The global random attractor for a class of stochastic porous media equations

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Abstract

We prove new L^2-estimates and regularity results for generalized porous media equations "shifted by" a function-valued Wiener path. To include Wiender paths with merely first spatial (weak) derivates we introduce the notion of "ζ-monotonicity" for the non-linear function in the equation. As a consequence we prove that stochastic porous media equations have global random attractors. In addition, we show that (in particular for the classical stochastic porous media equation) this attractor consists of a random point.

0 Introduction

In recent years there has been quite an interest in random attractors for stochastic partial differential equations. We refer e.g. to [13],[14],[23],[7],[9],[31],[18],[8], but this list is far from being complete. The study of a new class of stochastic partial differential equations, namely stochastic porous media equations was initiated in [15] and further developed in [16], as well as in a number of subsequent papers (see Sect. 1 below for a more complete list). So far, however, random attractors for stochastic porous media equations have not been investigated.

The purpose of this paper is to analyze or even determine the random attractor (in the sense of [10], [14], [13]) of a stochastic porous medium equation over a bounded open set $\Lambda \subset \mathbb{R}^d$ of type

e1.0 (0.0) $$dX_t = \Delta(\Phi(X_t))dt + QdW_t,\ t \geq s,$$

where $t, s \in \mathbb{R},\ \Phi : \mathbb{R} \to \mathbb{R}$ is continuous, $\Phi(0) = 0$, and Φ satisfies certain coercivity conditions and $(W_t)_{t\geq 0}$ is a function valued Wiener process on a probability space $(\Omega, \mathcal{F}, P)$.

To state our results precisely, we need to recall some of the underlying notions and describe the set-up. This we shall do in Section 1 below. Here we only briefly describe some of the main analytic results we have obtained and which are crucial for the probabilistic part, more precisely, for the proof of the existence of a global (compact) random attractor for (0.0).

As explained in detail in the next section a fundamental property to be established is the cocycle property for the random dynamical system given by the solutions to (0.0) for all $\omega \in \Omega$ (outside a set of $\mathbb{P}$-measure zero), all times $s, t \in \mathbb{R}$ and all initial conditions $x \in H$ (= the Hilbert space carrying the solution-paths to (0.0)).
Therefore, we have to restrict to additive noise and transform equation (0.0) by the usual change of variables

$$Z_t := X_t - QW_t(\omega)$$

to the equation

0.3 (0.1)
$$dZ_t = \Delta\Phi(Z_t + QW_t(\omega))dt, \quad t \geq s,$$

for $\omega \in \Omega$ fixed, i.e. to a deterministic partial differential equation with time dependent nonlinear coefficient and fixed parameter $\omega \in \Omega$. The analysis of this equation is hence purely analytic. Our main results are the regularity Lemma 3.3 and the estimate on the L_2-norm of the solution to (0.1) in Theorem 3.1. These results are crucial for the existence proof of a random attractor for (0.0) and in particular the latter gives an explicit control of the ω-dependence. To get this estimate on the L_2-norm of the solution to (0.1) we introduce the new notion of "ζ-weak monotonicity" (cf. Hypothesis 1.1 below) for the function Φ, which seems to be exactly appropriate for our purposes. We distinguish two cases, namely $QW_t \in H_0^{2,p+1}(\Lambda)$ and the much harder case when $QW_t \in H_0^{1,p+1}(\Lambda)$. For details we refer to Sections 2 and 3 below. We would, however, like to emphasize that these analytic results are of their own interest and bear potential for further applications besides merely the analysis of random attractors.
On the basis of the estimates obtained in Sections 2 and 3 we can then use a meanwhile standard result from [14] to prove the existence of a global (compact) random attractor for (0.0) in Section 4.
In Section 5 under a different (more restrictive) set of assumptions on Φ we prove that the random attractor exists and is just a random point by a different, but very direct technique.
We conclude this paper by some short remarks on computational methods in Section 6.

1 Basic notions and framework

Equation (0.0) has recently been extensively studied within the so-called variational approach to SPDE (cf. e.g. [28, Example 4.1.11],[16],[29],[3],[30],[4],[5],[6],[22],[32], we also refer to Aronson, Vazques and the references there in a background literature for the deterministic case). The underlying Gelfand triple is

eq-pp (1.0)
$$V \subset H \subset V^*,$$

where $V := L^{p+1}(\Lambda)$, $H := H_0^1(\Lambda)^*$, with $H_0^1(\Lambda)$ being the Sobolev space of order one on Λ with Dirichlet boundary conditions. We emphasize that the dualization in (1.0) is with respect to H, i.e. precisely

$$V \subset H \equiv H^*(= H_0^1(\Lambda)) \subset V^*,$$

where the identification of H and H^* is given by the Riesz isomorphism, $\|u\|_{H_0^1}^2 := \int_\Lambda |\nabla u|_{\mathbb{R}^d}^2 d\xi$, $u \in H_0^1(\Lambda)$, and $\|\cdot\|_H$ is its dual norm. Here $|\cdot|_{\mathbb{R}^d}$ denotes Euclidian norm on $\mathbb{R}^d$ and below $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ shall denote the corresponding inner product. By $\|\cdot\|_p$ we will denote the L^p-norm.

Here and below the notion of solution is the usual one (cf. [28, Definition 4.1]). We recall that in particular

(1.0')
$$\mathbb{E} \int_0^T ||X_t||_{p+1}^{p+1} \, dt < \infty \text{ , for all } T > 0.$$

We take Q and the Wiener process W_t of the following special type. $W = (\beta^{(1)}, \dots, \beta^{(m)})$ is a Brownian motion on $\mathbb{R}^m$ defined on the canonical Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, i.e. $\Omega := C(\mathbb{R}_+, \mathbb{R}^m)$, $W_t(\omega) := \omega(t)$, and $(\mathcal{F}_t)$ is the corresponding natural filtration. As usual we can extend W_t (and $\mathcal{F}_t$) for all $t \in \mathbb{R}$ (cf. e.g. [28, p. 99]). $Q : \mathbb{R}^m \to H$ is defined by

$$Qx = \sum_{j=1}^m x_j \varphi_j, \quad x = (x_1, \dots, x_m) \in \mathbb{R}^m,$$

for fixed $\varphi_1, \dots, \varphi_m \in \mathcal{C}_0^1(\Lambda)$ $(\subset L^2(\Lambda) \subset H)$. Here $\mathcal{C}_0^1(\Lambda)$ denotes the set of all continuously differentiable functions with compact support in Λ.

The existence and uniqueness of solutions for (0.0) under monotonicity and coercivity conditions on Φ is well-known even under much more general conditions than which will be used here (see [29], [4]). We will always assume the continuous function $\Phi : \mathbb{R} \to \mathbb{R}$ to satisfy the following conditions:

(A1) Weak monotonicity: For all $t, s \in \mathbb{R}$

$$(\Phi(t) - \Phi(s))(t - s) \geq 0.$$

litions_1_2 (A2) Coercivity: There are $p \in [1, \infty), a \in (0, \infty), c \in [0, \infty)$ such that for all $s \in \mathbb{R}$

$$\Phi(s)s \geq a|s|^{p+1} - c.$$

(A3) Polynomial boundedness: There are $c_1, c_2 \in [0, \infty)$ such that for all $s \in \mathbb{R}$

$$|\Phi(s)| \leq c_1 |s|^p + c_2,$$

where p is as in (A2).

In order to obtain the existence of a random attractor we need slightly more restrictive dissipativity and coercivity conditions on Φ. We will prove existence under two sets of assumptions. In the first case we need to assume stronger regularity of the noise, i.e. $QW_t \in C_0^2(\Lambda)$, while in the second we allow $QW_t \in C_0^1(\Lambda)$, but require stronger assumptions on the non-linearity Φ.

onditions_2 **Hypothesis 1.1.** *Assume $\varphi_j \in C_0^2(\Lambda)$, $1 \le j \le m$, thus $QW_t \in C_0^2(\Lambda)$. Let further $\zeta : \mathbb{R} \to \mathbb{R}$, $\zeta(0) = 0$ be a function such that we have*

(A1)' ζ-Weak monotonicity: For all $t, s \in \mathbb{R}$

$$(\Phi(t) - \Phi(s))(t - s) \ge (\zeta(t) - \zeta(s))^2.$$

(A2)' ζ-Coercivity: For p, a, c as in (A2) and for all $s \in \mathbb{R}$

$$\Phi(s)s \ge \zeta(s)^2 \ge a|s|^{p+1} - c.$$

Remark 1.2. *Note that we do not assume ζ (hence Φ) to be stricly monotone. Furthermore, we note that the first inequality in (A2)' follows from (A1)' since $\Phi(0) = 0 = \zeta(0)$.*

k:phi_diff **Remark 1.3.** *In case of a continuously differentiable nonlinearity Φ, (more precisely, it suffices to assume that $\Phi \in H^{1,1}_{loc}(\mathbb{R})$) it is easy to find a candidate for ζ. Namely, we simply define*

0.1 (1.2)
$$\zeta(s) := \int_0^s \sqrt{\Phi'(r)}dr, \; s \in \mathbb{R}.$$

Then by Hölder's inequality (A1)' holds and hence since $\Phi(0) = 0$, also the first inequality in (A2)' holds. Therefore, to ensure that also (A2)' holds we only need to assume that for some $a \in (0, \infty), c \in [0, \infty), p \in [1, \infty)$

