WEAK SOLUTIONS TO STOCHASTIC WAVE EQUATIONS WITH VALUES IN RIEMANNIAN MANIFOLDS

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Abstract. Let M be a compact Riemannian manifold. We prove existence of a global weak solution of the stochastic wave equation $\mathbf{D}_t\partial_t u = \mathbf{D}_x\partial_x u + (X_u + \lambda_0(u)\partial_t u + \lambda_1(u)\partial_x u)\,\dot{W}$ where X is a continuous tangent vector field on M, λ_0, λ_1 are continuous vector bundles homomorphisms from TM to TM and W is a spatially homogeneous Wiener process on $\mathbb{R}$ with finite spectral measure. A new general method of constructing weak solutions of SPDEs that does not rely on martingale representation theorem is used.

1. Introduction

Wave equations subject to random excitations have been intensively studied in the last forty years for their applications in physics, relativistic quantum mechanics or oceanography, see for instance [5, 6, 8, 9, 12, 13, 27, 28, 29, 34, 36, 37, 11, 23, 24, 30, 38, 39] and references therein. The mathematical research has mostly considered stochastic wave equations in Euclidean spaces. However many physical theories and models in modern physics such as harmonic gauges in general relativity, non-linear σ-models in particle systems, electro-vacuum Einstein equations or Yang-Mills field theory require the target space of the solutions to be a Riemannian manifold ([19, 44]). One then usually speaks about *geometric wave equations (GWE).*

Let us briefly outline the historical development of the deterministic theory of geometric wave equations, about which we refer the reader to nice surveys in [44] and [51]. The existence and the uniqueness of global solutions is known to hold for wave equations with an arbitrary target manifold provided that the Minkowski space of the equation is either $\mathbb{R}^{1+1}$ or $\mathbb{R}^{1+2}$, see [19, 20, 45, 53] or [10, 31]. In the case of $\mathbb{R}^{1+1}$, the global solutions are known to exist in the weak [53], respectively the strong form [19, 20, 45, 26] depending on the regularity of the initial conditions. In the case of $\mathbb{R}^{1+2}$, the existence of global weak solutions has been established in [10] and [31]. The case of $\mathbb{R}^{1+d}$ for $d \geq 3$ is more interesting since simple counterexamples were constructed to show that smooth solutions may explode in finite time or that weak solutions can be non-unique, see for example

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[45, 7, 44]. Existence (without uniqueness) of global solutions on $\mathbb{R}^{1+d}$ was proved for compact homogeneous spaces in [16].

Stochastic wave equations with values in Riemannian manifolds, also called *stochastic geometric wave equations (SGWE)*, see equation (1.2) below, were first studied by the present authours in [1] and [3]. In those papers the existence and uniqueness of global strong solutions were established for such equations defined on the one-dimensional Minkowski space $\mathbb{R}^{1+1}$ in the case when the target manifold M is an arbitrary compact Riemannian manifold, the (nonlinear) diffusion coefficient is a C^1-class map from T^2M to TM of a sub-linear growth and the spectral measure of the Wiener process has finite moments up to order 2. It was assumed that the initial data $u(0), u_t(0)$ are from the space $H^2_{loc} \times H^1_{loc}$ and it was proved that there exists an $H^2_{loc} \times H^1_{loc}$-valued continuous process $(u, \partial_t u)$ that is a solution to the SGWE (1.2). Finally, natural definitions of an intrinsic and extrinsic solution were proposed and and their equivalence was proved.

In a subsequent paper [2] the existence of solutions to the SGWE was investigated when the target manifold M is of a special form. To be precise the existence (but not uniqueness) of a global weak solution to (1.2) defined on a Minkowski space $\mathbb{R}^{1+d}$ with values in a **compact Riemannian homogeneous space**. In particular, the existence of a global weak solution defined on a Minkowski space $\mathbb{R}^{1+d}$ with values in a sphere was proved. It was assumed that the (nonlinear) diffusion coefficient is of the following form

$$X_u + \lambda_0(u)\partial_t u + \sum_{j=1}^{d} \lambda_j(u)\partial_{x_j} u \tag{1.1}$$

with X and λ_j, $j = 0, \cdots, d$ being respectively a continuous vector field on M and vector bundles homomorphisms from TM to TM, and with W being a spatially homogeneous Wiener process on $\mathbb{R}^d$ whose spectral measure is finite. On the other hand it was possible to weaken the assumptions on the spectral measure and on the space regularity of the initial data. The price that had to be paid was the lower space-time regularity of the solution $(u, \partial_t u)$ which is only an $H^1_{loc} \times L^2_{loc}$-valued weakly continuous process.

The aim of the present paper is to generalise results from all three papers [1, 3, 2]. We establish the existence of a solution under weak regularity assumptions on the data as in the third paper [2] for a general target manifold M as in the first two papers [1, 3]. Another generalisation of the previous results is that the diffusion coefficient is time dependent and the spectral measure of the spatially homogeneous Wiener process is assumed to be only finite. The main result of the present paper Theorem 3.4 generalizes [2] in the one-dimensional $\mathbb{R}^{1+1}$ case since no restriction on the target manifold imposed. This can be seen as an analogue of results from [53] for SGSE as far as the existence is concerned. One should point out that although uniqueness of was also proved in [53], the length of the present paper, relative distinction of both the problem and the methods have led us to postpone the question of uniqueness to a separate paper.

Towards this end we assume that M is a compact Riemannian manifold and we consider the following stochastic wave equation (SGWE)

$$\mathbf{D}_t \partial_t u = \mathbf{D}_x \partial_x u + (X_u + \lambda_0(u)\partial_t u + \lambda_1(u)\partial_x u)\, \dot{W} \tag{1.2}$$

with a random initial condition $(u(0,x,\omega),\partial_t u(0,x,\omega)) = (u_0(\omega,x),v_0(\omega,x)) \in TM$. We assume that X is a continuous tangent vector field on M, λ_i, $i=0,\cdots,d$ are continuous vector bundles homomorphisms from TM to TM and W is a spatially homogeneous Wiener process on $\mathbb{R}$ with finite spectral measure, see Section 2 for details. By $\mathbf{D}$ we denote the connection on the pull-back bundle $u^{-1}TM$ induced by the Riemannian connection on M, see for instance [44]. Note however that deep understanding of the covariant derivative $\mathbf{D}$ is not necessary for reading this paper, see however [3], where an attempt was made to present the theory in a self-contained way, in particular to introduce the "acceleration" operators $\mathbf{D}_t\partial_t$ and $\mathbf{D}_x\partial_x$.

The equation (1.2) is written in a formal way and it was showed in [1] that the two rigorous definitions of a strong solution, intrinsic and extrinsic, are equivalent. The proof relies on the use of the Nash embedding theorem [32] according to which M may be embedded by a metric-preserving diffeomorphism into a certain euclidean space $\mathbb{R}^n$ so that M can be identified with its image in $\mathbb{R}^n$. We show that also in the setting of the the present paper that a the definitions of a weak intrinsic and weak extrinsic solutions are equivalent, see Theorem 3.2. Finally, we prove existence of a global weak solution of (1.2) - our Theorem 3.4.

Finally, we remark that our proof of the main theorem is based on a new general method of constructing weak solutions of SPDEs that does not rely on any kind of martingale representation theorem and that might be of interest itself (it was succesfully used for the first time in [2]).

Notation, Definitions and Conventions

- $\mathscr{S}_{\mathbb{R}}$, respectively $\mathscr{S}_{\mathbb{C}}$ denote the Schwartz space of real, respectively complex, valued C^∞-class of rapidly decreasing functions on $\mathbb{R}$,
- $\mathscr{S}'_{\mathbb{R}}$, respectively $\mathscr{S}'_{\mathbb{C}}$ denote the corresponding spaces of tempered distributions on $\mathbb{R}$,
- $\mathcal{F}(S) = \widehat{S}$ denotes the Fourier transform of a tempered distribution S,
- $\mathcal{J}_2(H,X)$ denotes the space of Hilbert-Schmidt operators from a Hilbert space H to a Hilbert space X,
- M is a d-dimensional compact submanifold in $\mathbb{R}^n$,
- T_pM and N_pM denote the tangent and the normal space respectively at $p \in M$,
- TM, NM denote the tangent and the normal bundle, respectively,
- $A_p : T_pM \times T_pM \to N_pM$ is the second fundamental form of the submanifold M in $\mathbb{R}^n$ at $p \in M$,
- For $k \in [0,\infty)$, $H^{k+1}_{loc} \times H^k_{loc}(TM)$ denotes the closed subset of the metric space $H^{k+1}_{loc}(\mathbb{R};\mathbb{R}^n) \times H^k_{loc}(\mathbb{R};\mathbb{R}^n)$ consisting of the elements (f,g) such that $(f(x),g(x)) \in TM$ for a.e. $x \in \mathbb{R}$,
- $T^2(M)$ denotes a vector bundle over M whose fiber at $p \in M$ is $T_xm \times T_xm$,
- X is a continuous vector field on M and, for any $i \in \{0,\cdots,d\}$, λ_i is a continuous vector bundles homomorphism from TM to TM, i.e. $\lambda_i(p)$, is a linear endomorphism on T_pM for every $p \in M$.

- Y is a map from $T^2(M)$ to TM of the following form
$$Y(p,\xi,\eta) = X(p) + \lambda_0(p)\xi + \lambda_i(p)\eta,\ t\in\mathbb{R}_+, x\in\mathbb{R}, p\in M, \xi,\eta\in T_pM. \tag{1.3}$$
- Whenever E is a vector class of functions defined on $\mathbb{R}$, $E_{\rm comp}$ will denote the subclass of those $f\in E$ which have compact support, for instance $L_{\rm comp}$.
- A spectral measure on $\mathbb{R}$ is a positive symmetric Borel measure on $\mathbb{R}$. A spectral measure satisfying $\mu(\mathbb{R})<\infty$ will be called a finite spectral measure.
- We use the standard convention (which anyhow follows from the definition) that $\inf\emptyset=\infty$.
- By $\xi\cdot\eta$ we shall often denote the Euclidean scalar product of vectors ξ, η in $\mathbb{R}^n$.

2. Spatially homogeneous Wiener process

Following [37] and [4] let us assume that μ is a finite symmetric Borel measure on $\mathbb{R}$ and let $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P})$, $(\mathscr{F}_t)_{t\ge 0}$ be a stochastic basis. A spatially homogeneous Wiener process with *spectral measure* μ can be introduced in two equivalent ways. The first one is to consider a centered Gaussian random field $(\mathcal{W}(t,x): t\ge 0, x\in\mathbb{R})$ such that for every $x\in\mathbb{R}$, $(\mathcal{W}(t,x): t\ge 0)$ is an $\mathbb{F}$-Wiener process and
$$\mathbb{E}\left\{\mathcal{W}(s,x)\mathcal{W}(t,y)\right\} = \min\{s,t\}\Gamma(x-y), \qquad t,s\ge 0,\quad x,y\in\mathbb{R} \tag{2.1}$$
where $\Gamma:\mathbb{R}\to\mathbb{R}$ is the Fourier transform of the measure $(2\pi)^{-\frac{1}{2}}\mu$, i.e.
$$\Gamma(x) = (2\pi)^{-1}\int_{\mathbb{R}} e^{-ix\xi}\mu(d\xi),\ x\in\mathbb{R}.$$
The second is to consider an $\mathscr{S}'_{\mathbb{R}}$-valued $\mathbb{F}$-Wiener process satisfying the following condition, with $s\wedge t := \min\{s,t\}$,
$$\mathbb{E}\left\{\langle W(s),\varphi_0\rangle\langle W(t),\varphi_1\rangle\right\} = s\wedge t\,\langle\widehat{\varphi}_0,\widehat{\varphi}_1\rangle_{L^2(\mu)}, \qquad t,s\ge 0,\quad \varphi_0,\varphi_1\in\mathscr{S}_{\mathbb{R}}. \tag{2.2}$$
The equivalence between these two points of view is best seen from the following formula, see for instance [37, p. 190],
$$\langle W(t),\varphi\rangle = \int_{\mathbb{R}}\mathcal{W}(t,x)\varphi(x)\,dx, \qquad t\ge 0,\quad \varphi\in\mathscr{S}_{\mathbb{R}}. \tag{2.3}$$
Here however we will leave aside this question and use only the second approach.

The following result describes the reproducing kernel Hilbert space (RKHS) of a spatially homogeneous Wiener process and some of its properties, see Proposition 1.2 in [37] and Lemma 1 in [34].

Proposition 2.1. *Let W be a spatially homogeneous Wiener process with a finite spectral measure μ. Let H_μ the reproducing kernel Hilbert space*[1] *of W. Then the following equality holds*
$$\begin{aligned} H_\mu &= \{\widehat{\psi\mu}:\psi\in L^2_{\mathbb{C}}(\mathbb{R},\mu),\ \overline{\psi(x)}=\psi(-x)\}, && (2.4)\\ &\langle\widehat{\psi_0\mu},\widehat{\psi_1\mu}\rangle_{H_\mu} = \langle\psi_0,\psi_1\rangle_{L^2(\mu)},\ \widehat{\psi_0\mu},\widehat{\psi_1\mu}\in H_\mu. && (2.5)\end{aligned}$$

[1]More precisely, H_μ is the RKHS of the law of the $\mathscr{S}'_{\mathbb{R}}$-valued gaussian random variable $W(1)$.

Moreover, H_μ is continuously embedded in the space $C_b(\mathbb{R})$ of real continuous and bounded functions on $\mathbb{R}$ and there exists a constant $c \in \mathbb{R}_+$ such that

$$\|\xi \mapsto h\xi\|_{\mathcal{J}_2(H_\mu, L^2(\mathbb{R}))} = c\mu(\mathbb{R})]^{\frac{1}{2}}|h|_{L^2(\mathbb{R})}, \qquad h \in L^2_{\mathbb{R}}(\mathbb{R}). \tag{2.6}$$

2.1. **Stochastic integration.** If X and H are separable Hilbert spaces, the latter being real, and W is an H-cylindrical $\mathbb{F}$-Wiener process then the Itô integral $\int_0^T h\,dW$ can be constructed as an X-valued random variable provided that $h \in \mathcal{N}^2(0,T;\mathcal{J}_2(H,X))$, i.e. h is an $\mathbb{F}$-progressively measurable processes with values in $\mathcal{J}_2(H,X)$ and

$$\int_0^T \|h(s)\|^2_{\mathcal{J}_2(H,X)}\,ds < \infty. \tag{2.7}$$

See for instance [14] for details. Moreover, there exists an X-valued continuous $\mathbb{F}$-local martingale $M = (M(t))_{t\in[0,T]}$ such that for every $t \in [0,T]$, $\mathbb{P}$ almost surely,

$$M(t) = \int_0^t h(s)\,dW(s).$$

With a slight abuse of notation such martingales will often be denoted by $\int_0^t h(s)\,dW(s)$, $t \geq 0$.

We will often consider integrals, including Itô integrals, taking values in the local spaces, for instance $H^l_{\text{lok}}(\mathbb{R})$, where $l > 0$. We will thus write that integrals converge in $H^l_{\text{lok}}(\mathbb{R})$ whenever they converge in the Hilbert spaces $H^l(-R,R)$ for every $R > 0$.

Remark 2.2. It is known that reproducing kernel Hilbert space of a spatially homogeneous $\mathbb{F}$-Wiener process W with spectral measure μ is equal to H_μ.

A proof of the following proposition is based on the Garsia-Rodemich-Rumsey lemma [18] and can be found for instance in Lemma 4 in [36].

Proposition 2.3. *Let $p, r \in (2,\infty)$ and $\gamma \in (0,\frac{1}{2})$ satisfy $\gamma + \frac{1}{p} + \frac{1}{r} < \frac{1}{2}$ and let K be a separable Hilbert space. Then there exists a constant c_* such that*

$$\mathbb{E}\left|\int_0^\cdot \psi(s)\,dW\right|^p_{C^\gamma([0,t];K)} \leq c_*\mathbb{E}\left(\int_0^t \|\psi(s)\|^r_{\mathcal{J}_2(U,K)}\,ds\right)^{\frac{p}{r}}, \qquad t \geq 0$$

holds for every cylindrical Wiener process W on some real separable Hilbert space U and every progressively measurable process ψ with paths in $L^r_{loc}(\mathbb{R}_+;\mathcal{J}_2(U,K))$.

We assume that (X,d) is a separable Fréchet space, see paper [21] by Hamilton, satisfying the following properties. There exist sequences $(X_n)_{n=1}^\infty$, $(\overset{\circ}{X}_n)_{n=1}^\infty$, $(\pi_n)_{n=1}^\infty$ and $(\pi_{nm})_{1\leq n\leq m<\infty}$ such that for each $n \in \mathbb{N}^*$, H^n is a separable Hilbert space, $\pi_n : X \to X_n$ and $\pi_{nm} : X_m \to X_n$ are continuous linear surjections, $\pi_{nm} \circ \pi_m = \pi_n$ for $n \leq m$, $\overset{\circ}{X}_n$ is a dense subspace of X_n, and a family $(\varphi_n)_{n=1}$ defined by $\varphi_n(x) = |\pi_n(x)|_{X_n}$, form a family of semi-norms on X defining the topology on X. Deterministic calculus for functions taking values in X is described in the above mentioned work by Hamilton. What concerns the stochastic calculus, it is natural to generalise the Itô integral to the current setting.

Let K be separable real Hilbert space and W is an K-cylindrical $\mathbb{F}$-Wiener process. We say that a process h belongs to $\mathcal{N}^2(0,T;\mathcal{J}_2(K,X))$, if and only if for each $n\in\mathbb{N}$, $\pi_n\circ h$ is an $\mathbb{F}$-progressively measurable $\mathcal{J}_2(K,X_n)$-valued process. We say that $h\in\mathcal{M}^2(0,T;\mathcal{J}_2(K,X))$ iff $h\in\mathcal{N}^2(0,T;\mathcal{J}_2(K,X))$ and for each $n\in\mathbb{N}$

$$\mathbb{E}\int_0^T \|\pi_n\circ h(s)\|^2_{\mathcal{J}_2(K,X_n)}\,ds<\infty. \tag{2.8}$$

It is possible to prove that for each $h\in\mathcal{M}^2(0,T;\mathcal{J}_2(K,X))$ there exists a unique X-valued continuous process M such that $n\in\mathbb{N}$, $\pi_n\circ M$ is an X_n-valued local martingale and, for each $t\in[0,T]$,

$$\pi_n\circ M(t)=\int_0^t \pi_n\circ h(s)\,dW(s). \tag{2.9}$$

Moreover, if $h\in\mathcal{M}^2(0,T;\mathcal{J}_2(K,X))$, then the process M is such that $n\in\mathbb{N}$, $\pi_n\circ M$ is an X_n-valued martingale and,

$$\mathbb{E}|\pi_n\circ M(t)|^2_{X_n}=\mathbb{E}\int_0^T \|\pi_n\circ h(s)\|^2_{\mathcal{J}_2(K,X_n)}\,ds. \tag{2.10}$$

3. Statements of the main results

Definition 3.1. *Let W be a spatially homogeneous Wiener process with a finite spectral measure μ.*
An intrinsic *solution to problem* (1.2) *is an $\mathbb{F}$-adapted weakly continuous $H^1_{loc}\times L^2_{loc}(TM)$-valued process $z=(u,v)$ such that*
for every $\omega\in\Omega$ and every $\varphi\in L^2_{\rm comp}(\mathbb{R})$ the equality

$$\frac{d}{dt}\langle u(\cdot,\omega),\varphi\rangle_{L^2(\mathbb{R})}=\langle v(\cdot,\omega),\varphi\rangle_{L^2(\mathbb{R})} \tag{3.1}$$

holds on $\mathbb{R}_+$,
for every smooth vector field Z on M and every $\varphi\in H^1_{\rm comp}(\mathbb{R})$ the equality

$$\begin{aligned}\langle v(t)\cdot Z(u(t)),\varphi\rangle_{L^2(\mathbb{R})} &= \langle v(0)\cdot Z(u(0)),\varphi\rangle_{L^2(\mathbb{R})}-\int_0^t\langle\partial_x u(s)\cdot Z(u(s)),\partial_x\varphi\rangle_{L^2(\mathbb{R})}\,ds\\ &- \int_0^t\big\langle\partial_x u(s)\cdot(\nabla_{\partial_x u(s)}Z)|_{u(s)},\varphi\big\rangle_{L^2(\mathbb{R})}\,ds\\ &+ \int_0^t\big\langle v(s)\cdot(\nabla_{v(s)}Z)|_{u(s)},\varphi\big\rangle_{L^2(\mathbb{R})}\,ds\\ &+ \int_0^t\langle[Y(u(s),v(s),\partial_x u(s))\cdot Z(u(s))]\,dW,\varphi\rangle_{L^2(\mathbb{R})}\end{aligned} \tag{3.2}$$

holds $\mathbb{P}$-$\mathbb{P}$ almost surely, for every $t\geq 0$.
An extrinsic *solution to problem* (1.2) *is an $\mathbb{F}$-adapted weakly continuous $H^1_{loc}\times L^2_{loc}(TM)$-valued process $z=(u,v)$ satisfying the condition* (3.1) *and, instead of (3.2), the following one.*

For any $\varphi \in H^1_{\mathrm{comp}}(\mathbb{R})$ *the equality*

$$\begin{aligned}\langle v(t),\varphi\rangle_{L^2(\mathbb{R})} &= \langle v(0),\varphi\rangle_{L^2(\mathbb{R})} - \int_0^t \langle \partial_x u(s), \partial_x\varphi\rangle_{L^2(\mathbb{R})}\,ds \\ &+ \int_0^t \left\langle A_{u(s)}\left(v(s),v(s)\right) - A_{u(s)}\left(\partial_x u(s),\partial_x u(s)\right), \varphi\right\rangle_{L^2(\mathbb{R})} ds \\ &+ \int_0^t \langle Y(u(s),v(s),\partial_x u(s))\,dW,\varphi\rangle_{L^2(\mathbb{R})} \end{aligned} \tag{3.3}$$

holds $\mathbb{P}$ *almost surely for every* $t \geq 0$.

