

SPDE in Hilbert Space with Locally Monotone Coefficients

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Abstract

The aim of this paper is to extend the usual framework of SPDE with monotone coefficients to include a large class of cases with merely locally monotone coefficients. This new framework is conceptually not more involved than the classical one, but includes many more fundamental examples not included previously. Thus our main result can be applied to various types of SPDEs such as stochastic reaction-diffusion equations, stochastic Burgers type equation, stochastic 2-D Navier-Stokes equation, stochastic p -Laplace equation and stochastic porous media equation with non-monotone perturbations.

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1 Introduction

Let

$$V \subset H \equiv H^* \subset V^*$$

be a Gelfand triple, *i.e.* $(H, \langle \cdot, \cdot \rangle_H)$ is a separable Hilbert space and identified with its dual space by the Riesz isomorphism, V is a reflexive Banach space such that it is continuously and densely embedded into H . If ${}_{V^*}\langle \cdot, \cdot \rangle_V$ denotes the dualization between V and its dual space V^* , then it follows that

$${}_{V^*}\langle u, v \rangle_V = \langle u, v \rangle_H, \quad u \in H, v \in V.$$

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Let $\{W_t\}_{t \geq 0}$ be a cylindrical Wiener process on a separable Hilbert space U w.r.t a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and $(L_2(U; H), \|\cdot\|_2)$ denotes the space of all Hilbert-Schmidt operators from U to H . We consider the following stochastic evolution equation

$$(1.1) \quad dX_t = A(t, X_t)dt + B(t, X_t)dW_t,$$

where for some fixed time T

$$A : [0, T] \times V \times \Omega \rightarrow V^*; \quad B : [0, T] \times V \times \Omega \rightarrow L_2(U; H)$$

are progressively measurable, *i.e.* for every $t \in [0, T]$, these maps restricted to $[0, t] \times V \times \Omega$ are $\mathcal{B}([0, t]) \otimes \mathcal{B}(V) \otimes \mathcal{F}_t$ -measurable (where \mathcal{B} denotes the corresponding Borel σ -algebra).

It is well known that (1.1) has a unique solution if A, B satisfy the classical monotone and coercivity conditions (cf. [16, 26]), which we recall in the Appendix below. The theory of monotone operators starts from substantial work of Minty [23, 24] and Browder [6, 7] for PDE. We refer to [15, 36, 33] for a detailed exposition and references. In recent years, this variational approach has been also used intensively for analyzing SPDE driven by infinite-dimensional Wiener process. Unlike the semigroup approach (cf. [10]), it is not necessary to have a linear operator in the drift part which has to generate a semigroup. Hence the variational approach can be used to investigate nonlinear SPDE which are not necessarily of semilinear type. For general results on the existence and uniqueness of solutions to SPDE we refer to [25, 16, 12, 27, 35]. Within this framework many different types of properties have already been established, *e.g.* see [8, 19, 28, 32] for the small noise large deviation principle, [13, 14] for discretization approximation schemes to the solutions of SPDE, [34, 17, 18] for the dimension-free Harnack inequality and resulting ergodicity, compactness and contractivity properties of the associated transition semigroups, and [20, 5, 11] for the invariance of subspaces and existence of random attractors for corresponding random dynamical systems. As one typical example of SPDE in this framework, the stochastic porous media equation has been extensively studied in [1, 2, 3, 4, 9, 31].

The main aim of this paper is to provide a more general framework for the variational approach, being conceptually not more complicated than the classical one (cf. [16]), but including a large number of new applications as *e.g.* fundamental SPDE as the stochastic 2-D Navier-Stokes equation and stochastic Burgers type equation. The main changes consist of localizing the monotonicity condition and relaxing the growth condition. This new framework is, in addition, more stable with respect to perturbations. We refer to Section 3 below for details. In particular, we can simplify the related approach to the stochastic 2-D Navier-Stokes equation in the nice paper [22], which inspired us a lot to start this work. However, our approach also easily covers the case of arbitrary multiplicative noise, whereas in [22] only additive noise was considered. It is also straight forward to extend our new framework to more noise terms, *e.g.* Levy noise (cf. [12] for the classical case). This and further new applications will be the subject of future work.

Let us now state the precise conditions on the coefficients of (1.1):

Suppose there exist constants $\alpha > 1$, $\beta \geq 0$, $\theta > 0$, K and a positive adapted process $f \in L^1([0, T] \times \Omega; dt \times \mathbb{P})$ such that the following conditions hold for all $v, v_1, v_2 \in V$ and $(t, \omega) \in [0, T] \times \Omega$.

(H1) (Hemicontinuity) The map $s \mapsto {}_{V^*}\langle A(t, v_1 + sv_2), v \rangle_V$ is continuous on \mathbb{R} .

(H2) (Local monotonicity)

$$2{}_{V^*}\langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle_V + \|B(t, v_1) - B(t, v_2)\|_2^2 \leq (K + \rho(v_2)) \|v_1 - v_2\|_H^2,$$

where $\rho : V \rightarrow [0, +\infty)$ is a measurable function and locally bounded in V .

(H3) (Coercivity)

$$2{}_{V^*}\langle A(t, v), v \rangle_V + \|B(t, v)\|_2^2 + \theta \|v\|_V^\alpha \leq f_t + K \|v\|_H^2.$$

(H4) (Growth)

$$\|A(t, v)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \leq (f_t + K \|v\|_V^\alpha)(1 + \|v\|_H^\beta).$$

Remark 1.1. (1) (H2) is essentially weaker than the standard monotonicity (A2) (i.e. $\rho \equiv 0$). One typical form of (H2) in applications is

$$\rho(v) = C \|v\|^\gamma,$$

where $\|\cdot\|$ is some norm on V and C, γ are some constants.

One typical example is the stochastic 2-D Navier-Stokes equation on a bounded or unbounded domain, which satisfies (H2) but does not satisfy (A2) (see Section 3). In fact, if $A(t, v) = \nu P_H \Delta v - P_H [(v \cdot \nabla)v]$, we have

$$2{}_{V^*}\langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle_V \leq -\frac{\nu}{2} \|v_1 - v_2\|_V^2 + \left(\nu + \frac{16}{\nu^3} \|v_2\|_{L^4}^4 \right) \|v_1 - v_2\|_H^2.$$

(2) If the noise is zero or additive type in (1.1), then the existence and uniqueness of solutions to (1.1) can be established by replacing (H2) with the following more general type of local monotonicity:

$${}_{V^*}\langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle_V \leq (K + \eta(v_1) + \rho(v_2)) \|v_1 - v_2\|_H^2,$$

where $\eta, \rho : V \rightarrow [0, +\infty)$ are measurable functions and locally bounded in V .

