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DIMENSION-INDEPENDENT HARNACK INEQUALITIES FOR SUBORDINATED SEMIGROUPS

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Abstract. Dimension-independent Harnack inequalities are derived for a class of subordinate semigroups. In particular, for a diffusion satisfying the Bakry-Emery curvature condition, the subordinate semigroup with power α satisfies a dimension-free Harnack inequality provided $\alpha \in \left(\frac{1}{2}, 1\right)$, and it satisfies the log-Harnack inequality for all $\alpha \in (0,1)$. Some infinite-dimensional examples are also presented.

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1. Introduction

By using the gradient estimate for diffusion semigroups, the following dimension-free Harnack inequality was established in [19] for the diffusion semigroup P_t generated by $L = \Delta + Z$ on a complete Riemannian manifold M with curvature $\mathrm{Ric} - \nabla Z$ bounded below by $-K \in \mathbb{R}$

(1.1)
$$(P_t f(x))^p \leqslant \exp\left(\frac{pK\rho(x,y)^2}{2(p-1)(\mathrm{e}^{2Kt}-1)}\right) P_t f^p(y), \quad t > 0, x, y \in M, f \in \mathcal{B}_b^+(M),$$

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where $p > 1$, ρ is the Riemannian distance, and $\mathcal{B}_b^+(M)$ is the class of all bounded positive measurable functions on M. This inequality has been extended and applied in the study of contractivity properties, heat kernel bounds, strong Feller properties and cost-entropy properties for finite- and infinite-dimensional diffusions. In particular, using the coupling method and Girsanov transformations developed in [4], this inequality has been derived for diffusions without using curvature conditions, see e.g. [5, 6, 9, 13–15, 17, 20] and references therein. See also [1–3] for applications to the short time behavior of transition probabilities. On the other hand, however, due to absence of a chain rule for the "gradient estimate" argument and an explicit Girsanov theorem, this technique of proving dimension independent Harnack inequalities is not applicable to pure jump processes. The main purpose of this paper is to establish such inequalities for a class of α-stable like jump processes by using subordination.

Let (E, ρ) be a Polish space with the Borel σ-algebra $\mathcal{B}(E)$, and P_t the semigroup for a time-homogenous Markov process on E. Let $\{\mu_t\}_{t\geq 0}$ be a convolution semigroup of probability measures on $[0, \infty)$, i.e. one has $\mu_{t+s} = \mu_t * \mu_s$ for $s, t \geq 0$ and $\mu_t \to \mu_0 := \delta_0$ weakly as $t \to 0$. Thus, the Laplace transform for μ_t has the form

$$\int_0^\infty \mathrm{e}^{-xs}\, \mu_t(\mathrm{d}s) = \mathrm{e}^{-tB(x)}, \text{ for any } x \geqslant 0, t \geqslant 0 \tag{1.2}$$

for some Bernstein function B, see e.g. [12]. We shall study the Harnack inequality for the subordinated semigroup

$$P_t^B := \int_0^\infty P_s \mu_t(\mathrm{d}s), \quad t \geqslant 0. \tag{1.3}$$

Obviously, if P_t is generated by a negatively definite self-adjoint operator $(L, \mathcal{D}(L))$ on $L^2(\nu)$ for some σ-finite measure ν on E, then P_t^B is generated by $-B(-L)$. In particular, if $B(x) = x^\alpha$ for $\alpha \in (0, 1]$, we shall denote the corresponding μ_t by μ_t^α, and P_t^B by P_t^α respectively.

We shall use (1.3) and a known dimension independent Harnack inequality for P_t to establish the corresponding Harnack inequality for P_t^B. For instance, suppose we know that

$$(P_t f(x))^p \leqslant \exp\left(\Phi(p, t, x, y)\right) P_t f^p(y), x, y \in E, t > 0, p > 1, f \in \mathcal{B}_b^+(E)$$

for some $\Phi : (1, \infty) \times (0, \infty) \times E^2 \to [0, \infty)$. Then (1.3) implies

$$\begin{aligned}
(P_t^B f(x))^p &= \left(\int_0^\infty P_s f(x) \mu_t(\mathrm{d}s)\right)^p \\
&\leqslant \left(\int_0^\infty (P_s f^p(y))^{1/p} \exp\left(\frac{\Phi(p, s, x, y)}{p}\right) \mu_t(\mathrm{d}s)\right)^p \\
&\leqslant (P_t^B f^p(y)) \left(\int_0^\infty \exp\left(\frac{\Phi(p, s, x, y)}{p-1}\right) \mu_t(\mathrm{d}s)\right)^{p-1}.
\end{aligned} \tag{1.4}$$

In general, $\Phi(p,s,x,y) \to \infty$ as $s \to 0$, so we have to verify that $\exp[\Phi(p,s,x,y)/(p-1)]$ is integrable w.r.t. $\mu_t(\mathrm{d}s)$. Similarly to (1.1), for many specific models the singularity of $\Phi(p,s,x,y)$ at $s=0$ behaves like $\mathrm{e}^{\delta/s^\kappa}$ for some $\delta = \delta\,(p,x,y) > 0, \kappa \geqslant 1$ (see Section 3 below for specific examples). In this case, the following results say that the Harnack inequality provided by (1.4) is valid for P_t^α with $\alpha > \kappa/(\kappa+1)$.