The next result is about the relationship between two different types of solution. Its proof is postponed till section 8.

Theorem 3.2. *An* $\mathbb{F}$*-adapted weakly continuous* $H^1_{loc} \times L^2_{loc}(TM)$*-valued process* z *is an intrinsic solution if and only if it is an extrinsic solution.*

Hence the following definition is well posed.

Definition 3.3. *An intrinsic or an extrinsic solution is called a solution.*

Now we are ready to formulate the main result of our paper. Its proof will be preceded by some auxiliary results presented in the forthcoming sections.

Theorem 3.4. *Let* Θ *be a Borel probability measure on* $H^1_{loc} \times L^2_{loc}(TM)$ *and let* μ *be a finite spectral measure on* $\mathbb{R}$*. Then there exists a completely filtered probability space* $(\Omega,\mathscr{F},(\mathscr{F}_t),\mathbb{P})$*, a spatially homogeneous* $\mathbb{F}$*-Wiener process* W *with spectral measure* μ *and an* $\mathbb{F}$*-adapted process* z *with weakly continuous paths in* $H^1_{loc} \times L^2_{loc}(TM)$ *such that* (z,W) *is a solution and* Θ *is the law of* $z(0)$*. Morover, if* $q>0$ *and* $T>0$*, then there exists a constant* c *depending on* q*,* T*,* X*,* λ_0*,* λ_1 *and* $\mu(\mathbb{R})$ *such that*

$$\mathbb{E}\sup_{s\in[0,t]} I_{r,T}\mathbf{e}_{T,q}(s,z(s)) \leq 3e^{tc}\left\{\mathbb{E}\left[I_{r,T}\mathbf{e}_{T,2q}(0,z(0))\right]\right\}^{\frac{1}{2}} \tag{3.4}$$

holds for every $r>0$ *and* $t\in[0,T)$ *where* $I_{r,T} = \mathbf{1}_{[|z(0)|_{H^1(-T,T)\times L^2(-T,T)}\leq r]}$ *and*

$$\mathbf{e}_{T,q}(t,u,v) = \left(1 + \frac{1}{2}|u|^2_{L^2(-T+t,T-t)} + \frac{1}{2}|\partial_x u|^2_{L^2(-T+t,T-t)} + \frac{1}{2}|v|^2_{L^2(-T+t,T-t)}\right)^q \tag{3.5}$$

is defined for $(u,v) \in H^1_{loc}(\mathbb{R}) \times L^2_{loc}(\mathbb{R})$.

Remark 3.5. $\mathbf{e}_{T,q}$ is a local energy function and (3.5) is a uniform local energy estimation.

4. Approximation

Let us assume that $(\Omega,\mathscr{F},\mathbb{F},\mathbb{P})$, where $\mathbb{F}=(\mathscr{F}_t)_{t\geq 0}$ be a complete filtered probability space. Let $z_0=(u_0,v_0)$ be an $(\mathscr{F}_0)$-measurable $H^1_{loc} \times L^2_{loc}(TM)$-valued random variable whose law is Θ. Finally, let $(\beta^{ij} : i,j\in\mathbb{N})$ be a double sequence of i.i.d. standard $\mathbb{F}$-Wiener processes.

4.1. **Approximation of the initial condition.** By Lemma A.2 we can find a sequence $z_0^k = (u_0^k, v_0^k)$ $(\mathscr{F}_0)$-simple $H^2_{loc} \times H^1_{loc}(TM)$-valued random variables such that

$$z_0^k \to z_0 \text{ in } H^1_{loc}(\mathbb{R}) \times L^2_{loc}(\mathbb{R}) \text{ on } \Omega \tag{4.1}$$

and, for some $C_M > 0$ and $R > 0$ and $k \in \mathbb{N}$

$$|z_0^k|_{H^1(-R,R)\times L^2(-R,R)} \le C_M(R^{\frac{1}{2}} + |z_0|_{H^1(-R-1,R+1)\times L^2(-R-1,R+1)}) \text{ on } \Omega. \tag{4.2}$$

Remark 4.1. Approximation of the initial data z_0 by $H^2_{loc} \times H^1_{loc}(TM)$-valued random variables in $H^1_{loc} \times L^2_{loc}$-norm would be trivial and would follow from the density of $H^2_{loc} \times H^1_{loc}(TM)$ in $H^1_{loc} \times L^2_{loc}(TM)$ had we not required a sort of uniform approximation satisfying condition (4.2) which is not trivial and needs a justification.

4.2. **Approximation of the Wiener process.** It is well known, see [1], that there exists a strong solution of stochastic geometric wave equations available driven by spatially homogeneous Wiener processes with spectral measure having finite moments up to order 2. Since our assumptions on μ are much weaker, i.e. we only assume that μ is just a finite measure, a “localization” argument has to be emplyed. For this purpose we introduce the following sequence $(\nu_k)_{k=1}^\infty$ of symmetric Borel measures on $\mathbb{R}$

$$\nu_k(A) = \mu\big(A \cap (\bar{B}(0,k) \setminus \bar{B}(0,k-1))\big), k \in \mathbb{N}^*, \quad A \in \mathscr{B}(\mathbb{R}), \tag{4.3}$$

where $\bar{B}(0,k) := \{x \in A : k-1 < |x| \le k\}$ for $k \ge 1$ and $\bar{B}(0,0) := \emptyset$. We also introduce a corresponding sequence $(H_{\nu_k})_{k=1}^\infty$ of Hilbert spaces by

$$H_{\nu_k} = \{\widehat{\psi\nu_k} : \psi \in L^2_{\mathbb{C}}(\mathbb{R}, \nu_k), \overline{\psi(x)} = \psi(-x)\} \tag{4.4}$$

endowed with the following scalar products

$$\langle \widehat{\psi_0\nu_k}, \widehat{\psi_1\nu_k}\rangle_{H_{\nu_k}} = \langle \psi_0, \psi_1\rangle_{L^2(\nu_k)}.$$

We write

$$J_k := \begin{cases} \{1, \cdots, \dim(H_{\nu_k})\}, & \text{if } \dim(H_{\nu_k}) < \infty, \\ \mathbb{N}^*, & \text{otherwise.} \end{cases}$$

If $k \in \mathbb{N}^*$ then by $\{\xi_{kj} : j \in J_k\}$ we denote an orthonormal basis in H_{ν_k}. For each $k \in \mathbb{N}$ we consider a cylindrical $\mathbb{F}$-Wiener process W^k on H_{ν_k} of the following form

$$W^k(\varphi) = \sum_{i=1}^k \sum_{j\in J_i} \beta^{ij}\xi_{ij}(\varphi), \qquad \varphi \in \mathscr{S}_{\mathbb{R}}, \qquad k \in \mathbb{N}. \tag{4.5}$$

Note that W^k is a spatially homogeneous $\mathbb{F}$-Wiener processes with spectral measure $\mu_k := \sum_{i=1}^k \nu_i$.

The following results are simple and hence their proofs will be omitted.

Lemma 4.2. *For every $k \in \mathbb{N}$.*

$$\int_{\mathbb{R}} (1+y^2)\,\mu_k(dy) < \infty. \tag{4.6}$$

In particular, each measure μ_k, $k \in \mathbb{N}$, satisfies the condition (2.3) from [1].

Lemma 4.3. *The system* $\cup_{k=1}^{\infty}\{\xi_{kj} : j \in J_k\}$ *is an orthonormal basis in* H_μ.

4.3. **Approximation of the diffusion coefficient** Y**.** Theorem 11.1 in [1] requires the diffusion coefficient to be of C^1-class and to satisfy the growth conditions (2.1)-(2.2) from therein. In order to apply this result we have to approximate our Y in a suitable way. The following result shows that this is possible.

Proposition 4.4. *There exist the following objects.*

(i) *A sequence* $(X^k)_{k=1}^{\infty}$ *of smooth compactly supported functions* $X^k : \mathbb{R}^n \to \mathbb{R}^n$,
(ii) *a continuous compactly supported function* $\bar{X} : \mathbb{R}^n \to \mathbb{R}^n$,
(iii) *sequences* $\left(\lambda_0^k\right)_{k=1}^{\infty}$ *and* $\left(\lambda_1^k\right)_{k=1}^{\infty}$ *of smooth compactly supported* $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$*-valued functions* $\lambda_0^0, \lambda_1^k : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$
(iv) *continuous and compactly supported functions* $\bar{\lambda}_0, \bar{\lambda}_1 : \mathbb{R}^n \to \mathbb{R}^{n\times n}$

such that

- $X^k(p) \in T_pM$ *for every* $k \geq 1$ *and* $p \in M$,
- $\bar{X} = X$ *on* M,
- $\lambda_0^k(p)$, $\lambda_1^k(p)$ *map* T_pM *into itself for every* $p \in M$, $k \in \mathbb{N}^*$,
- $\bar{\lambda}_0 = \lambda_0$, $\bar{\lambda}_1 = \lambda_1$ *on* M,
- $X^k \to \bar{X}$, $\lambda_0^k \to \bar{\lambda}_0$, $\lambda_1^k \to \bar{\lambda}_1$ *uniformly on* $\mathbb{R}^n$.

In particular,

$$Y^k(p, \xi, \eta) = X^k(p) + \lambda_0^k(p)\xi + \lambda_1^k(p)\eta, \qquad p \in M, \quad \xi, \eta \in T_pM \tag{4.7}$$

satisfies the conditions (2.1) and (2.2) from [1] *for every* $k \in \mathbb{N}$ *and a map* $\bar{Y}$ *defined by*

$$\bar{Y}(q, \xi, \eta) = \bar{X}(q) + \bar{\lambda}_0(q)\xi + \bar{\lambda}_1(q)\eta, \qquad q, \xi, \eta \in \mathbb{R}^n \tag{4.8}$$

is an extension of the map Y.

Proof. Let U_P the neighbourhood of M introduced in Lemma A.1 and let P be the function also introduced in that Lemma. We define a a vector field $\tilde{X}$ on U_P by

$$\tilde{X}(q) = X(P(q)), \text{ for } q \in U_P.$$

Obviously, $\tilde{X}$ is an extension of the vector field X. Next, by employing the partition of unity, we can find a compactly supported continuous function $X^\circ : \mathbb{R}^n \to \mathbb{R}^n$ such that the restriction of X° to M equals X.

Now, let b be a smooth symmetric densities on $\mathbb{R}^n$ whose support is contained in a ball of radius 1. We put $b_k = k^{n/2}b(k\cdot)$, $k \in \mathbb{N}^*$. Let $\pi : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ be a C_0^∞ function such that

$$\pi_p \text{ is the orthogonal projection from } \mathbb{R}^n \text{ to } T_pM,\ p \in M.$$

Define next smooth compactly supported functions X^k by

$$X^k = \pi(b_k * X^\circ).$$

Then obviously, the restriction of X^k to M is a smooth vector field for each $k \in \mathbb{N}^*$. Moreover, X^k converges, as $k \to \infty$, uniformly on $\mathbb{R}^n$, to $\bar{X} := \pi(X^\circ)$.

Constructions of the approximation of the functions λ_0 and λ_1 are fully analogous. If $j \in \{0,1\}$ then $B_pZ := \lambda_j(p)\pi_pZ$, $Z \in \mathbb{R}^n$ defines a continuous $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$-valued function on M which can be extended to a compactly supported continuous $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$-valued function on $\mathbb{R}^n$, denoted again by B. We consider any smooth compactly supported $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$-valued functions B^k such that B^k converges uniformly to B on $\mathbb{R}^n$ and then we set $\lambda_j^k := \pi \circ B^k$ and $\bar\lambda_j := \pi \circ B$. □

4.4. **Solutions of the approximating problems.** It has been shown in [1, Theorem 11.1], that for each $k \in \mathbb{N}^*$ there exists an $\mathbb{F}$-adapted $H^2_{loc} \times H^1_{loc}(TM)$-valued continuous process $z^k = (u^k, v^k)$ such that

- every path of the process u^k belongs to $C^1\big(\mathbb{R}_+, H^1_{loc}(\mathbb{R})\big)$,
- $\frac{du^k}{dt}(t,\omega) = v^k(t,\omega)$ in $H^1_{loc}(\mathbb{R})$ for every $(t,\omega) \in \mathbb{R}_+ \times \Omega$,
- $z^k(0) = z^k_0$ $\mathbb{P}$-almost surely

and, for every $t \ge 0$ and $R > 0$, the following equality is satisfied in $L^2((-R,R);\mathbb{R}^n)$, $\mathbb{P}$ almost surely:

$$\begin{aligned} v^k(t) &= v^k_0 + \int_0^t \big[\partial_{xx}u^k(s) - A_{u^k(s)}(v^k(s),v^k(s)) + A_{u^k(s)}(\partial_x u^k(s),\partial_x u^k(s))\big]\,ds \\ &+ \int_0^t Y^k(u^k(s),v^k(s),\partial_x u^k(s))\,dW^k. \end{aligned} \tag{4.9}$$

Remark 4.5. By [1, Theorem 11.1] a strong solutions to problem (4.9) exists if the spectral measure μ_k satisfies condition (4.6) and the diffusion coefficient Y^k satisfies the growth and smoothness conditions (2.1)-(2.2) from therein.

Remark 4.6. A process $z^k = (u^k, v^k)$ satisfies the *extrinsic* equation (4.9) for every $t \ge 0$ $\mathbb{P}$-$\mathbb{P}$ almost surely if and only if it satisfies the following *intrinsic* equation (4.10) for every $t \ge 0$ $\mathbb{P}$ almost surely.

$$\begin{aligned} \langle v^k(t), Z(u^k(t))\rangle^n_{\mathbb{R}} &= \langle v^k_0, Z(u^k(0))\rangle^n_{\mathbb{R}} + \int_0^t \langle Y^k(u^k(s),v^k(s),\partial_x u^k(s)), Z(u^k(s))\rangle^n_{\mathbb{R}}\,dW^k \\ &+ \int_0^t \big[\langle \partial_{xx}u^k(s), Z(u^k(s))\rangle + \langle v^k(s), \nabla_{v^k(s)}Z|_{u^k(s)}\rangle^n_{\mathbb{R}}\big]\,ds \end{aligned} \tag{4.10}$$

whenever Z is a smooth vector field on M, see [1, Theorem 12.1].

In the following we will show that approximating processes (u^k, v^k) satisfy the following local energy inequality, where the local energy functional $\mathbf{e}_{T,q}$ has been defined in equality (3.5).

Theorem 4.7 (Uniform Local Energy Inequality)**.** *Let $q > 0$ and $T > 0$. Then there exists a constant c_* depending on q, T, X, λ_0, λ_1 and $\mu(\mathbb{R})$ such that for every $r > 0$, $t \in [0,T)$ and $k \in \mathbb{N}$*

$$\mathbb{E}\sup_{s\in[0,t]} I^k_{r,T}\mathbf{e}_{T,q}(s,z^k(s)) \le 3e^{tc_*}\left\{\mathbb{E}\,[I^k_{r,T}\mathbf{e}_{T,2q}(0,z^k(0))]\right\}^{\frac12} \tag{4.11}$$

where

$$I^k_{r,T} = \mathbf{1}_{\left\{z_0:|z^k(0)|_{H^1(-T,T)\times L^2(-T,T)}\leq r\right\}}. \tag{4.12}$$

Remark 4.8. The local energy inequality does not hold for any nonlinear wave equation unless, for instance, the drift nonlinearity $f(u,Du)$ in the equation either depends only on u and it is a gradient of a positive potential $f(u)=\nabla F(u)$, $F\geq 0$, see for instance [47, 49, 48], or it is orthogonal to the velocity, i.e. $\langle f(u,Du),\partial_t u\rangle = 0$ as it is so in our case. Indeed, u is manifold valued, $\partial_t u\in T_uM$ and the second fundamental form $A_u\in N_uM$.

Proof of Theorem 4.7. Let b be a smooth symmetric density on $\mathbb{R}$ with supports in $(-1,1)$. Define a sequence (b_j) by $b_j=\sqrt{j}b(\cdot/j)$.

Let us fix $k\in\mathbb{N}$. Define the following sequences of processes, $j\in\mathbb{N}$,

$$\begin{aligned} Z^j=(U^j,V^j) &:= z^k * b_j \\ a^j &= b_j * \left[A_{u^k}(\partial_x u^k,\partial_x u^k) - A_{u^k}(v^k,v^k)\right], \\ y^j\xi &= b_j * \left[Y^k(u^k,v^k,\partial_x u^k)\xi\right], \qquad \xi\in H_{\mu_k}. \end{aligned}$$

Then, for every $\omega\in\Omega$, we have

$$\begin{aligned} &\lim_{j\to\infty}\sup_{t\in[0,T]}|Z^j(t)-z^k(t)|_{H^2(-T,T)\times H^1(-T,T)}=0, \\ &\lim_{j\to\infty}\sup_{t\in[0,T]}\left|a^j(t)-\left[A_{u^k(t)}(\partial_x u^k(t),\partial_x u^k(t))-A_{u^k(t)}(v^k(t),v^k(t))\right]\right|_{H^1(-T,T)}=0. \end{aligned}$$

Let us choose and fix $q>0$, $R>0$ and $T>0$ and let us denote the local energy function $\mathbf{e}_{T,q}$ by $\mathbf{e}$.

Thus we infer that

$$\lim_{j\to\infty}\mathbf{e}(\cdot,Z^j(\cdot,\omega))=\mathbf{e}(\cdot,z^k(\cdot,\omega)) \text{ uniformly on } [0,T). \tag{4.13}$$

Moreover, since for $r=1$ is such that $\cup_j\mathrm{supp}\,(b_j)\subset(-r,r)$, for every i, every $s\in[0,T)$ and $\omega\in\Omega$,

$$\begin{aligned} &\lim_{j\to\infty}|y^j(s,\omega)\xi_i|^2_{L^2(-T+s,T-s)} = |Y^k(u^k(s,\omega),v^k(s,\omega),\partial_x u^k(s,\omega))\xi_i|^2_{L^2(-T+s,T-s)} \\ &|y^j(s,\omega)\xi_i|^2_{L^2(-T+s,T-s)} \leq |Y^k(u^k(s,\omega),v^k(s,\omega),\partial_x u^k(s,\omega))\xi_i|^2_{L^2(-T-r,T+r)}, \end{aligned}$$

Since by Proposition 2.1

$$\sum_i |Y^k(u^k,v^k,\partial_x u^k)\xi_i|^2_{L^2(-T-r,T+r)} = c\mu_k(\mathbb{R})|Y^k(u^k,v^k,\partial_x u^k)|^2_{L^2(-T-r,T+r)} \tag{4.14}$$

in view of the Lebesgue Dominated Convergence Theorem, we infer that for every $t\in[0,T)$,

$$\begin{aligned} &\lim_{j\to\infty}\int_0^t[\mathbf{e}(s,Z^j(s))]^{\frac{q-1}{q}}\sum_i|y^j(s)\xi_i|^2_{L^2(-T+s,T-s)}\,ds \qquad (4.15)\\ =\; &c\mu_k(\mathbb{R})\int_0^t[\mathbf{e}(s,z^k(s))]^{\frac{q-1}{q}}|Y^k(u^k(s),v^k(s),\partial_x u^k(s))|^2_{L^2(-T+s,T-s)}\,ds, \quad \text{on } \Omega. \end{aligned}$$

Finally, let us put

$$\begin{aligned} h_{ji}(s) &= [\mathbf{e}(s,Z^j(s))]^{\frac{q-1}{q}}\langle V^j(s),y^j(s)\xi_i\rangle_{L^2(-T+s,T-s)},\\ h_i(s) &= [\mathbf{e}(s,z^k(s))]^{\frac{q-1}{q}}\langle v^k(s),Y^k(u^k(s),v^k(s),\partial_x u^k(s))\xi_i\rangle_{L^2(-T+s,T-s)} \end{aligned}$$

Then, since $\lim_{j\to\infty} h_{ji} = h_i$ on $[0,T)\times\Omega$ and

$$|h_{ji}(s,\omega)-h_i(s,\omega)|^2 \le C_k(\omega)|Y^k(u^k(s),v^k(s),\partial_x u^k(s))\xi_i|^2_{L^2(-T-r,T+r)}.$$

and equality (4.14) we infer, by the Lebesgue Dominated Convergence Theorem, that on Ω,

$$\lim_{j\to\infty}\int_0^t \sum_i |h_{ji}(s)-h_i(s)|^2\,ds = 0, \quad \text{for every } t\in[0,T). \tag{4.16}$$

Let us observe that since for every $R\in\mathbb{N}$ the convolution operator b_j* is Hibert-Schmidt from the RKHS to $H^l(-R,R)$ by invoking Proposition 2.1 we infer that the following equality holds in $H^l_{\text{loc}}(\mathbb{R})$ for every $t\ge 0$ and every $l\in\mathbb{N}$, $\mathbb{P}$ almost surely,

$$\begin{aligned} U^j(t) &= U^j(0)+\int_0^t V^j(s)\,ds\\ V^j(t) &= V^j(0)+\int_0^t \left[\partial_{xx}U^j(s)+a^j(s)\right]\,ds+\int_0^t y^j(s)\,dW^k. \end{aligned}$$

Let us point out here that the stochastic integrals are convergent by Proposition 2.1.

