Explosive Solutions of Stochastic Reaction-Diffusion Equations in Mean L^p-Norm

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Abstract

The paper is concerned with the problem of non-existence of global solutions for a class of stochastic reaction-diffusion equations of Itô type. Under some sufficient conditions on the initial state, the nonlinear term and the multiplicative noise, it is proven that, in a bounded domain $\mathcal{D} \subset \mathbb{R}^d$, there exist positive solutions whose mean L^p-norm will blow up in finite time for $p \geq 1$, while, if $\mathcal{D} = \mathbb{R}^d$, the previous result holds in any compact subset of $\mathbb{R}^d$. Two examples are given to illustrate some application of the theorems.

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1 Introduction

Consider the initial-boundary problem for a reaction-diffusion equation in domain $\mathcal{D} \subset \mathbb{R}^d$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla^2 u + f(u), \quad t > 0, \\ u(x,0) = g(x), \quad x \in \mathcal{D}, \\ u(x,t) = 0, \quad x \in \partial\mathcal{D}, \end{cases} \tag{1.1}$$

where ∇^2 is the Laplacian operator, $\partial\mathcal{D}$ denotes the boundary of $\mathcal{D}$, and the functions f and g are given such that the problem (1.1) has a unique local solution. In 1963 it was first shown by S. Kaplan [10] that, for a certain class of nonlinear functions $f(u)$, the solution of equation (1.1) becomes infinite or explodes at a finite time, provided that the initial state $g(x)$ and the nonlinear function $f(u)$ satisfies appropriate conditions. His result was later extended by Fujita [6] and many others. Since then it has become known that solutions to more general nonlinear parabolic equations may develop singularities in finite time, see, e.g., the review article [7] and the book [15], where an extensive references can be found. Physically this phenomenon is manifested as the explosion in combustion, reaction diffusion and branching diffusion problems. It is therefore of interest to examine the effect of a random perturbation to equation (1.1) on the existence of an explosive solution. This consideration has led us to investigate the question of nonexistence of a global solution to the following type of parabolic Itô equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla^2 u + f(u) + \sigma(u)\partial_t W(x,t), \quad t > 0, \\ u(x,0) = g(x), \quad x \in \mathcal{D}, \\ u(x,t) = 0, \quad x \in \partial\mathcal{D}, \end{cases} \tag{1.2}$$

with a multiplicative noise, where σ is a given function and $W(x,t)$ is a Wiener random field. To study this type of problems, it is necessary to employ some analytical and probabilistic tools from the theory of stochastic partial differential equations (SPDEs) (see, e.g., [1], [5], among many papers on this subject). In contrast, for stochastic ordinary differential equations, the general results on the explosion and non-explosion of solutions have been well established, (see, e.g., [9]). However, so far, very little is known about such results for SPDEs due to some difficulty in infinite-dimensional stochastic analysis. Therefore one can only hope to resolve such questions for some special cases. Recently we studied the existence of explosive solutions for a class of nonlinear stochastic wave equations. Based on a stochastic energy method, we were able to obtain some sufficient conditions for the blow-up of the second moments of solutions in the L^2-norm [2]. In the paper [3] we considered the positive (nonnegative) solutions of nonlinear parabolic Itô equations such as (1.2). By extending Kaplan's approach to the deterministic case [10], we have shown that, if certain sufficient conditions for the explosion of the deterministic case ($\sigma \equiv 0$) are satisfied and $\sigma(u)$ is bounded in mean-square, then a positive solution can blow up in finite time, in the sense of mean L^p-norm defined by (2.10) for any $p \geq 1$ [3]. In the afore-mentioned paper, the nonlinear reaction function $f(u)$ plays a dominant role and the random perturbation term has only a secondary effect on the blow-up behavior. In a diametrically different case, Mueller [12] and, later, Mueller and Sowers [13] investigated the problem of a noise-induced explosion for a special case of equation (1.2) in one dimension, where $f(u) \equiv 0$, $\sigma(u) = u^\gamma$ with $\gamma > 0$ and $W(x,t)$ is a space-time white noise. It was shown that the solution will explode in finite time with positive probability for some $\gamma > 3/2$. In the present paper, to account for the possibility of a noise induced explosion, we will generalize the previous result in [3] by finding a new set of sufficient conditions for the solution to blow up in the mean L^p-norm. However this does not imply the path-wise explosion with a positive probability, which is an interesting open problem currently under investigation.

The paper is organized as follows. We shall first recall some basic results for nonlinear stochastic parabolic equations in Section 2. Here we also present a theorem (Theorem 2.1) on the positive solutions to a class of nonlinear stochastic parabolic equations. Since it will play a key role in the subsequent analysis, a sketch of proof will be provided. Section 3 contains the main results of the paper as presented in Theorems 3.1 and 3.2. Under some sufficient conditions, Theorem 3.1 shows the existence of positive solutions in a bounded domain that will explode within a finite time in the mean L^p-norm, while, in the case $\mathcal{D} = \mathbb{R}^d$, Theorem 3.2 affirms a similar result in any compact subset of $\mathbb{R}^d$. Finally in Section 4, we apply the theorems to two special problems to obtain some explicit conditions for explosive solutions.