1.3 (1.3)
$$\left(\int_0^s \sqrt{\Phi'(r)}dr\right)^2 \ge a|s|^{p+1} - c \quad \forall s \in \mathbb{R}.$$

Conversely, this produces a lot of examples for Φ satisfying (A1)',(A2)',(A3). Simply, take $\zeta : \mathbb{R} \to \mathbb{R}$ continuously differentiable and non-decreasing with $\zeta(0) = 0$ and such that for some $a \in (0, \infty); c, c_1, c_2 \in [0, \infty), p \in [1, \infty)$,

$$\zeta^2(s) \ge a|s|^{p+1} - c, \quad \zeta'(s) \le c_1|s|^{\frac{p-1}{2}} + c_2 \quad \forall s \in \mathbb{R}.$$

Then define

$$\Phi(s) := \int_0^s (\zeta'(r))^2 dr, \; s \in \mathbb{R}.$$

In particular, $\Phi(s) := s|s|^{p-1}$ arises this way (cf. also Section 5 below). In this case we have $\zeta(s) = \left(\frac{2\sqrt{p}}{p+1}\right) s|s|^{\frac{p-1}{2}}$.

nditions_3 **Hypothesis 1.4.** *Let* $\varphi_j \in C_0^1(\Lambda), 1 \leq j \leq m$, *only. Assume further that* $\Phi \in C^1(\mathbb{R})$, *satisfying* (1.3) *such that*

1.4 (1.4)
$$\Phi'(r) > 0 \text{ for almost all } r \in \mathbb{R},$$

and that for some $\tilde{c}_1 \in [0, \infty)$

1.5 (1.5)
$$\Phi'(s) \leq \tilde{c}_1(|s|^{p-1} + 1) \quad \forall s \in \mathbb{R},$$

where p *is as in* (1.3).

Remark 1.5. *Assume* $\varphi_j \in C_0^2(\Lambda), 1 \leq j \leq m$. *Then Remark* (1.3) *implies that Hypothesis* (1.4) *is stronger than Hypothesis* (1.1), *i.e. it implies that (A1)' and (A2)' hold.*

1.1 **Remark 1.6.** *(i.) There is a set* $\Omega_0 \subset \Omega$ *of full measure such that for each* $p \geq 1$, $\omega \in \Omega_0$ *and* $|t| \to \infty$, $\|QW_t(\omega)\|_p^p$, $\|\nabla(QW_t(\omega))\|_p^p$ *and (if* $QW_t \in C_0^2$*)* $\|\Delta(QW_t(\omega))\|_p^p$ *are asymptotically bounded by polynomials in* t *with* $\mathcal{F}$*-measurable coefficients.*

(ii.) We shall largely follow the strategy of [14], in which similar assumptions on Q*, hence on the noise* QW *are made. The condition that each* φ_i *should be in* $\mathcal{C}_0^1(\Lambda)$ *(*$\mathcal{C}_0^2(\Lambda)$ *resp.) can be easily relaxed to* $QW_t \in H_0^{1,p+1}(\Lambda)$ *(*$QW_t \in H_0^{2,p+1}(\Lambda)$ *resp.) and is imposed here for the sake of simplicity only.*

In the following for $r \in \mathbb{N}, p \geq 1$ let $H_0^{r,p}(\Lambda)$ denote the usual Sobolev space of order r in $L^p(\Lambda)$ with Dirichlet boundary conditions and λ_1 the constant appearing in Poincaré's inequality, i.e. for all $f \in H_0^{1,2}(\Lambda)$

$$\lambda_1 \int_\Lambda f(x)^2 dx \leq \int_\Lambda |\nabla f(x)|^2 dx.$$

For $t \geq s$ and $x \in H$, $X(t,s,x)$ will denote the value at time t of the solution X_t of (0.0) such that $X_s = x$.

We now recall the notions of a random dynamical system and a random attractor. For more details confer [2, 13, 14]. Let $((\Omega, \mathcal{F}, \mathbb{P}), (\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system over a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. $(t, \omega) \mapsto \theta_t(\omega)$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}/\mathcal{F}$–measurable, $\theta_0 =$ id, $\theta_{t+s} = \theta_t \circ \theta_s$ and θ_t is $\mathbb{P}$-preserving, for all $s, t \in \mathbb{R}$.

Definition 1.7. *Let* (H, d) *be a complete separable metric space. A random dynamical system (RDS) over* θ_t *is a measurable map*

$$\varphi : \mathbb{R}_+ \times H \times \Omega \to H$$
$$(t, x, \omega) \mapsto \varphi(t, \omega)x$$

such that $\varphi(0, \omega) = id$ *and* φ *satisfies the cocycle property, i.e.*

$$\varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega),$$

for all $t, s \in \mathbb{R}_+$ *and all* $\omega \in \Omega$. φ *is said to be a continuous RDS if* $\mathbb{P}$*-a.s.* $x \mapsto \varphi(t, \omega)x$ *is continuous for all* $t \in \mathbb{R}_+$.

With the notion of an RDS at our disposal we can now recall the stochastic generalization of notions of absorption, attraction and Ω-limit sets.

rds_basics **Definition 1.8.** *Let (H, d) be as in Definition 1.7*

(i.) A set-valued map $K : \Omega \to 2^H$ is called measurable if for all $x \in H$ the map $\omega \mapsto d(x, K(\omega))$ is measurable, where for nonempty sets $A, B \in 2^H$ we set $d(A, B) = \sup_{x\in A} \inf_{y\in B} d(x, y)$ and $d(x, B) = d(\{x\}, B)$. A measurable set-valued map is also called a random set.

(ii.) Let A, B be random sets. A is said to absorb B if $\mathbb{P}$-a.s. there exists an absorption time $t_B(\omega) \geq 0$ such that for all $t \geq t_B(\omega)$

$$\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subseteq A(\omega).$$

A is said to attract B if

$$d(\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega), A(\omega)) \xrightarrow[t\to\infty]{} 0, \ \mathbb{P}\text{-}a.s. \ .$$

(iii.) For a random set A we define the Ω-limit set to be

$$\Omega_A(\omega) := \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \varphi(t, \theta_{-t}\omega)A(\theta_{-t}\omega)}.$$

Definition 1.9. *A random attrator for an RDS φ is a compact random set A satisfying $\mathbb{P}$-a.s.*

i. A is invariant, i.e. $\varphi(t, \omega)A(\omega) = A(\theta_t\omega)$ for all $t > 0$.

ii. A attracts all deterministic bounded sets $B \subseteq H$.

The following proposition yields a sufficient criterion for the existence of a random attractor of an RDS φ.

_criterion **Proposition 1.10** (cf. [14], Theorem 3.11)**.** *Let φ be an RDS and assume the existence of a compact random set K absorbing every deterministic bounded set $B \subseteq H$. Then there exists a random attractor A, given by*

$$A(\omega) = \overline{\bigcup_{B\subseteq H,\ B \text{ bounded}} \Omega_B(\omega)}.$$

From now on we take $H := H_0^{1,2}(\Lambda)^*$ with metric determined by its norm $\|\cdot\|_H$. Since we aim to apply Proposition 1.10 to prove the existence of a random attractor for (0.0), we first need to define the RDS associated to (0.0). We take $(\Omega, \mathcal{F}, \mathbb{P})$ to be the canonical two-sided Wiener space, i.e. $\Omega = C_0(\mathbb{R}, \mathbb{R}^m)$ and θ_t to be the Wiener shift given by $\theta_t\omega := \omega(t+\cdot) - \omega(t)$.

As in [14, pp. 375–377] we consider $Y(t,s,x) := X(t,s,x) - QW_t$. Then we have for all $s \in \mathbb{R}, x \in H, \mathbb{P}$-a.s.:

$$Y(t,s,x) = x - QW_s + \int_s^t \Delta\Phi(Y(r,s,x) + QW_r)dr, \ \forall t \geq s.$$

We can rewrite this as an ω-wise equation:

eqn:o-wise

$$Z_t(\omega) = x - QW_s(\omega) + \int_s^t A_\omega(r, Z_r(\omega))dr, \ \forall t \geq s, \tag{1.6}$$

where $A_\omega(r,v) := \Delta\Phi(v + QW_r(\omega))$. Since for each fixed $\omega \in \Omega$, $A_\omega : V \to V^*$ is hemicontinuous, monotone, coercive and bounded we can apply [28, Theorem 4.2.4] to obtain the unique existence of a solution

eqn:lp-bdd

$$Z(t,s,x,\omega) \in L^{p+1}_{loc}([s,\infty); V) \cap C([s,\infty), H) \tag{1.7}$$

to (1.6) for all $x \in H$, $\omega \in \Omega$, $s \in \mathbb{R}$ and its continuous dependence on the initial condition x. We now define in analogy to [13]

eqn:def_rds

$$\begin{aligned} S(t,s,\omega)x &:= Z(t,s,x,\omega) + QW_t(\omega), \ s,t \in \mathbb{R}; \ s \leq t \\ \varphi(t,\omega)x &:= S(t,0,\omega)x = Z(t,0,x,\omega) + QW_t(\omega), \ t \geq 0. \end{aligned} \tag{1.8}$$

By uniqueness for (0.0) $S(t,s,\omega)x$ is a version of $X(t,s,x)(\omega)$, for each $x \in H$, $s \in \mathbb{R}$. For fixed s, ω, x we at times abbreviate $S(t,s,\omega)x$ by S_t and $Z(t,s,x,\omega)$ by Z_t. By the pathwise uniqueness of the solution to equation (1.6) we have for all $\omega \in \Omega$, $r,s,t \in \mathbb{R}$, $s \leq r \leq t$,