Note the local energy function $\mathbf{e}=\mathbf{e}_{T,q}$ is of $C^{1,2}$-class on $[0,T)\times H^2(-R,R)\times H^1(-R,R)$ and, for $z=(u,v)\in H^2(-R,R)\times H^1(-R,R)$,

$$\begin{aligned} \partial_t\mathbf{e}(t,z) &= -\frac{q}{2}[\mathbf{e}(t,z)]^{\frac{q-1}{q}}\sum_{h\in\{u,\partial_x u,v\}}\left(|h(-T+t))|^2+|h(T-T))|^2\right)\\ \partial_z\mathbf{e}(t,z)x &= q[\mathbf{e}(t,z)]^{\frac{q-1}{q}}\left(\langle u,x^1\rangle_{H^1(-T+t,T-t)}+\langle v,x^2\rangle_{L^2(-T+t,T-t)}\right)\\ \partial_{zz}\mathbf{e}(t,z)(x_1,x_2) &= q(q-1)[\mathbf{e}(t,z)]^{\frac{q-2}{q}}\left(\langle u,x_1^1\rangle_{H^1(-T+t,T-t)}+\langle v,x_1^2\rangle_{L^2(-T+t,T-t)}\right)\\ &\times\ \left(\langle u,x_2^1\rangle_{H^1(-T+t,T-t)}+\langle v,x_2^2\rangle_{L^2(-T+t,T-t)}\right)\\ &+\ q[\mathbf{e}(t,z)]^{\frac{q-1}{q}}\left(\langle x_1^1,x_2^1\rangle_{H^1(-T+t,T-t)}+\langle x_1^2,x_2^2\rangle_{L^2(-T+t,T-t)}\right). \end{aligned}$$

Therefore the Itô formula, see for instance [14, Theorem 4.17], is applicable.

Note also that by the by integration by parts

$$\partial_t\mathbf{e}(t,Z^j)+\partial_z\mathbf{e}(t,Z^j)\begin{pmatrix} V^j\\ \partial_{xx}U^j+a^j\end{pmatrix} \le q[\mathbf{e}(t,Z^j)]^{\frac{q-1}{q}}\langle U^j+a^j,V^j\rangle_{L^2(-T+t,T-t)}$$

and, for any ONB (ξ_i) in H_{μ_k},

$$\begin{aligned}\sum_i \partial_{zz}\mathbf{e}(t,Z^j)\left(\begin{pmatrix}0\\ y^j\xi_i\end{pmatrix},\begin{pmatrix}0\\ y^j\xi_i\end{pmatrix}\right) &= \sum_i q(q-1)[\mathbf{e}(t,Z^j)]^{\frac{q-2}{q}}\langle V^j, y^j\xi_i\rangle^2_{L^2(-T+t,T-t)}\\ &+ \sum_i q[\mathbf{e}(t,Z^j)]^{\frac{q-1}{q}}|y^j\xi_i|^2_{L^2(-T+t,T-t)}\\ &\le c_q\sum_i[\mathbf{e}(t,Z^j)]^{\frac{q-1}{q}}|y^j\xi_i|^2_{L^2(-T+t,T-t)}\end{aligned}$$

Applying finally the Itô formula yields that for every $t\in[0,T)$

$$\begin{aligned}\mathbf{e}(t,Z^j(t)) &\le \mathbf{e}(0,Z^j(0))+\int_0^t q[\mathbf{e}(s,Z^j(s))]^{\frac{q-1}{q}}\langle U^j(s)+a^j(s),V^j(s)\rangle_{L^2(-T+s,T-s)}\,ds\\ &+ c_q\int_0^t[\mathbf{e}(s,Z^j(s))]^{\frac{q-1}{q}}\sum_i|y^j(s)\xi_i|^2_{L^2(-T+s,T-s)}\,ds\\ &+ \int_0^t q[\mathbf{e}(s,Z^j(s))]^{\frac{q-1}{q}}\langle V^j(s),y^j(s)\,dW^k\rangle_{L^2(-T+s,T-s)}\end{aligned}\tag{4.17}$$

$\mathbb{P}$ almost surely.

Note that by (4.13),(4.15),(4.16), in view of [35, Proposition 4.1], we have

$$\begin{aligned}&\lim_{j\to\infty}\int_0^t q[\mathbf{e}(s,Z^j(s))]^{\frac{q-1}{q}}\langle V^j(s),y^j(s)\,dW^k\rangle_{L^2(-T+s,T-s)}\\ &= \int_0^t q[\mathbf{e}(s,z^k(s))]^{\frac{q-1}{q}}\langle v^k(s),Y^k(u^k(s),v^k(s),\partial_x u^k(s))\,dW^k\rangle_{L^2(-T+s,T-s)}\end{aligned}\tag{4.18}$$

in probability.

Since $v^k\in T_{u^k}M$ and A_{u^k} takes values in $N_{u^k}M$, by letting $j\to\infty$ in (4.17), we obtain for every $t\in[0,T)$, $\mathbb{P}$ almost surely,

$$\begin{aligned}\mathbf{e}(t,z^k(t)) &\le \mathbf{e}(0,z^k(0))+\int_0^t q[\mathbf{e}(s,z^k(s))]^{\frac{q-1}{q}}\langle u^k(s),v^k(s)\rangle_{L^2(-T+s,T-s)}\,ds\\ &+ c_q\mu_k(\mathbb{R})\int_0^t[\mathbf{e}(s,z^k(s))]^{\frac{q-1}{q}}|Y^k(u^k(s),v^k(s),\partial_x u^k(s))|^2_{L^2(-T+s,T-s)}\,ds\\ &+ M_{k,q}(t),\end{aligned}\tag{4.19}$$

where $M_{k,q}$ is a martingale defined by

$$M_{k,q}(t)=\int_0^t q[\mathbf{e}_{T,q}(s,z^k(s))]^{\frac{q-1}{q}}\langle v^k(s),Y^k(u^k(s),v^k(s),\partial_x u^k(s))\,dW^k\rangle_{L^2(-T+s,T-s)}.$$

Since by the assumptions on the coefficient Y, the definitions of Y^k and $\mathbf{e}_{T,q}$ we have

$$|Y^k(u^k(s),v^k(s),\partial_x u^k(s))|^2_{L^2(-T+s,T-s)}\le C_{X,\lambda_0,\lambda_1,T}[\mathbf{e}_{T,q}(s,z^k(s))]^{\frac{1}{q}}\tag{4.20}$$

by (4.19) we infer that there exists a constant c_* depending only on q, T, X, λ_0, λ_1 and $\mu(\mathbb{R})$, such that for every $t\in[0,T)$, $\mathbb{P}$ almost surely,

$$\mathbf{e}_{T,q}(t,z^k(t))\le \mathbf{e}_{T,q}(0,z^k(0))+c_*\int_0^t \mathbf{e}_{T,q}(s,z^k(s))\,ds+M_{k,q}(t). \tag{4.21}$$

Note that by Proposition 2.1 and (4.20), the quadratic variation of $\langle M_{k,q}\rangle$ of $M_{k,q}$ satisfies

$$\begin{aligned}\langle M_{k,q}\rangle_t &= \sum_i\int_0^t q^2[\mathbf{e}_{T,q}(s,z^k(s))]^{\frac{2(q-1)}{q}}\langle v^k(s),Y^k(u^k(s),v^k(s),\partial_x u^k(s))\xi_i\rangle^2_{L^2(-T+s,T-s)}\,ds\\ &\le c_*\int_0^t \mathbf{e}_{T,2q}(s,z^k(s))\,ds,\ t\in[0,T).\end{aligned} \tag{4.22}$$

Define, for each $j\in\mathbb{N}$, an $\mathbb{F}$-stopping time τ_j, by

$$\tau_j=\inf\{t\in[0,T):\mathbf{e}_{T,q}(t,z^k(t))\ge j\}.$$

Note that the function $I^k_{r,T}$ defined in (4.12) is $\mathscr{F}_0$-measurable.

By inequalities (4.21) and (4.22) in view of the Gronwall Lemma we infer that

$$\mathbb{E}\,I^k_{r,T}\mathbf{e}_{T,q}(t\wedge\tau_j,z^k(t\wedge\tau_j))\le e^{tc_*}\mathbb{E}\,I^k_{r,T}\mathbf{e}_{T,q}(0,z^k(0)),\quad\text{for every } t\in[0,T).$$

Hence, by the Fatou Lemma,

$$\mathbb{E}\,I^k_{r,T}\mathbf{e}_{T,q}(t,z^k(t))\le e^{tc_*}\mathbb{E}\,I^k_{r,T}\mathbf{e}_{T,q}(0,z^k(0)),\ t\in[0,T). \tag{4.23}$$

On the other hand, denoting

$$e(t)=\sup_{s\in[0,t]}\mathbf{e}_{T,q}(s,z^k(s)),\qquad t\in[0,T)$$

we infer from (4.21) that

$$\begin{aligned}\mathbb{E}\,I^k_{r,T}e(t\wedge\tau_j) &\le \mathbb{E}\,I_r\mathbf{e}_{T,q}(0,z^k(0))+c_*\int_0^t\mathbb{E}\,I^k_{r,T}e(s\wedge\tau_j)\,ds\\ &+\mathbb{E}\sup_{s\in[0,t]}I^k_{r,T}|M_{k,q}(s\wedge\tau_j)|\end{aligned} \tag{4.24}$$

Since by the maximal Doob inequality for martingales, (4.22) and (4.23),

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}I^k_{r,T}|M_{k,q}(s\wedge\tau_j)| &\le \left[\mathbb{E}\sup_{s\in[0,t]}I^k_{r,T}|M_{k,q}(s\wedge\tau_j)|^2\right]^{\frac12}\le 2\left[\mathbb{E}\,I^k_{r,T}\langle M_{k,q}\rangle_{t\wedge\tau_j}\right]^{\frac12}\\ &\le 2e^{tc_*}\left[\mathbb{E}\,I^k_{r,T}\mathbf{e}_{T,2q}(0,z^k(0))\right]^{\frac12}\end{aligned}$$

the Gronwall inequality applied to (4.24) yields

$$\mathbb{E}\,I^k_{r,T}e(t\wedge\tau_j))\le 3e^{tc_*}\left(\mathbb{E}\,I^k_{r,T}\mathbf{e}_{T,2q}(0,z^k(0))\right)^{\frac12},\qquad t\in[0,T)$$

and the result follows by again applying the Fatou Lemma. □

5. Pseudointrinsic equation

We will see later in this paper that we can find a subsequence of $z^k = (u^k, v^k)$ that converges (on another probability space) to a limit $z = (u, v)$ in the locally uniform weak topology of $H^1_{loc}(\mathbb{R}) \times L^2_{loc}(\mathbb{R})$. Unfortunately, the nonlinearities in both the extrinsic equation (4.9) and the intrinsic equation (4.10) do not allow to pass in the limit. This is due to the fact that the weak convergence of z^k to z does not imply the convergence of nonlinear the terms $A_{u^k(s)}(v^k(s), v^k(s))$, $A_{u^k(s)}(\partial_x u^k(s), \partial_x u^k(s))$ in (4.9) and $\langle v^k(s), \nabla_{v^k(s)} Z|_{u^k(s)}\rangle_{\mathbb{R}^n}$ in (4.10) to $A_{u(s)}(v(s), v(s))$, $A_{u(s)}(\partial_x u(s), \partial_x u(s))$ and $\langle v(s), \nabla_{v(s)} Z|_{u(s)}\rangle_{\mathbb{R}^n}$ in any topological sense, respectively.

In order to resolve this difficulty by a forced strengthening of weak convergence to strong convergence which is done by mollification of the solutions z^k by a smooth compactly supported density b. For example, $b * u^k$ converges to $b * u$ locally uniformly in the norm topology of $H^1_{loc}(\mathbb{R})$. This approach is only applicable for the intrinsic equation (4.10). In order to carry out this programme we will need the following result.

Lemma 5.1. *Let b be a smooth compactly supported symmetric density on $\mathbb{R}$. Let us assume that the functions $\varphi : \mathbb{R} \to \mathbb{R}$ and $Z : \mathbb{R}^n \to \mathbb{R}^n$ are of $C^\infty_)$-class. Then the processes (z^k) constructed in Section 4.4 satisfy the following. For every $t \geq 0$, $\mathbb{P}$ almost surely,*

$$
\begin{aligned}
&\langle v^k(t) \cdot Z(b * u^k(t)), \varphi\rangle_{L^2(\mathbb{R})} = \langle v^k_0 \cdot Z(b * u^k_0), \varphi\rangle_{L^2(\mathbb{R})} \qquad (5.1)\\
&- \int_0^t \langle \partial_x u^k(s) \cdot Z(b * u^k(s)), \partial_x \varphi\rangle_{L^2(\mathbb{R})}\, ds + \int_0^t \langle v^k(s) \cdot Z'_{b*u^k(s)}(b * v^k(s)), \varphi\rangle_{L^2(\mathbb{R})}\, ds\\
&- \int_0^t \langle \partial_x u^k(s) \cdot Z'_{b*u^k(s)}(b * \partial_x u^k(s)), \varphi\rangle_{L^2(\mathbb{R})}\, ds\\
&+ \int_0^t \langle \left[A_{u^k(s)}(\partial_x u^k(s), \partial_x u^k(s)) - A_{u^k(s)}(v^k(s), v^k(s))\right] \cdot Z(b * u^k(s)), \varphi\rangle_{L^2(\mathbb{R})}\, ds\\
&+ \int_0^t \langle \left[Y^k(u^k(s), v^k(s), \partial_x u^k(s)) \cdot Z(b * u^k(s)\right]\, dW^k, \varphi\rangle_{L^2(\mathbb{R})}.
\end{aligned}
$$

Proof of Lemma 5.1. Assume that both φ and b have support in some $(-r, r)$, put $R = 3r$ and define, with $K = L^2((-R, R); \mathbb{R}^n) \times L^2((-R, R); \mathbb{R}^n)$,

$$
\begin{aligned}
B(w_1, w_2) &= \int_{-R}^{R} w_2(x) \cdot Z(\bar{w}_1(x)) \varphi(x)\, dx,\\
\bar{w}_1(x) &= \int_{-R}^{R} w_1(y) b(x - y)\, dy, \ w = (w_1, w_2) \in K.
\end{aligned}
$$

Then obviously the mapping B is of C^2 class on K and, for $w,p,q\in K$, the following formulae holds.

$$\begin{aligned} B'(w)q &= \int_{-R}^{R} q_2(x)\cdot Z(\bar w_1(x))\varphi(x)\,dx + \int_{-R}^{R} w_2(x)\cdot Z'(\bar w_1(x))\bar q_1(x)\varphi(x)\,dx \\ B''(w)(q,p) &= \int_{-R}^{R} \big[q_2(x)\cdot Z'(\bar w_1(x))\bar p_1(x) + p_2(x)\cdot Z'(\bar w_1(x))\bar q_1(x)\big]\varphi(x)\,dx \\ &+ \int_{-R}^{R} w_2(x)\cdot Z''(\bar w_1(x))(\bar q_1(x),\bar p_1(x))\varphi(x)\,dx. \end{aligned}$$

Since $\bar h = b*h$ on $(-r,r)$, we get the result by applying the Itô formula [14, Theorem 4.17] to the process z^k. □

6. Tightness

For any fixed $m\in\mathbb{N}$ and $r>0$ let us define the following set

$$S_{m,r} := \left\{z_0 : |z_0|_{H^1(-2m-1,2m+1)\times L^2(-2m-1,2m+1)} \le r\right\}. \tag{6.1}$$

It follows from (4.2) in subsection 4.1, or rather from Lemma A.2, that there exist constants $C_{m,r}$ such that

$$S_{m,r} \subseteq \bigcap_{k=1}^{\infty} \left\{z_0 : |z_0^k|_{H^1(-2m,2m)\times L^2(-2m,2m)} \le C_{m,r}\right\} \tag{6.2}$$

Hence by Theorem 4.7 we infer that for every $m\in\mathbb{N}$, $r>0$ and $q\in(0,\infty)$

$$C_{m,r,q} := \sup_{k\in\mathbb{N}} \mathbb{E}\left[\mathbf{1}_{S_{m,r}} \sup_{t\in[0,m]} |z^k(t)|^q_{H^1(-m,m)\times L^2(-m,m)}\right] < \infty. \tag{6.3}$$

6.1. **Tightness of the sequence $(z^k)_{k\in\mathbb{N}}$ on $\mathbb{L}$.** Let us define a set

$$\mathbb{L} = \mathbb{L}^1 \oplus \mathbb{L}^0 := C_w(\mathbb{R}_+; H^1_{loc}(\mathbb{R})) \oplus C_w(\mathbb{R}_+; L^2_{loc}(\mathbb{R})), \tag{6.4}$$

i.e. $\mathbb{L}^1$, resp. $\mathbb{L}^0$, resp. $\mathbb{L}$ is a locally convex topological vector spaces of weakly continuous $H^1_{loc}(\mathbb{R})$-, resp. $L^2_{loc}(\mathbb{R})$-, resp. $H^1_{loc}(\mathbb{R})\times L^2_{loc}(\mathbb{R})$-valued functions defined on $\mathbb{R}_+$. The properties of these spaces are discussed in the Appendix B.

Lemma 6.1. *The sequence of laws of the random variables (z^k) constructed in Section 4.4 is tight on the space $\mathbb{L}$.*

Proof. We introduce open sets in $\mathbb{L}^i$

$$J^i_m(a) = \left\{h\in\mathbb{L}^i : \sup_{t\in[0,m]} |h(t)|_{H^i(-m,m)} > a\right\}, \qquad a>0$$

$$K^i_m(a) = \left\{h\in\mathbb{L}^i : \sup_{0\le s<t\le m}\left[\frac{|h(t)-h(s)|_{H^{-1}(-m,m)}}{(t-s)^{\frac18}}\right] > a\right\}, \qquad a>0.$$

The following inequalities consequencies either of the Sobolev embedding theorems or Lemma 2.1.

$$
\begin{aligned}
|h|_{H^{-1}(-m,m)} &\leq (2m)^{\frac{1}{2}}|h|_{L^1(-m,m)}, \quad h\in L^1(-m,m) && (6.5)\\
|h|_{H^{-1}(-m,m)} &\leq |h|_{L^2(-m,m)}, \quad h\in L^2(-m,m) && (6.6)\\
|\partial_{xx}h|_{H^{-1}(-m,m)} &\leq |h|_{H^1(-m,m)}, \quad h\in H^2(-m,m) && (6.7)\\
|\xi\mapsto h\xi|_{\mathcal{J}_2(H_{\mu_k},H^{-1}(-m,m))} &\leq c_\circ[\mu(\mathbb{R})]^{\frac{1}{2}}|h|_{L^2(-m,m)}, \quad h\in L^2(-m,m),\, k\in\mathbb{N}. && (6.8)
\end{aligned}
$$

Let us fix $\varepsilon>0$ and for each natural number $m\in\mathbb{N}$ find a corresponding positive number $r_m>0$ such that

$$\mathbb{P}\left(S_{m,r_m}\right)>1-\frac{\varepsilon}{2\cdot 8^m},\quad m\in\mathbb{N}.$$

Next we put

$$\alpha_m=\frac{6\cdot 8^m\cdot m^{\frac{7}{8}}}{\varepsilon}\left[C_{m,r_m,1}+(8m)^{\frac{1}{2}}\cdot C_A\cdot C_{m,r_m,2}\right]+\left[\frac{2\cdot 8^m\cdot m\cdot\beta_m\cdot(1+C_{m,r_m,8})}{\varepsilon}\right]^{\frac{1}{8}}$$

where the numbers $C_{m,r_m,q}$, $q=1,2$ have been defined in (6.3) and

$$
\begin{aligned}
\beta_m &= 3^{15}c_*C_Y^8c_\circ^8[\mu(\mathbb{R})]^4(2m)^5,\\
C_A &= \sup\left\{|A_p(\xi,\xi)|:\xi\in T_pM,\,|\xi|=1,\,p\in M\right\},\\
C_Y &= \sup\left\{\frac{|Y^k(p,\xi,\eta)|}{1+|\xi|+|\eta|}:k\in\mathbb{N},\,\xi,\eta\in T_pM,\,p\in M\right\}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\mathbb{P}\left[u^k\in J_m^1(\alpha_m)\right] &\leq \mathbb{P}\left(\Omega\setminus S_{m,r_m}\right)+\mathbb{P}\left[\mathbf{1}_{S_{m,r_m}}\sup_{t\in[0,m]}|u^k(t)|_{H^1(-m,m)}>\alpha_m\right]\\
&\leq \frac{\varepsilon}{2\cdot 8^m}+\frac{C_{m,r_m,1}}{\alpha_m}\leq\frac{\varepsilon}{8^m}, && (6.9)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}\left[v^k\in J_m^0(\alpha_m)\right] &\leq \mathbb{P}\left(\Omega\setminus S_{m,r_m}\right)+\mathbb{P}\left[\mathbf{1}_{S_{m,r_m}}\sup_{t\in[0,m]}|v^k(t)|_{L^2(-m,m)}>\alpha_m\right]\\
&\leq \frac{\varepsilon}{2\cdot 8^m}+\frac{C_{m,r_m,1}}{\alpha_m}\leq\frac{\varepsilon}{8^m}, && (6.10)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}\left[u^k\in K_m^1(\alpha_m)\right] &\leq \mathbb{P}\left(\Omega\setminus S_{m,r_m}\right)+\mathbb{P}\left[\mathbf{1}_{S_{m,r_m}}\sup_{0\leq s<t\leq m}\frac{\int_s^t|v^k(r)|_{L^2(-m,m)}\,dr}{(t-s)^{\frac{1}{8}}}>\alpha_m\right]\\
&\leq \mathbb{P}\left(\Omega\setminus S_{m,r_m}\right)+\mathbb{P}\left[m^{\frac{7}{8}}\mathbf{1}_{S_{m,r_m}}\sup_{t\in[0,m]}|v^k(t)|_{L^2(-m,m)}>\alpha_m\right]\\
&\leq \frac{\varepsilon}{2\cdot 8^m}+\frac{m^{\frac{7}{8}}C_{m,r_m,1}}{\alpha_m}\leq\frac{\varepsilon}{8^m} && (6.11)
\end{aligned}
$$

where we used (6.6) in (6.11). Concerning the term $\mathbb{P}\,[v^k \in K^0_m(\alpha_m)]$, we define processes

$$\begin{aligned} I^k_1(t) &= \int_0^t \partial_{xx} u^k(s)\,ds, \qquad t \geq 0 \\ I^k_2(t) &= \int_0^t \left[A_{u^k(s)}(\partial_x u^k(s), \partial_x u^k(s)) - A_{u^k(s)}(v^k(s), v^k(s))\right] ds, \qquad t \geq 0 \\ I^k_3(t) &= \int_0^t Y^k(u^k(s), v^k(s), \partial_x u^k(s))\,dW^k, \qquad t \geq 0 \end{aligned}$$

where the integrals are convergent in every $L^2(-m,m)$, $m \in \mathbb{N}$. By (6.7) the following inequality is satisfied.