2 Preliminaries

Let $\mathcal{D}$ be a domain in $\mathbb{R}^d$, which has a smooth boundary $\partial\mathcal{D}$ if it is bounded. We set $H = L^2(\mathcal{D})$ with the inner product and norm are denoted by $(.,.)$ and $\|.\|$ respectively. Let $H^1 = H^1(\mathcal{D})$ be the L^2-Sobolev space of first order and denote by H_0^1 the closure in H^1 of the space of C^1-functions with compact support in $\mathcal{D}$.

Let $W(x,t)$, for $x \in \mathbb{R}^d$, $t \geq 0$, be a continuous Wiener random field defined in a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathcal{F}_t$ (p.38, [1]). It has mean $E\,W(x,t) = 0$ and covariance function $q(x,y)$ defined by

$$E\,W(x,t)W(y,s) = (t \wedge s)q(x,y), \quad x, y \in \mathbb{R},$$

where $(t \wedge s) = \min\{t,s\}$ for $0 \leq t, s \leq T$.

Consider the initial-boundary value problem for the parabolic Itô equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = Au + f(u,x,t) + \sigma(u, \nabla u, x, t)\partial_t W(x,t), \\ u(x,0) = g(x), \quad x \in \mathcal{D}, \\ u(x,t)|_{\partial\mathcal{D}} = 0, \quad t \in (0,T), \end{cases} \tag{2.3}$$

where $A = \sum_{i,j=1}^{d} \dfrac{\partial}{\partial x_i}[a_{ij}(x)\dfrac{\partial}{\partial x_j}]$ is a symmetric, uniformly elliptic operator with smooth coefficients (say, in $C^3(\overline{\mathcal{D}})$), that is, there exists a constant $a_0 > 0$ such that

$$b(x,\xi) := \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2, \tag{2.4}$$

for all $x \in \overline{\mathcal{D}}$ and $\xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}$. Certain conditions will be imposed on the functions f, σ, g later.

Now, to regard the equation (2.3) with a homogeneous boundary condition as an Itô equation in the Hilbert space H, we set $u_t = u(\cdot, t)$, $F_t(u) = f(u, \cdot, t)$, $\Sigma_t(u) = \sigma(u, \nabla u, \cdot, t)$ and so on, and rewrite it as

$$\begin{cases} du_t = [A\, u_t + F_t(u_t)]\, dt + \Sigma_t(u_t)\, d\,W_t, \quad 0 < t < T, \\ \\ u_0 = g, \end{cases} \tag{2.5}$$

where A is now regarded as a linear operator from H^1 into H^{-1} with domain $H_0^1 \cap H^2$, $F_t : H \to H$ is continuous and, for $v \in H_1$, $\Sigma_t(v) : C(\overline{\mathcal{D}}) \to H$ can be defined as a multiplication operator. In this paper we assume that the covariance function $q(x, y)$ is bounded, continuous and there is $q_0 > 0$ such that

$$\sup_{x,y \in \mathcal{D}} |q(x, y)| \leq q_0, \quad \text{and} \int_{\mathbb{R}^d} q(x, x)\, dx < \infty. \tag{2.6}$$

Then we can rewrite equation (2.5) as

$$u_t = g + \int_0^t [A\, u_s + F_s(u_s)]\, ds + \int_0^t \Sigma_s(u_s)\, d\,W_s, \tag{2.7}$$

where the stochastic integral is well defined (see Theorem 2.4, [1]).

Under the usual conditions, such as the stochastic coercivity, Lipschitz continuity and monotonicity conditions, the equation (2.7) is known to have a unique global strong solution $u \in C([0, T]; H) \cap L^2((0, T); H_0^1)$ for any $T > 0$ (Theorem 7.4, [1]). Moreover, for a continuous C^2- functional Φ on H, the Itô formula holds [14], [8]

$$\begin{cases} \Phi(u_t) = \Phi(u_0) + \int_0^t [\langle A\, u_s, \Phi'(u_s) \rangle + (F_s(u_s), \Phi'(u_s)]\, ds \\ + \int_0^t (\Phi'(u_s), \Sigma_s(u_s)\, d\,W_s) + \frac{1}{2} \int_0^t Tr\, [\Phi''(u_s) \Sigma_s^\star(u_s) Q \Sigma_s(u_s)]\, ds, \end{cases} \tag{2.8}$$

where Φ', Φ'' denote the first and second Fréchet derivatives of Φ, Q is the covariance operator with kernel q, the star means the conjugate and Tr is the trace of an operator.

