Uniqueness for solutions of Fokker–Planck equations on infinite dimensional spaces

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Abstract

We develop a general technique to prove uniqueness of solutions for Fokker–Planck equations on infinite dimensional spaces. We illustrate this method by implementing it for Fokker–Planck equations in Hilbert spaces with Kolmogorov operators with irregular coefficients and both non-degenerate or degenerate second order part.

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1 Introduction

Fokker–Planck and transport equations with irregular coefficients in finite dimensions have been studied intensively in recent years (see e.g. [1], [2], [4], [5], [11], [12], [21], [22], [23], [24] and the references therein, and also the fundamental paper [20]). More recently

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transport and Fokker–Planck equations have also been studied in infinite dimensions (see, e.g., [3], [10] and [6], [7], [8], [9] respectively)

In this paper we further develop the method from [8], [9] to prove uniqueness of solutions of Fokker–Planck equations in infinite dimensions. Our main aim here is to give an independent general presentation of the single steps and to implement this method under considerably weakened assumptions on the coefficients. Though this method to prove uniqueness is more universal and can be applied to more general Kolmogorov operators, as e.g. those of nonlocal (i.e. pseudo-differential) type, thus allowing jumps for the corresponding stochastic dynamics, we here confine ourselves to the case where, at least on a heuristic level, there is an underlying stochastic differential equation in the background. More precisely our framework is as follows:

Let H be a separable real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and corresponding norm $|\cdot|$. $L(H)$ denotes the set of all bounded linear operators on H with its usual norm $\|\cdot\|$, $\mathcal{B}(H)$ its Borel σ-algebra, $\mathcal{B}_b(H)$ the set of all bounded $\mathcal{B}(H)$-measurable functions from H to $\mathbb{R}$ and $\mathcal{P}(H)$ the set of all probability measures on H, more precisely on $(H,\mathcal{B}(H))$.

Consider the following type of non-autonomous stochastic differential equations on H and time interval $[0,T]$:

$$\begin{cases} dX(t) = (AX(t) + F(t,X(t)))dt + \sqrt{C}dW(t), \\ \\ X(s) = x \in H,\ t \geq s. \end{cases} \tag{1.1}$$

Here $W(t),\ t\geq 0$, is a cylindrical Wiener process on H defined on a stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$, C is a symmetric positive operator in $L(H)$, $D(F)\in\mathcal{B}([0,T]\times H)$, $F\colon D(F)\subset[0,T]\times H\to H$ is a measurable map, and $A\colon D(A)\subset H\to H$ is the infinitesimal generator of a C_0-semigroup $e^{tA},\ t\geq 0$, in H.

Without further regularity assumptions on F it is, of course, not at all clear whether (1.1) has a solution in the strong or even in the weak sense. If, however, there is a weak solution to (1.1), then it is a well known consequence of Itô's formula that its transition probabilities $p_{s,t}(x,dy),\ x\in H,\ s\leq t$, solve the Fokker–Planck equation determined by the associated Kolmogorov operator, see e.g. [19]. But as shown in our earlier papers [6], [8], [9] one can describe very general conditions on F above for which one can solve the Fokker–Planck equation directly for Dirac initial conditions and thus to obtain the transition functions $p_{s,t},\ s\leq t$, corresponding to (1.1) though one might not have a solution to it.

The general motivation to study Fokker–Planck equations instead of Kolmogorov equations, as done in some of our former papers (see e.g. [13], [14], [16], [17], [18] and the references therein) is that the latter are equations for functions, whereas the first are equations for measures for which one has e.g. much better compactness criteria in our infinite dimensional situation. So, there is a good chance to obtain very general existence results. Uniqueness, however, is considerably harder to prove and this is in the centre of considerations in this paper.

Before we write down the Fokker–Planck equation precisely we recall that the Kolmogorov operator L_0 corresponding to (1.1) reads as follows:

$$\begin{aligned} L_0u(t,x) = D_tu(t,x) + \frac{1}{2}\ \mathrm{Tr}\ [CD_x^2u(t,x)] \\ + \langle x, A^*D_xu(t,x)\rangle + \langle F(t,x), D_xu(t,x)\rangle, \quad x\in H,\ t\in[0,T], \end{aligned} \tag{1.2}$$

where D_t denotes the derivative in time and D_x, D_x^2 denote the first and second order Fréchet derivatives in space, i.e. in $x \in H$, respectively. The operator L_0 is defined on the space $D(L_0) := \mathcal{E}_A([0,T] \times H)$, the linear span of all real parts of functions $u_{\phi,h}$ of the form

$$u_{\phi,h}(t,x) = \phi(t)e^{i\langle x,h(t)\rangle}, \quad t \in [0,T],\ x \in H, \tag{1.3}$$

where $\phi \in C^1([0,T]),\ \phi(T) = 0,\ h \in C^1([0,T]; D(A^*))$ and A^* denotes the adjoint of A.