1.6'

$$S(t,s,\omega) = S(t,r,\omega)S(r,s,\omega) \tag{1.8'}$$

1.6''

$$S(t,s,\omega) = S(t-s,0,\theta_s\omega). \tag{1.8''}$$

Hence φ defines an RDS. We can thus apply Proposition 1.10 to prove the existence of a random attractor for φ. For this we need to prove the existence of a compact set $K(\omega)$, which absorbs every bounded deterministic set in H, $\mathbb{P}$-almost surely. This set will be chosen as $K(\omega) := \overline{B_{L^2}(0,\kappa(\omega))}^H$, where $B_{L^2}(0,\kappa)$ denotes the ball with center 0 and radius κ in L^2. Note that since $\varphi(t,\theta_{-t}\omega) = S(t,0,\theta_{-t}\omega) = S(0,-t,\omega)$, this amounts to proving pathwise bounds on $S(0,-t,\omega)x$ in the L^2-norm, where we use the compactness of the embedding $L^2(\Lambda) \hookrightarrow H$. In order to get such estimates we consider norms $\|\cdot\|_{H_a}$ on H such that for $a \downarrow 0$, $\|\cdot\|_{H_a} \uparrow \|\cdot\|_{L^2}$. These are defined as the dual norms (via the Riesz isomorphism) of the norms

$$H^1_0(\Lambda) \ni u \mapsto \left(a \int_\Lambda |\nabla u|^2 d\xi + \int u^2 d\xi \right)^{1/2}.$$

Then for $s \leq t$ we have (see e.g. [30, Theorem 2.6 and Lemma 2.7 (i),(ii)]) for $a := \frac{1}{n}$

approx-ito

$$\|Z_t\|^2_{H_{\frac{1}{n}}} = \|Z_s\|^2_{H_{\frac{1}{n}}} + 2\int_s^t \langle \Phi(S_r), n(1 - \frac{1}{n}\Delta)^{-1} Z_r - nZ_r \rangle dr, \tag{1.9}$$

where for $f, g : \Lambda \to \mathbb{R}$ measurable we set

$$\langle f, g \rangle := \int_\Lambda f\, g\, d\xi$$

if $|fg| \in L^1(\Lambda)$. We shall use (1.9) in a crucial way several times below.

2 Estimates for $\|S_t\|_H$ and bounded absorption

hm:H-bound **Theorem 2.1.** *Let $\beta \in (0, \infty)$, with $\beta \le \frac{a}{2}$, if $p = 1$. Then there exists a function $p_1^{(\beta)} : \mathbb{R} \times \Omega \to \mathbb{R}_+$ with $\mathcal{F}$–measurable coefficients and for $|t| \to \infty$ of at most polynomial growth in t, such that for all $x \in H$, $\omega \in \Omega_0$ and $s \in \mathbb{R}$:*

qn:H-bound

$$\|Z(t_2, s, x, \omega)\|_H^2 \le \|Z(t_1, s, x, \omega)\|_H^2 - \beta \int_{t_1}^{t_2} \|Z(r, s, x, \omega)\|_2^2\, dr + \int_{t_1}^{t_2} p_1^{(\beta)}(r, \omega) dr, \quad \text{for all } s \le t_1 \le t_2. \tag{2.1}$$

Proof. We fix x, ω, s and set $Z_r := Z(r, s, x, \omega)$, $S_r := S(r, s, \omega)x$ for $r \ge s$. All constants appearing in the proof below are, however, independent of x, ω and s!
Since for $s \le t_1 \le t_2$

$$\|Z_{t_2}\|_H^2 = \|Z_{t_1}\|_H^2 - 2 \int_{t_1}^{t_2} \langle Z_r, \Phi(S_r) \rangle dr,$$

we have for dr–a.e. $r \in [s, \infty)$ by (A2)

$$\begin{aligned}
\frac{d}{dr} \|Z_r\|_H^2 &= -2\langle Z_r, \Phi(S_r)\rangle \\
&= -2\langle S_r - QW_r, \Phi(S_r)\rangle \\
&= -2\langle S_r, \Phi(S_r)\rangle + 2\langle QW_r, \Phi(S_r)\rangle \\
&\le -2a \int_\Lambda |S_r|^{p+1} d\xi + 2 \int_\Lambda \left(|QW_r \Phi(S_r)| + c\right) d\xi.
\end{aligned}$$

By Young's inequality, for arbitrary $\epsilon > 0$ and some $C_\epsilon (= C_\epsilon(p)), C_1, C_2 \in \mathbb{R}$ we have by (A3)

$$\begin{aligned}
\int_\Lambda |QW_r \Phi(S_r)| d\xi &\le \int_\Lambda \left(C_\epsilon |QW_r|^{p+1} + \epsilon |\Phi(S_r)|^{\frac{p+1}{p}} \right) d\xi \\
&\le \epsilon C_1 ||S_r||_{p+1}^{p+1} + C_\epsilon ||QW_r||_{p+1}^{p+1} + \epsilon C_2 |\Lambda|,
\end{aligned}$$

where $|\Lambda| := \int_\Lambda d\xi$. Thus by choosing $\epsilon = \frac{a}{C_1}$ we obtain for dr–a.e. $r \in [t_1, t_2]$

$$\frac{d}{dr} \|Z_r\|_H^2 \le -a ||S_r||_{p+1}^{p+1} + C_\epsilon ||QW_r||_{p+1}^{p+1} + 2|\Lambda|(c + C_3).$$

where $C_3 := \frac{aC_2}{C_1}$.
Now, if $p > 1$, then for each $\beta > 0$ we can find a C_β such that for all $y \in \mathbb{R}$ one has $a|y|^{p+1} \geq 2\beta|y|^2 - C_\beta$. If $p = 1$, then we have the same, provided $\beta \in (0, \frac{a}{2}]$. We obtain

$$a||S_r||_{p+1}^{p+1} \geq 2\beta||S_r||_2^2 - |\Lambda|C_\beta = 2\beta||Z_r + QW_r||_2^2 - |\Lambda|C_\beta \geq \beta||Z_r||_2^2 - 2\beta||QW_r||_2^2 - |\Lambda|C_\beta.$$

Hence for

$$p_1^{(\beta)}(r,\omega) := \begin{cases} 2\beta||QW_r||_2^2 + |\Lambda|C_\beta + C_\epsilon||QW_r||_{p+1}^{p+1} + 2|\Lambda|(c + C_3) & \text{, if } \omega \in \Omega_0 \\ 0 & \text{, else} \end{cases}$$

we obtain for dr-a.e. $r \in [t_1, t_2]$

$$\frac{d}{dr}\|Z_r\|_H^2 \leq -\beta||Z_r||_2^2 + p_1^{(\beta)}(r,\omega).$$

and the assertion follows. □

cor:H-bound **Corollary 2.2.** *Let $\beta \in (0,\infty)$, with $\beta \leq \frac{a}{2}$ if $p = 1$ and let $t \in \mathbb{R}$. Then there exists an $\mathcal{F}$–measurable function $q_1^{(\beta,t)} : \Omega \to \mathbb{R}$, such that for all $x \in H$, $\omega \in \Omega_0$ and $s \leq t$*

star
$$\|Z(t,s,x,\omega)\|_H^2 \leq q_1^{(\beta,t)}(\omega) + e^{-\frac{\beta}{c^2}(t-s)}\|Z(s,s,x,\omega)\|_H^2 . \tag{2.2}$$

Proof. Since the embedding $L^2 \hookrightarrow H$ is continuous, there is a constant $c > 0$ such that $\|v\|_H \leq c\|v\|_2$, for all $v \in L^2$. Hence by Theorem 2.1

$$\frac{d}{dr}(\|Z_r\|_H^2) \leq -\frac{\beta}{c^2}\|Z_r\|_H^2 + p_1^{(\beta)}(r,\omega) \quad dr\text{–a.e. on } [s,t].$$

Hence by Gronwall's Lemma the assertion follows with $q_1^{\beta,t}(\omega) := \int_{-\infty}^t e^{-\frac{\beta}{c^2}(t-r)}p_1(r,\omega)dr$. □

-absorption **Corollary 2.3** (Bounded absorption)**.** *Let $t \in \mathbb{R}$. Then there is an $\mathcal{F}$–measurable function $q_1^{(t)} : \Omega \to \mathbb{R}$ such that for each $\varrho > 0$ there is an $s(\varrho) \leq t$ such that for all $\omega \in \Omega_0$, $x \in H$ with $\|x\|_H \leq \varrho$*

$$Z(t,s,x,\omega) \in \bar{B}_H(0, q_1^{(t)}(\omega)), \quad \text{for all } s \leq s(\varrho)$$

i.e. there exists a bounded random set absorbing (Z_t) at time t.