This will be investigated in a separated paper [21].

(3) (H4) is also weaker than the standard growth condition (A4) (see the Appendix) assumed in the literature (cf. [16, 36, 26]). The advantage of (H4) is, *e.g.*, to include many semilinear type equations with nonlinear perturbation terms. For example, if we consider a reaction-diffusion type equation, *i.e.* $A(u) = \Delta u + F(u)$, then for verifying (H3) we have $\alpha = 2$. Hence (A4) would imply that F has at most linear growth. However, we can allow F to have some polynomial growth by using the weaker condition (H4) here. We refer to Section 3 for more details.

Definition 1.1. (Solution of SEE) A continuous H -valued (\mathcal{F}_t) -adapted process $\{X_t\}_{t \in [0, T]}$ is called a solution of (1.1), if for its $dt \otimes \mathbb{P}$ -equivalent class \bar{X} we have

$$\bar{X} \in L^\alpha([0, T] \times \Omega, dt \otimes \mathbb{P}; V) \cap L^2([0, T] \times \Omega, dt \otimes \mathbb{P}; H)$$

and $\mathbb{P} - a.s.$,

$$X_t = X_0 + \int_0^t A(s, \bar{X}_s) ds + \int_0^t B(s, \bar{X}_s) dW_s, \quad t \in [0, T].$$

Now we can state the main result.

Theorem 1.1. Suppose (H1)–(H4) hold for $f \in L^{p/2}([0, T] \times \Omega; dt \times \mathbb{P})$ with some $p \geq \beta + 2$, and there exists a constant C such that

$$(1.2) \quad \begin{aligned} \|B(t, v)\|_2^2 &\leq C(f_t + \|v\|_H^2), \quad t \in [0, T], v \in V; \\ \rho(v) &\leq C(1 + \|v\|_V^\alpha)(1 + \|v\|_H^\beta), \quad v \in V. \end{aligned}$$

Then for any $X_0 \in L^p(\Omega \rightarrow H; \mathcal{F}_0; \mathbb{P})$ (1.1) has a unique solution $\{X_t\}_{t \in [0, T]}$ and satisfies

$$\mathbb{E} \left(\sup_{t \in [0, T]} \|X_t\|_H^p + \int_0^T \|X_t\|_V^\alpha dt \right) < \infty.$$

Moreover, if $A(t, \cdot)(\omega), B(t, \cdot)(\omega)$ are independent of $t \in [0, T]$ and $\omega \in \Omega$, then the solution $\{X_t\}_{t \in [0, T]}$ of (1.1) is a Markov process.

2 Proof of the main theorem

The first step of the proof is mainly based on the Galerkin approximation. Let

$$\{e_1, e_2, \dots\} \subset V$$

be an orthonormal basis of H and let $H_n := \text{span}\{e_1, \dots, e_n\}$ such that $\text{span}\{e_1, e_2, \dots\}$ is dense in V . Let $P_n : V^* \rightarrow H_n$ be defined by

$$P_n y := \sum_{i=1}^n v^* \langle y, e_i \rangle_V e_i, \quad y \in V^*.$$

Obviously, $P_n|_H$ is just the orthogonal projection onto H_n in H and we have

$$v^* \langle P_n A(t, u), v \rangle_V = \langle P_n A(t, u), v \rangle_H = v^* \langle A(t, u), v \rangle_V, \quad u \in V, v \in H_n.$$

Let $\{g_1, g_2, \dots\}$ be an orthonormal basis of U and

$$W_t^{(n)} := \sum_{i=1}^n \langle W_t, g_i \rangle_U g_i = \tilde{P}_n W_t,$$

where \tilde{P}_n is the orthogonal projection onto $\text{span}\{g_1, \dots, g_n\}$ in U .

Then for each finite $n \in \mathbb{N}$ we consider the following stochastic equation on H_n

$$(2.1) \quad dX_t^{(n)} = P_n A(t, X_t^{(n)})dt + P_n B(t, X_t^{(n)})dW_t^{(n)}, \quad X_0^{(n)} = P_n X_0.$$

By the classical result for the solvability of SDE in finite-dimensional space (cf. [16, 26]) we know that (2.1) has a unique strong solution.

In order to construct the solution of (1.1), we need some a priori estimates for $X^{(n)}$. For convenience we use following notations:

$$\begin{aligned} K &= L^\alpha([0, T] \times \Omega \rightarrow V; dt \times \mathbb{P}); \\ K^* &= L^{\frac{\alpha}{\alpha-1}}([0, T] \times \Omega \rightarrow V^*; dt \times \mathbb{P}); \\ J &= L^2([0, T] \times \Omega \rightarrow L_2(U; H); dt \times \mathbb{P}). \end{aligned}$$

Lemma 2.1. *Under the assumptions in Theorem 1.1, there exists $C > 0$ such that for all $n \in \mathbb{N}$*

$$\|X^{(n)}\|_K + \sup_{t \in [0, T]} \mathbb{E} \|X_t^{(n)}\|_H^2 \leq C.$$

Proof. The conclusion follows from (H3) by using the same argument as in [26, Lemma 4.2.9]. Hence we omit the details here. \square

Lemma 2.2. *Under the assumptions in Theorem 1.1, there exists $C > 0$ such that for all $n \in \mathbb{N}$ we have*

$$(2.2) \quad \mathbb{E} \sup_{t \in [0, T]} \|X_t^{(n)}\|_H^p + \mathbb{E} \int_0^T \|X_t^{(n)}\|_H^{p-2} \|X_t^{(n)}\|_V^\alpha dt \leq C \left(\mathbb{E} \|X_0\|_H^p + \mathbb{E} \int_0^T f_t^{p/2} dt \right).$$