Theorem 1.1. *Let $p > 1, \kappa > 0$ and $\alpha \in \left(\frac{\kappa}{\kappa+1}, 1\right)$ be fixed. Suppose that P_t satisfies the Harnack inequality*

$$(1.5)\quad (P_t f(x))^p \leqslant \exp\left(H(x,y)(\varepsilon + t^{-\kappa})\right) P_t f^p(y), \quad x,y \in E, f \in \mathcal{B}_b^+(E), t > 0,$$

for some positive measurable function H on $E \times E$ and a constant $\varepsilon \geqslant 0$. Then there exists a constant $c > 0$ depending on α and κ such that

$$\begin{aligned}
&(P_t^\alpha f(x))^p \\
&\leqslant \mathrm{e}^{\varepsilon H(x,y)} \left(1 + \left[\exp\left(\left(\frac{cH(x,y)}{(p-1)t^{\kappa/\alpha}}\right)^{1/(1-(\alpha^{-1}-1)\kappa)}\right) - 1\right]^{(1-(\alpha^{-1}-1)\kappa)}\right)^{p-1} P_t^\alpha f^p(y) \\
&\leqslant 2^{p-1} \exp\left(\varepsilon H(x,y) + C_{p,\kappa,\alpha}\left(\frac{H(x,y)}{t^{\kappa/\alpha}}\right)^{1/(1-(\alpha^{-1}-1)\kappa)}\right) P_t^\alpha f^p(y), \quad t > 0, x,y \in E
\end{aligned}$$

holds for all $f \in \mathcal{B}_b^+(E)$, where

$$C_{p,\kappa,\alpha} = \frac{(1-(\alpha^{-1}-1)\kappa)c^{1/(1-(\alpha^{-1})\kappa)}}{(p-1)^{(\alpha^{-1}-1)\kappa/(1-(\alpha^{-1}-1)\kappa)}}.$$

Consequently, if P_t has an invariant probability measure μ, we have that

(i) for any $p, q > 1$,

$$\frac{\|P_t^\alpha\|_{p\to q}}{2^{(p-1)/p}} \leqslant \left(\int_E \frac{\mu(\mathrm{d}x)}{\left(\int_E \exp\left[-\varepsilon H(x,y) - C_{p,\kappa,\alpha}\left(\frac{H(x,y)}{t^{\kappa/\alpha}}\right)^{1/(1-(\alpha^{-1}-1)\kappa)}\right]\mu(\mathrm{d}y)\right)^{q/p}}\right)^{1/q};$$

(ii) P_t^α has a transition density $p_t^\alpha(x,y)$ w.r.t. μ such that for any $x \in \operatorname{supp}(\mu)$

$$\begin{aligned}
&\int_E p_t^\alpha(x,y)^2 \mu(\mathrm{d}y) \\
&\leqslant 2\left(\int_E \exp\left(-\varepsilon H(x,y) - C_{p,\kappa,\alpha}\left(\frac{H(x,y)}{t^{\kappa/\alpha}}\right)^{1/(1-(\alpha^{-1}-1)\kappa)}\right)\mu(\mathrm{d}y)\right)^{-1}.
\end{aligned}$$

As an application of Theorem 1.1 (*ii*), we have the following explicit heat kernel upper bounds for stable like processes.

Example 1.2. *Let P_t be generated by $L = \Delta + Z$ on a complete Riemannian manifold such that $\operatorname{Ric} - \nabla Z \geq -K$. By (1.1), (1.5) holds for $H(x,y) = \rho(x,y)^2$ and $\kappa = 1$. So, for $\alpha \in (1/2, 1]$, Theorem 1.1 (ii) implies*

$$p_{2t}^{\alpha}(x,x) \leqslant \frac{c}{\mu(\{y : \rho(x,y) \leqslant t^{1/2\alpha}\})}, \quad x \in M, \ t > 0$$

for some constant $c > 0$. In particular, for $L = \Delta$ on $\mathbb{R}^d$, $\mu(\mathrm{d}x) = \mathrm{d}x$ and $K = 0$, we have

$$\sup_{x,y\in\mathbb{R}^d} p_t^{\alpha}(x,y) = \sup_{x\in\mathbb{R}^d} p_t^{\alpha}(x,x) \leqslant ct^{-d/2\alpha}, t > 0,$$

for some constant $c > 0$. This is sharp due to the well known explicit bounds of heat kernels for the classical stable processes on $\mathbb{R}^d$.

Theorem 1.1 does not apply to $\alpha \in (0, \frac{\kappa}{\kappa+1}]$, since in this case $\int_0^\infty \mathrm{e}^{\delta/s^\kappa}\, \mu_t^{\alpha}(\mathrm{d}s) = \infty$ for large $\delta > 0$. A more careful analysis allows us to treat the case $\alpha = \frac{\kappa}{\kappa+1}$ under certain restrictions on x, y, t. Thus results of this type apply also to the Cauchy process.

Proposition 1.3 (The case $\alpha = \frac{\kappa}{\kappa+1}$)**.** *Suppose that P_t satisfies the Harnack inequality* (1.5) *for some positive measurable function H on $E \times E$ and a constant $\varepsilon \geqslant 0$. Then there exists a constant $C > 0$ depending on κ such that*

$$\begin{aligned}&(P_t^{\frac{\kappa}{\kappa+1}} f(x))^p \\ &\leqslant \mathrm{e}^{\varepsilon H(x,y)} \left(1 + \frac{C}{\frac{e(p-1)}{H(x,y)\kappa}\left(\frac{\kappa t}{\kappa+1}\right)^{\kappa+1} - 1}\right)^{p-1} P_t^{\frac{\kappa}{\kappa+1}} f^p(y), \quad f \in \mathcal{B}_b^+(E)\end{aligned}$$