For a fixed initial time $s \in [0,T]$ the Fokker–Planck equation is an equation for measures $\mu(dt, dx)$ on $[s,T] \times H$ of the type

$$\mu(dt, dx) = \mu_t(dx)dt, \tag{1.4}$$

with $\mu_t \in \mathcal{P}(H)$ for all $t \in [s,T]$, and $t \mapsto \mu_t(A)$ measurable on $[s,T]$ for all $A \in \mathcal{B}(H)$, i.e., $\mu_t(dx),\ t \in [s,T]$, is a probability kernel from $([s,T], \mathcal{B}([s,T])$ to $(H, \mathcal{B}(H))$. Then the equation for an initial condition $\zeta \in \mathcal{P}(H)$ reads as follows: $\forall\, u \in D(L_0)$ one has

$$\int_H u(t,y)\mu_t(dy) = \int_H u(s,y)\zeta(dy) + \int_s^t ds' \int_H L_0 u(s',y)\mu_{s'}(dy), \qquad \text{for } dt\text{-a.e. } t \in [s,T], \tag{1.5}$$

where the dt-zero set may depend on u. When writing (1.5) (or (1.8) below) we always implicitly assume that

$$\int_{[0,T]\times H} (|\langle y, A^*h(t)\rangle| + |F(t,y)|)\mu(dt,dy) < \infty \tag{1.6}$$

for all $h \in C^1([0,T]; D(A^*))$ with $|F(t,y)| := +\infty$ if $(t,y) \notin D(F)$, so that all involved integrals exist in the usual sense.

Remark 1.1 (i) Considering $D(L_0)$ as test functions and dualizing it is easy to see that (1.5) turns into the more familiar form of the Fokker–Planck equation

$$\frac{\partial}{\partial t}\mu_t = -L_0^*\mu_t, \quad \mu_s = \zeta. \tag{1.7}$$

(ii) Setting $t = T$ and recalling that $u(T,\cdot) \equiv 0$ for all $u \in D(L_0)$ we see that (under assumption (1.6)) equation (1.5) is obviously equivalent to

$$\int_{[s,T]\times H} L_0 u(s',y)\mu(ds',dy) = -\int_H u(s,y)\zeta(dy), \quad \forall\, u \in D(L_0). \tag{1.8}$$

Solving (1.5) (if this is possible) with $\zeta = \delta_x$ (:=Dirac measure in $x \in H$) for $x \in H$ and $s \in [0,T)$ and expressing the dependence on x, s in the notation, we obtain probability measures $p_{s,t}(x,dy),\ t \in [s,T]$, such that the measure $p_{s,t}(x,dy)dt$ on $[s,T] \times H$ is a solution of (1.5). It was proved in detail in Section 3 of [9] that if we have uniqueness for (1.5) and "sufficient continuity" of the functions $t \mapsto p_{s,t}(x,dy)$, then these measures satisfy the Chapman–Kolmogorov equations, i.e. for $0 \le r < s < t \le T$ and $x \in H$ (or in a properly chosen subset thereof)

$$\int_H p_{s,t}(x',dy)p_{r,s}(x,dx') = p_{r,t}(x,dy), \tag{1.9}$$

where the left hand side is a measure defined for $A \in \mathcal{B}(H)$ as

$$\int_{H\times H} 1\!\!1_A(y)p_{s,t}(x',dy)p_{r,s}(x,dx').$$

In all of this paper we shall concentrate on conditions on the coefficients A, C and F in (1.1) under which we can prove uniqueness, not caring about existence at all, since the last was studied in detail in [6], [8] and[9]. Unlike in the previous work we include both cases with Tr $C = +\infty$ and Tr $C < +\infty$.

The organization of the paper is as follows:

In Section 2 we explain the general argument, namely that "the dense range condition" (cf. (2.1) below) implies uniqueness of solutions to (1.3).

In the subsequent sections we show how to check the "the dense range condition". To this end in Section 3 we recall some known regularity results for the time dependent Ornstein–Uhlenbeck operator on Hilbert spaces from [7] and some of its consequences to be used below. In Section 4 we show that "the dense range condition" holds and hence that (1.3) has at most one solution in the case $C^{-1} \in L(H)$. This can be done just under an L^2-integrability condition on F. Section 5 is devoted to possibly degenerate cases where not necessarily C is invertible, including the deterministic case where $C = 0$. Section 6 contains applications.

Finally, we would like to mention that even if (1.1) has a unique solution, it is not clear at all why the Fokker–Planck equation (1.5) has a unique solution. For instance, there could be solutions $p_{s,t}(x,dy)$ to (1.5) for $\zeta = \delta_x$ for every $x \in H$ satisfying the Chapman–Kolmogorov equation (1.9), for which there exists no process with continuous or cadlag paths so that $p_{s,t}(x,dy),\ x \in H,\ s \le t$, are its transition probabilities.