In particular, there exists $C > 0$ such that for all $n \in \mathbb{N}$

$$\|A(\cdot, X^{(n)})\|_{K^*} \leq C.$$

Proof. By Itô's formula, Young's inequality and (1.2) we have

$$\begin{aligned}
(2.3) \quad & \|X_t^{(n)}\|_H^p = \|X_0^{(n)}\|_H^p + p(p-2) \int_0^t \|X_s^{(n)}\|_H^{p-4} \|(P_n B(s, X_s^{(n)}) \tilde{P}_n)^* X_s^{(n)}\|_H^2 ds \\
& + \frac{p}{2} \int_0^t \|X_s^{(n)}\|_H^{p-2} \left(2_{V^*} \langle A(s, X_s^{(n)}, X_s^{(n)}) \rangle_V + \|P_n B(s, X_s^{(n)}) \tilde{P}_n\|_2^2 \right) ds \\
& + p \int_0^t \|X_s^{(n)}\|_H^{p-2} \langle X_s^{(n)}, P_n B(s, X_s^{(n)}) dW_s^{(n)} \rangle_H \\
& \leq \|X_0\|_H^p - \frac{p\theta}{2} \int_0^t \|X_s^{(n)}\|_H^{p-2} \|X_s^{(n)}\|_V^\alpha ds \\
& + C \int_0^t (\|X_s^{(n)}\|_H^p + f_s \cdot \|X_s^{(n)}\|_H^{p-2}) ds \\
& + p \int_0^t \|X_s^{(n)}\|_H^{p-2} \langle X_s^{(n)}, P_n B(s, X_s^{(n)}) dW_s^{(n)} \rangle_H \\
& \leq \|X_0\|_H^p - \frac{p\theta}{2} \int_0^t \|X_s^{(n)}\|_H^{p-2} \|X_s^{(n)}\|_V^\alpha ds \\
& + C \int_0^t (\|X_s^{(n)}\|_H^p + f_s^{p/2}) ds \\
& + p \int_0^t \|X_s^{(n)}\|_H^{p-2} \langle X_s^{(n)}, P_n B(s, X_s^{(n)}) dW_s^{(n)} \rangle_H, \quad t \in [0, T],
\end{aligned}$$

where C is a generic constant (independent of n) and may change from line to line.

For any given n we define the stopping time

$$\tau_R^{(n)} = \inf\{t \in [0, T] : \|X_t^{(n)}\|_H > R\} \wedge T, \quad R > 0.$$

Here we take $\inf \emptyset = \infty$. It's obvious that

$$\lim_{R \rightarrow \infty} \tau_R^{(n)} = T, \quad \mathbb{P} - a.s., \quad n \in \mathbb{N}.$$

Then by the Burkholder-Davis-Gundy inequality we have

$$\begin{aligned}
(2.4) \quad & \mathbb{E} \sup_{r \in [0, t]} \left| \int_0^r \|X_s^{(n)}\|_H^{p-2} \langle X_s^{(n)}, P_n B(s, X_s^{(n)}) dW_s^{(n)} \rangle_H \right| \\
& \leq 3\mathbb{E} \left(\int_0^t \|X_s^{(n)}\|_H^{2p-2} \|B(s, X_s^{(n)})\|_2^2 ds \right)^{1/2} \\
& \leq 3\mathbb{E} \left(\sup_{s \in [0, t]} \|X_s^{(n)}\|_H^{2p-2} \cdot C \int_0^t (\|X_s^{(n)}\|_H^2 + f_s) ds \right)^{1/2} \\
& \leq 3\mathbb{E} \left[\varepsilon \sup_{s \in [0, t]} \|X_s^{(n)}\|_H^p + C_\varepsilon \left(\int_0^t (\|X_s^{(n)}\|_H^2 + f_s) ds \right)^{p/2} \right] \\
& \leq 3\varepsilon \mathbb{E} \sup_{s \in [0, t]} \|X_s^{(n)}\|_H^p + 3 \cdot (2T)^{p/2-1} C_\varepsilon \mathbb{E} \int_0^t (\|X_s^{(n)}\|_H^p + f_s^{p/2}) ds, \quad t \in [0, \tau_R^{(n)}],
\end{aligned}$$

where $\varepsilon > 0$ is a small constant and C_ε comes from Young's inequality.

Then by (2.3), (2.4) and Gronwall's lemma we have

$$\mathbb{E} \sup_{t \in [0, \tau_R^{(n)}]} \|X_t^{(n)}\|_H^p + \mathbb{E} \int_0^{\tau_R^{(n)}} \|X_s^{(n)}\|_H^{p-2} \|X_s^{(n)}\|_V^\alpha ds \leq C \left(\mathbb{E} \|X_0\|_H^p + \mathbb{E} \int_0^T f_s^{p/2} ds \right), \quad n \geq 1,$$

where C is a constant independent of n .

For $R \rightarrow \infty$, (2.2) follows from the monotone convergence theorem.

Moreover, by (H4) and $p \geq \beta + 2$ we have

$$\|A(\cdot, X^{(n)})\|_{K^*} \leq C, \quad n \geq 1,$$

where C is a constant independent of n . □

Proof of Theorem 1.1. (1) Existence: By Lemmas 2.1 and 2.2 there exists a subsequence $n_k \rightarrow \infty$ such that

- (i) $X^{(n_k)} \rightarrow \bar{X}$ weakly in K and weakly star in $L^p(\Omega; L^\infty([0, T]; H))$.
- (ii) $Y^{(n_k)} := A(\cdot, X^{(n_k)}) \rightarrow Y$ weakly in K^* .
- (iii) $Z^{(n_k)} := P_{n_k} B(\cdot, X^{(n_k)}) \rightarrow Z$ weakly in J and hence

$$\int_0^\cdot P_{n_k} B(s, X_s^{(n_k)}) dW_s^{(n_k)} \rightarrow \int_0^\cdot Z_s dW_s$$

weakly in $L^\infty([0, T], dt; L^2(\Omega, \mathbb{P}; H))$.

Now we define

$$(2.5) \quad X_t := X_0 + \int_0^t Y_s ds + \int_0^t Z_s dW_s, \quad t \in [0, T],$$

it is easy to show that $X = \bar{X}$ $dt \otimes \mathbb{P}$ -a.e.

Then by [26, Theorem 4.2.5] we know that X is an H -valued continuous (\mathcal{F}_t) -adapted process and

$$\mathbb{E} \left(\sup_{t \in [0, T]} \|X_t\|_H^p + \int_0^T \|X_t\|_V^\alpha dt \right) < \infty.$$

Therefore, it remains to verify that

$$A(\cdot, \bar{X}) = Y, \quad B(\cdot, \bar{X}) = Z \quad dt \otimes \mathbb{P} - a.e.$$

Define

$$\mathcal{M} = \left\{ \phi : \phi \text{ is } V\text{-valued } (\mathcal{F}_t)\text{-adapted process such that } \mathbb{E} \int_0^T \rho(\phi_s) ds < \infty \right\}.$$