$$\begin{aligned} \mathbb{P}\left[I^k_1 \in K^0_m\left(\frac{\alpha_m}{3}\right)\right] &\leq \mathbb{P}\,(\Omega \setminus S_{m,r_m}) + \mathbb{P}\left[\mathbf{1}_{S_{m,r_m}} \sup_{0 \leq s < t \leq m} \frac{\int_s^t |\partial_{xx} u^k(r)|_{H^{-1}(-m,m)}\,dr}{(t-s)^{\frac{1}{8}}} > \frac{\alpha_m}{3}\right] \\ &\leq \mathbb{P}\,(\Omega \setminus S_{m,r_m}) + \mathbb{P}\left[m^{\frac{7}{8}} \mathbf{1}_{S_{m,r_m}} \sup_{t \in [0,m]} |\partial_x u^k(t)|_{L^2(-m,m)} > \frac{\alpha_m}{3}\right] \\ &\leq \frac{\varepsilon}{2 \cdot 8^m} + \frac{3m^{\frac{7}{8}} C_{m,r_m,1}}{\alpha_m} \leq \frac{\varepsilon}{8^m} \end{aligned} \tag{6.12}$$

Moreover, we have

$$\mathbb{P}\left[I^k_2 \in K^0_m\left(\frac{\alpha_m}{3}\right)\right] \leq \mathbb{P}\,(\Omega \setminus S_{m,r_m})$$

$$+\mathbb{P}\left[\mathbf{1}_{S_{m,r_m}} \sup_{0 \leq s < t \leq m} \frac{\int_s^t |A_{u^k(r)}(\partial_x u^k(r), \partial_x u^k(r)) - A_{u^k(r)}(v^k(r), v^k(r))|_{H^{-1}(-m,m)}\,dr}{(t-s)^{\frac{1}{8}}} > \frac{\alpha_m}{3}\right].$$

Hence, by inequality (6.5), we deduce that

$$\begin{aligned} \mathbb{P}\left[I^k_2 \in K^0_m\left(\frac{\alpha_m}{3}\right)\right] &\leq \mathbb{P}\,(\Omega \setminus S_{m,r_m}) \\ &+ \mathbb{P}\left[(8m)^{\frac{1}{2}} C_A \mathbf{1}_{S_{m,r_m}} \sup_{0 \leq s < t \leq m} \frac{\int_s^t |z^k(r)|^2_{H^1(-m,m)\times L^2(-m,m)}\,dr}{(t-s)^{\frac{1}{8}}} > \frac{\alpha_m}{3}\right] \\ &\leq \frac{\varepsilon}{2 \cdot 8^m} + \mathbb{P}\left[m^{\frac{7}{8}} (8m)^{\frac{1}{2}} C_A \mathbf{1}_{S_{m,r_m}} \sup_{t \in [0,m]} |z^k(t)|^2_{H^1(-m,m)} > \frac{\alpha_m}{3}\right] \\ &\leq \frac{\varepsilon}{2 \cdot 8^m} + \frac{3m^{\frac{7}{8}} (8m)^{\frac{1}{2}} C_A C_{m,r_m,2}}{\alpha_m} \leq \frac{\varepsilon}{8^m}. \end{aligned} \tag{6.13}$$

Finally, by Proposition 2.3 we infer that

$$\begin{aligned}\mathbb{P}\left[I_3^k \in K_m^0\left(\frac{\alpha_m}{3}\right)\right] &\le \mathbb{P}\left(\Omega\setminus S_{m,r_m}\right) + \mathbb{P}\left[\left|\mathbf{1}_{S_{m,r_m}} I_3^k\right|_{C^{\frac{1}{8}}([0,m];H^{-1}(-m,m))} > \frac{\alpha_m}{3}\right] \le \frac{\varepsilon}{2\cdot 8^m}\\ &+ \frac{3^8 c_*}{\alpha_m^8}\mathbb{E}\int_0^m \mathbf{1}_{S_{m,r_m}}\left|Y^k(u^k(s),v^k(s),\partial_x u^k(s))\right|^8_{\mathcal{J}_2(H_{\mu_k},H^{-1}(-m,m))}\,ds\\ &\le \frac{\varepsilon}{2\cdot 8^m} + \frac{\beta_m(1+C_{m,r_m,8})}{\alpha_m^8} \le \frac{\varepsilon}{8^m}. \end{aligned} \tag{6.14}$$

Indeed, by (6.8) we have

$$\begin{aligned}\left|Y^k(u^k,v^k,\partial_x u^k)\right|^8_{\mathcal{J}_2(H_{\mu_k},H^{-1}(-m,m))} &\le c_\circ^8[\mu(\mathbb{R})]^4|Y^k(u^k,v^k,\partial_x u^k)|^8_{L^2(-m,m)}\\ &\le 3^7C_Y^8c_\circ^8[\mu(\mathbb{R})]^4[(2m)^4+|\partial_x u^k|^8_{L^2(-m,m)}+|v^k|^8_{L^2(-m,m)}]\\ &\le 3^7C_Y^8c_\circ^8[\mu(\mathbb{R})]^4(2m)^4(1+|z^k|^8_{H^1(-m,m)\times L^2(-m,m)}).\end{aligned}$$

The estimates (6.12)-(6.14) imply that

$$\mathbb{P}\left[v^k\in K_m^0(\alpha_m)\right] \le \sum_{j=1}^3 \mathbb{P}\left[I_j^k\in K_m^0\left(\frac{\alpha_m}{3}\right)\right] \le \frac{3\varepsilon}{8^m}. \tag{6.15}$$

On the other hand, by Proposition B.2 the set

$$C_\varepsilon = \left\{\bigcap_{m=1}^\infty\left[\mathbb{L}^1\setminus\left(J_m^1(\alpha_m)\cup K_m^1(\alpha_m)\right)\right]\right\}\times\left\{\bigcap_{m=1}^\infty\left[\mathbb{L}^0\setminus\left(J_m^0(\alpha_m)\cup K_m^0(\alpha_m)\right)\right]\right\}$$

is compact in $\mathbb{L}$ and, by inequalities (6.9)-(6.11) and (6.15) we infer that

$$\mathbb{P}\left[z^k\in C_\varepsilon\right]\ge 1-\varepsilon, \qquad k\in\mathbb{N}.$$

This completes the proof. □

6.2. **Tightness of the auxiliary processes.** We introduced the pseudointrinsic equation (5.1) in Section 5 in order to avoid lack of convergence when passing to a limit in the intrinsic equation (4.10). However, there are still terms in (5.1), denoted by $Q^k_{b,\varphi,Z}$ in the sequel, that might not converge to the corresponding term. Luckily, these terms form a tight sequence on one hand are "small" on the other.

Notation 6.2. *If $a,b\in\mathbb{R}$ are such that $a<b$, then by ${}_0\mathrm{Lip}\,[a,b]={}_0C^{0,1}[a,b]$ we will denote a Banach space of all Lipschitz continuous functions $h:[a,b]\to\mathbb{R}$ such that $h(a)=0$, equipped with a norm*

$$|h|_{{}_0\mathrm{Lip}\,[a,b]} = \sup_{a\le s<t\le b}\frac{|h(t)-h(s)|}{t-s}.$$

Lemma 6.3. *Let b be a smooth symmetric density on $\mathbb{R}$ with support in $(-1,1)$, φ a smooth real function on $\mathbb{R}$ with support in $(-r,r)$, $Z:\mathbb{R}^n\to\mathbb{R}^n$ be a smooth and compactly* *supported function such that $Z(p) \in T_pM$, $p \in M$ and let (z^k) be the processes constructed in Section 4.4. The sequence of processes $(Q^k_{b,\varphi,Z} : k \in \mathbb{N})$ where*

$$Q^k_{b,\varphi,Z}(t) = \int_0^t \langle \big[A_{u^k(s)}(\partial_x u^k(s), \partial_x u^k(s)) - A_{u^k(s)}(v^k(s), v^k(s))\big] \cdot Z(b * u^k(s)), \varphi\rangle_{L^2(\mathbb{R})}\, ds$$

is tight in $C(\mathbb{R}_+)$ and there exist a constant ζ depending on A, Z and φ such that

$$\begin{aligned} |Q^k_{b,\varphi,Z}|_{{}_0C^{0,1}[0,l]} &\leq \zeta |z^k|^2_{L^\infty((0,l);H^1(-r,r)\times L^2(-r,r))} |b * u^k - u^k|_{L^\infty((0,l);L^\infty(-r,r))} &(6.16)\\ &\leq \zeta |z^k|^2_{L^\infty((0,l);H^1(-r-1,r+1)\times L^2(-r-1,r+1))} &(6.17)\end{aligned}$$

holds for every $k, l \in \mathbb{N}$.

Proof. Since A_{u^k} takes values in $N_{u^k}M$ and $Z(u^k) \in T_{u^k}M$ the following identity holds

$$\big[A_{u^k(s)}(\partial_x u^k(s), \partial_x u^k(s)) - A_{u^k(s)}(v^k(s), v^k(s))\big] \cdot Z(u^k(s)) = 0$$

Hence, we get

$$\begin{aligned} |Q^k_{b,\varphi,Z}|_{{}_0C^{0,1}[0,l]} &\leq |\varphi|_{L^\infty(\mathbb{R})} |Z(b * u^k) - Z(u^k)|_{L^\infty((0,l),L^\infty(-r,r))}\\ &\times |A_{u^k}(\partial_x u^k, \partial_x u^k) - A_{u^k}(v^k, v^k)|_{L^\infty((0,l),L^1(-r,r))}\\ &\leq \zeta |z^k|^2_{L^\infty((0,l);H^1(-r,r)\times L^2(-r,r))} |b * u^k - u^k|_{L^\infty((0,l);L^\infty(-r,r))}\\ &\leq \zeta |z^k|^2_{L^\infty((0,l);H^1(-r-1,r+1)\times L^2(-r-1,r+1))}\end{aligned}$$

for some $\zeta > 0$ and (6.16), (6.17) are proved.

Now, let $m_l = \min\{m \in \mathbb{N} : m \geq r+1,\, m \geq l\}$, $l \in \mathbb{N}$, fix $\varepsilon > 0$, set

$$J^l = \left\{ h \in C(\mathbb{R}_+) : h(0) = 0,\, |h|_{C_0^{0,1}[0,l]} \leq \frac{3^l \cdot \zeta \cdot C_{m_l,r_l,2}}{\varepsilon} \right\}$$

and find $r_l > 0$ so that $\mathbb{P}(S_{m_l,r_l}) > 1 - \frac{\varepsilon}{3^l}$ where we use the notation (6.1) and (6.3). Then we have

$$\begin{aligned} \mathbb{P}\left[Q^k_{b,\varphi,Z} \notin J^l\right] &\leq \mathbb{P}(\Omega \setminus S_{m_l,r_l}) + \mathbb{P}\left[\mathbf{1}_{S_{m_l,r_l}} |Q^k_{b,\varphi,Z}|_{C_0^{0,1}[0,l]} > \frac{3^l \cdot \zeta \cdot C_{m_l,r_l,2}}{\varepsilon}\right]\\ &\leq \frac{\varepsilon}{3^l} + \mathbb{P}\left[\mathbf{1}_{S_{m_l,r_l}} |z^k|^2_{L^\infty((0,m_l);H^1(-m_l,m_l)\times L^2(-m_l,m_l))} > \frac{3^l \cdot C_{m_l,r_l,2}}{\varepsilon}\right] \leq \frac{2\varepsilon}{3^l}\end{aligned}$$

by (6.3). Hence

$$\mathbb{P}\left[Q^k_{b,\varphi,Z} \in \bigcap_{l=1}^\infty J^l\right] \geq 1 - \varepsilon$$

and $\bigcap_{l=1}^\infty J^l$ is compact in $C(\mathbb{R}_+)$ by the Arzela-Ascoli theorem. □

7. Skorokhod Representation Theorem

Let us consider the following objects.

- A smooth symmetric density b on $\mathbb{R}$ with support a sequence of in $(-1,1)$, $b_l = \sqrt{l}b(\cdot/l)$, $l \in \mathbb{N}^*$,
- the sequence (z^k) of processes constructed in Section 4.4,
- a family $(\beta^{ij})_{i,j\in\mathbb{N}}$ of i.i.d. Brownian Motions used in Section 4.2,
- the orthonormal bases $(\xi_{ij})_{j\in J_i,i\in\mathbb{N}}$ of H_{ν_i} introduced Section 4.2,
- $Q^k_{b,\varphi,Z}$ the processes from Lemma 6.3,
- (φ_m) the sequence in $C^\infty(\mathbb{R})$ with supports in $(-r_m, r_m)$ from Proposition D.2,
- the smooth vector fields $(Z^1,\dots,Z^N)$ satisfying (A.1),
- the spaces $\mathbb{L}^k$ introduced in Section B.2,
- the extension $\bar{Y}$ of Y from equality (4.8).

Remark 7.1. Using Proposition A.1, each Z^i can be extended to a compactly supported mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, denoted again by Z^i.

Let us recall that by Lemmata 6.1 and 6.3, the sequence of laws of random vectors

$$\left(u_0^k, v_0^k, u^k, v^k, (\beta^{ij})_{i,j}, (Q^k_{b_l,\varphi_m,Z^\gamma})_{l,m\in\mathbb{N},\gamma\in\{1,\dots,N\}}\right)_{k\in\mathbb{N}}$$

is tight on the space

$$H^1_{loc}(\mathbb{R}) \times L^2_{loc}(\mathbb{R}) \times \mathbb{L}^1 \times \mathbb{L}^0 \times \prod_{(i,j)\in\mathbb{N}^2} C(\mathbb{R}_+) \times \prod_{(l,m,\gamma)\in\mathbb{N}^2\times\{1,\dots,N\}} C(\mathbb{R}_+).$$

Moreover, as remarked in Section 4.1, the sequence (u_0^k, v_0^k) converges in $H^1_{loc}(\mathbb{R}) \times L^2_{loc}(\mathbb{R})$ to z_0 on Ω. Hence, by the Skorokhod-Jakubowski Theorem C.1, there exists a subsequence (k_α) and the following Borel measurable maps with σ-compact range

- $\mathbf{u} : [0,1] \to \mathbb{L}^1$, $\mathbf{u}^\alpha : [0,1] \to C(\mathbb{R}_+; H^2_{loc}(\mathbb{R}))$, $\alpha \in \mathbb{N}$,
- $\mathbf{v} : [0,1] \to \mathbb{L}^0$, $\mathbf{v}^\alpha : [0,1] \to C(\mathbb{R}_+; H^1_{loc}(\mathbb{R}))$, $\alpha \in \mathbb{N}$,
- $\mathbf{B}^\alpha_{ij} : [0,1] \to C(\mathbb{R}_+)$, $\mathbf{B}_{ij} : [0,1] \to C(\mathbb{R}_+)$, $\alpha, i, j \in \mathbb{N}$,
- $\mathbf{Q}^\alpha_{lm\gamma} : [0,1] \to C(\mathbb{R}_+)$, $\mathbf{Q}_{lm\gamma} : [0,1] \to C(\mathbb{R}_+)$, $\alpha, l, m \in \mathbb{N}$, $\gamma \in \{1,\dots,N\}$

where $[0,1]$ is equipped with the Borel σ-algebra $\mathcal{I}$ and Lebesgue measure Leb (which happens to be a the probability measure) such that

- $\mathbf{u}^\alpha(0)$ converges in H^1_{loc} to $\mathbf{u}(0)$ on $[0,1]$,
- $\mathbf{v}^\alpha(0)$ converges in L^2_{loc} to $\mathbf{v}(0)$ on $[0,1]$,
- $\mathbf{u}^\alpha$ converges in $\mathbb{L}^1$ to $\mathbf{u}$ on $[0,1]$,
- $\mathbf{v}^\alpha$ converges in $\mathbb{L}^0$ to $\mathbf{v}$ on $[0,1]$,
- $\mathbf{B}^\alpha_{ij}$ converges in $C(\mathbb{R}_+)$ to $\mathbf{B}_{ij}$ on $[0,1]$ for every $i,j \in \mathbb{N}$,
- $\mathbf{Q}^\alpha_{ml\gamma}$ converges in $C(\mathbb{R}_+)$ to $\mathbf{Q}_{ml\gamma}$ on $[0,1]$ for every $m,l \in \mathbb{N}$, $\gamma \in \{1,\dots,N\}$

and, for each $\alpha \in \mathbb{N}$, the laws on the Borel σ-algebra of

$$C(\mathbb{R}_+; H^2_{loc}(\mathbb{R})) \times C(\mathbb{R}_+; H^1_{loc}(\mathbb{R})) \times \prod_{(i,j)\in\mathbb{N}^2} C(\mathbb{R}_+) \times \prod_{(l,m,\gamma)\in\mathbb{N}^2\times\{1,\dots,N\}} C(\mathbb{R}_+),$$

of

$$\big(u^{k_\alpha}, v^{k_\alpha}, (\beta^{ij})_{i,j\in\mathbb{N}}, (Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma})_{l,m\in\mathbb{N},\gamma\in\{1,\dots,N\}}\big)$$

under the probability measure $\mathbb{P}$ and of

$$\big(\mathbf{u}^\alpha, \mathbf{v}^\alpha, (\mathbf{B}^\alpha_{ij})_{i,j\in\mathbb{N}}, (Q^\alpha_{lm\gamma})_{l,m\in\mathbb{N},\,\gamma\in\{1,\dots,N\}}\big)$$

under the Lebesgue measure Leb , are equal.

Remark 7.2. In fact, the Skorokhod-Jakubowski theorem implies that $\mathbf{u}^\alpha$ and $\mathbf{v}^\alpha$ are random variables in $\mathbb{L}^1$ and $\mathbb{L}^0$ respectively. Since the embeddings $C(\mathbb{R}_+; H^2_{loc}(\mathbb{R})) \subseteq \mathbb{L}^1$ and $C(\mathbb{R}_+; H^1_{loc}(\mathbb{R})) \subseteq \mathbb{L}^0$ are continuous, in view of Proposition C.2, we infer that these sets are Borel subsets of $\mathbb{L}^0$ and $\mathbb{L}^1$ respectively and that

$$\begin{aligned}
\text{Leb}\,\big(\{\mathbf{u}^\alpha \in C(\mathbb{R}_+; H^2_{loc}(\mathbb{R}))\}\big) &= \mathbb{P}\,\big\{u^{k_\alpha} \in C(\mathbb{R}_+; H^2_{loc}(\mathbb{R}))\}\big) = 1,\\
\text{Leb}\,\big(\{\mathbf{v}^\alpha \in C(\mathbb{R}_+; H^1_{loc}(\mathbb{R}))\}\big) &= \mathbb{P}\,\big\{v^{k_\alpha} \in C(\mathbb{R}_+; H^1_{loc}(\mathbb{R}))\}\big) = 1.
\end{aligned}$$

Hence, by the completeness of relevant probability spaces, we may assume that for every $\alpha \in \mathbb{N}$, $\mathbf{u}^\alpha$, respectively $\mathbf{v}^\alpha$, is a random variable with values in $C(\mathbb{R}_+; H^2_{loc}(\mathbb{R}))$, respectively $C(\mathbb{R}_+; H^1_{loc}(\mathbb{R}))$.

Remark 7.3. We will write $\mathbf{z}^\alpha = (\mathbf{u}^\alpha, \mathbf{v}^\alpha)$ and $\mathbf{z} = (\mathbf{u}, \mathbf{v})$.

Notation 7.4. *By $\mathcal{B}_t$, where $t \in [0,T)$, we will denote the σ-algebra on $[0,1]$ generated by the following random variables:*

$\mathbf{v}(0) : [0,1] \to L^2_{loc}(\mathbb{R})$,
$\mathbf{u}(s) : [0,1] \to H^1_{loc}(\mathbb{R})$,
$\mathbf{B}_{ij}(s) : [0,1] \to \mathbb{R}$,
$\mathbf{Q}_{ml\gamma} : [0,1] \to \mathbb{R}$ *for* $s \in [0,t]$, $i,j,m,l \in \mathbb{N}$, $\gamma \in \{1,\dots,N\}$.

By $\mathbb{B}$ we will denote the filtration $(\mathcal{B}_t)_{t\in[0,T)}$.