2 The general argument

Fix $\zeta \in \mathcal{P}(H)$, $s \in [0,T]$ and define $\mathcal{M}_{s,\zeta}$ the set of all finite nonnegative measures $\mu(dt\,dx)$ on $[s,T]\times H$ satisfying (1.4), (1.5) and (1.6).

Then we have the following general result:

Theorem 2.1 *Let $\mathcal{K} \subset \mathcal{M}_{s,\zeta}$ be a convex subset such that the following "dense range condition" is satisfied*

$$L_0(D(L_0)) \text{ is dense in } L^1([0,T]\times H,\mu), \tag{2.1}$$

for all $\mu \in \mathcal{K}$. Then $\mathcal{K}$ contains at most one element.

Proof. Let $\mu^{(i)}(dt\,dx) = \mu_t^{(i)}(dx)dt \in \mathcal{K},\ i = 1,2$. Then

$$\mu_t(dx)dt := \frac{1}{2}\,\mu_t^{(1)}(dx)dt + \frac{1}{2}\,\mu_t^{(2)}(dx)dt \in \mathcal{K} \tag{2.2}$$

and any $\mu(dt\,dx) := \mu_t(dx)dt$-zero set is a $\mu^{(i)}(dt\,dx) = \mu_t^{(i)}(dx)dt$-zero set for both $i = 1,2$. Hence for $i = 1,2$ by the Radon–Nikodym theorem there exist $\mathcal{B}([s,T]\times H)$-measurable functions $\rho_i\colon [s,T]\times H \to [0,\infty)$ such that

$$\mu_t^{(i)}(dx)dt := \rho^{(i)}(t,x)\mu_t(dx)dt \tag{2.3}$$

and it is easy to check from (2.2) that $\rho^{(i)} \le 2$. Furthermore, by Remark 1.1(ii) for all $u \in D(L_0)$

$$\int_s^T \int_H Lu(t,x)\mu_t^{(1)}(dx)dt = \int_s^T \int_H Lu(t,x)\mu_t^{(2)}(dx)dt,$$

hence by (2.3)

$$\int_s^T \int_H Lu(t,x)(\rho^{(1)}(t,x) - \rho^{(2)}(t,x))\mu_t(dx)dt = 0, \quad \forall\, u \in D(L_0).$$

Since μ satisfies (2.1) and $\rho^{(1)} - \rho^{(2)}$ is bounded, it follows that $\rho^{(1)} - \rho^{(2)} = 0$, i.e. $\mu^{(1)} = \mu^{(2)}$. $\square$

As we shall see in Sections 4 and 5 , sets as $\mathcal{K}$ arise very explicitly in the applications and are described by simple and natural integrability conditions.

Remark 2.2 Since we are in a parabolic situation Condition (2.1) holds, if it holds with $\lambda - L_0$ replacing L_0 for some $\lambda \in \mathbb{R}$.

3 Regularity results for time dependent Ornstein–Uhlenbeck operators

We need the following assumption on the coefficients A and C in (1.1), (1.2).

Hypothesis 3.1

(i) *There is $\omega \in \mathbb{R}$ such that $\langle Ax, x\rangle \le \omega |x|^2$, $\forall\, x \in D(A)$.*

(ii) *$C \in L(H)$ is symmetric, nonnegative and such that the linear operator*

$$Q_t := \int_0^t e^{sA} C e^{sA^*} ds$$

is of trace class for all $t > 0$.

(iii) *One has $e^{tA}(H) \subset Q_t^{1/2}(H)$ for all $t > 0$ and there is $\Lambda_t \in L(H)$ such that $Q_t^{1/2}\Lambda_t = e^{tA}$ and*

$$\gamma_\lambda := \int_0^{+\infty} e^{-\lambda t} \|\Lambda_t\| dt < +\infty,$$

where $\|\cdot\|$ denotes the operator norm in $L(H)$.

By R_t we denote the Ornstein–Uhlenbeck semigroup

$$R_t\varphi(x) := \int_H \varphi(e^{tA}x + y) N_{Q_t}(dy), \quad \varphi \in C_{u,2}(H),$$

where

$$Q_t x := \int_0^t e^{sA} C e^{sA^*} x ds, \quad x \in H, \quad t \ge 0,$$

and N_{Q_t} is the Gaussian measure in H with mean 0 and covariance operator Q_t.