holds for all $t > 0, x, y \in \mathbb{E}$ such that

$$e(p-1)(t\kappa)^{\kappa+1} > \kappa(\kappa+1)^{\kappa+1} H(x,y).$$

In other cases we can still prove the log-Harnack inequality. For diffusion semigroups, the known log-Harnack inequality looks like

$$P_t \log f(x) \leqslant \log P_t f(y) + H(x,y)(\varepsilon + t^{-\kappa}), x, y \in E, t > 0, f \geq 1, \tag{1.6}$$

for some positive measurable function H on $E \times E$ and some constants $\varepsilon \geq 0, \kappa \geq 1$. In many cases, one has $H(x,y) = c\rho(x,y)^2$ for a constant $c > 0$ and the intrinsic distance ρ induced by the diffusion (see e.g. [18]).

Theorem 1.4. *If* (1.6) *holds, then for any $\alpha \in (0, 1]$,*

$$P_t^{\alpha} \log f(x) \leqslant \log P_t^{\alpha} f(y) + H(x,y)\left(\varepsilon + \log P_t^{\alpha} f(y) + H(x,y)\left(\varepsilon + \frac{\Gamma\left(\frac{\kappa}{\alpha}\right)}{\alpha t^{\frac{\kappa}{\alpha}}\Gamma(\kappa)}\right)\right),$$

$t > 0, x, y \in E, f \geq 1.$

As observed in [6] and [18] , the log-Harnack inequality implies an entropy-cost inequality for the semigroup and an entropy inequality for the corresponding transition density. Let W_H be the Wasserstein distance induced by H, i.e.

$$W_H(\mu_1, \mu_2) = \inf_{\pi \in \mathcal{C}(\mu_1,\mu_2)} \int_{E\times E} H(x,y)\pi(\mathrm{d}x, \mathrm{d}y),$$

where μ_1, μ_2 are probability measures on E and $\mathcal{C}(\mu_1, \mu_2)$ is the set of all couplings for μ_1 and μ_2.

Corollary 1.5. *Assume that (1.6) holds and let P_t have an invariant probability measure μ. Then for any $\alpha \in (0,1]$:*

(1) *The entropy-cost inequality*

$$\mu(((P_t^\alpha)^* f)\log(P_t^\alpha)^* f) \leqslant W_H(f\mu,\mu)\left(\varepsilon + \log P_t^\alpha f(y) + H(x,y)\left(\varepsilon + \frac{\Gamma\left(\frac{\kappa}{\alpha}\right)}{\alpha t^{\frac{\kappa}{\alpha}}\Gamma(\kappa)}\right)\right),$$

$$t>0, f\geqslant 0, \mu(f)=1$$

holds for all $\alpha\in(0,1]$, where $(P_t^\alpha)^*$ is the adjoint of P_t^α in $L^2(E;\mu)$.

(2) *If $H(x,y)\to 0$ as $y\to x$ holds for any $x\in E$, then P_t^α is strong Feller and thus has a transition density $p_t(x,y)$ w.r.t. μ on $\mathrm{supp}\mu$, which satisfies the entropy inequality*

$$\int_E p_t(x,z)\log\frac{p_t(x,z)}{p_t(y,z)}\,\mu(\mathrm{d}z) \leqslant H(x,y)\left(\varepsilon + \frac{\Gamma\left(\frac{\kappa}{\alpha}\right)}{\alpha t^{\frac{\kappa}{\alpha}}\Gamma(\kappa)}\right),\quad t>0, x,y\in \mathrm{supp}\,\mu.$$

2. Proofs

Proof of Theorem 1.1. The consequences of the desired Harnack inequality are straightforward. Indeed, (i) follows by noting that the claimed Harnack inequality implies

$$\begin{aligned}&(P_t^\alpha f(x))^p\int_E \exp\Big[-\varepsilon H(x,y) - C_{p,\kappa,\alpha}\Big(\frac{H(x,y)}{t^{\kappa/\alpha}}\Big)^{1/(1-(\alpha^{-1}-1)\kappa)}\Big]\mu(\mathrm{d}y)\\ &\leqslant \mu(P_t^\alpha f^p) = \mu^\alpha(f^p),\end{aligned}$$

which also implies (ii) by taking $p=2$ and $f(z)=p_t^\alpha(x,z), z\in E$. Indeed, with $f=1_A$ for a μ-null set A, this inequality implies that the associated transition probability $P_t^\alpha(x,\cdot)$ is absolutely continuous w.r.t. μ and hence, has a density $p_t^\alpha(x,\cdot)$ for every $x\in E$. Then the desired upper bound for $\int_E p_t^\alpha(x,y)^2\mu(\mathrm{d}y)$ follows by first applying the above inequality with $p=2$ and $f(z)=p_t^\alpha(x,z)\wedge n$ then letting $n\to\infty$. So, it remains to prove the first assertion.

By (1.5), (1.4) holds for $\Phi(p,s,x,y)=H(x,y)(\varepsilon+s^{-\kappa})$, i.e.

$$(P_t^\alpha f(x))^p \leqslant \mathrm{e}^{\varepsilon H(x,y)}(P_t^\alpha f^p(y))\left(\int_0^\infty \exp\left[\frac{H(x,y)}{(p-1)s^\kappa}\right]\mu_t(\mathrm{d}s)\right)^{p-1}. \tag{2.1}$$

So it suffices to estimate the integral $\int_0^\infty \mathrm{e}^{\delta/s^\kappa}\,\mu_t(\mathrm{d}s)$ for $\delta := \frac{H(x,y)}{(p-1)}>0$.