On the other hand, if the nonlinear terms are only locally Lipschitz continuous and the monotonicity condition is dropped, one can only assert the existence of a unique local solution. In this case, by the conventional definition, the solution u_t in H is said to explode or blow up if the probability $Pr\{\tau < \infty\} = 1$, where τ is the explosion time defined by $\tau = \inf\{t > 0 : \|u_t\| = \infty\}$ [9]. In this paper we shall introduce an alternative definition which is closer to the deterministic case. For any $p \geq 1$, we let $L^p = L^p(\mathcal{D})$ denote the usual L^p space of functions v on $\mathcal{D}$ with norm $\|v\|_p$ defined by

$$\|v\|_p := \{ \int_{\mathcal{D}} |v(x)|^p \, dx\}^{1/p}. \tag{2.9}$$

Then we say that the solution u_t explodes in the mean L^p-norm if there exists a constant $T_p > 0$ such that the left limit

$$\lim_{t \to T_p^-} E\,\|u_t\|_p = \lim_{t \to T_p^-} E\,\{ \int_{\mathcal{D}} |u(x,t)|^p \, dx\}^{1/p} = \infty, \tag{2.10}$$

where T_p is called an explosion time. Clearly, by the Hölder inequality, the limit (2.10) implies that

$$\lim_{t \to T_p^-} E\,\|u_t\|_p^p = \infty.$$

To consider positive (nonnegative) solutions, we suppose that the parabolic Itô equation (2.3) has a unique strong solution $u(\cdot, t)$ for $t \leq T$. In addition, assume that the following conditions hold:

(P1) There exists a constant $\delta \geq 0$ such that

$$\frac{1}{2}\, q(x,x)\sigma^2(r,\xi,x,t) - \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \leq \delta\, r^2,$$

for all $r \in \mathbb{R},\ x \in \overline{\mathcal{D}},\ \xi \in \mathbb{R}^d$ and $t \in [0,T]$.

(P2) The function $f(r,x,t)$ is continuous on $\mathbb{R} \times \overline{\mathcal{D}} \times [0,T]$ such that $f(r,x,t) \geq 0$ for $r \geq 0$ and $x \in \overline{\mathcal{D}},\ \ t \in [0,T]$.

(P3) The initial datum $g(x)$ on $\overline{\mathcal{D}}$ is positive and continuous.

Then it can be shown that the solution of equation (2.3) is positive. Notice that, due to the lack of a maximum principle for parabolic equations as in the deterministic case [11], the proof will be quite different. In fact we shall make use of a regularization technique and the Itô formula (2.8). The following positivity theorem was proved in (Theorem 3.3, [3]). For the article to be more self-contained, the proof will be sketched here.

Theorem 2.1 Suppose that the conditions (P1), (P2) and (P3) hold true. Then the solution of the initial-boundary problem for the parabolic Itô equation (2.3) remains positive so that $u(x,t) \geq 0$, *a.s. for almost every* $x \in \mathcal{D}$, $\forall\, t \in [0,T]$.

Proof Let $\eta(r) = r^-$ denote the negative part of r for $r \in \mathbb{R}$, or $\eta(r) = 0$, if $r \geq 0$ and $\eta(r) = -r$, if $r < 0$. Set $k(r) = \eta^2(r)$ so that $k(r) = 0$ for $r \geq 0$ and $k(r) = r^2$ for $r < 0$. It can be shown that, for $\epsilon > 0$, there is a $\mathbf{C}^2-$regularization $k_\epsilon(r)$ of $k(r)$ which satisfies the conditions: $k'_\epsilon(r) = 0$ for $r \geq 0$; $k'_\epsilon(r) \leq 0$ and $k''_\epsilon(r) \geq 0$ for any $r \in \mathbb{R}$. Moreover, as $\epsilon \to 0$, we have

$$k_\epsilon(r) \to k(r), \quad k^{'}_\epsilon(r) \to -2\,\eta(r) \quad \text{and} \quad k^{''}_\epsilon(r) \to 2\theta(r), \tag{2.11}$$

for any $r \in \mathbb{R}$, where $\theta(r) = 0$ for $r \geq 0$, $\theta(r) = 1$ for $r < 0$.

Let $u_t = u(\cdot, t)$ denote the solution of the parabolic Itô equation (2.3). Define

$$\Phi_\epsilon(u_t) = (1, k_\epsilon(u_t)) = \int_{\mathcal{D}} k_\epsilon(u(x,t))dx. \tag{2.12}$$

Then, by applying the Itô formula, we can obtain the following formula

$$\begin{cases} \Phi_\epsilon(u_t) = \Phi_\epsilon(g) - \displaystyle\int_0^t \int_{\mathcal{D}} k''_\epsilon(u(x,s))\, b(x, \nabla u(x,s))\, dx\, ds \\ + \displaystyle\int_0^t \int_{\mathcal{D}} k'_\epsilon(u(x,s)) f(u,x,s)\, dx\, ds \\ + \displaystyle\int_0^t \int_{\partial\mathcal{D}} k'_\epsilon(h(x))\, \frac{\partial}{\partial\nu} u(x,s)\, dS\, ds \\ + \displaystyle\frac{1}{2}\int_0^t \int_{\mathcal{D}} k''_\epsilon(u(x,s)) q(x,x)\, \sigma^2(u, \nabla u, x, s)\, dx\, ds \\ + \displaystyle\int_0^t \int_{\mathcal{D}} k'_\epsilon(u(x,s)) \sigma(u, \nabla u, x, s)\, dW(x,s) dx, \end{cases} \tag{2.13}$$

where $b(x,\xi)$ is defined by (2.4), dS is the element of surface area on $\partial\mathcal{D}$, and $\frac{\partial}{\partial\nu}$ denotes the differentiation with respect to the conormal vector field $\nu = (\nu_1, \cdots, \nu_d)$ with

$$\nu_i(x) := \sum_{j=1}^{d} a_{ij}(x) n_j, \tag{2.14}$$

and $\mathbf{n} = (n_1, \cdots, n_d)$ being the unit outward normal vector to the boundary $\partial\mathcal{D}$.