We shall consider R_t acting in the Banach space $C_{u,2}(H)$, which consists of all functions $\varphi\colon H \to \mathbb{R}$ such that the function $x \mapsto \frac{\varphi(x)}{1+|x|^2}$ is uniformly continuous and bounded. Let us define the infinitesimal generator U of R_t through its resolvent by setting, following [15], $U := \lambda - \widetilde{G_\lambda}^{-1}$, $D(U) = \widetilde{G_\lambda}(C_{u,2}(H))$, where

$$\widetilde{G_\lambda} f(x) = \int_0^{+\infty} e^{-\lambda t} R_t f(x) dt, \quad x \in H,\ \lambda > 0,\ f \in C_{u,2}(H).$$

It is easy to see that for any $h \in D(A^*)$ the function $\varphi_h(x) = e^{i\langle x,h\rangle}$ belongs to the domain of U in $C_{u,2}(H)$ and we have

$$U\varphi_h = \frac{1}{2}\,\mathrm{Tr}\,[CD^2\varphi_h] + \langle x, A^* D\varphi_h\rangle. \tag{3.1}$$

As a consequence of Hypothesis 3.1 one gets (see [7, Lemma A.1])

Lemma 3.2 *Let Hypothesis 3.1 hold and let $\varphi \in D(U)$. Then there exists $c > 0$ such that*

$$|D_x\varphi(x)| \le c(\|\varphi\|_{C_{u,2}(H)} + \|U\varphi\|_{C_{u,2}(H)})(1 + |x|^2), \quad x \in H.$$

Now let us turn to the time-inhomogeneous case. Let

$$V_0 u(t,x) = D_t u(t,x) + Uu(t,x), \quad u \in \mathcal{E}_A([0,T] \times H).$$

It is clear that $V_0 u \in C([0,T]; C_{u,2}(H))$ (note that $Uu(t,x)$ contains a term growing as $|x|$). Let us introduce an extension of the operator V_0. For $\lambda \in \mathbb{R}$ set

$$G_\lambda f(t,x) = \int_t^T e^{-\lambda(s-t)} R_{t-s} f(s,x) ds, \quad f \in C([0,T]; C_{u,2}(H)).$$

It is easy to see that G_λ satisfies the resolvent identity, so that there exists a unique linear closed operator V in $C([0,T]; C_{u,2}(H))$ such that

$$G_\lambda = (\lambda - V)^{-1}, \quad D(V) = G_\lambda(C([0,T]; C_{u,2}(H))),\ \lambda \in \mathbb{R}. \tag{3.2}$$

It is clear that V is an extension of V_0.

Finally, it is easy to check that the semigroup $\mathcal{R}_\tau$, $\tau \ge 0$, generated by the operator V in the space $C_T([0,T]; C_{u,2}(H)) := \{u \in C_T([0,T]; C_{u,2}(H)) :\ u(T,x) = 0\}$ is given by

$$\mathcal{R}_\tau f(t,x) = \begin{cases} R_\tau f(t+\tau,\cdot)(x) & \text{if } t+\tau \le T \\ 0 \quad \text{otherwise.} \end{cases} \tag{3.3}$$

Arguing as in [25] one can show that $u \in D(V)$ and $Vu = f$ if and only if

$$\begin{cases} \text{(i)} \quad \lim\limits_{h\to 0} \dfrac{1}{h}\,(\mathcal{R}_h u(t,x) - u(t,x)) = f(t,x), \quad \forall\ (t,x) \in [0,T] \times H, \\[2ex] \text{(ii)} \quad \sup\limits_{h\in(0,1],(t,x)\in[0,T]\times H} \dfrac{(1+|x|^2)^{-1}}{h}\,|\mathcal{R}_h u(t,x) - u(t,x)| < +\infty. \end{cases} \tag{3.4}$$

We state now that $\mathcal{E}_A([0,T] \times H)$ is a core for V.

The following results are generalization of those in [18] and were proven in [7, Proposition A.2, Corollary A3].

Proposition 3.3 *Let Hypothesis 3.1 hold and let $u \in D(V)$. Let ν be a finite nonnegative Borel measure on $[0,t] \times H$. Then there exists a sequence $(u_n) \subset \mathcal{E}_A([0,T] \times H)$ such that for some $c_1 > 0$ one has*

$$|u_n(t,x)| + |V_0 u_n(t,x)| \le c_1(1+|x|^2), \quad \forall\ (t,x) \in [0,T] \times H$$

and $u_n \to u$, $V_0 u_n \to V_0 u$ in measure ν.

Corollary 3.4 *Let Hypothesis 3.1 hold. Let $u \in D(V)$ and let ν be a finite nonnegative Borel measure on $[0,T] \times H$. Then there exists a sequence $(u_n) \subset \mathcal{E}_A([0,T] \times H)$ such that for some $c > 0$ one has*

$$|u_n(t,x)| + |D_x u_n(t,x)| + |V_0 u_n(t,x)| \le c(1+|x|^2), \quad \forall\ (t,x) \in [0,T] \times H,$$

and $u_n \to u$, $D_x u_n \to D_x u$, $V_0 u_n \to Vu$ in measure ν.