We use the formula

$$s^{-r} = \frac{1}{\Gamma(r)}\int_0^\infty x^{r-1}\,\mathrm{e}^{-xs}\,\mathrm{d}x,\ \ r>0.$$

to obtain

$$\begin{aligned}&\int_0^\infty \frac{\mu_t^\alpha(ds)}{s^r} = \int_0^\infty \frac{1}{\Gamma(r)}\int_0^\infty x^{r-1}e^{-xs}dx\mu_t(ds) = \\ &\frac{1}{\Gamma(r)}\int_0^\infty x^{r-1}\int_0^\infty e^{-xs}\mu_t(ds)\,dx = \frac{1}{\Gamma(r)}\int_0^\infty x^{r-1}e^{-tB(x)}dx.\end{aligned}$$

In particular, for $B(x)=x^\alpha$ we have

(2.2)
$$\int_0^\infty \frac{\mu_t^\alpha (ds)}{s^r} = \frac{1}{\Gamma(r)} \int_0^\infty x^{r-1} e^{-tx^\alpha} dx = \frac{1}{\alpha \Gamma(r)} \int_0^\infty y^{\frac{r}{\alpha}-1} e^{-ty} dy = \frac{\Gamma\left(\frac{r}{\alpha}\right)}{\alpha \Gamma(r)} t^{-\frac{r}{\alpha}}.$$

We can use the generalization of Stirling's formula giving the asymptotic behavior of the Gamma function for large r

$$\Gamma(r) = \sqrt{2\pi} r^{r-\frac{1}{2}} e^{-r+\eta(r)},$$

where

$$\eta(r) = \sum_{n=0}^\infty \left(r + n + \frac{1}{2}\right) \ln\left(1 + \frac{1}{r+n}\right) - 1 = \frac{\theta}{12r}, 0 < \theta < 1.$$

We apply this estimate to $\Gamma(\kappa n)$, $\Gamma\left(\frac{\kappa n}{\alpha}\right)$ and $n!$. Thus

$$\begin{aligned}
&\int_0^\infty e^{\frac{\delta}{s^\kappa}} \mu_t^\alpha (\mathrm{d}s) = 1 + \sum_{n=1}^\infty \frac{\delta^n}{n!} \frac{\Gamma\left(\frac{\kappa n}{\alpha}\right)}{\alpha \Gamma(\kappa n)} t^{-\frac{\kappa n}{\alpha}} = \\
&1 + \frac{1}{\alpha} \sum_{n=1}^\infty \frac{\delta^n}{n!} (\kappa n)^{\kappa n\left(\frac{1}{\alpha}-1\right)} e^{-\kappa n\left(\frac{1}{\alpha}-1\right)} \alpha^{\frac{1}{2}-\frac{\kappa n}{\alpha}} e^{\frac{\theta_1 \alpha - \theta_2}{12\kappa n}} t^{-\frac{\kappa n}{\alpha}} \leqslant \\
&1 + \frac{1}{\sqrt{\alpha}} \sum_{n=1}^\infty \frac{\delta^n}{n!} (\kappa n)^{\kappa n\left(\frac{1}{\alpha}-1\right)} e^{-\kappa n\left(\frac{1}{\alpha}-1\right)} \alpha^{-\frac{\kappa n}{\alpha}} e^{\frac{\alpha}{12\kappa n}} t^{-\frac{\kappa n}{\alpha}} = \qquad (2.3) \\
&1 + \frac{1}{\sqrt{\alpha}} \sum_{n=1}^\infty \frac{n^{\kappa n\left(\frac{1}{\alpha}-1\right)}}{n!} \left(\delta \left(\frac{\kappa}{e}\right)^{\kappa\left(\frac{1}{\alpha}-1\right)} \alpha^{-\frac{\kappa}{\alpha}} t^{-\frac{\kappa}{\alpha}}\right)^n e^{\frac{\alpha}{12\kappa n}} \leqslant \\
&1 + \frac{1}{\sqrt{2\pi\alpha}} \sum_{n=1}^\infty n^{\kappa n\left(\frac{1}{\alpha}-1\right)-n-\frac{1}{2}} \left(\delta \left(\frac{\kappa}{e}\right)^{\kappa\left(\frac{1}{\alpha}-1\right)} \alpha^{-\frac{\kappa}{\alpha}} t^{-\frac{\kappa}{\alpha}}\right)^n e^{\frac{\alpha}{12\kappa n}}.
\end{aligned}$$

This series converges for $\alpha > \frac{\kappa}{\kappa+1}$, moreover, there is a constant c depending only on κ such that

$$\frac{1}{\sqrt{2\pi\alpha n}} \left(\left(\frac{\kappa}{e}\right)^{\kappa\left(\frac{1}{\alpha}-1\right)} \alpha^{-\frac{\kappa}{\alpha}} t^{-\frac{\kappa}{\alpha}}\right)^n e^{\frac{\alpha}{12\kappa n}} \leqslant c^n.$$

Denote

$$c(\delta, \alpha, \kappa) := 1 + \sum_{n=1}^\infty n^{n\left(\kappa\left(\frac{1}{\alpha}-1\right)-1\right)} \left(c\delta t^{-\frac{\kappa}{\alpha}}\right)^n,$$

then

$$(P_t^\alpha f(x))^p \leqslant \mathrm{e}^{\varepsilon H(x,y)} \left(c\left(\frac{H(x,y)}{p-1}, \alpha, \kappa\right)\right)^{p-1} P_t^\alpha f^p(y).$$