After taking an expectation over equation (2.13), we get

$$\left\{\begin{aligned} E\,\Phi_\epsilon(u_t) = \Phi_\epsilon(g) + E \int_0^t \int_{\mathcal{D}} \{\, k_\epsilon''(u(x,s))\, [\frac{1}{2}\, q(x,x)\, \sigma^2(u, \nabla u, x, s) \\ -b(x, \nabla u(x,s))] + \; k_\epsilon'(u(x,s)) f(u,x,s) \,\}\, dx\, ds \\ + E \int_0^t \int_{\partial\mathcal{D}} k_\epsilon'(h(x))\, \frac{\partial}{\partial\nu} u(x,s)\, dS\, ds. \end{aligned}\right. \tag{2.15}$$

By making use of condition (P1) and the properties of k_ϵ we can deduce from equation (2.15) that

$$\left\{\begin{aligned} & E\,\Phi_\epsilon(u_t) \le \Phi_\epsilon(g) + \delta\, E \int_0^t \int_{\mathcal{D}} k_\epsilon''(u(x,s))\, |u(x,s)|^2\, dx\, ds \\ & + E \int_0^t \int_{\mathcal{D}} k_\epsilon'(u(x,s)) f(u,x,s)\, dx\, ds \\ & + E \int_0^t \int_{\partial\mathcal{D}} k_\epsilon'(h(x))\, \frac{\partial}{\partial\nu} u(x,s)\, dS\, ds. \end{aligned}\right. \tag{2.16}$$

Note that $\lim_{\epsilon\to 0} E\,\Phi_\epsilon(u_t) = E\,\|\eta(u_t))\|^2$. By taking the limits termwise as $\epsilon \to 0$ and making use of (2.11), the equation (2.16) yields

$$\left\{\begin{aligned} & E \int_{\mathcal{D}} |\eta(u(x,t))|^2\, dx \le \int_{\mathcal{D}} |\eta(g(x))|^2\, dx \\ & + 2\,\delta\, E \int_0^t \int_{\mathcal{D}} \theta(u(x,s))\, |u(x,s)|^2\, dx\, ds \\ & - 2\, E \int_0^t \int_{\mathcal{D}} \eta(u(x,s)) f(u,x,s)\, dx\, ds \\ & - 2\, E \int_0^t \int_{\partial\mathcal{D}} \eta(h(x))\, \frac{\partial}{\partial\nu} u(x,s)\, dS\, ds. \end{aligned}\right. \tag{2.17}$$

By definition of η and conditions (P2) and (P3), we have $\eta(g) = \eta(h) = 0$, $\theta(u)u^2 = \eta^2(u)$ and $\eta(u)f(u, \xi, s) \geq 0$ so that equation (2.17) can be reduced simply to

$$E\,\|\eta(u_t))\|^2 \leq 2\delta \int_0^t E\,\|\eta(u_s))\|^2\,ds,$$

which, by means of the Gronwall inequality, implies that

$$E\,\|\eta(u_t))\|^2 = E\int_{\mathcal{D}} |\eta(u(x,t))|^2\,dx = 0 \quad \forall\, t \in [0,T].$$

It follows that $\eta(u(x,t)) = u^-(x,t) = 0$ a.s. for a.e. $x \in \mathbb{R}$ and $t \in [0,T]$. The theorem is thus proved. $\square$

Remark: The above theorem shows that, under appropriate conditions, a global solution is positive. This is also true for a local solution u_t before the explosion occurs. The proof can be carried out similarly as before by localization, that is, replacing t in the integrals by $\tau_t = (t \wedge \tau)$, where τ is a stopping time.

3 Existence of Explosive Solutions

Now we consider the unbounded solutions to the stochastic reaction-diffusion equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = Au + f(u,x,t) + \sigma(u,x,t)\partial_t W(x,t), \\ u(x,0) = g(x), \quad x \in \mathcal{D}, \\ u(x,t)|_{\partial\mathcal{D}} = 0, \quad t \in (0,T), \end{cases} \tag{3.18}$$

which is a special case of equation (2.3), where σ is independent of the gradient ∇u. Before proceeding to the key theorems, we consider the eigenvalue problem for the elliptic equation:

$$\begin{cases} Av = -\lambda v \quad \text{in} \quad \mathcal{D}, \\ v = 0 \quad \text{on} \quad \partial\mathcal{D}. \end{cases} \tag{3.19}$$

It is well known that all the eigenvalues are strictly positive, increasing, and the eigenfunction ϕ corresponding to the smallest eigenvalue λ_1 does not change sign in the domain $\mathcal{D}$ (see pp.451-455, [4]). Therefore we can normalize it in such a way that

$$\phi(x) \geq 0, \quad \int_{\mathcal{D}} \phi(x)\, dx = 1. \tag{3.20}$$

To prove the main theorems, we impose the following Conditions N on the nonlinear function f for the reaction rate:

(N1) There exists a constant $r_1 > 0$ and a continuous function $F(r)$ such that F is convex, positive and strictly increasing for $r \geq r_1$ and satisfy

$$f(r, x, t) \geq F(r),$$

for any $x \in \overline{\mathcal{D}}$, $t \in [0, \infty)$.