We need the following

Hypothesis 3.5 *$F\colon [0,T] \times H \to H$ is continuous together with $D_x F(t,\cdot)\colon H \to L(H)$ for all $t \in [0,T]$. Moreover, there exists $K > 0$ such that*

$$|F(t,x) - F(t,y)| \le K|x-y|, \ \ x,y \in H,\ t \in [0,T].$$

Let $C_u(H,H)$ denote the set of all bounded uniformly continuous maps from H to H and $C_u^1(H)$ the set of all functions from H to $\mathbb{R}$ which together with their first derivatives are bounded and uniformly continuous.

Then we have the following result from [7, Lemma 2.5].

Proposition 3.6 *Assume that Hypotheses 3.1 and 3.5 hold. Let $f \in C([0,T]; C_u^1(H))$ and $\lambda \in \mathbb{R}$. Then there exists $u \in D(V)$ such that*

(i) $D_x u \in C([0,T]; C_u(H,H))$,

(ii) $\lambda u - Vu - \langle F, D_x u\rangle = f$,

(iii) $\|u\|_\infty \le \frac{1}{\lambda}\, \|f\|_\infty, \quad$ if $\lambda > 0$.

4 The fully non-degenerate case

In this section we shall consider the case where $C^{-1} \in L(H)$. Fix $\zeta \in \mathcal{P}(H)$, $s \in [0,T]$ and let $\mathcal{M}_{s,\zeta}$ be defined as at the beginning of Section 2. The main result of this section is the following.

Theorem 4.1 *Assume that Hypothesis 3.1 holds and that*

$$C^{-1} \in L(H). \tag{4.1}$$

Define

$$\mathcal{K} := \left\{ \mu \in \mathcal{M}_{s,\zeta}\colon \ \int_s^T \int_H (|x|^4 + |F(t,x)|^2 + |x|^4 |F(t,x)|^2) \mu_t(dx)dt < \infty \right\}.$$

(where again we set $|F(t,x)| = +\infty$ if $(t,x) \in [0,T] \times H \setminus D(F)$). Then $\mathcal{K}$ contains at most one element.

To prove Theorem 4.1, by Theorem 2.1 we need to check that (2.1) holds for all $\mu \in \mathcal{K}$. In fact we shall prove (2.1) for an even larger class of measures $\mathcal{K}^\lambda$ to be introduced below. We need some preparations. First for $\lambda \in [0,\infty)$ we introduce the set of measures $\mathcal{M}_s^\lambda$ defined to be all finite nonnegative measures ν on $\mathcal{B}([0,T] \times H)$ satisfying (1.4) and (1.6) such that

$$\int_s^T \int_H L_0 u(t,x)\nu_t(dx)dt \le 2\lambda \int_s^T \int_H u(t,x)\nu_t(dx)dt, \quad \forall\, u \in D(L_0),\ u \ge 0. \tag{4.2}$$

Remark 4.2 By Remark 1.1(ii) we have that $\mathcal{M}_{s,\zeta} \subset \mathcal{M}_s^\lambda$ for every $\zeta \in \mathcal{P}(H)$ and all $\lambda \ge 0$. Furthermore, we note that the set $\mathcal{K}$ defined in Theorem 4.1 is convex. For a large class of examples where $\mathcal{K}$ is nonempty and thus consists of exactly one element we refer to Section 6.

Lemma 4.3 *Let $\lambda \ge 0$ and $\nu \in \mathcal{M}_s^\lambda$ such that*

$$\int_s^T \int_H |F(t,x)|^2 \nu_t(dx)dt < \infty. \tag{4.3}$$

Then

$$\begin{aligned}\int_s^T \int_H u(t,x)\, L_0 u(t,x)\nu_t(dx)dt &\le \lambda \int_s^T \int_H u(t,x)^2 \nu_t(dx)dt \\ -\frac{1}{2}\int_s^T \int_H |\sqrt{C}\ D_x u(t,x)|^2 \nu_t(dx)dt, &\quad \forall\, u \in D(L_0).\end{aligned} \tag{4.4}$$

In particular, $(L_0, D(L_0))$ is quasi-dissipative, hence closable in $L^2([0,T] \times H; \nu)$.

Proof. For all $u \in D(L_0)$ we have

$$L_0 u^2 = 2u L_0 u + |\sqrt{C}\ D_x u|^2, \tag{4.5}$$

which implies (4.4) by the definition of $\mathcal{M}_s^\lambda$. The proof of the last part of the assertion is standard. $\square$

For $\nu \in \mathcal{M}_s^\lambda$ satisfying (4.3), we denote the closure of $(L_0, D(L_0))$ on $L^2([0,T] \times H; \nu)$ by $(L^\nu, D(L^\nu))$. For $\lambda > 0$ define

$$\mathcal{K}^\lambda := \left\{ \nu \in \mathcal{M}_s^\lambda \colon \int_s^T \int_H (|x|^4 + |F(t,x)|^2 + |x|^4 |F(t,x)|^2) \nu_t(dx)dt < \infty \right\}. \tag{4.6}$$

Clearly, $\mathcal{K} \subset \mathcal{K}^\lambda$. Our aim is to prove that (2.1) holds for all $\nu \in \mathcal{K}^\lambda$.