Note that for $a > 0, 1 \geqslant b > 0$ we have the following estimate

$$\sum_{n=1}^{\infty} \frac{a^n}{n^{bn}} = \sum_{n=1}^{\infty} \frac{(2a)^n}{n^{bn}} \frac{1}{2^n} \leqslant \left(\sum_{n=1}^{\infty} \frac{(2a)^{\frac{n}{b}}}{n^n} \frac{1}{2^n} \right)^b \leqslant$$
$$\left(\sum_{n=1}^{\infty} \frac{(2a)^{\frac{n}{b}}}{n!} \frac{1}{2^n} \right)^b = \left(e^{\frac{(2a)^{1/b}}{2}} - 1 \right)^b,$$

where we used Jensen's inequality. Thus for any $\alpha \in \left(\frac{\kappa}{\kappa+1}, 1 \right)$ we use the above estimate with $b := \kappa \left(1 - \frac{1}{\alpha}\right) + 1 \leqslant 1$ to see that

$$c(\delta, \alpha, \kappa) = 1 + \sum_{n=1}^{\infty} n^{n\left(\kappa\left(\frac{1}{\alpha}-1\right)-1\right)} \left(c\delta t^{-\frac{\kappa}{\alpha}}\right)^n \leqslant$$
$$1 + \left(\exp\left(\frac{\left(2c\delta t^{-\frac{\kappa}{\alpha}}\right)^{\frac{1}{\kappa\left(1-\frac{1}{\alpha}\right)+1}}}{2} \right) - 1 \right)^{\kappa\left(1-\frac{1}{\alpha}\right)+1}.$$

Thus we can say that there is $c > 0$ depending on α and κ such that

$$\int_0^{\infty} e^{\frac{H(x,y)}{(p-1)s^{\kappa}}} \mu_t^{\alpha}(\mathrm{d}s) \leqslant 1 + \left(\exp\left(\left(\frac{cH(x,y)}{(p-1)\, t^{\frac{\kappa}{\alpha}}} \right)^{\frac{1}{\kappa\left(1-\frac{1}{\alpha}\right)+1}} \right) - 1 \right)^{\kappa\left(1-\frac{1}{\alpha}\right)+1}$$

Using the inequality

$$1 + (x-1)^a \leqslant 2x^a$$

for any $x \geqslant 1$ and $0 \leqslant a \leqslant 1$ we see that

$$\int_0^{\infty} e^{\frac{\delta}{s^{\kappa}}} \mu_t^{\alpha}(\mathrm{d}s) \leqslant 2 \exp\left(\left(\kappa \left(1 - \frac{1}{\alpha} \right) + 1 \right) \left(\frac{cH(x,y)}{(p-1)\, t^{\frac{\kappa}{\alpha}}} \right)^{\frac{1}{\kappa\left(1-\frac{1}{\alpha}\right)+1}} \right)$$

which completes the proof.

□

Proof of Proposition 1.3. In the case $\alpha = \frac{\kappa}{\kappa+1}$ the series in (2.3) converges for $t > 0$ and $x, y \in E$ such that

$$e(p-1)(t\kappa)^{\kappa+1} > \kappa(\kappa+1)^{\kappa+1} H(x,y). \tag{2.4}$$

Note that for $\delta := \frac{H(x,y)}{p-1}$ the last line of (2.3) reduces to

$$
\begin{aligned}
&1+\sqrt{\frac{\kappa+1}{2\pi\kappa}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left(\frac{\delta\kappa}{e}\left(\frac{\kappa+1}{\kappa t}\right)^{\kappa+1}\right)^{n}e^{\frac{1}{12(\kappa+1)n}}\\
&\leqslant 1+C\sum_{n=1}^{\infty}\left(\frac{\delta\kappa}{e}\left(\frac{\kappa+1}{\kappa t}\right)^{\kappa+1}\right)^{n}\\
&=1+\frac{C}{\frac{e}{\delta\kappa}\left(\frac{\kappa t}{\kappa+1}\right)^{\kappa+1}-1}.
\end{aligned}
$$

This completes the proof. □

Proof of Theorem 1.4. By (2.2) with $r=\kappa$, we have

$$
\int_0^\infty \frac{\mu_t^\alpha(ds)}{s^\kappa}=\frac{\Gamma\left(\frac{\kappa}{\alpha}\right)}{\alpha t^{\frac{\kappa}{\alpha}}\Gamma(\kappa)}.
$$

Using (1.2), (1.6) we obtain

$$
\begin{aligned}
P_t^\alpha \log f(x) &= \int_0^\infty P_s\log f(x)\mu_t^\alpha(\mathrm{d}s)\leqslant \int_0^\infty \big(\log P_s f(y)+H(x,y)(\varepsilon+s^{-\kappa})\big)\mu_t^\alpha(\mathrm{d}s)\\
&=\log P_t^\alpha f(y)+H(x,y)\left(\varepsilon+\frac{\Gamma\left(\frac{\kappa}{\alpha}\right)}{\alpha t^{\frac{\kappa}{\alpha}}\Gamma(\kappa)}\right).
\end{aligned}
$$

This completes the proof. □

Proof of Corollary 1.5. (1) It suffices to prove for $f\in\mathcal{B}_b^+(E)$ such that $\inf f>0$ and $\mu(f)=1$. In this case, there exists a constant $c>0$ such that $cf\geqslant 1$. By Theorem 1.4 for $cP_t^\alpha f$ in place of f, we obtain

$$
P_t^\alpha \log(P_t^\alpha)^*f(x)\leqslant \log P_t^\alpha(P_t^\alpha)^*f(y)+H(x,y)\left(\varepsilon+\frac{\Gamma\left(\frac{\kappa}{\alpha}\right)}{\alpha t^{\frac{\kappa}{\alpha}}\Gamma(\kappa)}\right).
$$

Since μ is invariant for P_t^α and $(P_t^\alpha)^*$, taking the integral for both sides w.r.t. $\pi\in(f\mu,\mu)$ and minimizing in π, we prove the first assertion.

