equations can only be used with extreme caution due to the presence of shocks in the solution. For the limiting equation ($\varepsilon = 0$), a shock is a jump discontinuity of the solution with the value u^\pm to the right/left of the jump satisfying $u^+ < u^-$. In other words, the jumps are always downwards jumps. For viscosity solutions² to the inviscid Burgers equation, shocks move through the system at velocity $\frac{1}{2}(u^+ + u^-)$.

If the limiting non-viscous Burgers equation is discretised as

$$\partial_t u_n = -\frac{1}{\delta} u_n (u_{n+1} - u_n), \quad (32)$$

then a simple linear stability analysis reveals that the solutions to this equation develop an *ultraviolet instability* (i.e. the mode $v_n = (-1)^n$ becomes unstable) in the regions where $u > 0$. Similarly, the corresponding left-handed discretisation reveals an instability in the regions where $u < 0$. However, in the case of a centred discretisation, the highly oscillatory modes are stable. A similar phenomenon appears if we consider the discretisation

$$\partial_t u_n = -\frac{1}{2\delta} (u_{n+1}^2 - u_n^2), \quad (33)$$

or variants thereof.

What happens at the formation of a shock? In this case, one expects to observe the correct behaviour only for discretisations that are conservative (i.e. of the type (33) rather than (32)) and that are ‘upwind’ in the sense that the direction of the discretisation coincides with the direction of propagation of the shock [CIR52, MRTB05]. In the case of a non-conservative discretisation of the type (32), we still expect the scheme to be stable when the discretisation is ‘upwind’, but we expect the shock to propagate at the wrong speed.

How is this picture modified for non-zero values of ε ? The instabilities discussed above grow at a speed $\mathcal{O}(\delta^{-1})$ and are therefore dominated by the stabilising effect of the viscosity (which is of the order $\varepsilon\delta^{-2}$) only if $\varepsilon \gg \delta$. Regarding the behaviour after the formation of a shock, a simple boundary layer analysis (see for example [EVE00a] for a more sophisticated boundary layer analysis that even goes to the next order in ε) shows that a typical shock for (31) has width $\mathcal{O}(\varepsilon)$, so that the caveats pointed out above are expected to become relevant as soon as $\varepsilon \lesssim \delta$. On the other hand, at least away from shocks, the analysis performed in Section 2 holds as soon as u can be approximated by the solution to the linearised equation at sufficiently small (but still much larger than δ) scales. Away from shocks, one expects this to be the case, provided that the linearised equation reaches equilibrium in times much smaller than 1. This is the case for wavenumbers higher than $\varepsilon^{-1/2}$, so that one expects the results from Section 2 to be relevant as long as $\varepsilon \gg \delta^2$. This leads to the following set of conjectures (for $\varepsilon \ll 1$):

1. If we take $\delta \ll \varepsilon$, then the solution to the finite difference approximation (32) of (31) converges to the viscosity solution of

$$\partial_t u = -\frac{1}{2} \partial_t u^2 + \frac{c}{4}, \quad (34)$$

where $c \in \{1, 0, -1\}$ depending on whether the discretisation is right-handed, centred, or left-handed.

²Also called entropy solutions, these are the solutions that are obtained as limits of (31) as $\varepsilon \rightarrow 0$

2. If we take $\varepsilon \ll \delta \ll \sqrt{\varepsilon}$, then we expect the finite difference approximation for (31) to converge to the solutions to (34), as long as solutions remain smooth and have the correct sign to prevent ultraviolet blow-up. After the occurrence of a shock, one expects to see the behaviour described above (i.e. stability only if the scheme is upwind, but wrong propagation speed even then).
3. If we take $\sqrt{\varepsilon} \ll \delta$, then we expect both the viscosity and the noise term to become irrelevant, so that the solution behaves like the corresponding approximation to the inviscid Burgers equation. In particular, as long as solutions remain smooth and have the correct sign to prevent ultraviolet blow-up, we expect to always converge to (34) with $c = 0$. (With shocks having the wrong propagation speed for discretisations of the type (32).)

A Simulations

To verify the heuristic arguments presented above, we perform a series of numerical simulations. This section describes some of the technical details of these simulations.

For the space discretisation we approximate $u \in L^2([0, 2\pi], \mathbb{R})$ by $u^N \in \mathbb{R}^N$. The space discretisation of the differential operators is already described above. The finite differences discretisation of the white noise process w is $W^N/\sqrt{\Delta x}$ where W^N is a standard Brownian motion on \mathbb{R}^N and $\Delta x = 2\pi/N$ is the grid size. This leads to \mathbb{R}^N -valued SDEs of the form

$$du^N = \nu L_N u^N dt + F_N(u^N) dt + \sigma \sqrt{\frac{1}{\Delta x}} dW^N(t)$$

where $L_N \in \mathbb{R}^{N \times N}$ is the discretisation of the linear part and $F_N: \mathbb{R}^N \rightarrow \mathbb{R}^N$ is the discretisation of the nonlinearity.

For discretising time we use the θ -method

$$u^{(n+1)} = u^{(n)} + \nu L_N (\theta u^{(n+1)} + (1 - \theta)u^{(n)}) \Delta t + F_N(u^{(n)}) \Delta t + \sigma \sqrt{\frac{\Delta t}{\Delta x}} \xi^{(n+1)}$$

where $\Delta t > 0$ is the time step size, $u^{(n)}$ is the discretised solution at time $n \Delta t$ and $\xi^{(n)}$ are i.i.d., N -dimensional standard normally distributed random variables. Rearranging this equation gives

$$(I - \nu \theta \Delta t L_N) u^{(n+1)} = (I + \nu(1 - \theta) \Delta t L_N) u^{(n)} + F_N(u^{(n)}) \Delta t + \sqrt{\frac{\Delta t}{\Delta x}} \xi^{(n+1)}. \quad (35)$$

Relation (35) allows to compute $u^{(n+1)}$ from $u^{(n)}$; since $I - \nu \theta \Delta t L_N$ is cyclic tridiagonal, this system can be solved efficiently.

Give link to code??

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