(N2) There exists a constant $M_1 > r_1$ such that $F(r) > \lambda_1 r$ for $r \geq M_1$.

(N3) The positive initial datum satisfies the condition

$$(\varphi, u_0) = \int_{\mathcal{D}} \varphi(x) g(x)\, dx > M_1.$$

(N4) The following integral is convergent so that

$$\int_{M_1}^{\infty} \frac{dr}{F(r) - \lambda_1 r} dr < \infty.$$

Alternatively we impose the following Conditions S on the noise term:

(S1) The correlation function $q(x, y)$ is continuous and positive for $x, y \in \overline{\mathcal{D}}$ such that

$$\int_{\mathcal{D}} \int_{\mathcal{D}} q(x, y) v(x) v(y) dx dy \geq q_1 \int_{\mathcal{D}} v^2(x) dx$$

for any positive $v \in H$ and for some $q_1 > 0$.

(S2) There exist a constant $r_2 > 0$, and continuous functions $\sigma_0(r)$ and $G(r)$ such that they are both convex, positive and strictly increasing for $r \geq r_2$ and satisfy

$$\sigma(r, x, t) \geq \sigma_0(r) \quad \text{and} \quad \sigma_0^2(r) \geq 2G(r^2),$$

for $r \geq r_2$, $x \in \overline{\mathcal{D}}$, $t \in [0, \infty)$.

(S3) There exists a constant $M_2 > r_2$ such that $q_1 G(r) > \lambda_1 r$, for $r > M_2$.

(S4) The positive initial datum satisfies the condition

$$(\varphi, u_0) = \int_{\mathcal{D}} \varphi(x) g(x)\, dx > M_2.$$

(S5) The following integral is convergent so that

$$\int_{M_2}^{\infty} \frac{dr}{q_1 G(r) - \lambda_1 r} dr < \infty.$$

The following theorem is concerned with explosive solutions under Conditions N or Conditions S, which will be called case N and case S, respectively. It is an extension of Theorem 3.3 in [3] to include case S for noise-induced explosion. Since the proof of case S is a generalization of that for case N, for the sake of continuity and completeness, the proofs for both cases will be given.

Theorem 3.1 Suppose the initial-boundary value problem (3.18) has a unique local solution and the conditions (P1)–(P3) are satisfied. In addition we assume that either the conditions (N1)–(N4) or the alternative conditions (S1)–(S5) given above hold true. Then, for an integer $p > 0$, there exists a constant $T_p > 0$ such that

$$\lim_{t \to T_p^-} E\, \|u_t\|_p = \lim_{t \to T_p^-} E\, \{ \int_{\mathcal{D}} |u(x,t)|^p\, dx \}^{1/p} = \infty, \tag{3.21}$$

or the solution explodes in the mean L^p-norm as shown by (3.21), where $p \geq 1$ under Conditions N, while $p \geq 2$ under Conditions S.

Proof Under conditions (P1)–(P3), by Theorem 2.1, the equation (3.18) has a unique positive solution. We will prove the theorem by contradiction. First we suppose conditions (N1)–(N4) are satisfied but the conclusion (3.21) is false. Then there exist a global positive solution u and a real number $p \geq 1$ such that

$$\sup_{0 \leq t \leq T} E \{ \int_{\mathcal{D}} |u(x,t)|^p \, dx \}^{1/p} < \infty, \tag{3.22}$$

for any $T > 0$. To reach a contradiction, let ϕ be the eigenfunction as given by (3.19) and define

$$\hat{u}(t) := \int_{\mathcal{D}} u(x,t)\phi(x) \, dx \geq 0. \tag{3.23}$$

Since ϕ is positive and normalized as in (3.20), it can be regarded as the probability density function of a random variable ξ in $\mathcal{D}$, independent of W_t, and the above integral can be interpreted as an expectation $\hat{u}(t) = E_\xi\{u(\xi,t)\}$ with respect to this random variable. Since $\hat{u}$ is a linear functional of u, we can deduce from (3.19) and (3.23) that

$$\left\{ \begin{aligned} \hat{u}(t) = (g,\phi) &+ \int_0^t \int_{\mathcal{D}} [Au(x,t)]\phi(x) \, dx \, ds \\ &+ \int_0^t \int_{\mathcal{D}} f(u,x,s) \, \phi(x) \, dx \, ds \\ &+ \int_0^t \int_{\mathcal{D}} \sigma(u,x,s) \, \phi(x) \, dW(x,s) \, dx. \end{aligned} \right. \tag{3.24}$$