(2) The strong Feller property follows from Theorem 1.4 according to [18, Proposition 2.3], while by [18, Proposition 2.4] the desired entropy inequality for the transition density is equivalent to the log-Harnack inequality for P_t^α provided by Theorem 1.4. □

3. Some infinite-dimensional examples

As explained in Section 1, Theorems 1.1 and 1.4 hold for $\kappa=1$ if P_t is a diffusion semigroup on a Riemannian manifold with the Ricci curvature bounded below. In this section we present some infinite dimensional examples where these theorems can be used.

3.1. **Stochastic porous medium equation.** Let Δ be the Dirichlet Laplace operator on a bounded interval (a,b) and W_t the cylindrical Brownian motion on $L^2((a,b);\mathrm{d}x)$. Since the eigenvalues $\{\lambda_i\}$ of $-\Delta$ satisfies $\sum_{i=1}^{\infty}\lambda_i^{-1}<\infty$, W_t is a continuous process on $\mathbb{H}$, the completion of $L^2((a,b);\mathrm{d}x)$ under the inner product

$$\langle x,y\rangle := \sum_{i=1}^{\infty}\frac{1}{\lambda_i}\langle x,e_i\rangle\langle y,e_i\rangle,$$

where e_i is the unit eigenfunction corresponding to λ_i for each $i\geq 1$. Let $\|\cdot\|$ denote the norm on $\mathbb{H}$, and suppose $r>1$. Then the following stochastic porous medium equation has a unique strong solution on $\mathbb{H}$ for any $X_0\in\mathbb{H}$ (see e.g. [7]):

$$\mathrm{d}X_t = \Delta X_t^r\mathrm{d}t + \mathrm{d}W_t.$$

Let P_t be the corresponding Markov semigroup. According to [20, Remark 1.1 and Theorem 1.2], Theorem 1.1 in [20] holds for $\theta=r-1$ and some constant $\gamma,\delta,\xi>0$. Thus, there exist two constants $c_1,c_2>0$ depending on r such that

$$(P_tf)^p(x)\leqslant (P_tf^p(y))\exp\left[\frac{c_1p\|x-y\|^{4/(1+r)}}{(p-1)(1-\mathrm{e}^{-c_2t})^{(3+r)/(1+r)}}\right],\quad p>1, t>0, x,y\in\mathbb{H}$$

holds for all $f\in\mathcal{B}_b^+(\mathbb{H})$. By [18, Proposition 2.2] for $\rho(x,y)^2=\|x-y\|^{2/(1+r)}$, this implies the log-Harnack inequality

$$P_t\log f(x)\leqslant \log P_tf(x)+\frac{c_1\|x-y\|^{4/(1+r)}}{(1-\mathrm{e}^{-c_2t})^{(3+r)/(1+r)}},\quad x,y\in\mathbb{H}, f\geq 1.$$

Therefore, Theorems 1.1 and 1.4 apply to P_t^{α} for

$$\kappa=\frac{r+r}{1+r}$$

and some constant ε depending on r.

3.2. **Singular stochastic semi-linear equations.** Let $\mathbb{H}$ be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and W_t the cylindrical Brownian motion on $\mathbb{H}$. Consider the stochastic equation

$$\mathrm{d}X_t=(AX_t+F(X_t))\mathrm{d}t+\sigma\mathrm{d}W_t,\quad X_0\in H. \tag{3.1}$$

Let A,F and σ satisfy the following hypotheses:

(H1) $(A,\mathcal{D}(A))$ is the generator of a C_0-semigroup, $T_t=e^{tA}$, $t\geq 0$, on $\mathbb{H}$ and for some $\omega\in\mathbb{R}$

$$\langle Ax,x\rangle\leq\omega\|x\|^2,\quad \forall\, x\in\mathcal{D}(A). \tag{3.2}$$

(H2) σ is a bounded positively definite, self-adjoint operator on $\mathbb{H}$ such that σ^{-1} is bounded and $\int_0^{\infty}\|T_t\sigma\|_{HS}^2dt<\infty$, where $\|\cdot\|_{HS}$ denotes the norm on the space of all Hilbert–Schmidt operators on $\mathbb{H}$.
(H3) $F:\mathcal{D}(F)\subset\mathbb{H}\to\mathbb{H}$ is an m-dissipative map, i.e.,

$$\langle F(x)-F(y),x-y\rangle\leqslant 0,\qquad x,y\in\mathcal{D}(F),\ u\in F(x),\ v\in F(y),$$

("dissipativity") and

$$\text{Range}\,(I-F) := \bigcup_{x\in\mathcal{D}(F)} (x - F(x)) = \mathbb{H}.$$

Furthermore, $F_0(x) \in F(x),\ x \in \mathcal{D}(F)$, is such that

$$\|F_0(x)\| = \min_{y\in F(x)} \|y\|.$$

Here we recall that for F as in (H3) we have that $F(x)$ is closed, non empty and convex.