Denote finally by $\bar{\mathbb{B}} = \big(\bar{\mathcal{B}}(t)\big)_{t\in[0,T)}$ the natural augmentation of the filtration $\mathbb{B} = \big(\mathcal{B}(t)\big)_{t\in[0,T)}$.

Let us here point out that as in [52], in view of [15, p. 75], in order to show that a process an $\bar{\mathbb{B}}$-martingale it is enough to show that it is an $\mathbb{B}$-martingale.

7.1. **Uniform Local Energy and other Inequalities.** The following results are an immediate consequence of Theorem 4.7and the equality of the laws of $\mathbf{z}^\alpha$ and z^{k_α} on the Borel σ-algebra over $C(\mathbb{R}_+; H^2_{loc}(\mathbb{R})) \times C(\mathbb{R}_+; H^1_{loc}(\mathbb{R}))$. Let us first introduce some useful notation.

$$\begin{aligned}
\mathbf{I}^\alpha_{r,T} &= \mathbf{1}_{\{z_0 : |\mathbf{z}^\alpha(0)|_{H^1(-T,T)\times L^2(-T,T)} \le r\}},\\
\mathbf{I}_{r,T} &= \mathbf{1}_{\{|\mathbf{z}(0)|_{H^1(-T,T)\times L^2(-T,T)} \le r\}}.
\end{aligned}$$

The constant c_* is taken from in Theorem 4.7) (and hence does not depend on q, T, X, λ_0, λ_1 and $\mu(\mathbb{R})$).

Corollary 7.5. *For any $q>0$, $T>0$, $r>0$, $t\in[0,T)$ and $\alpha\in\mathbb{N}$, the following inequality holds*

$$\mathbb{E}\sup_{s\in[0,t]}\mathbf{I}^{\alpha}_{r,T}\mathbf{e}_{T,q}(s,\mathbf{z}^{\alpha}(s))\le 3e^{tc_*}\left\{\mathbb{E}\left[\mathbf{I}^{\alpha}_{r,T}\mathbf{e}_{T,2q}(0,\mathbf{z}^{\alpha}(0))\right]\right\}^{\frac{1}{2}}. \tag{7.1}$$

In particular, for every $m\in\mathbb{N}$, $r>0$ and $q\in(0,\infty)$,

$$\mathbf{C}_{m,r,q}=\sup_{\alpha\in\mathbb{N}}\mathbb{E}\left[\mathbf{I}^{\alpha}_{r,2m}\sup_{t\in[0,m]}|\mathbf{z}^{\alpha}(t)|^{q}_{H^1(-m,m)\times L^2(-m,m)}\right]<\infty. \tag{7.2}$$

Corollary 7.6. *For any $q>0$, $T>0$, $r>0$ and $t\in[0,T)$ the following inequality holds*

$$\mathbb{E}\sup_{s\in[0,t]}\mathbf{I}_{r,T}\mathbf{e}_{T,q}(s,\mathbf{z}(s))\le 3e^{tc_*}\left\{\mathbb{E}\left[\mathbf{I}_{r,T}\mathbf{e}_{T,2q}(0,\mathbf{z}(0))\right]\right\}^{\frac{1}{2}}.$$

In particular, for all $m\in\mathbb{N}$ and $q\in(0,\infty)$ and any $r>0$ such that $\mathrm{Leb}\left(\{|\mathbf{z}(0)|_{H^1(-2m,2m)\times L^2(-2m,2m)}=r\}\right)=0$, *the following holds*

$$\mathbb{E}\left[\mathbf{I}_{r,2m}\sup_{t\in[0,m]}|\mathbf{z}(t)|^{q}_{H^1(-m,m)\times L^2(-m,m)}\right]\le\mathbf{C}_{m,r,q}. \tag{7.3}$$

Proof of Corollary 7.6. We can apply the Fatou Lemma to the inequality (7.1) in Corollary 7.5 for $r>0$ such that $\mathrm{Leb}\left(\{z_0:|\mathbf{z}(0)|_{H^1(-T,T)\times L^2(-T,T)}=r\}\right)=0$, and then use Beppo Levi's monotone convergence theorem to get the result for other $r>0$. □

Corollary 7.7. *For every $m\in\mathbb{N}$ then there exists a constant ζ°_m such that for every $\gamma\in\{1,\dots,N\}$, $T>0$, $\kappa\in[\max\{r_m+1,T\},\infty)\cap\mathbb{N}$, $\alpha,l\in\mathbb{N}$ and for every $r>0$ such that* $\mathrm{Leb}\left(\{z_0:|\mathbf{z}(0)|_{H^1(-2\kappa,2\kappa)\times L^2(-2\kappa,2\kappa)}=r\}\right)=0$,

$$\mathbb{E}\left\{\mathbf{I}_{r,2\kappa}|\mathbf{Q}_{lm\gamma}|_{{}_0C^{0,1}[0,T]}\right\}\le\zeta^{\circ}_m\mathbf{C}^{\frac{1}{2}}_{\kappa,r,4}\left\{\mathbb{E}\left[\mathbf{I}_{r,2\kappa}|b_l*\mathbf{u}-\mathbf{u}|^2_{L^\infty((0,T);L^\infty(-r_m,r_m))}\right]\right\}^{\frac{1}{2}}. \tag{7.4}$$

Proof of Corollary 7.7. Since the laws on the Borel σ-algebra on $C(\mathbb{R}_+;H^2_{loc}(\mathbb{R}))\times C(\mathbb{R}_+;H^1_{loc}(\mathbb{R}))\times C(\mathbb{R}_+)$ of the random variables $(u^{k_\alpha},v^{k_\alpha},Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma})$ and $(\mathbf{u}^\alpha,\mathbf{v}^\alpha,\mathbf{Q}^\alpha_{lm\gamma})$ are equal, by (6.16) we infer that for every $\gamma\in\{1,\dots,N\}$ and $\alpha,l\in\mathbb{N}$, $\mathbb{P}$ almost surely,

$$|\mathbf{Q}^\alpha_{lm\gamma}|_{{}_0C^{0,1}[0,T]}\le\zeta^{\circ}_m|\mathbf{z}^\alpha|^2_{L^\infty((0,T);H^1(-r_m,r_m)\times L^2(-r_m,r_m))}|b_l*\mathbf{u}^\alpha-\mathbf{u}^\alpha|_{L^\infty((0,T);L^\infty(-r_m,r_m))}. \tag{7.5}$$

The inequality (7.2) now implies that

$$\mathbb{E}\left\{\mathbf{I}^\alpha_{r,2\kappa}|\mathbf{Q}^\alpha_{lm\gamma}|_{{}_0C^{0,1}[0,T]}\right\}\le\zeta^{\circ}_m\mathbf{C}^{\frac{1}{2}}_{\kappa,r,4}\left\{\mathbb{E}\left[\mathbf{I}^\alpha_{r,2\kappa}|b_l*\mathbf{u}^\alpha-\mathbf{u}^\alpha|^2_{L^\infty((0,T);L^\infty(-r_m,r_m))}\right]\right\}^{\frac{1}{2}}. \tag{7.6}$$

Since the weak convergence in $H^1_{loc}(\mathbb{R})$ implies the strong convergence in $L^\infty_{loc}(\mathbb{R})$ we infer that on $[0,1]$

$$\begin{aligned}\lim_{\alpha\to\infty}|b_l*\mathbf{u}^\alpha-\mathbf{u}^\alpha|_{L^\infty((0,T);L^\infty(-r_m,r_m))}&=|b_l*\mathbf{u}-\mathbf{u}|_{L^\infty((0,T);L^\infty(-r_m,r_m))}\\ \mathbb{E}\left[\mathbf{I}^\alpha_{r,2\kappa}|b_l*\mathbf{u}^\alpha-\mathbf{u}^\alpha|^4_{L^\infty((0,T);L^\infty(-r_m,r_m))}\right]&\le 16\mathbb{E}\left[\mathbf{I}^\alpha_{r,2\kappa}|\mathbf{u}^\alpha|^4_{L^\infty((0,\kappa);L^\infty(-\kappa,\kappa))}\right]\\ &\le c_\kappa\mathbb{E}\left[\mathbf{I}^\alpha_{r,2\kappa}|\mathbf{u}^\alpha|^4_{L^\infty((0,\kappa);H^1(-\kappa,\kappa))}\right]\\ &\le c_\kappa\mathbf{C}_{\kappa,r,4}.\end{aligned}$$

Hence the the final result follows by letting $\alpha \to \infty$ in inequality (7.6) and applying the Fatou Lemma. $\square$

7.2. **Identification of the random variables on** $[0,1]$**.** In this whole subsection we assume that the sequence (φ_m) is as in Proposition D.2.

Lemma 7.8. *There exists a set* $\Omega \subset [0,1]$ *of full* Leb *-measure such that for every* $\hat\omega \in \hat\Omega$*, and for all* $R > 0$ *and* $t \geq 0$*, the following equality holds in* $L^2(-R,R)$

$$\mathbf{u}(t) = \mathbf{u}(0) + \int_0^t \mathbf{v}(s)\,ds.$$

Proof. Let us note that the sequence (φ_m) separate points of $L^1_{loc}(\mathbb{R})$. Hence, it is enough to find set $\Omega \subset [0,1]$ of full Leb-measure such that for every $\hat\omega \in \hat\Omega$, the following equality holds on $C(\mathbb{R}_+)$

$$\langle u(\cdot), \varphi_m\rangle_{L^2(\mathbb{R})} - \langle u(0), \varphi_m\rangle_{L^2(\mathbb{R})} - \int_0^{\cdot} \langle v(s), \varphi_m\rangle_{L^2(\mathbb{R})}\,ds. \tag{7.7}$$

For this aim we introduce the following sequence of continuous mappings

$$\begin{aligned} B_m : \mathbb{L}^1 \times \mathbb{L}^0 \quad &\ni \quad (u,v) \mapsto \langle u(\cdot), \varphi_m\rangle_{L^2(\mathbb{R})} \\ &- \quad \langle u(0), \varphi_m\rangle_{L^2(\mathbb{R})} - \int_0^{\cdot} \langle v(s), \varphi_m\rangle_{L^2(\mathbb{R})}\,ds \in C(\mathbb{R}_+). \end{aligned}$$

Since, for every $\alpha \in \mathbb{N}$, the laws on the Borel σ-algebra on $\mathbb{L}^1 \times \mathbb{L}^0$ of $(u^{k_\alpha}, v^{k_\alpha})$ and $(\mathbf{u}^\alpha, \mathbf{v}^\alpha)$ are equal and $(\mathbf{u}^\alpha, \mathbf{v}^\alpha)$ converges in $\mathbb{L}^1 \times \mathbb{L}^0$ to $(\mathbf{u}, \mathbf{v})$ on $[0,1]$, $B_m(u^{k_\alpha}, v^{k_\alpha}) = 0$ Leb almost surely and

$$B_m(\mathbf{u}, \mathbf{v}) = \lim_{\alpha\to\infty} B_m(\mathbf{u}^\alpha, \mathbf{v}^\alpha) = 0 \quad \text{Leb almost surely}$$

we infer that $B_m(\mathbf{u}^\alpha, \mathbf{v}^\alpha) = 0$ Leb almost surely. This completes the proof of (7.7) and so the result follows. $\square$

Corollary 7.9. *The process* $\mathbf{v}$ *has* $L^2_{loc}(\mathbb{R})$*-valued weakly continuous paths. Moreover it is* $\bar{\mathbb{B}}$*-adapted.*

Proof of Corollary 7.9. Let $t > 0$, $m \in \mathbb{N}$ and $j \in \mathbb{N}$. Then a function

$$a_j(t) = j\big(\langle \mathbf{u}(t), \varphi_m\rangle_{L^2(\mathbb{R})} - \langle \mathbf{u}(t - \frac{1}{j}), \varphi_m\rangle_{L^2(\mathbb{R})}\big)$$

is $\mathcal{B}_t$-measurable and $a_j(t) \to \langle \mathbf{v}(t), \varphi_m\rangle_{L^2(\mathbb{R})}$ Leb almost surely by Lemma 7.8. Hence $\langle \mathbf{v}(t), \varphi_m\rangle_{L^2(\mathbb{R})}$ is $\mathcal{B}_t$-measurable. Finally, (φ_m) generate the Borel σ-algebra of $L^2_{loc}(\mathbb{R})$ by Proposition D.2. $\square$

Lemma 7.10. *For every* $t \geq 0$*,* $\mathbf{z}(t) \in H^1_{loc} \times L^2_{loc}(TM)$ Leb *-almost surely.*

Proof of Lemma 7.10. The set $H^1_{loc} \times L^2_{loc}(TM)$ is sequentially closed in the space $(H^1_{loc}(\mathbb{R}), \text{weak}) \times (L^2_{loc}(\mathbb{R}), \text{weak})$ and $z^{k_\alpha}(t)$ has the same law as $\mathbf{z}^\alpha(t)$ on the Borel σ-algebra over $(H^1_{loc}(\mathbb{R}), \text{weak}) \times (L^2_{loc}(\mathbb{R}), \text{weak})$ for every $\alpha \in \mathbb{N}$ and $t \geq 0$. Hence $\mathbf{z}^\alpha(t) \in H^1_{loc} \times L^2_{loc}(TM)$ $\mathbb{P}$-almost surely and so $\mathbf{z}(t) \in H^1_{loc} \times L^2_{loc}(TM)$ $\mathbb{P}$-almost surely. Since the paths of $\mathbf{z}$ are $(H^1_{loc}(\mathbb{R}), \text{weak}) \times$

$(L^2_{loc}(\mathbb{R}),\text{weak})$-valued continuous, we can exchange the order of *"$\mathbb{P}$ almost surely"* and *"$t \geq 0$"*. $\square$

Lemma 7.11. *The processes $(\mathbf{B}_{ij})_{i,j\in\mathbb{N}}$ are independent standard $\mathbb{B}$-Wiener processes.*

Proof. The random variable $(\beta^{ij})_{i,j\in\mathbb{N}}$ has the same law as $(\mathbf{B}^\alpha_{ij})_{i,j\in\mathbb{N}}$ on the Borel σ-algebra over $\prod_{i,j\in\mathbb{N}} C(\mathbb{R}_+)$ for every α and $(\mathbf{B}^\alpha_{ij})_{i,j\in\mathbb{N}}$ converges to $(\mathbf{B}_{ij})_{i,j\in\mathbb{N}}$ as $\alpha \to \infty$ in the topology of $\prod_{i,j\in\mathbb{N}} C(\mathbb{R}_+)$. Hence $(\mathbf{B}_{ij})_{i,j\in\mathbb{N}}$ are independent processes such that for all $0 \leq t_0 < t_1$, the random variable $\frac{\mathbf{B}_{ij}(t_1)-\mathbf{B}_{ij}(t_0)}{(t_1-t_0)^{\frac{1}{2}}}$ is $N(0,1)$, i.e. a standard centered Gaussian. Let us fix a natural number $\kappa \in \mathbb{N}$ and real numbers $0 \leq r_1 \leq \cdots \leq r_\kappa \leq t_0 < t_1$. Put $\bar{\kappa} = \kappa + \kappa\cdot\kappa + \kappa\cdot\kappa\cdot\kappa + \kappa\cdot\kappa\cdot N\cdot\kappa$ and consider the following $\mathbb{R}^{\bar{\kappa}}$-valued random vectors:

$$\begin{aligned}
O^k &= \left(\langle v^k_0, \varphi_m\rangle_{L^2(\mathbb{R})}, \langle u^k(r_\delta), \varphi_m\rangle_{L^2(\mathbb{R})}, \beta^{ij}(r_\delta), Q^k_{b_l,\varphi_m,Z^\gamma}(r_\delta)\right)_{i,j,l,m,\delta\leq\kappa,\,\gamma\leq N} \\
\mathbf{O}^\alpha &= \left(\langle \mathbf{v}^\alpha(0), \varphi_m\rangle_{L^2(\mathbb{R})}, \langle \mathbf{u}^\alpha(r_\delta), \varphi_m\rangle_{L^2(\mathbb{R})}, (\mathbf{B}^\alpha_{ij}(r_\delta))_{i,j}, \mathbf{Q}^\alpha_{lm\gamma}(r_\delta)\right)_{l,m,\delta\leq\kappa,\,\gamma\leq N} \qquad (7.8)\\
\mathbf{O} &= \Big(\left(\langle \mathbf{v}(0), \varphi_m\rangle_{L^2(\mathbb{R})}\right)^\kappa_{m=1}, \left(\langle \mathbf{u}(r_\delta), \varphi_m\rangle_{L^2(\mathbb{R})}\right)^\kappa_{m,\delta=1}, \left((B_{ij}(r_\delta))_{i,j}\right)^\kappa_{\delta=1}, \\
&\qquad \left(\mathbf{Q}_{lm\gamma}(r_\delta)\right)^\kappa_{l,m,\delta=1}{}^k_{\gamma=1}\Big).
\end{aligned}$$

The law of O^{k_α} under $\mathbb{P}$ coincides with the law of $\mathbf{O}^\alpha$ under Leb for every α and $\mathbf{O}^\alpha$ converges to $\mathbf{O}$ on $[0,1]$. Hence, if g_0 is a continuous bounded function on $\mathbb{R}^{\bar{\kappa}}$ and g_1 is a continuous real bounded function, there is

$$\begin{aligned}
[\mathbb{E}\, g_0(\mathbf{O})]\,[\mathbb{E}\, g_1(\mathbf{B}_{IJ}(t_1) - \mathbf{B}_{IJ}(t_0))] &= \lim_{\alpha\to\infty} [\mathbb{E}\, g_0(\mathbf{O}^\alpha)]\,[\mathbb{E}\, g_1(\mathbf{B}^\alpha_{IJ}(t_1) - \mathbf{B}^\alpha_{IJ}(t_0))] \\
&= \lim_{\alpha\to\infty} \left[\mathbb{E}\, g_0(O^{k_\alpha})\right] \left[\mathbb{E}\, g_1(\beta^{IJ}(t_1) - \beta^{IJ}(t_0))\right] \\
&= \lim_{\alpha\to\infty} \mathbb{E}\, \left[g_0(O^{k_\alpha}) g_1(\beta^{IJ}(t_1) - \beta^{IJ}(t_0))\right] \\
&= \lim_{\alpha\to\infty} \mathbb{E}\, [g_0(\mathbf{O}^\alpha) g_1(\mathbf{B}^\alpha_{IJ}(t_1) - \mathbf{B}^\alpha_{IJ}(t_0))] \\
&= \mathbb{E}\, [g_0(\mathbf{O}) g_1(\mathbf{B}_{IJ}(t_1) - \mathbf{B}_{IJ}(t_0))]
\end{aligned}$$

so $\mathbf{B}_{IJ}(t_1) - \mathbf{B}_{IJ}(t_0)$ is independent from $\mathcal{B}_{t_0}$ for every $I, J \in \mathbb{N}$. $\square$

Remark 7.12. The process

$$\mathbf{W}(\varphi) = \sum_{i=1}^\infty \sum_{j\in J_i} \mathbf{B}_{ij}\xi_{ij}(\varphi), \qquad \varphi \in \mathscr{S}_\mathbb{R} \tag{7.9}$$

is a spatially homogeneous $\mathbb{B}$-Wiener process with the spectral measure μ.

Let us recall that $\bar{Y}$ is an the extension of Y from equality (4.8). Let us introduce auxiliary operators

$$\mathcal{V}_{lm\gamma}, \mathcal{V}^k_{lm\gamma}, \mathcal{V}^\infty_{lm\gamma}, \mathcal{V}^{ijk}_{lm\gamma}, \mathcal{V}^{ij\infty}_{lm\gamma} : \mathbb{L}^1 \times \mathbb{L}^0 \to C(\mathbb{R}_+) \tag{7.10}$$

$$\begin{aligned} \mathcal{V}_{lm\gamma}(z)(t) &= \langle v(t)\cdot Z^\gamma(b_l * u(t)),\varphi_m\rangle_{L^2(\mathbb{R})} - \langle v(0)\cdot Z^\gamma(b_l*u(0)),\varphi_m\rangle_{L^2(\mathbb{R})} \\ &+ \int_0^t \langle \partial_x u(s)\cdot Z^\gamma(b_l*u(s)),\partial_x\varphi_m\rangle_{L^2(\mathbb{R})}\,ds \\ &+ \int_0^t \left\langle \partial_x u(s)\cdot (Z^\gamma)'_{b_l*u(s)}(b_l*\partial_x u(s)),\varphi_m\right\rangle_{L^2(\mathbb{R})}\,ds \\ &- \int_0^t \left\langle v(s)\cdot (Z^\gamma)'_{b_l*u(s)}(b_l*v(s)),\varphi_m\right\rangle_{L^2(\mathbb{R})}\,ds \end{aligned} \tag{7.11}$$

$$\mathcal{V}^k_{lm\gamma}(z)(t) = \sum_{i=1}^k\sum_{j\in J_i}\int_0^t \left|\left\langle \left[Y^k(u(s),v(s),\partial_x u(s))\cdot Z^\gamma(b_l*u(s))\right]\xi_{ij},\varphi_m\right\rangle_{L^2(\mathbb{R})}\right|^2 ds \tag{7.12}$$

$$\mathcal{V}^\infty_{lm\gamma}(z)(t) = \sum_{i=1}^\infty\sum_{j\in J_i}\int_0^t \left|\left\langle \left[\bar Y(u(s),v(s),\partial_x u(s))\cdot Z^\gamma(b_l*u(s))\right]\xi_{ij},\varphi_m\right\rangle_{L^2(\mathbb{R})}\right|^2 ds$$

$$\mathcal{V}^{ijk}_{lm\gamma}(z)(t) = \int_0^t \left\langle \left[Y^k(u(s),v(s),\partial_x u(s))\cdot Z^\gamma(b_l*u(s))\right]\xi_{ij},\varphi_m\right\rangle_{L^2(\mathbb{R})} ds \tag{7.13}$$

if $i\le k$ and $j\in J_i$, otherwise $\mathcal{V}^{ijk}_{lm\gamma}(z)=0$,

$$\mathcal{V}^{ij\infty}_{lm\gamma}(z)(t) = \int_0^t \left\langle \left[\bar Y(u(s),v(s),\partial_x u(s))\cdot Z^\gamma(b_l*u(s))\right]\xi_{ij},\varphi_m\right\rangle_{L^2(\mathbb{R})} ds$$

if $j\in J_i$, otherwise $\mathcal{V}^{ijk}_{lm\gamma}(z)=0$.