Recall that A is self-adjoint. So we have $\langle Au, \phi \rangle = (u, A\phi) = -\lambda_1(u,\phi)$. After taking the expectation $E\{\cdot\}$ over equation (3.24) and changing the order of the expectation and an integration by appealing to Fubini's theorem, we obtain

$$\left\{ \begin{aligned} &E\,\hat{u}(t) = (g,\phi) - \lambda_1 \int_0^t E\,\hat{u}(s) \, ds \\ &+ \int_0^t E \int_{\mathcal{D}} f(u,x,s) \, \phi(x) \, dx \, ds, \end{aligned} \right.$$

or, in the differential form,

$$\begin{cases} \dfrac{d\mu(t)}{dt} = -\lambda_1 \mu(t) + E\displaystyle\int_{\mathcal{D}} f(u,x,t)\,\phi(x)\,dx, \\ \mu(0) = \mu_0, \end{cases} \tag{3.25}$$

where we set $\mu(t) = E\,\hat{u}(t)$ and $\mu_0 = (g, \phi)$. In view of condition (N1), the equation (3.25) yields

$$\begin{cases} \dfrac{d\mu(t)}{dt} \geq -\lambda_1 \mu(t) \,+\, E\displaystyle\int_{\mathcal{D}} F(u(x,t))\,\phi(x)\,dx, \\ \mu(0) = \mu_0. \end{cases} \tag{3.26}$$

By condition (N1), $F(r)$ is convex and positive for $r > r_1$ so that Jensen's inequality gives us

$$\begin{cases} E\displaystyle\int_{\mathcal{D}} F(u(x,t))\,\phi(x)\,dx = E\,E_\xi\,F(u(\xi,t)) \\ \qquad\qquad \geq F(E\,E_\xi\,u(\xi,t)) = F(\mu(t)). \end{cases} \tag{3.27}$$

By taking (3.26), (3.27) and conditions (N2)–(N3) into account, we find

$$\begin{cases} \dfrac{d\mu(t)}{dt} \geq F(\mu(t)) - \lambda_1 \mu(t), \\ \mu(0) = \mu_0 = (\varphi, g), \end{cases} \tag{3.28}$$

which implies, for $\mu_0 > M_1$, $F(\mu(t)) - \lambda_1 \mu(t) > 0$ and $\mu(t) > \mu_0$ for $t > 0$. An integration of equation (3.28) yields

$$T \leq \int_{\mu_0}^{\mu(T)} \frac{dr}{F(r) - \lambda_1 r} \leq \int_{M_1}^{\infty} \frac{dr}{F(r) - \lambda_1 r}. \tag{3.29}$$

But, by condition (N4), the last integral is bounded. Hence the inequality (3.29) cannot hold for a sufficiently large T. This contradiction shows that $\mu(t) = E\displaystyle\int_{\mathcal{D}} u(x,t)\phi(x)\,dx$ must blow up at a time $T_e \leq \displaystyle\int_{\mu_0}^{\infty} \frac{dr}{F(r) - \lambda_1 r}$.

Since ϕ is bounded and continuous on $\overline{\mathcal{D}}$, we apply Hölder's inequality for each $p \geq 1$ to get

$$\mu(t) \leq C_p\, E\, \{ \int_{\mathcal{D}} |u(x,t)|^p\, dx \}^{1/p},$$

where $C_p = \{ \int_{\mathcal{D}} |\phi(x)|^q\, dx \}^{1/q}$ with $q = p/(p-1)$. Therefore we can conclude that the positive solution explodes at some time $T_p \leq T_e$ in the mean L^p-norm for each $p \geq 1$, as asserted by equation (3.21).

Now we suppose that the alternative conditions (S1)-(S5) hold true but the assertion (3.21) is false. Then the solution u exists and, for some $p \geq 2$, $E\, \|u_t\|_p < \infty$ for any $T > 0$.

Let $\hat{u}(t) = (\varphi, u_t)$ as defined as before. By applying Itô's formula to $\hat{u}^2(t)$ and making use of (3.24), we can obtain

$$\left\{ \begin{aligned} \hat{u}^2(t) = {} & (g,\phi)^2 - 2\lambda_1 \int_0^t \int_{\mathcal{D}} \hat{u}^2(s)\, ds + 2 \int_0^t \int_{\mathcal{D}} \hat{u}(s) f(u,x,s)\, \phi(x)\, dx\, ds \\ & +2 \int_0^t \int_{\mathcal{D}} \hat{u}(s)\sigma(u,x,s)\, \phi(x)\, dW(x,s)\, dx \\ & + \int_0^t \int_{\mathcal{D}} \int_{\mathcal{D}} q(x,y)\phi(x)\phi(y)\sigma(u,x,s)\, \sigma(u,y,s)\, dxdyds. \end{aligned} \right. \tag{3.30}$$

Let $\eta(t) = E\hat{u}^2(t)$. By taking an expectation over equation (3.30), it yields

$$\left\{ \begin{aligned} \eta(t) = {} & (g,\phi)^2 - 2\lambda_1 \int_0^t \eta(s)\, ds + 2\, E \int_0^t \int_{\mathcal{D}} \hat{u}(s) f(u,x,s)\, \phi(x)\, dx\, ds \\ & +E \int_0^t \int_{\mathcal{D}} \int_{\mathcal{D}} q(x,y)\phi(x)\phi(y)\sigma(u,x,s)\, \sigma(u,y,s)\, dxdyds. \end{aligned} \right. \tag{3.31}$$

or, in the differential form,

$$\left\{ \begin{aligned} & d\eta(t) = [-2\lambda_1 \eta(t) + 2\, E\, \hat{u}(t) \int_{\mathcal{D}} f(u,x,t)\, \phi(x)\, dx \\ & \qquad +E \int_{\mathcal{D}} \int_{\mathcal{D}} q(x,y)\phi(x)\phi(y)\sigma(u,x,t)\, \sigma(u,y,t)\, dxdy\,]\, dt, \\ & \eta(0) = \eta_0 = (g,\phi)^2. \end{aligned} \right. \tag{3.32}$$