For $\phi \in K \cap \mathcal{M} \cap L^p(\Omega; L^\infty([0, T]; H))$,

$$\begin{aligned}
(2.6) \quad & \mathbb{E} \left(e^{-\int_0^t (K + \rho(\phi_s)) ds} \|X_t^{(n_k)}\|_H^2 \right) - \mathbb{E} \left(\|X_0^{(n_k)}\|_H^2 \right) \\
& = \mathbb{E} \left[\int_0^t e^{-\int_0^s (K + \rho(\phi_r)) dr} \left(2_{V^*} \langle A(s, X_s^{(n_k)}), X_s^{(n_k)} \rangle_V + \|P_{n_k} B(s, X_s^{(n_k)}) \tilde{P}_{n_k}\|_2^2 \right. \right. \\
& \quad \left. \left. - (K + \rho(\phi_s)) \|X_s^{(n_k)}\|_H^2 \right) ds \right] \\
& \leq \mathbb{E} \left[\int_0^t e^{-\int_0^s (K + \rho(\phi_r)) dr} \left(2_{V^*} \langle A(s, X_s^{(n_k)}), X_s^{(n_k)} \rangle_V + \|B(s, X_s^{(n_k)})\|_2^2 \right. \right. \\
& \quad \left. \left. - (K + \rho(\phi_s)) \|X_s^{(n_k)}\|_H^2 \right) ds \right] \\
& = \mathbb{E} \left[\int_0^t e^{-\int_0^s (K + \rho(\phi_r)) dr} \left(2_{V^*} \langle A(s, X_s^{(n_k)}) - A(s, \phi_s), X_s^{(n_k)} - \phi_s \rangle_V \right. \right. \\
& \quad \left. \left. + \|B(s, X_s^{(n_k)}) - B(s, \phi_s)\|_2^2 - (K + \rho(\phi_s)) \|X_s^{(n_k)} - \phi_s\|_H^2 \right) ds \right] \\
& \quad + \mathbb{E} \left[\int_0^t e^{-\int_0^s (K + \rho(\phi_r)) dr} \left(2_{V^*} \langle A(s, X_s^{(n_k)}) - A(s, \phi_s), \phi_s \rangle_V + 2_{V^*} \langle A(s, \phi_s), X_s^{(n_k)} \rangle_V \right. \right. \\
& \quad \left. \left. - \|B(s, \phi_s)\|_2^2 + 2 \langle B(s, X_s^{(n_k)}), B(s, \phi_s) \rangle_{L_2(U, H)} \right. \right. \\
& \quad \left. \left. - 2(K + \rho(\phi_s)) \langle X_s^{(n_k)}, \phi_s \rangle_H + (K + \rho(\phi_s)) \|\phi_s\|_H^2 \right) ds \right].
\end{aligned}$$

Let $k \rightarrow \infty$, by (H2) and the lower semicontinuity (cf. e.g. [26, (4.2.27)] for details) we have for every nonnegative $\psi \in L^\infty([0, T]; dt)$,

$$\begin{aligned}
(2.7) \quad & \mathbb{E} \left[\int_0^T \psi_t \left(e^{-\int_0^t (K + \rho(\phi_s)) ds} \|X_t\|_H^2 - \|X_0\|_H^2 \right) dt \right] \\
& \leq \liminf_{k \rightarrow \infty} \mathbb{E} \left[\int_0^T \psi_t \left(e^{-\int_0^t (K + \rho(\phi_s)) ds} \|X_t^{(n_k)}\|_H^2 - \|X_0^{(n_k)}\|_H^2 \right) dt \right] \\
& \leq \mathbb{E} \left[\int_0^T \psi_t \left(\int_0^t e^{-\int_0^s (K + \rho(\phi_r)) dr} \left(2_{V^*} \langle Y_s - A(s, \phi_s), \phi_s \rangle_V \right. \right. \right. \\
& \quad \left. \left. + 2_{V^*} \langle A(s, \phi_s), \bar{X}_s \rangle_V - \|B(s, \phi_s)\|_2^2 + 2 \langle Z_s, B(s, \phi_s) \rangle_{L_2(U, H)} \right. \right. \\
& \quad \left. \left. - 2(K + \rho(\phi_s)) \langle X_s, \phi_s \rangle_H + (K + \rho(\phi_s)) \|\phi_s\|_H^2 \right) ds \right) dt \right].
\end{aligned}$$

By Itô's formula we have for $\phi \in K \cap \mathcal{M} \cap L^p(\Omega; L^\infty([0, T]; H))$,

$$\begin{aligned}
(2.8) \quad & \mathbb{E} \left(e^{-\int_0^t (K + \rho(\phi_s)) ds} \|X_t\|_H^2 \right) - \mathbb{E} \left(\|X_0\|_H^2 \right) \\
& = \mathbb{E} \left[\int_0^t e^{-\int_0^s (K + \rho(\phi_r)) dr} \left(2_{V^*} \langle Y_s, \bar{X}_s \rangle_V + \|Z_s\|_2^2 - (K + \rho(\phi_s)) \|X_s\|_H^2 \right) ds \right].
\end{aligned}$$

By inserting (2.8) into (2.7) we obtain

$$(2.9) \quad 0 \geq \mathbb{E} \left[\int_0^T \psi_t \left(\int_0^t e^{-\int_0^s (K + \rho(\phi_r)) dr} (2_{V^*} \langle Y_s - A(s, \phi_s), \bar{X}_s - \phi_s \rangle_V + \|B(s, \phi_s) - Z_s\|_2^2 - (K + \rho(\phi_s)) \|X_s - \phi_s\|_H^2) ds \right) dt \right].$$

Note that (1.2), Lemmas 2.1 and 2.2 imply that

$$\bar{X} \in K \cap \mathcal{M} \cap L^p(\Omega; L^\infty([0, T]; H)).$$

By taking $\phi = \bar{X}$ we obtain that $Z = B(\cdot, \bar{X})$. Next, we first take $\phi = \bar{X} - \varepsilon \tilde{\phi} v$ for $\tilde{\phi} \in L^\infty([0, T] \times \Omega; dt \otimes \mathbb{P}; \mathbb{R})$ and $v \in V$, then we divide by ε and let $\varepsilon \rightarrow 0$ to derive that

$$(2.10) \quad 0 \geq \mathbb{E} \left[\int_0^T \psi_t \left(\int_0^t e^{-\int_0^s (K + \rho(\bar{X}_r)) dr} \tilde{\phi}_s v^* \langle Y_s - A(s, \bar{X}_s), v \rangle_V ds \right) dt \right].$$

By the arbitrariness of ψ and $\tilde{\phi}$, we conclude that $Y = A(\cdot, \bar{X})$.

Hence \bar{X} is a solution of (1.1).