Lemma 7.13. *$\mathcal{V}_{lm\gamma}$, $\mathcal{V}^k_{lm\gamma}$, $\mathcal{V}^\infty_{lm\gamma}$, $\mathcal{V}^{ijk}_{lm\gamma}$ and $\mathcal{V}^{ij\infty}_{lm\gamma}$ are sequentially continuous mappings from $\mathbb{L}^1\times\mathbb{L}^0$ to $C(\mathbb{R}_+)$ for every $k,l,m\in\mathbb{N}$ and $\gamma\in\{1,\dots,N\}$. Moreover, if z^k converges to z in $\mathbb{L}^1\times\mathbb{L}^0$ then $\mathcal{V}^k_{lm\gamma}(z^k)$ converges to $\mathcal{V}^\infty_{lm\gamma}(z)$ and $\mathcal{V}^{ijk}_{lm\gamma}(z^k)$ converges to $\mathcal{V}^{ij\infty}_{lm\gamma}(z)$ in $C(\mathbb{R}_+)$ for every $k,l,m\in\mathbb{N}$ and $\gamma\in\{1,\dots,N\}$.*

Proof. It is enough to apply the Lebesgue Dominated Convergence Theorem. Indeed, if $z^k=(u^k,v^k)$ converges to $z=(u,v)$ in $\mathbb{L}^1\times\mathbb{L}^0$ then, for every $R>0$,

$$\lim_{k\to\infty}\left(\sup_{t\in[0,R]}|b_l*v^k(t)-b_l*v(t)|_{C([-R,R])}+\sup_{t\in[0,R]}|u^k(t)-u(t)|_{C([-R,R])}\right)=0,$$

if h^k converge to h uniformly on $[0,R]\times[-R,R]$ then

$$\lim_{k\to\infty}\sup_{t\in[0,R]}|\langle v^k(t),h^k(t)\rangle_{L^2(-R,R)}-\langle v(t),h(t)\rangle_{L^2(-R,R)}|=0$$

and

$$\sum_{i=1}^\infty\sum_{j\in J_i}|\xi_{ij}(x)|^2=\sum_{i=1}^\infty\frac{\nu_i(\mathbb{R})}{2\pi}=\frac{\mu(\mathbb{R})}{2\pi},\qquad x\in\mathbb{R}. \tag{7.14}$$

□

Lemma 7.14. *Under the above assumptions, the following identity*

$$\begin{aligned}
&\langle \mathbf{v}(t)\cdot Z^\gamma(b_l*\mathbf{u}(t)),\varphi_m\rangle_{L^2(\mathbb{R})}\\
&\quad= \mathbf{Q}_{lm\gamma}(t)+\langle \mathbf{v}(0)\cdot Z^\gamma(b_l*\mathbf{u}(0)),\varphi_m\rangle_{L^2(\mathbb{R})}\\
&\quad- \int_0^t \langle \partial_x\mathbf{u}(s)\cdot Z^\gamma(b_l*\mathbf{u}(s)),\partial_x\varphi_m\rangle_{L^2(\mathbb{R})}\,ds\\
&\quad- \int_0^t \big\langle \partial_x\mathbf{u}(s)\cdot (Z^\gamma)'_{b_l*\mathbf{u}(s)}(b_l*\partial_x\mathbf{u}(s)),\varphi_m\big\rangle_{L^2(\mathbb{R})}\,ds\\
&\quad+ \int_0^t \big\langle \mathbf{v}(s)\cdot (Z^\gamma)'_{b_l*\mathbf{u}(s)}(b_l*\mathbf{v}(s)),\varphi_m\big\rangle_{L^2(\mathbb{R})}\,ds\\
&\quad+ \int_0^t \langle [Y(\mathbf{u}(s),\mathbf{v}(s),\partial_x\mathbf{u}(s))\cdot Z^\gamma(b_l*\mathbf{u}(s))]\,d\mathbf{W},\varphi_m\rangle_{L^2(\mathbb{R})}
\end{aligned}$$

holds for every $t\geq 0$, $m,l\in\mathbb{N}$ and $\gamma\in\{1,\dots,N\}$ $\mathbb{P}$ almost surely.

Remark 7.15. Notice that the equation in Lemma 7.14 is only similar to the pseudointrinsic equation (5.1) since the term $\mathbf{Q}_{lm\gamma}$ is not identified as indefinite integral of the corresponding integrand as in Lemma 6.3 and plays just a role of a "small" remainder that will eventually disappear as $l\to\infty$.

Proof. Let us begin with fixing $T>0$, $m,l\in\mathbb{N}$, $\gamma\in\{1,\dots,N\}$, $\mathbb{N}\ni\kappa\geq\max\{t,r_m+1\}$. Let us take $0\leq t_0<t_1\leq T$. Then we observe that using the notation of Corollaries 7.5 and 7.6, the following equality

$$\begin{aligned}
&\mathcal{V}_{lm\gamma}(z^{k_\alpha})(t)-Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma}(t)\\
&\quad= \int_0^t \big\langle \big[Y^{k_\alpha}(u^{k_\alpha}(s),v^{k_\alpha}(s),\partial_x u^{k_\alpha}(s))\cdot Z^\gamma(b_l*u^{k_\alpha})\big]\,dW^{k_\alpha},\varphi_m\big\rangle_{L^2(\mathbb{R})}
\end{aligned}$$

holds for every $\alpha\in\mathbb{N}$ and $t\geq 0$ $\mathbb{P}$ almost surely by (5.1). Since

$$\begin{aligned}
&\sup_{t\in[0,\kappa]}\Big(\big|\mathcal{V}_{lm\gamma}(z^{k_\alpha})(t)\big|+\big|\mathcal{V}^{k_\alpha}_{lm\gamma}(z^{k_\alpha})(t)\big|+\Big|\mathcal{V}^{ijk_\alpha}_{lm\gamma}(z^{k_\alpha})(t)\Big|+\big|Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma}(t)\big|\Big)\\
&\quad\leq c_{\kappa,m}\Big(1+|z^{k_\alpha}|^2_{L^\infty((0,\kappa);H^1(-\kappa,\kappa)\times L^2(-\kappa,\kappa))}\Big)
\end{aligned}$$

by (6.17) and (7.14), we infer that also

$$\begin{aligned}
&\sup_{t\in[0,\kappa]}\Big(|\mathcal{V}_{lm\gamma}(\mathbf{z}^\alpha)(t)|+\big|\mathcal{V}^{k_\alpha}_{lm\gamma}(\mathbf{z}^\alpha)(t)\big|+\Big|\mathcal{V}^{ijk_\alpha}_{lm\gamma}(\mathbf{z}^\alpha)(t)\Big|+\big|\mathbf{Q}^\alpha_{lm\gamma}(t)\big|\Big)\\
&\quad\leq c_{\kappa,m}\Big(1+|\mathbf{z}^\alpha|^2_{L^\infty((0,\kappa);H^1(-\kappa,\kappa)\times L^2(-\kappa,\kappa))}\Big)
\end{aligned}$$

$\mathbb{P}$ almost surely and that the processes

$$
\begin{aligned}
&I^{k_\alpha}_{r,2\kappa}\left[\mathcal{V}_{lm\gamma}(z^{k_\alpha})-Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma}\right],\\
&I^{k_\alpha}_{r,2\kappa}\left\{\left[\mathcal{V}_{lm\gamma}(z^{k_\alpha})-Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma}\right]^2-\mathcal{V}^{k_\alpha}_{lm\gamma}(z^{k_\alpha})\right\},\\
&I^{k_\alpha}_{r,2\kappa}\left\{\left[\mathcal{V}_{lm\gamma}(z^{k_\alpha})-Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma}\right]\beta^{ij}-\mathcal{V}^{ijk_\alpha}_{lm\gamma}(z^{k_\alpha})\right\},
\end{aligned}
$$

are $\mathbb{F}$-martingales on $[0,\kappa]$ for every $r>0$ and $i,j,\alpha\in\mathbb{N}$ by Theorem 4.7. Hence, with the same notation as in (7.8),

$$
\begin{aligned}
&\mathbb{E}\,g_0(\mathbf{O})\mathbf{I}_{r,2\kappa}\left[\mathcal{V}_{lm\gamma}(\mathbf{z})(t_1)-\mathbf{Q}_{lm\gamma}(t_1)\right]=\lim_{\alpha\to\infty}\mathbb{E}\,g_0(\mathbf{O}^\alpha)\mathbf{I}^\alpha_{r,2\kappa}\left[\mathcal{V}_{lm\gamma}(\mathbf{z}^\alpha)(t_1)-\mathbf{Q}^\alpha_{lm\gamma}(t_1)\right]\\
&=\lim_{\alpha\to\infty}\mathbb{E}\,g_0(O^{k_\alpha})I^{k_\alpha}_{r,2\kappa}\left[\mathcal{V}_{lm\gamma}(z^{k_\alpha})(t_1)-Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma}(t_1)\right]\\
&=\lim_{\alpha\to\infty}\mathbb{E}\,g_0(O^{k_\alpha})I^{k_\alpha}_{r,2\kappa}\left[\mathcal{V}_{lm\gamma}(z^{k_\alpha})(t_0)-Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma}(t_0)\right]\\
&=\lim_{\alpha\to\infty}\mathbb{E}\,g_0(\mathbf{O}^\alpha)\mathbf{I}^\alpha_{r,2\kappa}\left[\mathcal{V}_{lm\gamma}(\mathbf{z}^\alpha)(t_0)-\mathbf{Q}^\alpha_{lm\gamma}(t_0)\right]\\
&=\mathbb{E}\,g_0(\mathbf{O})\mathbf{I}_{r,2\kappa}\left[\mathcal{V}_{lm\gamma}(\mathbf{z})(t_0)-\mathbf{Q}_{lm\gamma}(t_0)\right],
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\,g_0(\mathbf{O})\mathbf{I}_{r,2\kappa}\left\{\left[\mathcal{V}_{lm\gamma}(\mathbf{z})(t_1)-\mathbf{Q}_{lm\gamma}(t_1)\right]^2-\mathcal{V}^\infty_{lm\gamma}(\mathbf{z})(t_1)\right\}\\
&=\lim_{\alpha\to\infty}\mathbb{E}\,g_0(\mathbf{O}^\alpha)\mathbf{I}^\alpha_{r,2\kappa}\left\{\left[\mathcal{V}_{lm\gamma}(\mathbf{z}^\alpha)(t_1)-\mathbf{Q}^\alpha_{lm\gamma}(t_1)\right]^2-\mathcal{V}^{k_\alpha}_{lm\gamma}(\mathbf{z}^\alpha)(t_1)\right\}\\
&=\lim_{\alpha\to\infty}\mathbb{E}\,g_0(O^{k_\alpha})I^{k_\alpha}_{r,2\kappa}\left\{\left[\mathcal{V}_{lm\gamma}(z^{k_\alpha})(t_1)-Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma}(t_1)\right]^2-\mathcal{V}^{k_\alpha}_{lm\gamma}(z^{k_\alpha})(t_1)\right\}\\
&=\lim_{\alpha\to\infty}\mathbb{E}\,g_0(O^{k_\alpha})I^{k_\alpha}_{r,2\kappa}\left\{\left[\mathcal{V}_{lm\gamma}(z^{k_\alpha})(t_0)-Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma}(t_0)\right]^2-\mathcal{V}^{k_\alpha}_{lm\gamma}(z^{k_\alpha})(t_0)\right\}\\
&=\lim_{\alpha\to\infty}\mathbb{E}\,g_0(\mathbf{O}^\alpha)\mathbf{I}^\alpha_{r,2\kappa}\left\{\left[\mathcal{V}_{lm\gamma}(\mathbf{z}^\alpha)(t_0)-\mathbf{Q}^\alpha_{lm\gamma}(t_0)\right]^2-\mathcal{V}^{k_\alpha}_{lm\gamma}(\mathbf{z}^\alpha)(t_0)\right\}\\
&=\mathbb{E}\,g_0(\mathbf{O})\mathbf{I}_{r,2\kappa}\left\{\left[\mathcal{V}_{lm\gamma}(\mathbf{z})(t_0)-\mathbf{Q}_{lm\gamma}(t_0)\right]^2-\mathcal{V}^\infty_{lm\gamma}(\mathbf{z})(t_0)\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\,g_0(\mathbf{O})\mathbf{I}_{r,2\kappa}\left\{\left[\mathcal{V}_{lm\gamma}(\mathbf{z})(t_1)-\mathbf{Q}_{lm\gamma}(t_1)\right]\mathbf{B}_{ij}(t_1)-\mathcal{V}^{ij\infty}_{lm\gamma}(\mathbf{z})(t_1)\right\}\\
&=\lim_{\alpha\to\infty}\mathbb{E}\,g_0(\mathbf{O}^\alpha)\mathbf{I}^\alpha_{r,2\kappa}\left\{\left[\mathcal{V}_{lm\gamma}(\mathbf{z}^\alpha)(t_1)-\mathbf{Q}^\alpha_{lm\gamma}(t_1)\right]\mathbf{B}^\alpha_{ij}(t_1)-\mathcal{V}^{ijk_\alpha}_{lm\gamma}(\mathbf{z}^\alpha)(t_1)\right\}\\
&=\lim_{\alpha\to\infty}\mathbb{E}\,g_0(O^{k_\alpha})I^{k_\alpha}_{r,2\kappa}\left\{\left[\mathcal{V}_{lm\gamma}(z^{k_\alpha})(t_1)-Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma}(t_1)\right]\beta^{ij}(t_1)-\mathcal{V}^{ijk_\alpha}_{lm\gamma}(z^{k_\alpha})(t_1)\right\}\\
&=\lim_{\alpha\to\infty}\mathbb{E}\,g_0(O^{k_\alpha})I^{k_\alpha}_{r,2\kappa}\left\{\left[\mathcal{V}_{lm\gamma}(z^{k_\alpha})(t_0)-Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma}(t_0)\right]\beta^{ij}(t_0)-\mathcal{V}^{ijk_\alpha}_{lm\gamma}(z^{k_\alpha})(t_0)\right\}\\
&=\lim_{\alpha\to\infty}\mathbb{E}\,g_0(\mathbf{O}^\alpha)\mathbf{I}^\alpha_{r,2\kappa}\left\{\left[\mathcal{V}_{lm\gamma}(\mathbf{z}^\alpha)(t_0)-\mathbf{Q}^\alpha_{lm\gamma}(t_0)\right]\mathbf{B}^\alpha_{ij}(t_0)-\mathcal{V}^{ijk_\alpha}_{lm\gamma}(\mathbf{z}^\alpha)(t_0)\right\}\\
&=\mathbb{E}\,g_0(\mathbf{O})\mathbf{I}_{r,2\kappa}\left\{\left[\mathcal{V}_{lm\gamma}(\mathbf{z})(t_0)-\mathbf{Q}_{lm\gamma}(t_0)\right]\mathbf{B}_{ij}(t_0)-\mathcal{V}^{ij\infty}_{lm\gamma}(\mathbf{z})(t_0)\right\}
\end{aligned}
$$

holds for every $r>0$ such that Leb $\left(\{|\mathbf{z}(0)|_{H^1(-2\kappa,2\kappa)\times L^2(-2\kappa,2\kappa)}=r\}\right)=0$ by Lemma 7.13 and by the Lebesgue Dominated Convergence Theorem as (7.2) holds. In particular,

$$\begin{aligned}
&\mathcal{V}_{lm\gamma}(\mathbf{z})-\mathbf{Q}_{lm\gamma},\\
&[\mathcal{V}_{lm\gamma}(\mathbf{z})-\mathbf{Q}_{lm\gamma}]^2-\mathcal{V}^{\infty}_{lm\gamma}(\mathbf{z}),\\
&[\mathcal{V}_{lm\gamma}(\mathbf{z})-\mathbf{Q}_{lm\gamma}]\,\mathbf{B}_{ij}-\mathcal{V}^{ij\infty}_{lm\gamma}(\mathbf{z})
\end{aligned}$$

are local $\mathbb{B}$-martingales for every $l,m,i,j\in\mathbb{N}$, $\gamma\in\{1,\dots,N\}$. Hence, the quadratic variation satisfies

$$\begin{aligned}
&\left\langle \mathcal{V}_{lm\gamma}(\mathbf{z})-\mathbf{Q}_{lm\gamma}-\int_0^{\cdot}\left\langle\left[\bar{Y}(\mathbf{u}(s),\mathbf{v}(s),\partial_x\mathbf{u}(s))\cdot Z^{\gamma}(b_l*\mathbf{u}(s))\right]\,d\mathbf{W},\varphi_m\right\rangle_{L^2(\mathbb{R})}\right\rangle\\
=\;&\langle \mathcal{V}_{lm\gamma}(\mathbf{z})-\mathbf{Q}_{lm\gamma}\rangle+\left\langle\int_0^{\cdot}\left\langle\left[\bar{Y}(\mathbf{u}(s),\mathbf{v}(s),\partial_x\mathbf{u}(s))\cdot Z^{\gamma}(b_l*\mathbf{u}(s))\right]\,d\mathbf{W},\varphi_m\right\rangle_{L^2(\mathbb{R})}\right\rangle\\
-\;&2\left\langle \mathcal{V}_{lm\gamma}(\mathbf{z})-\mathbf{Q}_{lm\gamma},\int_0^{\cdot}\left\langle\left[\bar{Y}(\mathbf{u}(s),\mathbf{v}(s),\partial_x\mathbf{u}(s))\cdot Z^{\gamma}(b_l*\mathbf{u}(s))\right]\,d\mathbf{W},\varphi_m\right\rangle_{L^2(\mathbb{R})}\right\rangle\\
=\;&\mathcal{V}^{\infty}_{lm\gamma}(\mathbf{z})+\mathcal{V}^{\infty}_{lm\gamma}(\mathbf{z})-2\sum_{i=1}^{\infty}\sum_{j\in J_i}\\
&\left\langle \mathcal{V}_{lm\gamma}(\mathbf{z})-\mathbf{Q}_{lm\gamma},\int_0^{\cdot}\left\langle\left[\bar{Y}(\mathbf{u}(s),\mathbf{v}(s),\partial_x\mathbf{u}(s))\cdot Z^{\gamma}(b_l*\mathbf{u}(s))\right]\xi_{ij},\varphi_m\right\rangle_{L^2(\mathbb{R})}\,d\mathbf{B}_{ij}\right\rangle\\
=\;&2\mathcal{V}^{\infty}_{lm\gamma}(\mathbf{z})-2\sum_{i=1}^{\infty}\sum_{j\in J_i}\int_0^{\cdot}\left\langle\left[\bar{Y}(\mathbf{u}(s),\mathbf{v}(s),\partial_x\mathbf{u}(s))\cdot Z^{\gamma}(b_l*\mathbf{u}(s))\right]\xi_{ij},\varphi_m\right\rangle^2_{L^2(\mathbb{R})}\,ds\\
=\;&0.
\end{aligned}$$

The result now follows from equality $\bar{Y}(\mathbf{u},\mathbf{v},\partial_x\mathbf{u})=Y(\mathbf{u},\mathbf{v},\partial_x\mathbf{u})$ $\mathbb{P}$ which in view of Lemma 7.10 holds almost surely. ☐

Lemma 7.16. *In the above framework the following identity*

$$\begin{aligned}
\langle \mathbf{v}(t)\cdot Z^{\gamma}(\mathbf{u}(t)),\varphi_m\rangle_{L^2(\mathbb{R})}\;=\;&\langle \mathbf{v}(0)\cdot Z^{\gamma}(\mathbf{u}(0)),\varphi_m\rangle_{L^2(\mathbb{R})}\\
-\;&\int_0^t\langle\partial_x\mathbf{u}(s)\cdot Z^{\gamma}(\mathbf{u}(s)),\partial_x\varphi_m\rangle_{L^2(\mathbb{R})}\,ds\\
-\;&\int_0^t\left\langle\partial_x\mathbf{u}(s)\cdot(\nabla_{\partial_x\mathbf{u}(s)}Z^{\gamma})|_{\mathbf{u}(s)},\varphi_m\right\rangle_{L^2(\mathbb{R})}\,ds\\
+\;&\int_0^t\left\langle\mathbf{v}(s)\cdot(\nabla_{\mathbf{v}(s)}Z^{\gamma})|_{\mathbf{u}(s)},\varphi_m\right\rangle_{L^2(\mathbb{R})}\,ds\\
+\;&\int_0^t\langle[Y(\mathbf{u}(s),\mathbf{v}(s),\partial_x\mathbf{u}(s))\cdot Z^{\gamma}(\mathbf{u}(s))]\,d\mathbf{W},\varphi_m\rangle_{L^2(\mathbb{R})}
\end{aligned}\tag{7.15}$$

holds for every $t\geq 0$, $m\in\mathbb{N}$ and $\gamma\in\{1,\dots,N\}$ $\mathbb{P}$ almost surely.