Proof. Let $\beta := \frac{a}{2}$. By Corollary 2.2, we have for $\tilde{\beta} := \frac{\beta}{c^2}$

$$\begin{aligned} \|Z_t\|_H^2 &\leq e^{-\tilde{\beta}(t-s)}\|Z_s\|_H^2 + q_1^{(\beta,t)} \\ &\leq 2e^{-\tilde{\beta}(t-s)}(\|x\|_H^2 + \|QW_s\|_H^2) + q_1^{(\beta,t)} \\ &\leq 2\varrho^2 e^{-\tilde{\beta}(t-s)} + 2e^{-\tilde{\beta}(t-s)}\|QW_s\|_H^2 + q_1^{(\beta,t)}, \end{aligned}$$

for all $t \geq s$. Hence the result follows with

$$q_1^{(t)} := 1 + q_1^{(\beta,t)} + 2\sup_{s \leq t}(e^{-\tilde{\beta}(t-s)}\|QW_s\|_H^2)$$

and $s(\varrho) \leq t$ chosen so that $2\varrho^2 e^{-\tilde{\beta}(t-s)} \leq 1$ for all $s \leq s(\varrho)$. □

We will need the following auxiliary estimate.

ax_estimate **Corollary 2.4.** *There is an $\mathcal{F}$–measurable function $q:\Omega\to\mathbb{R}_+$ such that for each $\varrho>0$ there exists $s(\varrho)\le -1$ such that for all $\omega\in\Omega_0,\ x\in H$ with $\|x\|_H\le\varrho$*

$$\int_{-1}^{0}\|S(r,s,\omega)x\|_2^2\,dr\le q(\omega)\ \textit{for all}\ s\le s(\varrho).$$

Proof. Using (2.1) in Theorem 2.1 with $t_1=-1,t_2=0$ and then using Corollary 2.3 for $t=-1$ yields for $\beta=\frac{a}{2}$ and $s\le s(\varrho)$, where $s(\varrho)\le -1$ is as in Corollary 2.3,

$$\begin{aligned}\beta\int_{-1}^{0}\|S(r,s,\omega)x\|_2^2\,dr &\le 2\,\|Z(-1,s,x,\omega)\|_H^2+2\int_{-1}^{0}p_1^{(\beta)}(r,\omega)dr+2\beta\int_{-1}^{0}\|QW_r(\omega)\|_2^2\,dr\\ &\le\beta q(\omega),\end{aligned}$$

where $q(\omega):=\frac{2}{\beta}q_1^{(-1)}(\omega)+\frac{2}{\beta}\int_{-1}^{0}p_1^{(\beta)}(r,\omega)dr+2\int_{-1}^{0}\|QW_r(\omega)\|_2^2\,dr.$ □

3 Estimate for $\|S_t\|_2$ and compact absorption

m:L2-bound **Theorem 3.1.** *Suppose that either Hypothesis 1.1 or Hypothesis 1.4 holds. Let $\alpha>0$, with $\alpha\in(0,\frac{\alpha\lambda_1}{2}]$ if $p=1$. Then there is a function $p_2^{(\alpha)}:\mathbb{R}\times\Omega\to\mathbb{R}$ with $\mathcal{F}$-measurable coefficients and for $|t|\to\infty$ of at most polynomial growth in t such that for all $x\in L^2(\Lambda),\omega\in\Omega_0$, and $s\in\mathbb{R}$*

n:L2-bound (3.1)
$$\|Z(t_2,s,x,\omega)\|_2^2\le\|Z(t_1,s,x,\omega)\|_2^2-\alpha\int_{t_1}^{t_2}\|Z(r,s,x,\omega)\|_2^2\,dr+\int_{t_1}^{t_2}p_2^{(\alpha)}(r,\omega)dr$$
$$\textit{for all}\ s\le t_1\le t_2.$$

In particular, $t\to Z_t$ is strongly right continuous in $L^2(\Lambda)$.

Proof. Again we fix x,ω,s and use the abbreviation $Z_r:=Z(r,s,x,\omega),\ S_r:=S(r,s,\omega)x$ for $r\in[s,\infty)$. But all constants appearing in the proof below are independent of x,ω and s.
Case 1: Assume Hypothesis 1.1.

Let $t_1\ge s$ such that $Z_{t_1}\in L^2(\Lambda)$ and $t_2\ge t_1$. (1.9) implies

eqn_L^2_1 (3.2)
$$\begin{aligned}\|Z_{t_2}\|_{H_{\frac1n}}^2=\|Z_{t_1}\|_{H_{\frac1n}}^2&+2\int_{t_1}^{t_2}\langle\Phi(S_r),n(1-\tfrac1n\Delta)^{-1}S_r-nS_r\rangle dr\\ &-2\int_{t_1}^{t_2}\langle\Phi(S_r),\Delta(1-\tfrac1n\Delta)^{-1}QW_r\rangle dr.\end{aligned}$$

A calculation analogous to the calculation following formula (5.6) in [30] yields for dr-a.e. $r \in [s, \infty)$

$$\begin{aligned}
\langle \Phi(S_r), n(1-\tfrac{1}{n}\Delta)^{-1}S_r - nS_r\rangle &= -n\langle \Phi(S_r), S_r - (1-\tfrac{1}{n}\Delta)^{-1}S_r\rangle \\
&= -\frac{n}{2}\int_\Lambda\int_\Lambda [\Phi(S_r(\tilde{\xi})) - \Phi(S_r(\xi))][S_r(\tilde{\xi}) - S_r(\xi)]p_n(\xi, d\tilde{\xi})d\xi \\
&\quad - n\int_\Lambda (1-(1-\frac{1}{n}\Delta)^{-1}1)\Phi(S_r)S_r d\xi \\
&\le -\frac{n}{2}\int_\Lambda\int_\Lambda (\zeta(S_r(\tilde{\xi})) - \zeta(S_r(\xi)))^2 p_n(\xi, d\tilde{\xi})d\xi \\
&\quad - n\int_\Lambda (1-(1-\frac{1}{n}\Delta)^{-1})\zeta(S_r)^2 d\xi \\
&= -n\langle \zeta(S_r), (1-(1-\frac{1}{n}\Delta)^{-1})\zeta(S_r)\rangle \\
&= -\mathcal{E}^{(n)}(\zeta(S_r), \zeta(S_r)),
\end{aligned}$$

where $p_n(\xi, d\tilde{\xi})$ is the kernel corresponding to $(1-\frac{1}{n}\Delta)^{-1}$ (cf. Lemma 5.1 in [30]) and $(\mathcal{E}^{(n)}, \mathcal{D}(\mathcal{E}^{(n)}))$ is the closed coercive form on $L^2(\Lambda)$ with $\mathcal{D}(\mathcal{E}^{(n)}) = H_0^1(\Lambda)$ and generator $n(1-(1-\frac{1}{n}\Delta)^{-1}) = \Delta(1-\frac{1}{n}\Delta)^{-1}$. We obtain:

eqn_L^2_3

$$\begin{aligned}
&\|Z_{t_2}\|^2_{H_{\frac{1}{n}}} + 2\int_{t_1}^{t_2} \mathcal{E}^{(n)}(\zeta(S_r), \zeta(S_r))dr \\
&\le \|Z_{t_1}\|^2_{H_{\frac{1}{n}}} - 2\int_{t_1}^{t_2} \langle \Phi(S_r), \Delta(1-\tfrac{1}{n}\Delta)^{-1}QW_r\rangle dr.
\end{aligned} \tag{3.3}$$

Next we prove an upper bound for the second term on the right hand side of (3.3). Note that we shall make use of the assumption $QW_t \in C_0^2$ here. Using Young's inequality, for all $\epsilon > 0$ and some $C_\epsilon, C_1, C_2 > 0$ we obtain for dr–a.e. $r \in [s, \infty)$

$$\begin{aligned}
|\langle \Phi(S_r), \Delta(1-\frac{1}{n}\Delta)^{-1}QW_r\rangle| &= |\langle \Phi(S_r), (1-\frac{1}{n}\Delta)^{-1}\Delta QW_r\rangle| \\
&\le \epsilon\int_\Lambda |\Phi(S_r)|^{\frac{p+1}{p}}d\xi + C_\epsilon\int_\Lambda |((1-\frac{\Delta}{n})^{-1}\Delta QW_r)|^{p+1}d\xi \\
&\le \epsilon C_1||S_r||^{p+1}_{p+1} + C_\epsilon||\Delta QW_r||^{p+1}_{p+1} + C_2.
\end{aligned}$$

Hence

eqn_L^2_4

$$\begin{aligned}
&\|Z_{t_2}\|^2_{H_{\frac{1}{n}}} + 2\int_{t_1}^{t_2} \mathcal{E}^{(n)}(\zeta(S_r), \zeta(S_r))dr \\
&\le \|Z_{t_1}\|^2_2 - 2\int_{t_1}^{t_2} \left[\epsilon C_1||S_r||^{p+1}_{p+1} + C_\epsilon||\Delta QW_r||^{p+1}_{p+1} + C_2\right] dr < \infty.
\end{aligned} \tag{3.4}$$