Remark 4.4 Once (2.1) is proved for all $\nu \in \mathcal{K}^\lambda$, it follows that $(L^\nu, D(L^\nu))$ is m-dissipative on $L^2([0,T] \times H; \nu)$, hence by the Lumer–Phillips Theorem it generates a C_0-semigroup on $L^2([0,T] \times H; \nu)$. However, we shall not use this fact below.

Lemma 4.5 *Assume that Hypothesis 3.1 holds. Let $\lambda > 0$ and $\nu \in \mathcal{K}^\lambda$ and let a map $F_0\colon [0,T] \times H \to H$ satisfy Hypothesis 3.5. Let $f \in C_u^1(H)$ and let u_0 be as in Proposition 3.3, applied with F_0 replacing F, i.e. $u_0 \in D(V)$, $\|u_0\|_\infty \le \frac{1}{\lambda}\|f\|_\infty$ and*

$$\lambda u_0 - V u_0 - \langle F_0, D_x u_0 \rangle = f.$$

Then:

(i) $u_0 \in D(L^\nu)$ *and*

$$\lambda u_0 - L^\nu u_0 = f + \langle F_0 - F, D_x u_0\rangle, \tag{4.7}$$

as elements in $L^2([0,T]\times H;\nu)$.

(ii) *Suppose that* (4.1) *holds. Then*

$$\begin{aligned}&\int_s^T\int_H |D_x u(t,x)|^2\nu_t(dx)dt\\ &\qquad\le \frac{4}{\lambda}\,\|C^{-1}\|\,\|f\|_\infty^2\left((T-s)+\frac{\|C^{-1}\|}{\lambda}\int_s^T\int_H |F_0(t,x)-F(t,x)|^2\nu_t(dx)dt\right).\end{aligned}$$

Proof. By Corollary 3.4 there exists $u_n \in D(L_0)$, $n\in\mathbb{N}$ such that

$$u_n \to u_0, \quad D_x u_n \to D_x u_0, \quad V_0 u_n \to V u_0$$

as $n\to\infty$ in ν-measure and there exists $c\in(0,\infty)$ such that for all $(t,x)\in[0,T]\times H$

$$|u_n(t,x)| + |V_0 u_n(t,x)| + |D_x u_n(t,x)| \le c(1+|x|^2).$$

Hence $L_0 u_n \to V u_0 + \langle F, D_x u_0\rangle$ as $n\to\infty$ in ν-measure and

$$|L_0 u_n| \le c(1+|F(t,x)|)(1+|x|^2).$$

Hence by assumption on ν, Lebesgue's dominated convergence theorem implies that

$$L_0 u_n \to V u_0 + \langle F, D_x u_0\rangle \quad \text{as } n\to\infty \text{ in } L^2([0,T]\times H;\nu).$$

Since $(L^\nu, D(L^\nu))$ is the closure of $(L^0, D(L^0))$, assertion (i) follows.

To prove (ii) we first note that by the above approximation and the assumptions on ν, (4.4) also holds for u_0. Hence multiplying (4.7) by u_0 and integrating with respect to ν, by the assumption on ν this implies

$$\begin{aligned}\frac{1}{2}\int_s^T\int_H |\sqrt{C}\,D_x u_0(t,x)|^2\nu_t(dx)dt &\le \int_s^T\int_H |f(t,x)|\,|u_0(t,x)|\nu_t(dx)dt\\ &+\int_s^T\int_H |F_0(t,x)-F(t,x)|\,|D_x u_0(t,x)|\,|u_0(t,x)|\nu_t(dx)dt.\end{aligned}$$

Since $\|u_0\|_\infty \le \frac{1}{\lambda}\,\|f\|_\infty$, this implies assertion (ii). $\square$

By Remark 4.2 and Theorem 2.1 the following result implies Theorem 4.1.

Proposition 4.6 *Let Hypothesis 3.1 and assumption* (4.1) *hold. Let* $\lambda>0$ *and* $\nu\in\mathcal{K}^\lambda$. *Then*

$$L_0(D(L_0)) \text{ is dense in } L^1([0,T]\times H,\nu), \tag{4.8}$$

Proof. There exist $F_n\colon [0,T]\times H$, $n\in\mathbb{N}$, satisfying Hypothesis 3.5 such that

$$\lim_{n\to\infty}\int_{[s,T]\times H}|F_n-F|^2 d\nu = 0. \tag{4.9}$$

Let $f\in C_u^1(H)$ and u_n as in Proposition 3.3, applied with F_n replacing F, i.e. $u_n\in D(V)$, $\|u_n\|_\infty \le \frac{1}{\lambda}\,\|f\|_\infty$ and

$$\lambda u_n - V u_n - \langle F_n, D_x u_n\rangle = f.$$

Then by Lemma 4.5(i)

$$\lambda u_n - L^\nu u_n = f + \langle F_n - F, D_x u_n\rangle \tag{4.10}$$

and by Lemma 4.5(ii) and (4.9)

$$\sup_{n\in\mathbb{N}} \int_s^T \int_H |D_x u_n(t,x)|^2 \nu_t(dx)dt < \infty. \tag{4.11}$$