The corresponding Kolmogorov operator is then given as follows: Let $\mathcal{E}_A(H)$ denote the linear span of all real parts of functions of the form $\varphi = e^{i\langle h,\cdot\rangle}$, $h \in D(A^*)$, where A^* denotes the adjoint operator of A, and define for any $x \in \mathcal{D}(F)$,

$$L_0\varphi(x) = \frac{1}{2}\,\text{Tr}\,(\sigma^2 D^2\varphi(x)) + \langle x, A^* D\varphi(x)\rangle + \langle F_0(x), D\varphi(x)\rangle, \quad \varphi \in \mathcal{E}_A(H).$$

Additionally, we assume:

(H4) There exists a probability measure μ on H (equipped with its Borel σ-algebra $\mathcal{B}(H)$) such that

(i) $\mu(\mathcal{D}(F)) = 1$,
(ii) $\int_H (1 + \|x\|^2)(1 + \|F_0(x)\|)\mu(dx) < \infty$,
(iii) $\int_H L_0\varphi d\mu = 0$ for all $\varphi \in \mathcal{E}_A(H)$.

By [8], the closure of $(L_0, \mathcal{E}_A(\mathbb{H}))$ in $L^1(\mathbb{H};\mu)$ generates a Markov semigroup P_t with μ as an invariant probability measure, which is point-wisely determined on $\mathbb{H}_0 := \text{supp}\mu$. If moreover the following hypotheses holds:

(H5) (i) $(1+\omega-A, \mathcal{D}(A))$ satisfies the weak sector condition: there exists a constant $K > 0$ such that

$$\langle(1+\omega-A)x, y\rangle \leqslant K\langle(1+\omega-A)x, x\rangle^{1/2}\langle(1+\omega-A)y, y\rangle^{1/2}, \quad \forall\, x, y \in \mathcal{D}(A). \tag{3.3}$$

(ii) There exists a sequence of A-invariant finite dimensional subspaces $\mathbb{H}_n \subset \mathcal{D}(A)$ such that $\bigcup_{n=1}^\infty \mathbb{H}_n$ is dense in $\mathbb{H}$.

Then (see [9, Theorem 1.6])

$$(P_t f(x))^p \leqslant P_t f^p(y) \exp\left[\|\sigma^{-1}\|^2\ \frac{p\omega\|x-y\|^2}{(p-1)(1-\mathrm{e}^{-2\omega t})}\right], \quad t > 0,\ x, y \in \mathbb{H}_0.$$

As mentioned above, according to [18, Proposition 2.2] this implies the corresponding log-Harnack inequality. Therefore, our Theorems 1.1 and 1.4 apply to P_t^p for $\kappa = 1$.

3.3. **The Ornstein–Uhlenbeck type semigroups with jumps.** Consider the following stochastic differential equation driven by a Lévy process

$$\mathrm{d}X_t = AX_t\mathrm{d}t + \mathrm{d}Z_t, \quad X_0 = x \in \mathbb{H}, \tag{3.4}$$

where A is the infinitesimal generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$ on $\mathbb{H}$, $Z_t := \{Z_t^u,\ u \in \mathbb{H}\}$ is a cylindrical Lévy process with characteristic triplet

(a, R, M) on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, that is, for every $u \in \mathbb{H}$ and $t \geq 0$

$$\mathbb{E}\exp(\mathrm{i}\langle Z_t, u\rangle) = \exp\Big(\mathrm{i}t\langle a, u\rangle - \frac{t}{2}\langle Ru, u\rangle - \int_{\mathbb{H}}\left[1 - \exp(\mathrm{i}\langle x, u\rangle) + \mathrm{i}\langle x, u\rangle 1_{\{\|x\|\leqslant 1\}}(x)\right], M(\mathrm{d}x)\Big),$$

where $a \in \mathbb{H}$, R is a symmetric linear operator on $\mathbb{H}$ such that

$$R_t := \int_0^t T_s R T_s^* \,\mathrm{d}s$$

is a trace class operator for each $t > 0$, and M is a Lévy measure on $\mathbb{H}$. (For simplicity, we shall write $Z_t^u = \langle Z_t, u\rangle$ for every $u \in \mathbb{H}$.) In this case, (3.4) has a unique mild solution

$$X_t = T_t x + \int_0^t T_{t-s}\mathrm{d}Z_s, t \geqslant 0.$$

Let

$$P_t f(x) = \mathbb{E}f(X_t), \quad x \in \mathbb{H},\ f \in \mathbb{B}_b(\mathbb{H}).$$

If

$$\|R^{-1/2}T_t Rx\| \leqslant \sqrt{h(t)}\,\|R^{1/2}x\|, \quad x \in \mathbb{H},\ t \geq 0$$

holds for some positive function $h \in C([0, \infty))$. Then by [16, Theorem 1.2] (see also [17] for the diffusion case),

$$(P_t f)^\alpha(x) \leqslant \exp\left[\frac{\alpha\|R^{-1/2}(x-y)\|^2}{2(\alpha-1)\int_0^t h(s)^{-1}\mathrm{d}s}\right] P_t f^\alpha(y), \quad t > 0, x - y \in R^{1/2}\mathbb{H}$$

holds for all $f \in \mathcal{B}_b^+(\mathbb{H})$. By this and [18, Proposition 2.2] which implies the corresponding log-Harnack inequality, Theorems 1.1 and 1.4 apply to some $\varepsilon \geq 0$ and $\kappa \geq 1$ if

$$\limsup_{t\to 0} \frac{1}{t^\kappa}\int_0^t \frac{\mathrm{d}s}{h(s)} > 0.$$

3.4. **Infinite-dimensional Heisenberg groups.** In [10] an integrated Harnack inequality similar to (1.1) has been established for a Brownian motion on infinite-dimensional Heisenberg groups modeled on an abstract Wiener space. The inequality is the consequence of the Ricci curvature bounds for both finite-dimensional approximations to these groups and the group itself, and the results established for inductive limits of finite-dimensional Lie groups in [11]. Even though the methods described in that paper are applicable to inductive and projective limits of finite-dimensional Lie groups, the infinite-dimensional Heisenberg groups provide a very concrete setting. We follow the exposition in [10].