We note that by (1.7) the right hand side of (3.4) is indeed finite. Since $\mathcal{E}^{(n)}(\zeta(S_r), \zeta(S_r))$ is increasing in n, we conclude that $\sup_{n\in\mathbb{N}} \mathcal{E}^{(n)}(\zeta(S_r), \zeta(S_r)) < \infty$ for dr-a.e. $r \in [t_1, \infty)$. By (A2)' and (A3) we know that for some $c_1, c_2 \geq 0$

$$\zeta(s)^2 \leq \Phi(s)s \leq c_1|s|^{p+1} + c_2|s|.$$

Since $S_r \in L^{p+1}(\Lambda)$ this implies $\zeta(S_r) \in L^2(\Lambda)$ for dr-a.e. $r \in [t_1, \infty)$. We now recall the following result from the theory of Dirichlet forms: Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the closed coercive form on $L^2(\Lambda)$ given by $\mathcal{E}(f,g) = \int_\Lambda \langle \nabla f, \nabla g\rangle_{\mathbb{R}^d} d\xi$ for $f, g \in \mathcal{D}(\mathcal{E}) = H_0^1(\Lambda)$. From [24, Chap. I, Theorem 2.13] we know for $f \in L^2(\Lambda)$, that $f \in \mathcal{D}(\mathcal{E}) = H_0^1(\Lambda)$ iff $\sup_{n\in\mathbb{N}} \mathcal{E}^{(n)}(f,f) < \infty$ and $\lim_{n\to\infty} \mathcal{E}^{(n)}(f,g) = \mathcal{E}(f,g) = \int_\Lambda \langle \nabla f, \nabla g\rangle_{\mathbb{R}^d} d\xi$ for $f, g \in \mathcal{D}(\mathcal{E})$. Hence we obtain for d$r$-a.e. $r \in [t_1, \infty)$ that $\zeta(S_r) \in \mathcal{D}(\mathcal{E}) = H_0^1(\Lambda)$ and that

$$\lim_{n\to\infty} \mathcal{E}^{(n)}(\zeta(S_r), \zeta(S_r)) = \mathcal{E}(\zeta(S_r), \zeta(S_r)) = \int_\Lambda |\nabla\zeta(S_r)|^2_{\mathbb{R}^d} d\xi.$$

Using Fatou's lemma and taking $n \to \infty$ in (3.4) yields

2.6

$$\begin{aligned}
&\|Z_{t_2}\|_2^2 + 2\int_{t_1}^{t_2}\int_\Lambda |\nabla\zeta(S_r)|^2_{\mathbb{R}^d} d\xi\ dr \\
&\leq \|Z_{t_1}\|_2^2 + 2\epsilon C_1 \int_{t_1}^{t_2} ||S_r||_{p+1}^{p+1} dr + \int_{t_1}^{t_2} (C_\epsilon ||\Delta QW_r||_{p+1}^{p+1} + C_2) dr.
\end{aligned} \tag{3.5}$$

Since $Z_s = x - QW_s \in L^2(\Lambda)$, for all $t_1 \geq s$ we obtain $Z_{t_1} \in L^2(\Lambda)$ and thus (3.5) holds for all $t_2 \geq t_1 \geq s$.

Choosing $\epsilon = \frac{a\lambda_1}{2C_1}$, applying Poincaré's inequality and using the fact that if $p > 1$ for each $\alpha > 0$ we can find $\tilde{C}_\alpha \geq 0$ such that for all $y \in \mathbb{R}$ one has $a\lambda_1|y|^{p+1} \geq 2\alpha|y|^2 - \tilde{C}_\alpha$, and that the same is true for $p = 1$, if $\alpha \in (0, \frac{\alpha\lambda_1}{2}]$. We obtain from (A2)' that

$$\begin{aligned}
\|Z_{t_2}\|_2^2 &\leq \|Z_{t_1}\|_2^2 - 2\lambda_1 \int_{t_1}^{t_2} ||\zeta(S_r)||_2^2 dr + a\lambda_1 \int_{t_1}^{t_2} ||S_r||_{p+1}^{p+1} dr + \int_{t_1}^{t_2} (C_\epsilon ||\Delta QW_r||_{p+1}^{p+1} + C_2) dr \\
&\leq \|Z_{t_1}\|_2^2 - a\lambda_1 \int_{t_1}^{t_2} ||S_r||_{p+1}^{p+1} dr + \int_{t_1}^{t_2} (C_\epsilon ||\Delta QW_r||_{p+1}^{p+1} + C_2 + c) dr \\
&\leq \|Z_{t_1}\|_2^2 - 2\alpha \int_{t_1}^{t_2} ||S_r||_2^2 dr + \int_{t_1}^{t_2} (C_\epsilon ||\Delta QW_r||_{p+1}^{p+1} + C_2 + c + \tilde{C}_\alpha) dr.
\end{aligned}$$

Now

$$\|Z_r\|_2^2 = \|S_r - QW_r\|_2^2 \leq 2\left(\|S_r\|_2^2 + \|QW_r\|_2^2\right),$$

whence

inal_bound

$$\|Z_{t_2}\|_2^2 \leq \|Z_{t_1}\|_2^2 - \alpha \int_{t_1}^{t_2} ||Z_r||_2^2 dr + \int_{t_1}^{t_2} p_2^\alpha(r,\omega) dr, \tag{3.6}$$

for $\alpha > 0$ arbitrary and

$$p_2^{(\alpha)}(r,\omega) := \begin{cases} C_\epsilon ||\Delta Q W_r||_{p+1}^{p+1} + C_2 + c + \tilde{C}_\alpha + 2\alpha \|QW_r\|_2^2 & \text{, if } \omega \in \Omega_0 \\ 0 & \text{, else.} \end{cases}$$

To obtain right continuity of Z_t in $L^2(\Lambda)$ first note that by (3.5) applied for $t_1 = s$ and continuity of Z_t in H we obtain weak continuity in $L^2(\Lambda)$. Now for $t_n \downarrow t$ by (3.5) applied to $t_1 = t$ we obtain

$$\limsup_{n\to\infty} \|Z_{t_n}\|_2^2 \le \|Z_t\|_2^2 ,$$

which implies the right continuity of Z_t in $L^2(\Lambda)$.

Case 2: Assume Hypothesis 1.4.

Let ζ be as defined in Remark 1.3 and again let $t_1 \ge s$ such that $Z_{t_1} \in L^2(\Lambda)$ and $t_2 \ge t_1$. In order to prove (3.1) in the case $QW_t \in C_0^1(\Lambda)$ we need to be more careful when bounding the second term on the right hand side of (3.3). For this we need the regularity result proved in Lemma 3.3 below, which implies that for every $\epsilon > 0$ there exist constants $C_\epsilon, \tilde{C}_\epsilon (= C_\epsilon(p), \tilde{C}_\epsilon(p))$ such that for dr-a.e. $r \in [s,\infty)$

3.6'

$$\begin{aligned} -\langle \Phi(S_r), \Delta(1-\frac{1}{n}\Delta)^{-1} QW_r\rangle &= \langle \nabla\Phi(S_r), \nabla(1-\frac{1}{n}\Delta)^{-1} QW_r\rangle \\ &\le \epsilon ||\nabla\Phi(S_r)||_{\frac{p+1}{p}}^{\frac{p+1}{p}} + C_\epsilon ||\nabla(1-\frac{1}{n}\Delta)^{-1} QW_r||_{p+1}^{p+1} \\ &\le \epsilon ||\nabla\Phi(S_r)||_{\frac{p+1}{p}}^{\frac{p+1}{p}} + \tilde{C}_\epsilon ||\nabla QW_r||_{p+1}^{p+1}. \end{aligned} \tag{3.6'}$$

Now using Lemma 3.3 and (3.6') with $\varepsilon = 1$ in (3.3) yields for some constants $c, C \in \mathbb{R}$

$$\begin{aligned} &\|Z_{t_2}\|_{H_{\frac{1}{n}}}^2 + 2\int_{t_1}^{t_2} \mathcal{E}^{(n)}(\zeta(S_r),\zeta(S_r))dr \\ &\le \|Z_{t_1}\|_{H_{\frac{1}{n}}}^2 + 2\int_{t_1}^{t_2} \left[||\nabla\Phi(S_r)||_{\frac{p+1}{p}}^{\frac{p+1}{p}} + \tilde{C}_1 ||\nabla QW_r||_{p+1}^{p+1} \right] dr \\ &\le c\,\|Z_{t_1}\|_2^2 + C\int_{t_1}^{t_2} (||\nabla QW_r||_{p+1}^{p+1} + 1) dr < \infty \ . \end{aligned}$$

Now we can proceed as after (3.4) to deduce $\zeta(S_r) \in \mathcal{D}(\mathcal{E}) = H_0^1(\Lambda)$ and

$$\lim_{n\to\infty} \mathcal{E}^{(n)}(\zeta(S_r),\zeta(S_r)) = \int_\Lambda |\nabla\zeta(S_r)|_{\mathbb{R}^d}^2 d\xi,$$

for dr-a.e. $r \in [s,\infty)$. Since $\Phi'(r) > 0$, $\zeta = \int_0^\cdot \sqrt{\Phi'(r)}dr$ is $C^1(\mathbb{R})$ with continuous inverse

ζ^{-1}. Thus

$$\Phi(x) = \int_0^x \Phi'(r)dr = \int_0^x \sqrt{\Phi'(r)}\sqrt{\Phi'(r)}dr$$
$$= \int_0^x \zeta'(r)\sqrt{\Phi'(r)}dr = \int_0^{\zeta(x)} \sqrt{\Phi'(\zeta^{-1}(r))}dr = F(\zeta(x)),$$

where $F := \int_0^{\cdot} \sqrt{\Phi'(\zeta^{-1}(r))}dr$. Since $F \in C^1(\mathbb{R})$, $\zeta(S_r) \in H_0^1(\Lambda)$ for dr-a.e. $r \in [s,\infty)$ and $F'(\zeta(S_r))\nabla\zeta(S_r) = \sqrt{\Phi'(S_r)}\nabla\zeta(S_r) \in L^1(\Lambda)$ (by (1.4)), we have $\Phi(S_r) = F(\zeta(S_r)) \in H_0^{1,1}(\Lambda)$ for dr-a.e. $r \in [s,\infty)$ with