(2) Uniqueness: Suppose X_t, Y_t are the solutions of (1.1) with initial conditions X_0, Y_0 respectively, i.e.

$$(2.11) \quad \begin{aligned} X_t &= X_0 + \int_0^t A(s, X_s) ds + \int_0^t B(s, X_s) dW_s, \quad t \in [0, T]; \\ Y_t &= Y_0 + \int_0^t A(s, Y_s) ds + \int_0^t B(s, Y_s) dW_s, \quad t \in [0, T]. \end{aligned}$$

Then by the product rule, Itô's formula and (H2) we have

$$\begin{aligned} & e^{-\int_0^t (K + \rho(Y_s)) ds} \|X_t - Y_t\|_H^2 \\ & \leq \|X_0 - Y_0\|_H^2 + 2 \int_0^t e^{-\int_0^s (K + \rho(Y_r)) dr} \langle X_s - Y_s, B(s, X_s) dW_s - B(s, Y_s) dW_s \rangle_H, \quad t \in [0, T]. \end{aligned}$$

By a standard localization argument we have

$$\mathbb{E} \left[e^{-\int_0^t (K + \rho(Y_s)) ds} \|X_t - Y_t\|_H^2 \right] \leq \mathbb{E} \|X_0 - Y_0\|_H^2, \quad t \in [0, T].$$

If $X_0 = Y_0, \mathbb{P} - a.s.$, then

$$\mathbb{E} \left[e^{-\int_0^t (K + \rho(Y_s)) ds} \|X_t - Y_t\|_H^2 \right] = 0, \quad t \in [0, T].$$

Since (1.2) and Lemmas 2.1, 2.2 imply that

$$\int_0^t (K + \rho(Y_s)) ds < \infty, \quad \mathbb{P} - a.s., \quad t \in [0, T],$$

we have

$$X_t = Y_t, \quad \mathbb{P} - a.s., \quad t \in [0, T].$$

Therefore, the pathwise uniqueness follows from the path continuity of X, Y in H .

(3) Markov property: the proof of Markov property is standard, we refer to [26, Proposition 4.3.5] or [16, Theorem II.2.4]. \square

3 Application to examples

Obviously, the main result can be applied to stochastic evolution equations with monotone coefficients (cf. [26] for the stochastic porous medium equation and stochastic p -Laplace equation) and non-monotone perturbations (*e.g.* some locally Lipschitz perturbation) in the drift. Below we present some examples where the coefficients are only locally monotone, hence the classical result of monotone operators cannot be applied.

In this section we use the notation D_i to denote the spatial derivative $\frac{\partial}{\partial x_i}$, $\Lambda \subseteq \mathbb{R}^d$ is an open bounded domain with smooth boundary. For the standard Sobolev space $W_0^{1,p}(\Lambda)$ ($p \geq 2$) we always use the following (equivalent) Sobolev norm:

$$\|u\|_{1,p} := \left(\int_{\Lambda} |\nabla u(x)|^p dx \right)^{1/p}.$$

For simplicity we only consider examples where the coefficients are time independent, but one can easily adapt those examples to the time dependent case.

Lemma 3.1. *Consider the Gelfand triple*

$$V := W_0^{1,2}(\Lambda) \subseteq H := L^2(\Lambda) \subseteq W^{-1,2}(\Lambda)$$

and the operator

$$A(u) = \Delta u + \sum_{i=1}^d f_i(u) D_i u,$$

where f_i ($i = 1, \dots, d$) are bounded Lipschitz functions on \mathbb{R} .

(1) *If $d < 3$, then there exists a constant K such that*

$$2_{V^*} \langle A(u) - A(v), u - v \rangle_V \leq -\|u - v\|_V^2 + (K + K\|v\|_V^2) \|u - v\|_H^2, \quad u, v \in V.$$

(2) *If $d = 3$, then there exists a constant K such that*

$$2_{V^*} \langle A(u) - A(v), u - v \rangle_V \leq -\|u - v\|_V^2 + (K + K\|v\|_V^4) \|u - v\|_H^2, \quad u, v \in V.$$

(3) *If f_i are independent of u for $i = 1, \dots, d$, i.e.*

$$A(u) = \Delta u + \sum_{i=1}^d f_i \cdot D_i u,$$

then for any $d \geq 1$ we have

$$2_{V^*} \langle A(u) - A(v), u - v \rangle_V \leq -\|u - v\|_V^2 + K\|u - v\|_H^2, \quad u, v \in V.$$

Proof. (1) Since all f_i are bounded and Lipschitz, we have

$$\begin{aligned}
& v^* \langle A(u) - A(v), u - v \rangle_V \\
&= -\|u - v\|_V^2 + \sum_{i=1}^d \int_{\Lambda} (f_i(u)D_i u - f_i(v)D_i v) (u - v) dx \\
&= -\|u - v\|_V^2 + \sum_{i=1}^d \int_{\Lambda} (f_i(u)(D_i u - D_i v) + D_i v(f_i(u) - f_i(v))) (u - v) dx \\
(3.1) \quad &\leq -\|u - v\|_V^2 + \sum_{i=1}^d \left[\left(\int_{\Lambda} (D_i u - D_i v)^2 dx \right)^{1/2} \left(\int_{\Lambda} f_i^2(u) (u - v)^2 dx \right)^{1/2} \right. \\
&\quad \left. + \left(\int_{\Lambda} (D_i v)^2 dx \right)^{1/2} \left(\int_{\Lambda} (f_i(u) - f_i(v))^2 (u - v)^2 dx \right)^{1/2} \right] \\
&\leq -\|u - v\|_V^2 + K\|u - v\|_V \left(\int_{\Lambda} (u - v)^2 dx \right)^{1/2} + K\|v\|_V \left(\int_{\Lambda} (u - v)^4 dx \right)^{1/2} \\
&\leq -\frac{3}{4}\|u - v\|_V^2 + K\|u - v\|_H^2 + K\|v\|_V\|u - v\|_{L^4}^2, \quad u, v \in V,
\end{aligned}$$

where K is a generic constant that may change from line to line.

For $d < 3$, we have the following well-known estimate on \mathbb{R}^2 (see [22, Lemma 2.1])

$$(3.2) \quad \|u\|_{L^4}^4 \leq 2\|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2, \quad u \in W_0^{1,2}(\Lambda).$$

Hence combining with (3.1) we have

$$v^* \langle A(u) - A(v), u - v \rangle_V \leq -\frac{1}{2}\|u - v\|_V^2 + (K + K\|v\|_V^2) \|u - v\|_H^2, \quad u, v \in V.$$

(2) For $d = 3$ we use the following estimate (cf.[22])

$$(3.3) \quad \|u\|_{L^4}^4 \leq 4\|u\|_{L^2} \|\nabla u\|_{L^2}^3, \quad u \in W_0^{1,2}(\Lambda),$$

then the second assertion can be derived similarly from (3.1) and Young's inequality.