Proof. Since for every $R>0$

$$\lim_{l\to\infty}\left[\sup_{t\in[0,R]}|b_l*\mathbf{u}(t)-\mathbf{u}(t)|_{C([-R,R])}\right]=0$$

we infer that

$$\lim_{l\to\infty}\left[\sup_{t\in[0,R]}|Z^\gamma(b_l*\mathbf{u}(t))-Z^\gamma(\mathbf{u}(t))|_{C([-R,R])}\right]=0$$

for every $R>0$ and $\gamma\in\{1,\dots,N\}$ on $[0,1]$. On the other hand, the terms $\partial_x\mathbf{u}*b_l$ and $\mathbf{v}*b_l$ converge to $\partial_x\mathbf{u}$ and $\mathbf{v}$ in $L^2_{loc}(\mathbb{R})$ for every $(t,\omega)\in\mathbb{R}_+\times[0,1]$. Moreover, we also have

$$\lim_{l\to\infty}\sum_{i=1}^\infty\sum_{j\in J_i}\int_0^T\langle[Y(\mathbf{u}(s),\mathbf{v}(s),\partial_x\mathbf{u}(s))\xi_{ij}]\cdot[Z^\gamma(\mathbf{u}(s)*b_l)-Z^\gamma(\mathbf{u}(s))],\varphi_m\rangle^2_{L^2(\mathbb{R})}\,ds$$

$$\begin{aligned}&\le\ \lim_{l\to\infty}\frac{r_m\mu(\mathbb{R})}{\pi}|\varphi_m|^2_{L^\infty(\mathbb{R})}\sup_{t\in[0,T]}|Z^\gamma(b_l*u(t))-Z^\gamma(u(t))|^2_{C([-r_m,r_m])}\\&\times\ \sup_{t\in[0,T]}|Y(\mathbf{u}(t),\mathbf{v}(t),\partial_x\mathbf{u}(t))|^2_{L^2(-r_m,r_m)}=0\end{aligned}$$

on $[0,1]$ for every $T>0$, $m\in\mathbb{N}$ and $\gamma\in\{1,\dots,N\}$ by (7.14), hence by for instance Proposition 4.1 in [35]

$$\begin{aligned}&\lim_{l\to\infty}\int_0^t\langle[Y(\mathbf{u}(s),\mathbf{v}(s),\partial_x\mathbf{u}(s))\cdot Z^\gamma(\mathbf{u}(s)*b_l)]\,d\mathbf{W},\varphi_m\rangle_{L^2(\mathbb{R})}\\&=\int_0^t\langle[Y(\mathbf{u}(s),\mathbf{v}(s),\partial_x\mathbf{u}(s))\cdot Z^\gamma(\mathbf{u}(s))]\,d\mathbf{W},\varphi_m\rangle_{L^2(\mathbb{R})}\end{aligned}$$

locally uniformly in t in probability. Finally,

$$\lim_{l\to\infty}|\mathbf{Q}_{lm\gamma}|_{C_0^{0,1}[0,T]}=0$$

in probability for every $m\in\mathbb{N}$ and $\gamma\in\{1,\dots,N\}$ by Corollary 7.7 and (7.3), thus the result follows from Lemma 7.14. □

8. Relations between intrinsic and extrinsic equations

Lemma 8.1. *Let $z=(u,v)$ be an adapted process with weakly continuous paths in $H^1_{loc}\times L^2_{loc}(TM)$ such that*

$$\frac{d}{dt}\langle u(\cdot,\omega),\varphi\rangle_{L^2(\mathbb{R})}=\langle v(\cdot,\omega),\varphi\rangle_{L^2(\mathbb{R})}$$

holds on $\mathbb{R}_+$ for every $\omega\in\Omega$ and every compactly supported $\varphi\in L^2(\mathbb{R})$, and let W be a spatially homogeneous Wiener process with a finite spectral measure μ. Then the following is equivalent:

- *(i) (z,W) satisfies the equation (7.15) for every $t\ge0$, $m\in\mathbb{N}$ and $\gamma\in\{1,\dots,N\}$ $\mathbb{P}$ almost surely.*

(*ii*) (z,W) *satisfies the intrinsic equation (3.2).*
(*iii*) (z,W) *satisfies the extrinsic equation (3.3).*

Proof. If (i) holds then (3.2) is satisfied $\mathbb{P}$ almost surely. for every $t\geq 0$, $\gamma\in\{1,\dots,N\}$ and every compactly supported $\varphi\in H^1(\mathbb{R})$ by Proposition D.2. Let (b_l) be smooth densities on $\mathbb{R}$ with supports in $(-\frac{1}{l},\frac{1}{l})$ and define

$$\begin{aligned} h_l^\gamma(t) &= b_l*[v(t)\cdot Z^\gamma(u(t))] \\ H_l^\gamma(t) &= [\partial_x b_l]*[\partial_x u(t)\cdot Z^\gamma(u(t))]-b_l*\left[\partial_x u(t)\cdot\nabla_{\partial_x u(t)}Z^\gamma|_{u(t)}\right]+b_l*\left[v(t)\cdot\nabla_{v(t)}Z^\gamma|_{u(t)}\right] \\ g_l^\gamma(t)\xi &= b_l*\{[Y(u(t),v(t),\partial_x u(t))\cdot Z^\gamma(u(t))]\,\xi\}, \qquad \xi\in H_\mu. \end{aligned}$$

Since by Lemma 2.1,

$$\begin{aligned} \sup_{t\in[0,R]}|g_l^\gamma(t)|_{\mathcal{J}_2(H_\mu,H^m(-R,R))} &\leq c_{b_l}\sup_{t\in[0,R]}|Y(z(t),\partial_x u(t))\cdot Z^\gamma(u(t))|_{\mathcal{J}_2(H_\mu,L^2(-1-R,R+1))} \\ &= c_{b_l}[\mu(\mathbb{R})]^{\frac{1}{2}}\sup_{t\in[0,R]}|Y(z(t),\partial_x u(t))\cdot Z^\gamma(u(t))|_{L^2(-1-R,R+1)} \\ &\leq c_{l,\gamma,R,\mu,Y}\left[1+\sup_{t\in[0,R]}|z(t)|_{H^1(-1-R,R+1)\times L^2(-1-R,R+1)}\right] \\ &\leq c_{l,\gamma,R,\mu,Y,z}(\omega)<\infty \end{aligned}$$

we infer that

$$h_l^\gamma(t)=h_l^\gamma(0)+\int_0^t H_l^\gamma(s)\,ds+\int_0^t g_l^\gamma(s)\,dW,\ t\in[0,T)$$

where the integrals converge in every $H^m(-R,R)$ for any $R>0$, $m\in\mathbb{N}$. On the other hand, as $Z^\gamma:\mathbb{R}^n\to\mathbb{R}^n$ is a smooth compactly supported function, there is

$$Z^\gamma(u(t))=Z^\gamma(u(0))+\int_0^t (Z^\gamma)'(u(s))v(s)\,ds$$

in every $L^2(-R,R)$ for every $t\geq 0$ and $R>0$. The mapping

$$H^1(-R,R)\times L^2((-R,R);\mathbb{R}^n)\to\mathbb{R}^n:(u,v)\mapsto\int_{\mathbb{R}}u(x)v(x)\varphi(x)\,dx$$

is C^2-smooth for every compactly supported $\varphi\in H^1(\mathbb{R})$ so, by the Itô formula (see for instance Theorem 4.17 in [14]),

$$\begin{aligned} \langle h_l^\gamma(t)Z^\gamma(u(t)),\varphi\rangle_{L^2(\mathbb{R})} &= \langle h_l^\gamma(t)Z^\gamma(u(0)),\varphi\rangle_{L^2(\mathbb{R})} \\ &+ \int_0^t\langle h_l^\gamma(s)(Z^\gamma)'(u(s))v(s),\varphi\rangle_{L^2(\mathbb{R})}\,ds \\ &+ \int_0^t\langle H_l^\gamma(s)Z^\gamma(u(s)),\varphi\rangle_{L^2(\mathbb{R})}\,ds \\ &+ \int_0^t\langle[g_l^\gamma(s)\,dW]\,Z^\gamma(u(s)),\varphi\rangle_{L^2(\mathbb{R})}\,ds \end{aligned}$$

for every $t \geq 0$ and $R > 0$, $\mathbb{P}$ almost surely. Now h_l^γ converges in L^2_{loc} to $v(t) \cdot Z^\gamma(u(t))$,

$$\begin{aligned}\langle\{[\partial_x b_l] * [\partial_x u \cdot Z^\gamma(u)]\} Z^\gamma(u), \varphi\rangle_{L^2(\mathbb{R})} &= \langle\partial_x \{b_l * [\partial_x u \cdot Z^\gamma(u)]\} Z^\gamma(u), \varphi\rangle_{L^2(\mathbb{R})} \\ &= -\langle\{b_l * [\partial_x u \cdot Z^\gamma(u)]\} (Z^\gamma)'(u)\partial_x u, \varphi\rangle_{L^2(\mathbb{R})} \\ &- \langle\{b_l * [\partial_x u \cdot Z^\gamma(u)]\} Z^\gamma(u), \partial_x\varphi\rangle_{L^2(\mathbb{R})}\end{aligned}$$

so $\langle H_l^\gamma Z^\gamma(u), \varphi\rangle_{L^2(\mathbb{R})}$ converges to

$$-\langle[\partial_x u \cdot Z^\gamma(u)] (Z^\gamma)'(u)\partial_x u, \varphi\rangle_{L^2(\mathbb{R})} - \langle[\partial_x u \cdot Z^\gamma(u)] Z^\gamma(u), \partial_x\varphi\rangle_{L^2(\mathbb{R})}$$

$$-\langle[\partial_x u \cdot \nabla_{\partial_x u} Z^\gamma|_u] Z^\gamma(u), \varphi\rangle_{L^2(\mathbb{R})} + \langle[v \cdot \nabla_v Z^\gamma|_u] Z^\gamma(u), \varphi\rangle_{L^2(\mathbb{R})}.$$

Finally,

$$\lim_{l\to\infty} \sum_{i=1}^\infty \sum_{j\in J_i} \int_0^t \langle\{b_l * [w(s)\xi_{ij}] - w(s)\xi_{ij}\} Z^\gamma(u(s)), \varphi\rangle^2_{L^2(\mathbb{R})} \, ds = 0$$

where $w = Y(u, v, \partial_x u) \cdot Z^\gamma(u)$ by the Lebesgue Dominated Convergence Theorem and (7.14), hence, by for instance Proposition 4.1 in [35],

$$\begin{aligned}&\langle[v(t) \cdot Z^\gamma(u(t))] Z^\gamma(u(t)), \varphi\rangle_{L^2(\mathbb{R})} \\ &\quad= \langle[v(0) \cdot Z^\gamma(u(0))] Z^\gamma(u(0)), \varphi\rangle_{L^2(\mathbb{R})} \\ &\quad+ \int_0^t \left\langle[v(s) \cdot Z^\gamma(u(s))] (Z^\gamma)'(u(s))v(s), \varphi\right\rangle_{L^2(\mathbb{R})} ds \\ &\quad- \int_0^t \langle[\partial_x u(s) \cdot Z^\gamma(u(s))] (Z^\gamma)'(u(s))\partial_x u(s), \varphi\rangle_{L^2(\mathbb{R})} ds \\ &\quad- \int_0^t \langle[\partial_x u(s) \cdot Z^\gamma(u(s))] Z^\gamma(u(s)), \partial_x\varphi\rangle_{L^2(\mathbb{R})} ds \\ &\quad- \int_0^t \langle\left[\partial_x u(s) \cdot \nabla_{\partial_x u(s)} Z^\gamma|_{u(s)}\right] Z^\gamma(u(s)), \varphi\rangle_{L^2(\mathbb{R})} \\ &\quad+ \int_0^t \langle\left[v(s) \cdot \nabla_{v(s)} Z^\gamma|_{u(s)}\right] Z^\gamma(u(s)), \varphi\rangle_{L^2(\mathbb{R})} ds \\ &\quad+ \int_0^t \langle[Y(u(s), v(s), \partial_x u(s)) \cdot Z^\gamma(u(s))] Z^\gamma(u(s)) \, dW, \varphi\rangle_{L^2(\mathbb{R})}\end{aligned}$$

holds for every $t \geq 0$ $\mathbb{P}$ almost surely. whenever $\varphi \in H^1(\mathbb{R})$ is compactly supported. Now (Z^γ) satisfy

$$A_p(\xi, \xi) = \sum_{\gamma=1}^N \left[(\xi \cdot \nabla_\xi Z^\gamma|_p) Z^\gamma(p) + (\xi \cdot Z^\gamma(p)) \nabla_\xi Z^\gamma|_p\right], \qquad p \in M, \quad \xi \in T_p M$$

by the equality on p. 479 in [1], so

$$\begin{aligned}
\langle v(t),\varphi\rangle_{L^2(\mathbb{R})} &= \langle v(0),\varphi\rangle_{L^2(\mathbb{R})} - \int_0^t \langle \partial_x u(s),\partial_x\varphi\rangle_{L^2(\mathbb{R})}\,ds \\
&+ \int_0^t \left\langle A_{u(s)}\left(v(s),v(s)\right) - A_{u(s)}\left(\partial_x u(s),\partial_x u(s)\right),\varphi\right\rangle_{L^2(\mathbb{R})}\,ds \\
&+ \int_0^t \langle Y(u(s),v(s),\partial_x u(s))\,dW,\varphi\rangle_{L^2(\mathbb{R})}
\end{aligned}$$

holds for every $t\geq 0$ $\mathbb{P}$ almost surely. whenever $\varphi\in H^1(\mathbb{R})$ is compactly supported and (3.3) holds and (i) implies (iii).

To prove (iii) implies (ii), we define processes

$$\begin{aligned}
u_l(t) &= b_l * u(t) \\
v_l(t) &= b_l * v(t) \\
a_l(t) &= b_l * \left[A_{u(t)}\left(v(t),v(t)\right) - A_{u(t)}\left(\partial_x u(t),\partial_x u(t)\right)\right] \\
g_l(t)\xi &= b_l * \left[Y(u(t),v(t),\partial_x u(t))\xi\right], \qquad \xi\in H_\mu.
\end{aligned}$$

Proceeding analogously as in the first part of the proof, there is

$$v_l(t) = v_l(0) + \int_0^t \left[\partial_{xx}u_l(s) + a_l(s)\right]\,ds + \int_0^t g_l(s)\,dW$$

$\mathbb{P}$ almost surely. for every $t\geq 0$ in every $H^m(-R,R)$ whenever $l,m\in\mathbb{N}$, $R>0$. Hence, by the Itô formula,

$$\begin{aligned}
\langle v_l(t)\cdot Z(u(t)),\varphi\rangle_{L^2(\mathbb{R})} &= \langle v_l(0)\cdot Z(u(0)),\varphi\rangle_{L^2(\mathbb{R})} \\
&+ \int_0^t \langle [\partial_{xx}u_l(s)+a_l(s)]\cdot Z(u(s)),\varphi\rangle_{L^2(\mathbb{R})}\,ds \\
&+ \int_0^t \langle v_l(s)\cdot \nabla_{v(s)}Z|_{u(s)},\varphi\rangle_{L^2(\mathbb{R})}\,ds \\
&+ \int_0^t \langle [g_l(s)\,dW]\cdot Z(u(s)),\varphi\rangle_{L^2(\mathbb{R})}
\end{aligned}$$

holds $\mathbb{P}$ almost surely. for every $t\geq 0$, $l\in\mathbb{N}$ and compactly supported $\varphi\in H^1(\mathbb{R})$. Since

$$\langle \partial_{xx}u_l\cdot Z(u),\varphi\rangle_{L^2(\mathbb{R})} = -\langle \partial_x u_l\cdot\nabla_{\partial_x u}Z|_u,\varphi\rangle_{L^2(\mathbb{R})} - \langle \partial_x u_l\cdot Z(u),\partial_x\varphi\rangle_{L^2(\mathbb{R})},$$

$v_l(t)$ and $\partial_x u_l(t)$ converge in $L^2_{loc}(\mathbb{R})$ to $v(t)$ and $\partial_x u(t)$, $a_l(t)$ converges in $L^1_{loc}(\mathbb{R})$ to $A_{u(t)}(v(t),v(t)) - A_{u(t)}(\partial_x u(t),\partial_x u(t))$ and

$$\lim_{l\to\infty}\sum_{i=1}^\infty\sum_{j\in J_i} |\langle [g_l\xi_{ij} - Y(z,\partial_x u)\xi_{ij}]\cdot Z(u),\varphi\rangle_{L^2(\mathbb{R})}|^2_{\mathcal{J}_2(H_\mu,\mathbb{R})} = 0,$$

we get that

$$\begin{aligned}\langle v(t)\cdot Z(u(t)),\varphi\rangle_{L^2(\mathbb{R})} &= \langle v(0)\cdot Z(u(0)),\varphi\rangle_{L^2(\mathbb{R})}\\ &- \int_0^t \langle \partial_x u(s)\cdot \nabla_{\partial_x u(s)} Z|_{u(s)},\varphi\rangle_{L^2(\mathbb{R})}\,ds\\ &- \int_0^t \langle \partial_x u(s)\cdot Z(u(s)),\partial_x\varphi\rangle_{L^2(\mathbb{R})}\,ds\\ &+ \int_0^t \langle \big[A_{u(s)}(v(s),v(s)) - A_{u(s)}(\partial_x u(s),\partial_x u(s))\big]\cdot Z(u(s)),\varphi\rangle_{L^2(\mathbb{R})}\,ds\\ &+ \int_0^t \langle v(s)\cdot \nabla_{v(s)} Z|_{u(s)},\varphi\rangle_{L^2(\mathbb{R})}\,ds\\ &+ \int_0^t \langle [Y(u(s),v(s),\partial_x u(s)\cdot Z(u(s))]\,dW,\varphi\rangle_{L^2(\mathbb{R})}\end{aligned}$$

holds $\mathbb{P}$ almost surely. for every $t \geq 0$ and every compactly supported $\varphi \in H^1(\mathbb{R})$ by the Lebesgue Dominated Convergence Theorem and Proposition 4.1 in [35]. The result now follows from perpendicularity $A_p \cdot Z_p = 0$, $p \in M$. □

Appendix A. Some useful facts about riemannian geometry

Lemma A.1. *There exists a smooth compactly supported mapping $P:\mathbb{R}^n \to \mathbb{R}^n$ such that $P \in M$ on a neighbourhood U_P about M and $P(p) = p$ for every $p \in M$.*

Proof. We will use a suitable smooth projection of the ambient space on the manifold. There exists an open neighbourhood V of the set $\{(p,0) : p \in M\}$ in the normal bundle NM such that the mapping $\mathcal{E} : V \to O \subseteq \mathbb{R}^n : (p,\xi) \mapsto p+\xi$ is a diffeomorphism between open sets V and O (see Proposition 7.26, p. 200 in [33]). We define a smooth mapping $\tilde{P} : O \to M$ as a composition of $NM \to M : (p,\xi) \mapsto p$ and $\mathcal{E}^{-1}$. Derived from $\tilde{P}$, there apparently exists a smooth compactly supported mapping $P : \mathbb{R}^n \to \mathbb{R}^n$ such that $P \in M$ on a neighbourhood $U \subseteq O$ of M (where P coincides with $\tilde{P}$), hence $P(p) = p$ for every $p \in M$. □

Lemma A.2. *There exists a sequence of Borel measurable mappings $\Theta_k : H^1_{loc}\times L^2_{loc}(TM) \to H^2_{loc} \times H^1_{loc}(TM)$ and a finite constant C_M depending on M such that*

$$\lim_{k\to\infty} |\Theta_k(z) - z|_{H^1_{loc}(\mathbb{R})\times L^2_{loc}(\mathbb{R})} = 0, \qquad z \in H^1_{loc} \times L^2_{loc}(TM)$$

and

$$|\Theta_k(z)|_{H^1(-R,R)\times L^2(-R,R)} \leq C_M(R^{\frac{1}{2}} + |z|_{H^1(-R-1,R+1)\times L^2(-R-1,R+1)})$$

holds for every $R > 0$, $k \in \mathbb{N}$ and $z \in H^1_{loc} \times L^2_{loc}(TM)$.