2.9 (3.7)
$$\nabla\Phi(S_r) = \sqrt{\Phi'(S_r)}\nabla\zeta(S_r) \in L^1(\Lambda).$$

By (A2)' and (1.5) there are some constants C_1, C_2 such that

$$\zeta'(r)^{2\frac{p+1}{p-1}} \leq C_1\zeta(r)^2 + C_2.$$

Using (3.7) and then Young's and Poincaré's inequalities, for some constants C_1, C_2 (which may change from line to line) we have for dr-a.e. $r \in [s,\infty)$

d_phi_zeta (3.9)
$$||\nabla\Phi(S_r)||_{\frac{p+1}{p}}^{\frac{p+1}{p}} = \int_\Lambda |\nabla\Phi(S_r)|^{\frac{p+1}{p}} d\xi = \int_\Lambda |\sqrt{\Phi'(S_r)}\nabla\zeta(S_r)|^{\frac{p+1}{p}} d\xi$$
$$= \int_\Lambda |\zeta'(S_r)\nabla\zeta(S_r)|^{\frac{p+1}{p}} d\xi \leq ||\nabla\zeta(S_r)||_2^2 + C_1 \int_\Lambda |\zeta'(S_r)|^{2\frac{p+1}{p-1}} d\xi$$
$$\leq ||\nabla\zeta(S_r)||_2^2 + C_1||\zeta(S_r)||_2^2 + C_2 \leq C_1||\nabla\zeta(S_r)||_2^2 + C_2.$$

We can now go on with bounding the second term on the right hand side of (3.3) as follows: (3.6') and (3.9) imply that for dr-a.e. $r \in [s,\infty)$

oise-bound (3.10)
$$\langle\Phi(S_r), \Delta(1 - \frac{1}{n}\Delta)^{-1}QW_r\rangle \leq \varepsilon||\nabla\Phi(S_r)||_{\frac{p+1}{p}}^{\frac{p+1}{p}} + \tilde{C}_\varepsilon||\nabla QW_r||_{p+1}^{p+1}$$
$$\leq \varepsilon C_1||\nabla\zeta(S_r)||_2^2 + \varepsilon C_2 + \tilde{C}_\varepsilon||\nabla QW_r||_{p+1}^{p+1}.$$

Using this with $\varepsilon = \frac{1}{C_1}$ in (3.3) and letting $n \to \infty$ yields for some constant C

(3.11)
$$\|Z_{t_2}\|_2^2 + \int_{t_1}^{t_2} ||\nabla\zeta(S_r)||_2^2 dr \leq \|Z_{t_1}\|_2^2 + 2C\int_{t_1}^{t_2}(1 + ||\nabla QW_r||_{p+1}^{p+1})dr.$$

Now we can proceed as done in the proof of Case 1 after (3.5). □

Remark 3.2. *As indicated before the arguments in the proof can easily be generalized to noise* $QW_t \in H_0^{2,p+1}(\Lambda)$ *(*$QW_t \in H_0^{1,p+1}(\Lambda)$ *resp.).*

regularity **Lemma 3.3.** *Let $x \in L^2(\Lambda)$, $s \in \mathbb{R}$ and $\omega \in \Omega$. Then $\Phi(S(\cdot, s, \omega)x) \in L_{loc}^{\frac{p+1}{p}}([s, \infty); H_0^{1,\frac{p+1}{p}})$ and there exist constants $c > 0, C \in \mathbb{R}$, independent of x, s and ω, such that*

$$\begin{aligned}
&||Z(t_2, s, x, \omega)||_2^2 + c \int_{t_1}^{t_2} ||\nabla\Phi(S(r, s, \omega)x)||_{\frac{p+1}{p}}^{\frac{p+1}{p}} dr \\
\leq &||Z(t_1, s, x, \omega)||_2^2 + C \int_{t_1}^{t_2} (||\nabla QW_r(\omega)||_{p+1}^{p+1} + 1) dr, \quad \forall t_2 \geq t_1 \geq s.
\end{aligned}$$

Proof. We use the Galerkin approximation and the notation used in the proof of unique existence of a solution to (1.6) in [28, Theorem 4.2.4]). Let $\{e_i | i \in \mathbb{N}\}$ be the orthonormal basis of H consisting of eigenfunctions of Δ on $L^2(\Lambda)$ with Dirichlet boundary. Then $e_i \in C_0^\infty(\Lambda) \subseteq V$. Furthermore, let $H_n = \text{span}\{e_1, ..., e_n\}$ and define $P_n : V^* \to H_n \subseteq C_0^\infty(\Lambda)$ by

$$P_n y := \sum_{i=1}^n {}_{V^*}\!< y, e_i >_V \; e_i.$$

Note that via the embedding $L^2(\Lambda) \subseteq H \subseteq V^*$, $P_{n|L^2(\Lambda)} : L^2(\Lambda) \to H_n$ is just the orthogonal projection in $L^2(\Lambda)$ onto H_n. Let $t_1 \geq s$ such that $Z_{t_1} \in L^2(\Lambda)$, let Z_t^n denote the solution of

$$Z_t^n = P_n Z_{t_1} + \int_{t_1}^t P_n A_\omega(r, Z_r^n) dr, \; \forall t \geq t_1$$

and let $S_t^n := Z_t^n + QW_t$. By the chain rule, for all $t_2 \geq t_1$

galerkin_1
$$\begin{aligned}
(3.12) \qquad ||Z_{t_2}^n||_2^2 &= ||P_n Z_{t_1}||_2^2 + 2 \int_{t_1}^{t_2} \langle A_\omega(r, Z_r^n), Z_r^n \rangle \, dr \\
&= ||P_n Z_{t_1}||_2^2 + 2 \int_{t_1}^{t_2} \langle \Delta\Phi(S_r^n), S_r^n \rangle \, dr - 2 \int_{t_1}^{t_2} \langle \Delta\Phi(S_r^n), QW_r \rangle \, dr.
\end{aligned}$$

By the same argument as for (3.7) we get

$$\langle \Delta\Phi(S_r^n), S_r^n \rangle = - \langle \nabla\Phi(S_r^n), \nabla S_r^n \rangle = - \left\langle \sqrt{\Phi'(S_r^n)} \nabla\zeta(S_r^n), \nabla S_r^n \right\rangle = -||\nabla\zeta(S_r^n)||_2^2$$

and using Young's inequality

$$- \langle \Delta\Phi(S_r^n), QW_r \rangle = \langle \nabla\Phi(S_r^n), \nabla QW_r \rangle \leq \varepsilon ||\nabla\Phi(S_r^n)||_{\frac{p+1}{p}}^{\frac{p+1}{p}} + C_\varepsilon ||\nabla QW_r||_{p+1}^{p+1},$$

for all $\varepsilon > 0$ and some $C_\varepsilon \in \mathbb{R}$. By (3.12) this yields

galerkin_2
$$\begin{aligned}
(3.13) \qquad ||Z_{t_2}^n||_2^2 \leq &||P_n Z_{t_1}||_2^2 - 2 \int_{t_1}^{t_2} ||\nabla\zeta(S_r^n)||_2^2 dr \\
&+ 2\varepsilon \int_{t_1}^{t_2} ||\nabla\Phi(S_r^n)||_{\frac{p+1}{p}}^{\frac{p+1}{p}} dr + 2C_\varepsilon \int_{t_1}^{t_2} ||\nabla QW_r||_{p+1}^{p+1} dr.
\end{aligned}$$

By the same argument as for (3.9) we realize

$$||\nabla\Phi(S_r^n)||_{\frac{p+1}{p}}^{\frac{p+1}{p}} \leq C_1||\nabla\zeta(S_r^n)||_2^2 + C_2,$$

for some constants C_1, C_2. Using this in (3.13), with $\varepsilon = \frac{1}{2C_1}$ yields for some $c > 0, C \in \mathbb{R}$

galerkin_3

$$||Z_{t_2}^n||_2^2 + c\int_{t_1}^{t_2} ||\nabla\Phi(S_r^n)||_{\frac{p+1}{p}}^{\frac{p+1}{p}} dr \leq ||Z_{t_1}||_2^2 + C\int_{t_1}^{t_2} (||\nabla QW_r||_{p+1}^{p+1} + 1) dr. \tag{3.14}$$

Both C_1, C_2 and c, C are independent of x, s and ω.