(3) This assertion can be easily derived as (3.1). \square

Example 3.2. (*Semilinear stochastic equations*)

Let Λ be an open bounded domain in \mathbb{R}^d with smooth boundary. We consider the following triple

$$V := W_0^{1,2}(\Lambda) \subseteq H := L^2(\Lambda) \subseteq (W_0^{1,2}(\Lambda))^*$$

and the semilinear stochastic equation

$$(3.4) \quad dX_t = \left(\Delta X_t + \sum_{i=1}^d f_i(X_t) D_i X_t + g(X_t) \right) dt + B(X_t) dW_t,$$

where W_t is a Wiener process on $L^2(\Lambda)$ and f_i, g, B satisfy the following conditions:

- (i) f_i are bounded Lipschitz functions on \mathbb{R} for $i = 1, \dots, d$;
- (ii) g is a continuous function on \mathbb{R} such that

$$(3.5) \quad \begin{aligned} |g(x)| &\leq C(|x|^r + 1), \quad x \in \mathbb{R}; \\ (g(x) - g(y))(x - y) &\leq C(1 + |y|^s)(x - y)^2, \quad x, y \in \mathbb{R}. \end{aligned}$$

where C, r, s are some positive constants.

- (iii) $B : V \rightarrow L_2(L^2(\Lambda))$ is Lipschitz.

Then we have the following result:

(1) If $d = 1, r = 3, s = 2$, then for any $X_0 \in L^6(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, (3.4) has a unique solution $\{X_t\}_{t \in [0, T]}$ and this solution satisfies

$$(3.6) \quad \mathbb{E} \left(\sup_{t \in [0, T]} \|X_t\|_H^6 + \int_0^T \|X_t\|_V^2 dt \right) < \infty.$$

(2) If $d = 2, r = \frac{7}{3}, s = 2$, then for any $X_0 \in L^6(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, (3.4) has a unique solution $\{X_t\}_{t \in [0, T]}$ and this solution satisfies (3.6).

(3) If $d = 3, r = \frac{7}{3}, s = \frac{4}{3}$, $f_i, i = 1, \dots, d$ are bounded measurable functions and independent of u , then for any $X_0 \in L^6(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, (3.4) has a unique solution $\{X_t\}_{t \in [0, T]}$ and this solution satisfies (3.6).

Proof. (1) We define the operator

$$A(u) = \Delta u + \sum_{i=1}^d f_i(u) D_i u + g(u), \quad u \in V.$$

The hemicontinuity (H1) follows easily from the continuity of f and g .

Note that (3.5) and (3.2) imply

$$(3.7) \quad \begin{aligned} v^* \langle g(u) - g(v), u - v \rangle_V &\leq C(1 + \|v\|_{L^{2s}}^s) \|u - v\|_{L^4}^2 \\ &\leq \frac{1}{4} \|u - v\|_V^2 + C(1 + \|v\|_{L^{2s}}^{2s}) \|u - v\|_H^2, \quad u, v \in V. \end{aligned}$$

Therefore, by Lemma 3.1 we have for $d < 3$

$$2v^* \langle A(u) - A(v), u - v \rangle_V \leq -\frac{1}{2} \|u - v\|_V^2 + C(1 + \|v\|_V^2 + \|v\|_{L^{2s}}^{2s}) \|u - v\|_H^2, \quad u, v \in V,$$

i.e. (H2), (H3) hold with $\rho(v) = \|v\|_V^2 + \|v\|_{L^{2s}}^{2s}$ and $\alpha = 2$.

For $d = 1, r = 3$, by the Sobolev embedding theorem we have

$$\|g(u)\|_{V^*} \leq C(1 + \|u\|_{L^3}^3) \leq C(1 + \|u\|_V \|u\|_H^2), \quad u \in V.$$

Then it is easy to show that

$$\|A(u)\|_{V^*} \leq C(1 + \|u\|_V + \|u\|_V \|u\|_H^2), \quad u \in V.$$

Hence (H4) holds with $\beta = 4$.

Therefore, all assertions follow from Theorem 1.1 by taking $p = 6$.

(2) For $d = 2, 3$ we have

$$\|g(u)\|_{V^*} \leq C (1 + \|u\|_{L^{6r/5}}^r), \quad u \in V.$$

For $r = \frac{7}{3}$, by the interpolation theorem we have

$$\|u\|_{L^{6r/5}} \leq \|u\|_{L^2}^{4/7} \|u\|_{L^6}^{3/7}, \quad u \in W_0^{1,2}(\Lambda) \subseteq L^6(\Lambda).$$

Then

$$(3.8) \quad \|g(u)\|_{V^*} \leq C (1 + \|u\|_{L^{6r/5}}^r) \leq C \left(1 + \|u\|_H^{4/3} \|u\|_V\right), \quad u \in V.$$

Hence (H4) holds for $d = 2, 3$ with $\beta = 8/3$.

Therefore, for $d = 2$, all assertions follow from Theorem 1.1 by taking $p = 6$ (in fact, $p = 14/3$ is enough).

(3) If $d = 3$ and $f_i, i = 1, 2, 3$ are bounded measurable functions and independent of u , then by Lemma 3.1 and (3.3) we have

$$2_{V^*} \langle A(u) - A(v), u - v \rangle_V \leq -\frac{1}{2} \|u - v\|_V^2 + K (1 + \|v\|_{L^{2s}}^{4s}) \|u - v\|_H^2, \quad u, v \in V.$$

Hence (H2), (H3) hold with $\rho(v) = \|v\|_{L^{2s}}^{4s}$ and $\alpha = 2$.

Since $s = \frac{4}{3}$, by the interpolation inequality we have

$$\|u\|_{L^{2s}} \leq \|u\|_{L^2}^{5/8} \|u\|_{L^6}^{3/8}, \quad u \in V.$$

Therefore,

$$\|u\|_{L^{2s}}^{4s} \leq C \|u\|_H^{10/3} \|u\|_V^2, \quad u \in V,$$

i.e. (1.2) holds with $\beta = 10/3$.

Hence combining with (3.8) we can take $p = 6$ (in fact, $p \geq 16/3$ is enough).