Let (W, H, μ) be an abstract Wiener space over $\mathbb{R}(\mathbb{C})$, $\mathbf{C}$ be a real(complex) finite dimensional inner product space, and $\omega : W \times W \to \mathbf{C}$ be a continuous skew symmetric bilinear quadratic form on W. Further, let

$$\|\omega\|_0 := \sup\left\{\|\omega(w_1, w_2)\|_{\mathbf{C}} : w_1, w_2 \in W \text{ with } \|w_1\|_W = \|w_2\|_W = 1\right\} \tag{3.5}$$

be the uniform norm on ω which is finite since ω is assumed to be continuous. We will need the Hilbert-Schmidt norm of ω which is defined as

$$\|\omega\|_2^2 = \|\omega\|_{H^*\otimes H^*\otimes \mathbf{C}} := \sum_{i,j=1}^{\infty} \|\omega(e_i, e_j)\|_{\mathbf{C}}^2,$$

which is finite by Proposition 3.14 in [10].

Definition 3.1. Let $\mathfrak{g}$ denote $W \times \mathbf{C}$ when thought of as a Lie algebra with the Lie bracket operation given by

$$[(A,a),(B,b)] := (0, \omega(A,B)). \tag{3.6}$$

Let $G := G(\omega)$ denote $W \times \mathbf{C}$ when thought of as a group with the multiplication law given by

$$g_1 g_2 = g_1 + g_2 + \frac{1}{2}[g_1, g_2] \text{ for any } g_1, g_2 \in G. \tag{3.7}$$

It is easily verified that $\mathfrak{g}$ is a Lie algebra and G is a group. The identity of G is the zero element, $\mathbf{e} := (0,0)$.

Notation 3.2. *Let $\mathfrak{g}_{CM}$ denote $H \times \mathbf{C}$ when viewed as a Lie subalgebra of $\mathfrak{g}$ and G_{CM} denote $H \times \mathbf{C}$ when viewed as a subgroup of $G = G(\omega)$. We will refer to $\mathfrak{g}_{CM}$ (G_{CM}) as the **Cameron–Martin subalgebra (subgroup)** of $\mathfrak{g}$ (G). (For explicit examples of such $(W, H, \mathbf{C}, \omega)$, see [10].)*

We equip $G = \mathfrak{g} = W \times \mathbf{C}$ with the Banach space norm

$$\|(w,c)\|_{\mathfrak{g}} := \|w\|_W + \|c\|_{\mathbf{C}} \tag{3.8}$$

and $G_{CM} = \mathfrak{g}_{CM} = H \times \mathbf{C}$ with the Hilbert space inner product,

$$\langle (A,a),(B,b)\rangle_{\mathfrak{g}_{CM}} := \langle A, B\rangle_H + \langle a, b\rangle_{\mathbf{C}}. \tag{3.9}$$

The associated Hilbertian norm is given by

$$\|(A,\delta)\|_{\mathfrak{g}_{CM}} := \sqrt{\|A\|_H^2 + \|\delta\|_{\mathbf{C}}^2}. \tag{3.10}$$

As was shown in [10, Lemma 3.3], these Banach space topologies on $W \times \mathbf{C}$ and $H \times \mathbf{C}$ make G and G_{CM} into topological groups.

Then we can define a Brownian motion on G starting at $\mathbf{e} = (0,0) \in G$ to be the process

$$g(t) = \left(B(t), B_0(t) + \frac{1}{2}\int_0^t \omega(B(\tau), dB(\tau)) \right). \tag{3.11}$$

We denote by ν_t the corresponding heat kernel measure on G. The following estimate was used in the proof of Theorem 8.1 in [10]. For any $h \in G_{CM}$, $1 < p < \infty$

$$\int_G |f(xh)|\, d\nu_t(x) \leqslant \|f\|_{L^p(G,\nu_t)} \exp\left(\frac{c(-k(\omega)t)(p-1)}{2t} d_{G_{CM}}^2(e,h) \right). \tag{3.12}$$

where

$$c(t) = \frac{t}{e^t - 1} \quad \text{for all} \quad t \in \mathbb{R}$$

with the convention that $c(0) = 1$ and

$$k(\omega) := \frac{1}{2} \sup_{\|A\|_H = 1} \|\omega(\cdot, A)\|_{H^*\otimes \mathbf{C}}^2 \leqslant \frac{1}{2}\|\omega\|_2^2 < \infty.$$

Equation (3.12) implies the corresponding L^p-estimates of Radon-Nikodym derivatives of ν_t relative to the left and right multiplication by elements in G_{CM}. This in turn is equivalent to the Harnack inequality (1.1) following an argument similar to Lemma D.1 in [11]

$$[(P_t f)(x)]^p \leq C^p (P_t f^p)(y) \text{ for all } f \geqslant 0.$$

Thus we are in position to apply our results to the heat kernel measure ν_t subordinated as described in Section 1.

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