(4.9)-(4.11) imply that f is the closure of $(\lambda - L_0)(D(L_0))$ in $L^1([0,T]\times H, \nu)$. Since $C_u^1(H)$ is dense in $L^1([0,T]\times H,\nu)$, (4.8) now follows from Remark 2.2. □

5 Possibly degenerate cases

In case C is not invertible and the noise is allowed to be very degenerate (including the deterministic $C = 0$), more restrictive conditions on F are needed to prove uniqueness of solutions to (1.5). Just for comparison with the results in the non degenerate case of the previous section we here recall the results from [7] which have no conditions on the noise and in particular include the case $C = 0$.

Hypothesis 5.1 *For each $t \in [0,T]$, $F(t,\cdot)$ is the minimal section of an m–dissipative graph*

$$\overline{F}(t,\cdot)\colon\ D(\overline{F}(t,\cdot)) \subset H \to 2^H,\ t\in[0,T],$$

i.e. for all $t\in[0,T]$, $D(\overline{F}(t,\cdot)) \in \mathcal{B}(H)$ and there exists $K > 0$ independent of t such that

$$\langle u - v, x - y\rangle \le K|x-y|^2, \quad \forall\, x,y \in D(\overline{F}(t,\cdot)),\ u\in \overline{F}(t,x),\ v\in\overline{F}(t,y),$$

and for every $\lambda > K$ one has

$$\text{Range}\ (\lambda - \overline{F}(t,\cdot)) := \bigcup_{x\in D(\overline{F}(t,\cdot))} (\lambda x - \overline{F}(t,x)) = H,$$

such that for all $t\in[0,T]$, $D(F(t,\cdot)) = D(\overline{F}(t,\cdot))$ and for all $x \in D(\overline{F}(t,\cdot))$, $F(t,x) \in \overline{F}(t,x)$ and $|F(t,x)| = \min_{y\in\overline{F}(t,x)} |y|$. Furthermore, $0 \in D(F(t,\cdot))$ and $F(t,0) = 0$ for all $t\in[0,T]$.

For $s \in [0,T]$, $\lambda > 0$ let $\mathcal{M}_s^\lambda$ be as defined at the beginning of the previous section. Then the following result is proved in [7, Theorem 3.3].

Theorem 5.2 *Assume that Hypotheses 3.1 and 5.1 hold and let $s\in[0,T]$, $\lambda > 0$, $\nu\in\mathcal{M}_s^\lambda$ such that*

$$\int_s^T \int_H (|x|^2 + |F(t,x)| + |x|^2|F(t,x)|)\nu_t(dx)dt < \infty.$$

Then

$$L_0(D(L_0)) \text{ is dense in } L^1([0,T]\times H,\nu), \tag{5.1}$$

Corollary 5.3 *Assume Hypotheses 3.1 and 5.1. Let $s\in[0,T]$, $\zeta\in\mathcal{P}(H)$ and $M_{s,\zeta}$ be defined as at the beginning of Section 2. Define*

$$\mathcal{K}_1 := \left\{\mu\in\mathcal{M}_{s,\zeta}\colon\ \int_s^T\int_H (|x|^2 + |F(t,x)| + |x|^2|F(t,x)|)\nu_t(dx)dt < \infty\right\} \tag{5.2}$$

Then $\mathcal{K}_1$ contains at most one element.

Proof. Since $\mathcal{K}_1$ is convex the assertion follows immediately from Theorems 5.2 and 2.1. $\square$

Remark 5.4 We note that once one assumes F to satisfy Hypothesis 5.1 one can prove uniqueness in $\mathcal{K}_1$ which is a larger set than $\mathcal{K}$ in Theorem 4.1, because of the weaker integrability condition. For large classes of examples where $\mathcal{K}_1$ is non empty we refer to [6], [8] and [9].

6 Applications

Let $H = L^2(0,1) := L^2((0,1), d\xi)$ (with $|\cdot| := |\cdot|_{L^2(0,1)}$) and let $A\colon D(A) \subset H \to H$ be defined by

$$Ax(\xi) = \partial_\xi^2 x(\xi),\ \xi \in (0,1), \quad D(A) = H^2(0,1) \cap H_0^1(0,1),$$

where $\partial_\xi = \frac{d}{d\xi}$, $\partial_\xi^2 = \frac{d^2}{d\xi^2}$.

We would like to mention here that what is done below generalizes to the case where $(0,1)$ is replaced by an open set $\mathcal{O}$ in $\mathbb{R}^d$, $d \geq 1$. One has only to replace the operator C below by $A^{-\delta}$ with properly chosen $\delta > 0$, depending on the dimension d.