Then all assertions follow from Theorem 1.1. □

Remark 3.1. (1) For some specific examples, one might derive the local monotonicity without assuming the boundedness of $f_i, i = 1, \dots, d$. For instance, Wilhelm Stannat (whom we like to thank for this at this point) pointed out to us that our local monotonicity condition is also fulfilled by the classical stochastic Burgers equation. Since the remaining conditions hold anyway in this case, all our results apply to the classical stochastic Burgers equation as well. More precisely, for the classical stochastic Burgers equation we have

$$d = 1, \quad \Lambda = [0, 1], \quad A(u) = \Delta u + u \frac{\partial u}{\partial x},$$

then we can derive the following local monotonicity:

$$\begin{aligned}
& v^* \langle A(u) - A(v), u - v \rangle_V \\
&= -\|u - v\|_V^2 + \int_{\Lambda} \left(u \frac{\partial u}{\partial x} - v \frac{\partial v}{\partial x} \right) (u - v) dx \\
&= -\|u - v\|_V^2 - \frac{1}{2} \int_{\Lambda} (u - v + 2v) (u - v) \frac{\partial}{\partial x} (u - v) dx \\
(3.9) \quad &= -\|u - v\|_V^2 - \int_{\Lambda} v (u - v) \frac{\partial}{\partial x} (u - v) dx \\
&\leq -\|u - v\|_V^2 + \|v\|_{L^4} \|u - v\|_{L^4} \|u - v\|_V \\
&\leq -\|u - v\|_V^2 + K \|v\|_{L^4} \|u - v\|_H^{1/2} \|u - v\|_V^{3/2} \\
&\leq -\frac{3}{4} \|u - v\|_V^2 + K \|v\|_{L^4}^4 \|u - v\|_H^2, \quad u, v \in V,
\end{aligned}$$

where K is some constant that may change from line to line.

(2) One obvious generalization is that one can replace Δ in (3.4) by the p -Laplace operator $\mathbf{div}(|\nabla u|^{p-2} \nabla u)$ or the more general quasi-linear differential operator

$$\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D_{\alpha} A_{\alpha}(Du),$$

where $Du = (D_{\beta} u)_{|\beta| \leq m}$. Under certain assumptions (cf. [36, Proposition 30.10]) this operator satisfies the monotonicity and coercivity condition. Then, according to Theorem 1.1, we can obtain the existence and uniqueness of solutions to this type of quasi-linear SPDE with non-monotone perturbations (*e.g.* some locally Lipschitz lower order terms).

Now we apply Theorem 1.1 to the stochastic 2-D Navier-Stokes equation.

Let Λ be a bounded domain in \mathbb{R}^2 with smooth boundary. Define

$$V = \left\{ v \in W_0^{1,2}(\Lambda, \mathbb{R}^2) : \nabla \cdot v = 0 \text{ a.e. in } \Lambda \right\}, \quad \|v\|_V := \left(\int_{\Lambda} |\nabla v|^2 dx \right)^{1/2},$$

and H is the closure of V in the following norm

$$\|v\|_H := \left(\int_{\Lambda} |v|^2 dx \right)^{1/2}.$$

The linear operator P_H (Helmholtz-Hodge projection) and A (Stokes operator with viscosity constant ν) are defined by

$$P_H : L^2(\Lambda, \mathbb{R}^2) \rightarrow H \quad \text{orthogonal projection;}$$

$$A : W^{2,2}(\Lambda, \mathbb{R}^2) \cap V \rightarrow H, \quad Au = \nu P_H \Delta u.$$

It is well known then the Navier-Stokes equation can be reformulated as follows

$$(3.10) \quad u' = Au + F(u) + f, \quad u(0) = u_0 \in H,$$

where $f \in L^2(0, T; V^*)$ denotes some external force and

$$F : \mathcal{D}_F \subset H \times V \rightarrow H, \quad F(u, v) = -P_H [(u \cdot \nabla) v], \quad F(u) = F(u, u).$$

It is standard that using the Gelfand triple

$$V \subseteq H \equiv H^* \subseteq V^*,$$

we see that the following mappings

$$A : V \rightarrow V^*, \quad F : V \times V \rightarrow V^*$$

are well defined. In particular, we have

$${}_{V^*}\langle F(u, v), w \rangle_V = -{}_{V^*}\langle F(u, w), v \rangle_V, \quad {}_{V^*}\langle F(u, v), v \rangle_V = 0, \quad u, v, w \in V.$$

Now we consider the stochastic 2-D Navier-Stokes equation

$$(3.11) \quad dX_t = (AX_t + F(X_t) + f_t) dt + B(X_t)dW_t,$$

where W_t is a Wiener process on H .

Example 3.3. (*Stochastic 2-D Navier-Stokes equation*) Suppose that $X_0 \in L^4(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and $B : V \rightarrow L_2(H)$ satisfies

$$\|B(v_1) - B(v_2)\|_2^2 \leq K \left(1 + \|v_2\|_{L^4(\Lambda; \mathbb{R}^2)}^4\right) \|v_1 - v_2\|_H^2, \quad v_1, v_2 \in V,$$

where K is some constant. Then (3.11) has a unique solution $\{X_t\}_{t \in [0, T]}$ and this solution satisfies

$$\mathbb{E} \left(\sup_{t \in [0, T]} \|X_t\|_H^4 + \int_0^T \|X_t\|_V^2 dt \right) < \infty.$$

Proof. The hemicontinuity (H1) is obvious since F is a bilinear map.

Note that ${}_{V^*}\langle F(v), v \rangle_V = 0$, it is also easy to show (H3) with $\alpha = 2$:

$$\begin{aligned} {}_{V^*}\langle Av + F(v) + f_t, v \rangle_V &\leq -\nu \|v\|_V^2 + \|f_t\|_{V^*} \|v\|_V \leq -\frac{\nu}{2} \|v\|_V^2 + C \|f_t\|_{V^*}^2, \quad v \in V, \\ \|B(v)\|_2^2 &\leq 2K \|v\|_H^2 + 2\|B(0)\|_2^2, \quad v \in V. \end{aligned}$$

Recall the following estimates (cf. e.g.[22, Lemmas 2.1, 2.2])

$$(3.12) \quad \begin{aligned} |{}_{V^*}\langle F(w), v \rangle_V| &\leq 2\|w\|_{L^4(\Lambda; \mathbb{R}^2)} \|v\|_V; \\ |{}_{V^*}\langle F(w), v \rangle_V| &\leq 2\|w\|_V^{3/2} \|w\|_H^{1/2} \|v\|_{L^4(\Lambda; \mathbb{R}^2)}, \quad v, w \in V. \end{aligned}$$

Then we have

$$(3.13) \quad \begin{aligned} {}_{V^*}\langle F(u) - F(v), u - v \rangle_V &= -{}_{V^*}\langle F(u, u - v), v \rangle_V + {}_{V^*}\langle F(v, u - v), v \rangle_V \\ &= -{}_{V^*}\langle F(u - v), v \rangle_V \\ &\leq 2\|u - v\|_V^{3/2} \|u - v\|_H^{1/2} \|v\|_{L^4(\Lambda; \mathbb{R}^2)} \\ &\leq \frac{\nu}{2} \|u - v\|_V^2 + \frac{32}{\nu^3} \|v\|_{L^4(\Lambda; \mathbb{R}^2)}^4 \|u - v\|_H^2, \quad u, v \in V. \end{aligned}$$

Hence we have the local monotonicity (H2) with $\rho(v) = \|v\|_{L^4(\Lambda; \mathbb{R}^2)}^4$:

$${}_{V^*}\langle Au + F(u) - Av - F(v), u - v \rangle_V \leq -\frac{\nu}{2}\|u - v\|_V^2 + \frac{32}{\nu^3}\|v\|_{L^4(\Lambda; \mathbb{R}^2)}^4\|u - v\|_H^2.$$

(3.12) and (3.2) imply that (H4) and (1.2) hold with $\beta = 2$.