Making use of conditions (S1) and (S2), the Jensen inequality as well as the Cauchy-Schwarz inequality, we can get

$$\begin{cases} \int_{\mathcal{D}}\int_{\mathcal{D}} q(x,y)\phi(x)\phi(y)\sigma(u,x,t)\,\sigma(u,y,t)\,dxdy \\ \geq q_1\int_{\mathcal{D}} \phi^2(x)\sigma_0^2(u)\,dx \geq q_1[\int_{\mathcal{D}} \phi(x)\sigma_0(u)\,dx]^2 \\ \geq q_1\,\sigma_0^2(\hat{u}(t)) \geq 2\,q_1\,G(\hat{u}^2(t)). \end{cases} \tag{3.33}$$

In view of (3.33) and the fact that the second term on the right-hand side of equation (3.32) is positive, we can deduce from (3.32) that

$$\begin{cases} d\eta(t) \geq [-2\lambda_1\eta(t) + 2q_1\,E\,G(\hat{u}^2(t))]dt \\ \geq [-2\lambda_1\eta(t) + 2q_1\,G(\eta_t)]dt, \end{cases} \tag{3.34}$$

where the Jensen inequality was used one more time. Similar to the previous case (3.29), it follows from (3.34) that, for $\eta_0 > M_2$,

$$T \leq \frac{1}{2}\int_{\eta_0}^{\eta(T)} \frac{dr}{q_1G(r) - \lambda_1 r} \leq \int_{M_2}^{\infty} \frac{dr}{q_1G(r) - \lambda_1 r} < \infty,$$

where the integrals are well defined as ensured by conditions (S3) and (S5). Again this shows that T cannot be arbitrarily large. Hence the mean-square $\eta_t = E\,(\varphi, u_t)^2$ must blow up at some finite time $T_e > 0$. It follows from the Hölder inequality that the assertion (3.21) must hold for any $p \geq 2$. □

Now we consider the Cauchy problem for equation (2.3) in an unbounded domain $\mathcal{D} = \mathbb{R}^d$, where the boundary condition is omitted. Let $B(R) = \{x \in \mathbb{R}^d : |x| < R\}$ be an open ball of radius R in $\mathbb{R}^d$. In this case Theorem 3.1 still holds in the mean L^p-norm on $B(R)$ for any $R > 0$ as indicated in the following theorem.

Theorem 3.2 Suppose the conditions for Theorem 3.1 hold with $\mathcal{D} = \mathbb{R}^d$. Then the solution u of the Cauchy problem for equation (3.18) explodes in the mean $L^p(B(R))-$norm, or, for a positive p, there is a constant $T_p(R) > 0$ such that

$$\lim_{t \to T_p^-(R)} E\{\int_{B(R)} |u(x,t)|^p \, dx\}^{1/p} = \infty, \tag{3.35}$$

for any $R > 0$, where $p \geq 1$ under Conditions N, while $p \geq 2$ under Conditions S.

Proof We will only sketch the proof under Conditions S. The proof under Conditions N is similar.

Consider the eigenvalue problem (3.19) with $\mathcal{D} = B(R)$ and let ϕ be the eigenfunction normalized as in (3.20). By restricting the solution u to $\overline{B}(R)$, let $\hat{u}(t) = \int_{B(R)} u(x,t)\phi(x)\, dx$ as defined by (3.23). Then, as in the proof of Theorem 3.1, one can proceed to obtain equation (3.30) with an additional boundary integral term. This is the case because $u \geq 0$ on the boundary $\partial B(R)$. By Green's identity, instead of $\langle Au, \phi\rangle = -\lambda_1 \hat{u}(t)$, one would get

$$\langle Au_t, \phi\rangle = -\lambda_1 \hat{u}(t) + \int_{\partial B(R)} u(x,t)[-\frac{\partial \phi(x)}{\partial \nu}]\, dS. \tag{3.36}$$

Since the matrix $[a_{ij}(x)]$ is uniformly positive definite, in view of equation (2.4), $\nu \cdot \mathbf{n} = \Sigma_{i,j} a_{ij} n_i n_j \geq 0$. Hence the conormal $\nu(\mathbf{x})$ is an exterior direction field. Due to the fact that $\phi > 0$ in $B(R)$ and $\phi = 0$ on $\partial B(R)$, we have $\frac{\partial \phi(x)}{\partial \nu} \leq 0$. Therefore the extra term in (3.30):

$$2\, \hat{u}(t) \int_{\partial B(R)} u(x,t)[-\frac{\partial \phi(x)}{\partial \nu}]\, dS$$

is positive so that the differential inequality (3.34) remains valid and the rest of proof can be completed as in Theorem 3.1. □