Let $D(F) := [0,T] \times L^{2m}(0,1)$ and for $(t,\xi) \in D(F)$

$$F(t,x)(\xi) := f(\xi, t, x(\xi)) + h(\xi, t, x(\xi)), \quad \xi \in (0,1).$$

Here $f, h\colon (0,1) \times [0,T] \times \mathbb{R} \to \mathbb{R}$ are functions such that for every $\xi \in (0,1)$ the maps $f(\xi,\cdot,\cdot)$, $h(\xi,\cdot,\cdot)$ are continuous on $(0,T) \times \mathbb{R}$ and have the following properties:

(f1) ("polynomial bound"). There exist $m \in \mathbb{N}$ and a nonnegative function $c_1 \in L^2(0,T)$ such that for all $t \in (0,T)$, $z \in \mathbb{R}$, $\xi \in (0,1)$ one has

$$|f(\xi,t,z)| \leq c_1(t)(1 + |z|^m),$$

also assuming without loss of generality that m is odd.

(f2) ("quasi-dissipativity"). There is a nonnegative function $c_2 \in L^1(0,T)$ such that for all $t \in [0,T]$, $z_1, z_2 \in \mathbb{R}$, $\xi \in (0,1)$ one has

$$(f(\xi,t,z_2) - f(\xi,t,z_1))(z_2 - z_1) \leq c_2(t)|z_2 - z_1|^2.$$

(h1) ("linear growth"). There exists a nonnegative function $c_3 \in L^2(0,T)$ such that for all $t \in [0,T]$, $z \in \mathbb{R}$, $\xi \in (0,1)$, one has

$$|h(\xi,t,z)| \leq c_3(t)(1 + |z|).$$

Finally, let $C \in L(H)$ be symmetric, nonnegative and such that $C^{-1} \in L(H)$.

It is worth noting that it is not known whether under these assumptions the stochastic differential equation (1.1) has a solution.

Set

$$V_N(t,x) := \begin{cases} 2(c_1(t) + c_3(t) + 1)(1 + |x|_{L^{2N}(0,1)}^N) & \text{if } (t,x) \in [0,T] \times L^{2N}(0,1), \\ +\infty \quad \text{otherwise.} \end{cases} \tag{6.1}$$

Observe, that by (f1) and (h1) one has

$$|F(t,x)| \leq V_m(t,x) < \infty \quad \forall\, (t,x) \in D(F). \tag{6.2}$$

Let $N \geq m$. It was proved in [9, Section 4] that for every $\zeta \in \mathcal{P}(H)$ such that

$$\int_H |x|^{2N}_{L^{2N}(0,1)} \zeta(dx) < +\infty \tag{6.3}$$

(in particular for any Dirac measure with mass in $L^{2N}(0,1)$) there exists a solution $\mu(dt\,dx) = \mu_t(dx)dt$ to (1.5) satisfying (1.4), (1.6) and in addition having the following properties

$$\sup_{t\in[s,T]} \int_H |x|^2_{L^2(0,1)} \mu_t(dx) < \infty, \tag{6.4}$$

$$t \mapsto \int_H u(t,x)\mu_t(dx) \text{ is continuous } \forall\, u \in D(L_0), \tag{6.5}$$

$$\begin{aligned} &\exists\, C > 0:\quad \int_s^T \int_H (V_N^2(r,x) + |(-A)^\delta x|^2_{L^2(0,1)})\mu_r(dx)dr \\ &\leq C \int_s^T \int_H V_N^2(r,x)\zeta(dx)dr < \infty, \quad \forall\, \delta \in (\tfrac{1}{4}, \tfrac{1}{2}). \end{aligned} \tag{6.6}$$

Since by (6.2) for $N := m+2$ and some constant $C_1 > 0$ we have

$$\begin{aligned} &|x|^4_{L^2(0,1)} + |F(t,x)|^2_{L^2(0,1)} + |x|^4_{L^2(0,1)}\, |F(t,x)|^2_{L^2(0,1)} \\ &\leq |x|^4_{L^{2N}(0,1)} + V_m^2(t,x) + |x|^4_{L^{2N}(0,1)}\, V_m^2(t,x) \\ &\leq C_1 V_N^2(t,x), \end{aligned}$$

it follows that, if

$$\int_H |x|^{2(m+2)}_{L^{2(m+2)}(0,1)} \zeta(dx) < \infty,$$

then the corresponding solution $\mu(dt\,dx) = \mu_t(dx)dt$ to (1.5) is in the set $\mathcal{K}$ defined in Theorem 4.1 which in turn implies that it is the unique solution to (1.5) with A, C, F as above such that

$$\int_s^T \int_H (|x|^4 + |F(t,x)|^2 + |x|^4 |F(t,x)|^2)\mu_t(dx)dt < \infty.$$

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