Hence we obtain the existence of a $\bar{\phi} \in L^{\frac{p+1}{p}}([t_1, t_2]; H_0^{1,\frac{p+1}{p}})$ such that (selecting a subsequence if necessary)

$$\Phi(S_r^n) \rightharpoonup \bar{\phi},$$

in $L^{\frac{p+1}{p}}([t_1, t_2]; H_0^{1,\frac{p+1}{p}})$ and thus in $L^{\frac{p+1}{p}}([t_1, t_2]; L^{\frac{p+1}{p}}(\Lambda))$. By the proof of unique existence of a solution we also know that (again selecting a subsequence if necessary)

$$\Delta\Phi(S_r^n) \rightharpoonup \Delta\Phi(S_r),$$

in $L^{\frac{p+1}{p}}([t_1, t_2]; V^*)$ and by definition of $\Delta\Phi : V \to V^*$ this is equivalent to $\Phi(S_r^n) \rightharpoonup \Phi(S_r)$, in $L^{\frac{p+1}{p}}([t_1, t_2]; L^{\frac{p+1}{p}}(\Lambda))$. Hence $\bar{\phi} = \Phi(S_r)$. An analogous argument applied to $Z_{t_2}^n$ yields $Z_{t_2}^n \rightharpoonup Z_{t_2}$ in $L^2(\Lambda)$. Letting $n \to \infty$ in (3.14) we arrive at

galerkin_4

$$||Z_{t_2}||_2^2 + c\int_{t_1}^{t_2} ||\nabla\Phi(S_r)||_{\frac{p+1}{p}}^{\frac{p+1}{p}} dr \leq ||Z_{t_1}||_2^2 + C\int_{t_1}^{t_2} (||\nabla QW_r||_{p+1}^{p+1} + 1) dr. \tag{3.15}$$

Since $Z_s = x - QW_s \in L^2(\Lambda)$, for all $t_1 \geq s$ we obtain $Z_{t_1} \in L^2(\Lambda)$ and thus (3.15) holds for all $t_2 \geq t_1 \geq s$. □

-absorption

Corollary 3.4 (Compact absorption)**.** *There is an $\mathcal{F}$–measurable function $\kappa : \Omega \to \mathbb{R}_+$ such that for each $\varrho > 0$ there exists $s(\varrho) \leq -1$ such that for all $x \in H$ with $\|x\|_H \leq \varrho$ and all $\omega \in \Omega_0$*

$$\|S(0, s, \omega)x\|_2 \leq \kappa(\omega), \text{ for all } s \leq s(\varrho).$$

Remark 3.5. *This is analogous to [14, Lemma 5.5, p. 380].*

Proof. (3.1) in Theorem 3.1 with $t_2 = 0 \geq t_1 \geq s$ implies

$$\|Z_0\|_2^2 \leq \|Z_{t_1}\|_2^2 - \alpha\int_{t_1}^0 \left(\|Z_r\|_2^2 + p_2^{(\alpha)}(r, \omega)\right) dr.$$

Integrating over $t_1 \in [-1, 0]$ yields

$$\begin{aligned}\|Z_0\|_2^2 &\leq \int_{-1}^0 \left(\|Z_r\|_2^2 + |p_2^{(\alpha)}(r, \omega)|\right) dr \\ &\leq \int_{-1}^0 (2\|S_r\|_2^2 + 2\|QW_r\|_2^2 + |p_2^{(\alpha)}(r, \omega)|) dr.\end{aligned}$$

Hence using Corollary 2.4 and recalling that $Z_0 = S(0, s, \omega)x$ we obtain the assertion. □

4 Existence of the global random attractor

_attractor **Theorem 4.1.** *The random dynamical system associated with* (0.0) *and defined by* (1.8) *admits a random attractor.*

Proof. We show that the assumptions of Proposition 1.10 are satisfied. Since the embedding $L^2(\Lambda) \hookrightarrow H$ is compact, for each $\omega \in \Omega$ the set

$$K(\omega) := \overline{\bar{B}_{L^2}(0, \kappa(\omega))}^H$$

is nonempty and compact in H.

For the reader's convenience, we prove that it is a random set (cf. Definition 1.8 (i)) in the Polish space H. According to [11, Proposition 2.4], it is enough to check that for each open set $O \subset H$, $C_O := \{\omega \in \Omega | O \cap K(\omega) \neq \emptyset\}$ is measurable. But

$$\begin{aligned} O \cap K(\omega) =& O \cap \overline{\bar{B}_{L^2}(0, \kappa(\omega))}^H = O \cap \bar{B}_{L^2}(0, \kappa(\omega)) \\ =& O \cap L^2(\Lambda) \cap \bar{B}_{L^2}(0, \kappa(\omega)). \end{aligned}$$

For $C \subseteq L^2(\Lambda)$ and $x \in L^2(\Lambda)$ let $d_{L^2}(x, C) := \inf_{y \in C} ||x - y||_2$. If $O \cap L^2(\Lambda) = \emptyset$, then $C_O = \emptyset$ is measurable and if $O \cap L^2(\Lambda) \neq \emptyset$, then

$$C_O = \{\omega \in \Omega | d_{L^2}\left(0, O \cap L^2(\Lambda)\right) \leq \kappa(\omega)\}$$

is measurable as κ is.

Let B be a bounded subset of H. Then $B \subset \bar{B}_H(0, \varrho)$, for some $\varrho > 0$. By Corollary 3.4 there exists a $t_B := -s(\varrho) \geq 1$ such that for all $x \in B$, $t \geq t_B$ and $\omega \in \Omega_0$

$$\varphi(t, \theta_{-t}\omega)(x) = S(t, 0, \theta_{-t}\omega)x = S(0, -t, \omega)x \leq \kappa(\omega).$$

Hence for all $t \geq t_B, \omega \in \Omega_0$, $\varphi(t, \theta_{-t}\omega)(B) \subset K(\omega)$, i.e. the random compact set K absorbs all deterministic bounded sets.

Now we may apply Proposition 1.10 to get the existence of a global compact attractor A, given by:

$$A(\omega) = \overline{\bigcup_{B \subset H, B \text{ bounded}} \Omega_B(\omega)}^H,$$

where $\Omega_B(\omega) := \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} \varphi(t, \theta_{-t}\omega)B}$ denotes the Ω-limit set of B. □

Remark 4.2. *By [14, Proposition 4.5] the existence of a random attractor as constructed in the proof of Theorem 4.1 implies the existence of an invariant Markov measure* $\mu_\cdot \in \mathcal{P}_\Omega(H)$ *for* φ *(in the sense of [14, Definition 4.1]), supported by* A*. Hence using [12] there exists an invariant measure for the Markovian semigroup defined by* $P_t\varphi(x) = \mathbb{E}[\varphi(S(t, 0, x))]$ *and it is given by*

$$\mu(B) = \int_\Omega \mu_\omega(B) P(d\omega),$$

where $B \subseteq H$ is a Borel set. If the invariant measure μ for P_t is unique, then the invariant Markov measure $\mu.$ for φ is unique and given by

$$\mu_\omega = \lim_{t\to\infty} \varphi(t, \theta_{-t}\omega)\mu.$$

5 Attraction by a single point

So far we obtained the existence of the random attractor A for (0.0), but we did not deduce any information about its finer structure. Under a stronger monotonicity condition which was first introduced in [16] we will now prove that A consists of a single random point. While we had to restrict to noise of regularity at least $H_0^{1,p+1}(\Lambda)$ before, we can now allow Q to be a Hilbert-Schmidt operator from $L^2(\Lambda) \to H$.

Let W_t denote a cylindrical Brownian Motion on $L^2(\Lambda)$ and define $\Phi : \mathbb{R} \to \mathbb{R}$ to be a continuous function such that there exist some constants $c \geq 0$, $p \in (1, \infty)$, $\eta > 0$ such that

n:monotone (5.16)
$$\begin{aligned} &|\Phi(s)| \leq c(1 + |s|^p) \\ &(s - t)(\Phi(s) - \Phi(t)) \geq \eta |s - t|^{p+1}, \quad s, t \in \mathbb{R}. \end{aligned}$$

It has been shown in [16] that (5.16) holds if $\Phi \in C^1(\mathbb{R})$, $\Phi(0) = 0$ and if there exist constants $\kappa, \eta > 0$ such that

g_monotone (5.17)
$$\frac{(p+1)^2}{4}\eta |s|^{p-1} \leq \Phi'(s) \leq \kappa(1 + |s|^{p-1}), \qquad s \in \mathbb{R}.$$

This, for example is true for $\Phi(s) = s|s|^{p-1}$. By Remark 1.3 it is easy to see that (5.17) implies the weaker monotonicity assumption (A1)'. Also note that (5.16) implies the coercivity property (A2). Thus (A1)-(A3) are satisfied and we can define Z_t, S_t and the RDS φ as before (cf. (1.8)).

thm:main **Theorem 5.1.** *For $s_1 \leq s_2 < t$, $\omega \in \Omega$ and $x, y \in H$ we have:*

$$\begin{aligned} ||S(t, s_1, \omega)x - S(t, s_2, \omega)y||_H^2 &\leq \left\{ ||S(s_2, s_1, \omega)x - y||_H^{1-p} + \eta \lambda_1^{\frac{p+1}{2}} (p-1)(t - s_2) \right\}^{-\frac{2}{p-1}} \\ &\leq \left\{ \eta \lambda_1^{\frac{p+1}{2}} (p-1)(t - s_2) \right\}^{-\frac{2}{p-1}}. \end{aligned}$$

In particular for each $t \in \mathbb{R}$, $\lim_{s\to-\infty} S(t, s, \omega)x = \eta_t(\omega)$ exists independently of x and uniformly in x, ω.