4 Examples

As the first example, let us consider the following problem in a spherical domain $\mathcal{D} = B(R)$ in $\mathbb{R}^3$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla^2 u + |u|^{1+\alpha} + \gamma\, u\, \partial_t W(x,t), \\ u(x,0) = a_0\, e^{-\beta |x|}, \\ u(x,t)|_{|x|=R} = 0, \end{cases} \tag{4.37}$$

where $W(x,t)$ is a continuous Wiener random field with the covariance function

$$q(x,y) = b_0 \exp\{-\rho\ (x \cdot y)\}, \quad \text{for} \quad x, y \in \mathbb{R}^3. \tag{4.38}$$

All of the above constants $a_0, b_0, \alpha, \beta, \rho, \gamma$ are strictly positive and $x \cdot y = \sum_{i=1}^{3} x_i y_i$. Obviously the functions $f = |u|^{1+\alpha}$, $\sigma(t,x,u) = \gamma\, u$ and $g = a_0\, e^{-\beta|x|}$ satisfy conditions (P1)–(P3). By Theorem 2.1, the solution u of equation (4.37) is positive.

To determine sufficient conditions for explosion, consider the associated eigenvalue problem for the Laplace equation in $B(R)$. It is not hard to find the smallest eigenvalue $\lambda_1 = (\frac{\pi}{R})^2$ and the corresponding normalized eigenfunction $\phi(x) = \frac{C}{|x|} \sin \frac{\pi |x|}{R}$ for $|x| \leq R$ and $C = \frac{1}{4R^2}$. Let $F(r) = f(r) = r^{1+\alpha}$ with $\alpha > 0$ so that condition (N1) holds for any $r > 0$. Let M_1 be any number greater than $\lambda_1^{1/\alpha} = (\frac{\pi}{R})^{2/\alpha}$. For definiteness, take $M_1 = (\frac{2\pi}{R})^{2/\alpha}$. Then, for $r \geq M_1$,

$$F(r) - \lambda_1 r = r^{1+\alpha} - \lambda_1 r\ >\ 0,$$

so that condition (N2) is met. By some simple calculations, we can show that condition (N3) is satisfied if the initial amplitude a_0 is large enough such that

$$\frac{a_0}{R} \int_0^R r\ \exp\{-\beta r\}\ \sin \frac{\pi r}{R}\, dr > (\frac{2\pi}{R})^{2/\alpha}, \tag{4.39}$$

where the integral can be evaluated exactly but will not be given for brevity. For any $\alpha > 0$, the integral $\int_{M_1}^{\infty} \frac{dr}{r^{1+\alpha} - \lambda_1 r}$ is convergent so that condition (N4) holds. Therefore, by Theorem 3.1, the solution of the equation (4.37) will blow up in finite time in the mean L^p-norm for any $p \geq 1$. In view of Theorem 3.2, this is also true for the corresponding Cauchy problem in $\mathbb{R}^3$. Of course, in this case, the mean L^p-norm is restricted to any ball $B(R) \subset \mathbb{R}^3$.

As the second example, we consider the following initial-boundary value problem in a spherical domain as in the previous example:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla^2 u + k\, u + \gamma\, u^{1+\alpha}\, \partial_t W(x,t), \\ u(x,0) = a_0\, e^{-\,\beta|x|}, \\ u(x,t)|_{|x|=R} = 0, \end{cases} \tag{4.40}$$

where k and γ are constants, and the rest of parameters are the same as in (4.37). Let the correlation function $q(x,y)$ be given by (4.38) as before. Then we have

$$q(x,y) \geq q_1 = b_0 \exp\{-\rho\, R^2\}$$

for all $x, y \in B(R)$. Then, for any positive $v \in H$, we have

$$\int_{\mathcal{D}}\int_{\mathcal{D}} q(x,y)v(x)v(y)dxdy \geq q_1\, [\int_{\mathcal{D}} v(x)dx]^2$$

so that the condition (S1) is met. Clearly $\sigma(r,x,t) = \sigma_0(r) = \gamma\, r^{1+\alpha}$ is convex and so is $G(r)$ for any $r > 0$, where $2\, G(r^2) = \sigma_0^2(r) = \gamma^2\, (r^2)^{1+\alpha}$. So the condition (S2) is easily verified. Condition (S3) requires that

$$\frac{1}{2} q_1\, \gamma^2\, r^{1+\alpha} - \lambda_1 r > 0,$$

which holds for $r > M_2 = (\frac{4\lambda_1}{q_1 \gamma^2})^{1/\alpha}$. Similar to (4.39), condition (S4) is satisfied if the initial amplitude a_0 is large enough such that

$$\frac{a_0}{R} \int_0^R r \,\exp\{-\beta r\}\, \sin\frac{\pi r}{R}\, dr > (\frac{4\pi^2}{b_0 \gamma^2 R^2})^{1/\alpha} \exp\{-\rho\, R^2/\alpha\}. \tag{4.41}$$

For any $\alpha > 0$, the integral in condition (S5) is convergent so that, by Theorem 3.1, the mean L^p-norm of the solution will blow up in finite time for any $p \geq 2$. For $\mathcal{D} = \mathbb{R}^3$, by Theorem 3.2, this is also true for the corresponding Cauchy problem, for which the mean L^p-norm is restricted to any ball $B(R) \subset \mathbb{R}^3$.

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