Therefore, the existence and uniqueness of solutions to (3.11) follow from Theorem 1.1 by taking $p = 4$. \square

Remark 3.2. (1) If the noise in (3.11) is additive type, then the existence and uniqueness of solutions to (3.11) have been established in [22]. Here we can conclude the same result for (3.11) with general multiplicative noise by a direct application of our main result.

(2) For the 3-D Navier-Stokes equation, we recall the following well-known estimate (cf. e.g. [22, (2.5)])

$$\|\psi\|_{L^4}^4 \leq 4\|\psi\|_{L^2}\|\nabla\psi\|_{L^2}^3, \quad \psi \in W_0^{1,2}(\Lambda; \mathbb{R}^3).$$

Then one can show that

$$\begin{aligned} {}_{V^*}\langle F(u) - F(v), u - v \rangle_V &= -{}_{V^*}\langle F(u - v), v \rangle_V \\ &\leq 2\|u - v\|_V^{7/4}\|u - v\|_H^{1/4}\|v\|_{L^4(\Lambda; \mathbb{R}^3)} \\ &\leq \frac{\nu}{2}\|u - v\|_V^2 + \frac{2^{12}}{\nu^7}\|v\|_{L^4(\Lambda; \mathbb{R}^3)}^8\|u - v\|_H^2, \quad u, v \in V. \end{aligned}$$

Hence we have the following local monotonicity (H2):

$${}_{V^*}\langle Au + F(u) - Av - F(v), u - v \rangle_V \leq -\frac{\nu}{2}\|u - v\|_V^2 + \frac{2^{12}}{\nu^7}\|v\|_{L^4(\Lambda; \mathbb{R}^3)}^8\|u - v\|_H^2.$$

Another form of local monotonicity can be derived similarly:

$$\begin{aligned} {}_{V^*}\langle F(u) - F(v), u - v \rangle_V &= -{}_{V^*}\langle F(u - v), v \rangle_V \\ &\leq 2\|u - v\|_V^{3/2}\|u - v\|_H^{1/2}\|v\|_{L^6(\Lambda; \mathbb{R}^3)} \\ &\leq \frac{\nu}{2}\|u - v\|_V^2 + \frac{32}{\nu^3}\|v\|_{L^6(\Lambda; \mathbb{R}^3)}^4\|u - v\|_H^2, \quad u, v \in V. \end{aligned}$$

(3) Concerning the growth condition, we have in the 3-D case that

$$\|F(u)\|_{V^*} \leq 2\|u\|_{L^4(\Lambda; \mathbb{R}^3)}^2 \leq 4\|u\|_H^{1/2}\|u\|_V^{3/2}, \quad u \in V.$$

Unfortunately, this is not enough to verify (H4) in Theorem 1.1.

(4) One should note that the only role of (H4) is to assure that $\|A(\cdot, X^{(n)})\|_{K^*}$ is uniformly bounded for all n (see Lemma 2.2). Therefore, one can replace (H4) by some weaker growth condition once we can derive some stronger a priori estimate for $X^{(n)}$ in the Galerkin approximation (e.g. as in [30]). One good example of a further generalization of our main result is that we can apply Theorem 1.1 (with a revised version of (H4)) to derive the existence

and uniqueness of solutions to the following stochastic tamed 3-D Navier-Stokes equation with smooth enough initial condition:

$$dX_t = (AX_t + F(X_t) + f_t - P_H(g_N(|X_t|^2)X_t)) dt + B(X_t)dW_t,$$

where the taming function $g_N : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is smooth and satisfies for some $N > 0$

$$\begin{cases} g_N(r) = 0, & \text{if } r \leq N, \\ g_N(r) = (r - N)/\nu, & \text{if } r \geq N + 1, \\ 0 \leq g'_N(r) \leq C, & r \geq 0. \end{cases}$$

We refer to [29, 30] for more details on the stochastic tamed 3-D Navier-Stokes equation.

4 Appendix: The classical monotone and coercivity conditions

For the existence and uniqueness of the solution to (1.1) we recall the following classical monotone and coercivity conditions on A and B .

Suppose there exist constants $\alpha > 1$, $\theta > 0$, K and a positive adapted process $f \in L^1([0, T] \times \Omega; dt \times \mathbb{P})$ such that the following conditions hold for all $v, v_1, v_2 \in V$ and $(t, \omega) \in [0, T] \times \Omega$.

(A1) (Hemicontinuity) The map $s \mapsto {}_{V^*V^*}\langle A(t, v_1 + sv_2), v \rangle_V$ is continuous on \mathbb{R} .

(A2) (Monotonicity)

$$2{}_{V^*}\langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle_V + \|B(t, v_1) - B(t, v_2)\|_2^2 \leq K\|v_1 - v_2\|_H^2.$$

(A3) (Coercivity)

$$2{}_{V^*}\langle A(t, v), v \rangle_V + \|B(t, v)\|_2^2 + \theta\|v\|_V^\alpha \leq f_t + K\|v\|_H^2.$$

(A4) (Growth)

$$\|A(t, v)\|_{V^*} \leq f_t^{(\alpha-1)/\alpha} + K\|v\|_V^{\alpha-1}.$$

Theorem 4.1. ([16] Theorems II.2.1, II.2.2) *Suppose (A1) – (A4) hold, then for any $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ (1.1) has a unique solution $\{X_t\}_{t \in [0, T]}$ and this solution satisfies*

$$\mathbb{E} \sup_{t \in [0, T]} \|X_t\|_H^2 < \infty.$$

Moreover, we have the following Itô formula

$$\begin{aligned} \|X_t\|_H^2 &= \|X_0\|_H^2 + \int_0^t (2{}_{V^*}\langle A(s, X_s), X_s \rangle_V + \|B(s, X_s)\|_2^2) ds \\ &\quad + 2 \int_0^t \langle X_s, B(s, X_s) dW_s \rangle_H, \quad t \in [0, T], \quad \mathbb{P} - a.s. \end{aligned}$$

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