Proof. Let $s_1 \leq s_2 < t$. Then for all $s_2 \leq s \leq t$

$$S(t, s_1, \omega)x - S(t, s_2, \omega)y = S(s, s_1, \omega)x - S(s, s_2, \omega)y + \int_s^t A(S(r, s_1, \omega)x) - A(S(r, s_2, \omega)y) dr.$$

By Itô's-Formula and since $||u||_{p+1}^{p+1} \geq \lambda_1^{\frac{p+1}{2}} ||u||_H^{p+1}$, for all $s_2 \leq s \leq t$:

eqn:diff2

$$
\begin{aligned}
&||S(t,s_1,\omega)x - S(t,s_2,\omega)y||_H^2 \\
&= ||S(s,s_1,\omega)x - S(s,s_2,\omega)y||_H^2 \\
&\quad + 2\int_s^t {}_{V^*}\langle A(S(r,s_1,\omega)x) - A(S(r,s_2,\omega)y), S(r,s_1,\omega)x - S(r,s_2,\omega)y\rangle_V \, dr \\
&= ||S(s,s_1,\omega)x - S(s,s_2,\omega)y||_H^2 \qquad (5.18)\\
&\quad - 2\int_s^t \langle \Phi(S(r,s_1,\omega)x) - \Phi(S(r,s_2,\omega)y), S(r,s_1,\omega)x - S(r,s_2,\omega)y\rangle \, dr \\
&\leq ||S(s,s_1,\omega)x - S(s,s_2,\omega)y||_H^2 - 2\eta \int_s^t ||S(r,s_1,\omega)x - S(r,s_2,\omega)y||_{p+1}^{p+1} dr \\
&\leq ||S(s,s_1,\omega)x - S(s,s_2,\omega)y||_H^2 - \tilde{\eta} \int_s^t ||S(r,s_1,\omega)x - S(r,s_2,\omega)y||_H^{p+1} dr,
\end{aligned}
$$

where for notational convenience we have set $\tilde{\eta} := 2\eta\lambda_1^{\frac{p+1}{2}}$. Thus formally $||S(t,s_1,\omega)x - S(t,s_2,\omega)y||_H^2$ is a subsolution of the ordinary differential equation

def:ode

$$
\begin{aligned}
h'(t) &= -\tilde{\eta} h(t)^{\frac{p+1}{2}}, \ \forall t \geq s_2 \qquad (5.19)\\
h(s_2) &= ||S(s_2,s_1,\omega)x - y||_H^2.
\end{aligned}
$$

Let

$$
h_\epsilon(t) = \left\{ (||S(s_2,s_1,\omega)x - y||_H + \epsilon)^{1-p} + \frac{\tilde{\eta}}{2}(p-1)(t-s_2) \right\}^{-\frac{2}{p-1}}, \ t \geq s_2.
$$

h_ϵ is a solution of (5.19) with $h_\epsilon(s_2) = (||S(s_2,s_1,x) - y||_H + \epsilon)^2$, which suggests $||S(t,s_1,\omega)x - S(t,s_2,\omega)y||_H^2 \leq h_\epsilon(t)$. This will be proved next.

Let $\Phi_\epsilon(t) := h_\epsilon(t) - ||S(t,s_1,\omega)x - S(t,s_2,\omega)y||_H^2$ and $\tau_\epsilon = \inf\{t \geq s_2 | \ 0 \geq \Phi_\epsilon(t)\}$. Using $0 < \Phi_\epsilon(s_2)$ and continuity of Φ_ϵ we realize $\tau_\epsilon > s_2$. Further note that by definition we have $h_\epsilon(t) \geq ||S(t,s_1,\omega)x - S(t,s_2,\omega)x||_H^2$ on $[s_2,\tau_\epsilon]$ and that

$$
h_\epsilon(t) \leq (||S(s_2,s_1,\omega)x - y||_H + \epsilon)^2 =: c_\epsilon.
$$

Assume $\tau_\epsilon < \infty$. Then $\Phi_\epsilon(\tau_\epsilon) \leq 0$ and for all $s_2 \leq s \leq t \leq \tau_\epsilon$, by the mean value theorem and (5.18):

$$
\begin{aligned}
\Phi_\epsilon(t) &= h_\epsilon(t) - ||S(t,s_1,\omega)x - S(t,s_2,\omega)y||_H^2 \\
&\geq \Phi_\epsilon(s) - \tilde{\eta} \int_s^t (h_\epsilon(r)^{\frac{p+1}{2}} - \left(||S(r,s_1,\omega)x - S(r,s_2,\omega)||_H^2\right)^{\frac{p+1}{2}}) dr \\
&\geq \Phi_\epsilon(s) - \tilde{\eta} \left(\frac{p+1}{2}\right) c_\epsilon^{\frac{p-1}{2}} \int_s^t \Phi_\epsilon(r) dr.
\end{aligned}
$$

Using the Gronwall Lemma we obtain

$$\Phi_\epsilon(\tau_\epsilon) \geq \Phi_\epsilon(s_2) e^{-\tilde{\eta}\left(\frac{p+1}{2}\right) c_\epsilon^{\frac{p-1}{2}} (\tau_\epsilon - s_2)} > 0.$$

This contradiction proves $\tau_\epsilon = \infty$ and since this is true for all $\epsilon > 0$ we conclude:

$$\begin{aligned} ||S(t,s_1,\omega)x - S(t,s_2,\omega)y||_H^2 &\leq \left\{ (||S(s_2,s_1,\omega)x - y||_H)^{1-p} + \frac{\tilde{\eta}}{2}(p-1)(t-s_2) \right\}^{-\frac{2}{p-1}} \\ &\leq ||S(s_2,s_1,\omega)x - y||_H^2 \wedge \left\{ \frac{\tilde{\eta}}{2}(p-1)(t-s_2) \right\}^{-\frac{2}{p-1}} \\ &\leq \left\{ \frac{\tilde{\eta}}{2}(p-1)(t-s_2) \right\}^{-\frac{2}{p-1}}, \end{aligned}$$

for each $t > s_2$.

□

Theorem 5.2. *The random dynamical system given by $\varphi(t,\omega)x = S(t,0,\omega)x$ has a compact global attractor $A(\omega)$ consisting of one point*

$$A(\omega) = \{\eta_0(\omega)\}.$$

Proof. Since $\eta_0(\omega)$ is measurable, $A(\omega)$ is a random compact set. We need to check invariance and attraction for $A(\omega)$. Let $t > 0$. Then for any $x \in H$, by continuity of $x \mapsto S(t,0,\omega)x$ and (1.8'), (1.8")

$$\begin{aligned} \varphi(t,\omega)A(\omega) &= \left\{ S(t,0,\omega) \lim_{s\to-\infty} S(0,s,\omega)x \right\} = \left\{ \lim_{s\to-\infty} S(t,s,\omega)x \right\} \\ &= \left\{ \lim_{s\to-\infty} S(0,s-t,\theta_t\omega)x \right\} = \{\eta_0(\theta_t\omega)\} = A(\theta_t\omega). \end{aligned}$$

Since the convergence in Theorem 5.1 is uniform with respect to $x \in H$, for any bounded set $B \subseteq H$ we have (again using (1.8"))

$$\begin{aligned} d(\varphi(t,\theta_{-t}\omega)B, A(\omega)) &= \sup_{x\in B} ||S(t,0,\theta_{-t}\omega)x - \eta_0(\omega)||_H \\ &= \sup_{x\in B} ||S(0,-t,\omega)x - \eta_0(\omega)||_H \to 0, \end{aligned}$$

for $t \to \infty$. Hence $A(\omega)$ attracts all deterministic bounded sets.

□

It is easy to see that the convergence $\lim_{s\to-\infty} S(t,s,\omega)x = \eta_t(\omega)$ implies the existence and uniqueness of an invariant measure for the associated Markovian semigroup, defined by $P_t\varphi(x) := \mathbb{E}[\varphi(S(t,0,\cdot)x)]$ (cf. [16]). This invariant measure is given by $\mu = \mathbb{P}\circ\eta_0^{-1}$. In fact we can deduce much more. Since evidently η_0 is measurable with respect to $\mathcal{F}^-$ by [12] $\mu_\omega := \lim_{t\to\infty} \varphi(t,\theta_{-t}\omega)\mu$ exists $\mathbb{P}$-a.s. and defines an invariant measure for the random dynamical system φ (for more details on invariant random measures cf. [14]). Moreover by [11, Theorem 2.12] every invariant measure for φ is supported by $A = \{\eta_0\}$, i.e. $\mu_\omega(\{\eta_0(\omega)\}) = 1$ for $\mathbb{P}$-a.a. ω. Hence we have proved the following

Corollary 5.3. *There exists a unique invariant random measure $\mu_\cdot \in \mathcal{P}_\Omega(H)$ for the random dynamical system φ and it is given by*

$$\mu_\omega = \delta_{\eta_0(\omega)}, \quad \mathbb{P}\text{-}a.s.\ .$$

6 Concluding remarks on computational methods

The porous medium equation considered here is a model case for a general type of equations that include more details of the permeable medium and that has important applications to the simulation of oil reservoirs. We refer to [1] for such an application and for an up-to-date finite element method that can be used for solving the deterministic version of (1.1). One of the major difficulties here is to account for the spatial variations (represented by the functions φ_j in the operator Q) by introducing different scales in the finite element subspace. For the quasilinear steady state equation suitable finite element approximations have been set up, cf. [27],[26] and the references therein.

It seems, however, that computational methods for random attractors in infinite dimensional systems (except for the case of a singleton) are well beyond today's computational capabilities.

There are a few approaches to approximate random attractors in stochastic ordinary differential equations [21],[20]. These are based on the subdivision and box covering techniques developed over the last years by Dellnitz and coworkers (see [17] for a survey). However, these methods are essentially still limited to lower dimensions. In order to proceed to high-dimensional or even infinite-dimensional cases (see e.g. [33]) one will need reduction principles as they are well established in the theory of inertial manifolds for deterministic PDEs. The corresponding properties of squeezing and flattening (cf. [19],[25]) have been generalized to random dynamical systems in [23]. It is also shown in [23] that squeezing is a stronger condition than flattening, but that the latter one is sufficient to establish the existence of a compact random attractor. The determining modes occuring in these properties should form the basis of a reduced space to which numerical methods apply.

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