Proof. The mappings Θ_k are constructed as follows. Let (b_k) be a sequence of smooth symmetric densities on $\mathbb{R}$ with supports in $(-\frac{1}{k},\frac{1}{k})$, let $z = (u,v) \in H^1_{loc} \times L^2_{loc}(TM)$ and set

$$\bar{u}_k(x) = \begin{cases} u(-k), & x < -k \\ u(x), & |x| \leq k \\ u(k), & x > k \end{cases}$$

which is M-valued and belongs to $H^1_{loc}(\mathbb{R})$. Let $\varepsilon > 0$ be such that $p + z \in U_P$ whenever $p \in M$ and $|z| \leq \varepsilon$ where U_P is the same as in Lemma A.1. We set

$$m_k(u) = \min\{j \geq k : j \geq \varepsilon^{-2}|\partial_x u|^2_{L^2(-k,k)}\}, \qquad w_{k,u} = \bar{u}_k * b_{m_k(u)}.$$

Since $\partial_x \bar{u}_k = \mathbf{1}_{(-k,k)}\partial_x u$ we infer that

$$\sup_{x \neq y} \frac{|\bar{u}_k(x) - \bar{u}_k(y)|}{|x-y|^{\frac{1}{2}}} \leq |\partial_x u|_{L^2(-k,k)}.$$

Therefore,

$$\sup_{x \in \mathbb{R}} |w_{k,u}(x) - \bar{u}_k(x)| \leq [m_k(u)]^{-\frac{1}{2}} |\partial_x u|_{L^2(-k,k)} \leq \varepsilon.$$

Hence we infer that $w_{k,u}$ is a smooth function taking values in a compact set $M + \overline{B_\varepsilon} \subseteq U_P$. We set $u_k = P(w_{k,u})$ and $\Theta_k(u) = (u_k, \pi_{u_k}(b_k * v))$ where $\pi_p : \mathbb{R}^n \to T_pM$ is the orthogonal projection at $p \in M$. $\square$

Proposition A.3. *There exist smooth vector fields $Z^1, \dots, Z^N$ on M such that*

$$\xi = \sum_{i=1}^N \langle \xi, Z^i(p)\rangle Z^i(p), \qquad \xi \in T_pM, \quad p \in M. \tag{A.1}$$

Proof. Every point on M has a neighbourhood where some vector fields are an ONB in the tangent space, hence we may assume existence of open sets $O_1, \dots, O_m$ covering M and smooth vector fields $(E^i_j : i \leq m, j \leq d)$ such that $E^i_1(p), \dots, E^i_d(p)$ is an ONB in T_pM for every $p \in O^i$ whenever $i \leq m$. Let $\varphi_1, \dots, \varphi_k$ is a decomposition of unity on M and each φ_l is a smooth function with support in some O^{i_l}. If we set

$$X^{lj} = \psi_l E^{i_l}_j, \qquad \psi_l = L^{-\frac{1}{2}}\varphi_l, \qquad L = \sum_{l=1}^k \varphi_l^2$$

then

$$\psi_l^2(p)\xi = \sum_{j=1}^d \psi_l^2(p) \langle \xi, Z^{i_l}_j(p)\rangle Z^{i_l}_j(p), \qquad \xi \in T_p, \quad p \in M$$

holds for every $l \leq k$ since either $\psi_l^2(p) = 0$ or $p \in O^{i_l}$. Summing the above formula over the range $l \in \{1, \dots, k\}$ leads to the conclusion. $\square$

Appendix B. Some useful facts about function spaces

Let us denote by $\mathbb{L}^k = C_w(\mathbb{R}_+; H^k_{loc}(\mathbb{R};\mathbb{R}^n))$, for $k \in \mathbb{N}$, the vector space of all weakly continuous functions $h : \mathbb{R}_+ \to H^k_{loc}(\mathbb{R};\mathbb{R}^n)$. The space $\mathbb{L}^k$ becomes a locally convex topological vector space when equipped with the locally convex topology generated by the seminorms

$$|h|_{m,\varphi} = \sup_{t\in[0,m]} \sum_{j=0}^{k} \big|\big\langle \frac{\partial^j h}{\partial x^j}(t), \varphi_j \big\rangle_{L^2(-m,m)}\big|, \tag{B.1}$$

where $m \in \mathbb{N}^*$ and $\varphi \in \prod_{j=0}^k L^2(\mathbb{R};\mathbb{R}^n)$. By $\mathbb{L}$ we denote the following locally convex topological vector space

$$\mathbb{L} = \mathbb{L}^1 \oplus \mathbb{L}^0 \tag{B.2}$$

Proposition B.1. *Let $a = (a_m)$ be a sequence of positive real numbers and $k \in \mathbb{N}$. Then the set*

$$\mathbb{L}^k(a) = \left\{ h \in \mathbb{L}^k : \sup_{t\in[0,m]} |h(t)|_{H^k(-m,m)} \le a_m,\, m \in \mathbb{N} \right\} \tag{B.3}$$

is convex and closed in $\mathbb{L}^k$. Moreover, the trace topology of $\mathbb{L}^k$ on $\mathbb{L}^k(a)$ is metrizable.

Proof of Proposition B.1. Obviously the set $\mathbb{L}^k(a)$ is convex.

Since $\mathbb{L}(a) = \cap_{m=1}^\infty \mathbb{L}_m(a)$, where for $m \in \mathbb{N}$, $\mathbb{L}_m(a) := \big\{h \in C(\mathbb{R}_+, X_w) : \|\pi_m h\|_{C([0,m];X_m)} \le a_m\big\}$ and each set $\mathbb{L}_m(a)$ is closed in $\mathbb{L}$, we infer that $\mathbb{L}(a)$ is closed as well.

Let us choose and fix a countable dense subset $\mathcal{L}$ of $\prod_{j=0}^k L^2(\mathbb{R};\mathbb{R}^n)$. Let us denote by τ^1 be the topology on $\mathbb{L}^u$ introduced before the Proposition B.1 was formulated. Let τ^2 be a locally convex topology on $\mathbb{L}^u$ generated by a family of seminorms $\big\{|\cdot|_{m,\varphi} : m \in \mathbb{N}, \varphi \in \mathcal{L}\big\}$. The τ^2 topology is metrizable since it is generated by a countable number of seminorms, see for instance [43, Theorem 1.24]. Hence, it is enough to show that on $\mathbb{L}^k(a)$ the trace topologies of τ^1 and τ^2 coincide.

Since the topology τ^2 is generated by a smaller family of seminorms than the topology τ^2 we infer that τ^2 is smaller than τ^1 and hence the trace of τ^2 is smaller than the trace of τ^1.

In order to show that the trace of τ^1 is smaller that the trace of τ^2 we need to show that an intersection of $\mathbb{L}^k(a)$ and an arbitrary neighborhood of 0 from τ^1 contains an intersection of $\mathbb{L}^k(a)$ and some neighborhood of 0 from τ^2.

For simplicity let us put $k = 0$.

Let us choose $\psi \in L^2$, $m \in \mathbb{N}^*$ and $\varepsilon > 0$. Consider a set

$$V := \{h \in \mathbb{L}^0 : |h|_{m,\psi} < \varepsilon\}.$$

which is a neighborhood of 0 from τ^1.

We want to find $\phi \in \mathcal{L}$ such that $U \cap \mathbb{L}^k(a) \subset V \cap \mathbb{L}^k(a)$, where

$$U := \{h \in \mathbb{L}^0 : |h|_{m,\phi} < \frac{\varepsilon}{2}\}.$$

For this aim we observe that since $\mathcal{L}$ is dense in $H^0(\mathbb{R}) = L^2(\mathbb{R})$ we can find $\phi \in \mathcal{L}$ such that $|\psi - \phi|_{L^2(-m,m)} < \frac{\varepsilon}{2a_m}$. Hence, if $h \in U \cap \mathbb{L}^k(a)$,

$$\begin{aligned} |h|_{m,\psi} &= \sup_{t\in[0,m]} \left|\left\langle h(t),\psi\right\rangle_{L^2(-m,m)}\right| \leq \sup_{t\in[0,m]} \left|\left\langle h(t),\phi\right\rangle_{L^2(-m,m)}\right| \\ &+ \sup_{t\in[0,m]} \left|\left\langle h(t),\psi-\phi\right\rangle_{L^2(-m,m)}\right| \leq |h|_{m,\psi} \\ &+ \sup_{t\in[0,m]} \left|h(t)\right|_{L^2(-m,m)}|\psi-\phi|_{L^2(-m,m)} < \frac{\varepsilon}{2} + a_m\frac{\varepsilon}{2a_m} = \varepsilon. \end{aligned}$$

This proves, as required, that $U \cap \mathbb{L}^k(a) \subset V \cap \mathbb{L}^k(a)$.

The proof of closedness of $\mathbb{L}^k(a)$ of is standard.

□

We will use the following family of Sobolev spaces,

$$\begin{aligned} H^1_0(-R,R) &= \{\varphi \in H^1(-R,R) : \varphi(-R) = \varphi(R) = 0\}, \\ H^{-1}(-R,R) &= \left[H^1_0(-R,R)\right]^* \end{aligned}$$

where $R > 0$. We remark that all these spaces are separable Hilbert spaces.

Proposition B.2. *Let $a = (a_m)$ and $b = (b_m)$ be two sequences of positive real numbers, $\alpha \in (0,1]$ and $k \in \mathbb{N}$. Then the set*

$$\mathbb{L}^k(a,b) = \left\{ h \in \mathbb{L}^k(a) : \sup_{0\leq s<t\leq m} \left[\frac{|h(t)-h(s)|_{H^{-1}(-m,m)}}{(t-s)^\alpha}\right] \leq b_m,\ m \in \mathbb{N} \right\} \tag{B.4}$$

is a convex metrizable compact subset of $\mathbb{L}^k$.

Proof. Since $\mathbb{L}^k(a,b)$ is a subset of $\mathbb{L}^k(a)$, metrizability of the former follows from Proposition B.1. Since $L^2(-m,m)$ is compactly embedded in $H^{-1}(-m,m)$ for every $m \in \mathbb{N}$ one can prove that the function

$$\mathbb{L}^k(a) \ni h \mapsto \sup_{0\leq s<t\leq m} \frac{|h(t)-h(s)|_{H^{-1}(-m,m)}}{(t-s)^\alpha} \in [0,\infty]$$

is sequentially weakly lower semi-continuous, see [46], we infer that $\mathbb{L}^k(a,b)$ is a closed subset of $\mathbb{L}^k(a)$ and hence, by Proposition B.1, a closed subset of $\mathbb{L}^k$.

Hence it remains to prove the relative compactness $\mathbb{L}^k(a,b)$. For this aim let us consider an $\mathbb{L}^k(a,b)$-valued sequence (h_j). Since $|h_j(t)|_{H^k(-m,m)} \leq b_m$ for every $t \in [0,m]$ and $m \in \mathbb{N}$, by employing the Helly's diagonalisation procedure, we can find a subsequence j_l and a function $h : \mathbb{Q}_+ \to H^k_{loc}(\mathbb{R})$ such that $h_{j_l}(q)$ converges weakly in $H^k_{loc}(\mathbb{R})$ to $h(q)$ for every $q \in \mathbb{Q}_+$.

Let us fix $m \in \mathbb{N}$ and write $X = H^k(-m,m)$, $Y = H^{-1}(-m,m)$. Then the Hilbert space X is continuously embedded in the Hilbert space Y and hence the dual Y^* is dense in the dual X^*. Let us notice that for all $h_1, h_2 \in \mathbb{L}^k(a,b)$, $\varphi \in X^*$, $\psi \in Y^*$, $i, j \in \mathbb{N}$ and

$t \in [0,m]$, $q \in [0,m] \cap \mathbb{Q}$,

$$\begin{aligned} |\varphi(h_1(t)) - \varphi(h_2(t))| &\le 4a_m|\varphi - \psi|_{X^*} \\ &+ 2b_m|\psi|_{Y^*}|t-q|^\alpha + |\varphi(h_1(q)) - \varphi(h_2(q))|. \end{aligned} \tag{B.5}$$

Hence, since the weak topology on $\bar{B}_Y(0,a_m)$, the closed ball of radius b_m in the space Y, is metrisable, we infer that for every $t \in [0,m]$, the sequence $(h_{j_i}(t))$ is Cauchy in $\bar{B}_Y(0,b_m)$. By the completeness of the the last set we infer that there exists $h(t;m) \in H^k(-m,m)$ such that the sequence $(h_{j_i}(t))$ is weakly convergent in $H^k(-m,m)$ to $h(t;m)$. Consequently, by again employing the Helly's diagonalisation procedure, we can find $h : \mathbb{R}_+ \to H^k_{loc}(\mathbb{R})$ such that for every $t \in \mathbb{R}_+$, the sequence $(h_{j_i}(t))$ is weakly convergent in $H^k_{loc}(\mathbb{R})$ to $h(t)$.

Finally, in a classical way we can verify that $h \in \mathbb{L}^k$ and that h_{j_l} converges to h in the topology of $\mathbb{L}^k$. □

Appendix C. Skorokhod-Jakubowski Representation Theorem

Let X be a topological space such that here exists a sequence (f_j) of real continuous functions on X that separate points of X. Then, by [22], every compact set in X is metrizable and a Borel probability measure is Radon iff it is supported by a σ-compact set. The following result has also been proved by Jakubowski in [22].

Theorem C.1. *Let (ν_j) be a tight sequence of Borel probability measures on X. Then there exist a subsequence (j_k) and Borel measurable mappings $\theta, \theta_k : [0,1] \to X$, $k \ge 1$ with a σ-compact range such that ν_{j_k} is the law of θ_k, $k \ge 1$ and $\theta_k(t)$ converges in X to $\theta(t)$ for every $t \in [0,1]$.*

The following result claims, in particular, that Borel σ-algebra of a Polish space Z continuously embedded in X coincides with the trace σ-algebra of X on Z.

Proposition C.2. *If Z is a Polish space and $b : Z \to X$ is a continuous injection, then $b[B]$ is a Borel set whenever B is Borel in Z.*

Proof. Since the map $F = (f_1, f_2, \dots) : X \to \mathbb{R}^{\mathbb{N}}$ is a continuous injection, $F \circ b : Z \to \mathbb{R}^{\mathbb{N}}$ is also a continuous injection. Let us take a Borel set $B \subseteq Z$. Since both Z and $\mathbb{R}^{\mathbb{N}}$ are Polish spaces, we infer that $(F \circ b)[B]$ is a Borel set. Therefore $b[B] = F^{-1}[(F \circ b)[B]] \subseteq X$ is Borel set too. □

Appendix D. A measurability lemma

Proposition D.1. *Let X be a separable Fréchet space (with a countable system of pseudonorms $(|\cdot|_k)_{k\in\mathbb{N}}$, let X_k be separable Hilbert spaces and $i_k : X \to X_k$ linear mappings such that $|i_k(x)|_{X_k} = |x|_k$, $k \ge 1$. Let $\varphi_{k,j} \in X_k^*$, $j \in \mathbb{N}$ separate points of X_k. Then the mappings $(\varphi_{k,j} \circ i_k)_{k,j\in\mathbb{N}}$ generate the Borel σ-algebra on X.*

Proof. Since the map $(\varphi_{k,j} \circ i_k)_{k,j\in\mathbb{N}} : X \to \prod_{k,j\in\mathbb{N}} X_k$ is injective, the result follows from [50, Theorem 3] which for the convenience of the reader we formulate right now.

Theorem 3. Let X, $(Y_\alpha)_{\alpha\in I}$ be Polish spaces equipped with their Borel σ-algebras $\mathcal{B}(X)$, $(\mathcal{B}_\alpha)_{\alpha\in I}$, respectively. Let $f_\alpha : X \to Y_\alpha$, $\alpha \in I$ be functions. Then, $\sigma(\{f_\alpha : \alpha \in I\}) = \mathcal{B}(X)$

if and only if for each $\alpha \in I$ the function f_α is Borel measurable and there exists a countable $J \subset I$ such that the map $\prod_{\alpha\in J} f_\alpha : X \to \prod_{\alpha\in J} X_\alpha$ is injective. $\square$

The next result is a direct consequence of the previous one.

Proposition D.2. *There exists a countable system of compactly supported functions $\varphi_k \in C^\infty(\mathbb{R})$ such that, for every $L \in \mathbb{N}$, there is a subsequence k_j such that φ_{k_j} have support in $(-L, L)$ for every $j \in \mathbb{N}$, $\{\varphi_{k_j}\}$ is dense in $H^m(-L, L)$ and the mappings*

$$H^m_{loc}(\mathbb{R}) \ni h \mapsto \langle h, \varphi_k\rangle_{L^2} \in \mathbb{R}, \qquad k \in \mathbb{N}$$

generate the Borel σ-algebra on $H^m_{loc}(\mathbb{R})$ whenever $m \geq 0$.

References

[1] Brzeźniak Z. and M. Ondreját; *Strong solutions to stochastic wave equations with values in Riemannian manifolds*. J. Funct. Anal. **253**, no. 2, 449–481 (2007)

[2] Brzeźniak Z. and Ondreját, M., Stochastic geometric wave equations with values in compact homogeneous spaces, submitted

[3] Brzeźniak Z. and Ondreját, M., *Stochastic wave equations with values in Riemannian manifolds*, to appear in Stochastic Partial Differential Equations and Applications v. VIII, Quaderni di Matematica,

[4] Brzeźniak Z. and Peszat, S.; *Space-time continuous solutions to SPDE's driven by a homogeneous Wiener process*. Studia Math. **137**, no. 3, 261–299 (1999)

[5] Cabaña, E.; *On barrier problems for the vibrating string*, Z. Wahrsch. Verw. Gebiete **22**, 13-24 (1972)

[6] Carmona, R. and Nualart, D.; *Random nonlinear wave equations: propagation of singularities*, Ann. Probab. **16**, no. 2, 730-751

[7] Cazenave, Thierry; Shatah, Jalal; Tahvildar-Zadeh, A. Shadi *Harmonic maps of the hyperbolic space and development of singularities in wave maps and Yang-Mills fields*. Ann. Inst. H. Poincar Phys. Thor. **68**, no. 3, 315–349 (1998)

[8] Carmona, R. and Nualart, D., *Random nonlinear wave equations: smoothness of the solutions*, Probab. Theory Related Fields **79**, no. 4, 469-508 (1988)

[9] Chow, P.-L., *Stochastic wave equations with polynomial nonlinearity*, Ann. Appl. Probab. **12**, no. 1, 361-381 (2002)

[10] Christodoulou, Demetrios; Tahvildar-Zadeh, A. Shadi *On the regularity of spherically symmetric wave maps*, Comm. Pure Appl. Math. **46**, no. 7, 1041–1091 (1993)

[11] Dalang, R. C. and Frangos, N. E., *The stochastic wave equation in two spatial dimensions*, Ann. Probab. **26**, no. 1 (1998) 187-212.

[12] Dalang, R. C., *Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s,* Electron. J. Probab. **4**, no. 6, 1-29 (1999)

[13] Dalang, R. C. and Lévêque, O., *Second-order linear hyperbolic SPDEs driven by isotropic Gaussian noise on a sphere*, Ann. Probab. **32**, no. 1B, 1068-1099 (2004)

[14] Da Prato, G. and Zabczyk, J., Stochastic equations in infinite dimensions, Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge, 1992.

[15] Dellacherie, C. and Meyer, P-A, Probabilits et potentiel. Chapitres V VIII. Thorie des martingales. Revised edition. Actualits Scientifiques et Industrielles, 1385. Hermann, Paris, 1980.

[16] Freire, A., *Global weak solutions of the wave map system to compact homogeneous spaces*, Manuscripta Math. **91**, no. 4, 525–533 (1996)

[17] Fukutake, T., *Inductive and projective limits of W^m_p spaces*, Bull. Fukuoka Univ. Ed. III **23**, 27–33 (1973)

[18] Garsia, A. M.; Rodemich E. and Rumsey, H. Jr.; *A real variable lemma and the continuity of paths of some Gaussian processes, Indiana Univ. Math. J.*, **20**, 565-578 (1970)

[19] Ginibre, J.; Velo, G.: *The Cauchy problem for the* $\mathrm{O}(N)$, $C\mathrm{P}(N-1)$, *and* $G_C(N,p)$ *models*, Ann. Physics **142**, no. 2, 393–415 (1982)

[20] Gu, Chao Hao *On the Cauchy problem for harmonic maps defined on two-dimensional Minkowski space*, Comm. Pure Appl. Math. **33**, no. 6, 727–737 (1980)

[21] Hamilton Richard S., *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. (N.S.) **7**, no. 1, 65–222 (1982)

[22] Jakubowski, A., *The almost sure Skorokhod representation for subsequences in nonmetric spaces*, Teor. Veroyatnost. i Primenen. 42 (1997), no. 1, 209–216; translation in Theory Probab. Appl. 42 (1997), no. 1, 167–174 (1998)

[23] Karczewska, A. and J. Zabczyk, J., *A note on stochastic wave equations*, in Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998), 501-511, Lecture Notes in Pure and Appl. Math. **215**, Dekker, New York, 2001.

[24] Karczewska, A. and Zabczyk, J., *Stochastic PDE's with function-valued solutions*, in Infinite dimensional stochastic analysis (Amsterdam, 1999), 197-216, Verh. Afd. Natuurkd. 1. Reeks. K. Ned. Akad. Wet., 52, R. Neth. Acad. Arts Sci., Amsterdam, 2000.

[25] Köthe, G., Topological vector spaces. I. Translated from the German by D. J. H. Garling. Die Grundlehren der mathematischen Wissenschaften, Band **159**, Springer-Verlag New York Inc., New York 1969

[26] Ladyzhenskaya, O. A.; Shubov, V. I. *On the unique solvability of the Cauchy problem for equations of two-dimensional relativistic chiral fields with values in complete Riemannian manifolds* (Russian) Boundary value problems of mathematical physics and related questions in the theory of functions, 13. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **110**, 81–94, 242–243 (1981)

[27] Marcus, M. and Mizel, V. J., *Stochastic hyperbolic systems and the wave equation*, Stochastics Stochastic Rep. **36**, 225-244 (1991)

[28] Maslowski, B.; Seidler, J. and Vrkoč, I.; *Integral continuity and stability for stochastic hyperbolic equations*, Differential Integral Equations **6**, no. 2 355-382 (1993)

[29] Millet, A. and Morien, P.-L.; *On a nonlinear stochastic wave equation in the plane: existence and uniqueness of the solution*, Ann. Appl. Probab. **11**, no. 3, 922-951 (2001)

[30] Millet, A. and Sanz-Solé, M.; *A stochastic wave equation in two space dimension: smoothness of the law*, Ann. Probab. **27**, no. 2 803-844 (1999)

[31] Müller, Stefan; Struwe, Michael *Global existence of wave maps in* $1+2$ *dimensions with finite energy data*, Topol. Methods Nonlinear Anal. **7** (1996), no. 2, 245–259.

[32] Nash, J.; *The imbedding problem for Riemannian manifolds*, Ann. of Math. (2) **63**, 20-63 (1956)

[33] O'Neill,B.; Semi-Riemannian geometry. With applications to relativity, Pure and Applied Mathematics, 103. Academic Press, Inc., New York, 1983.

[34] Ondreját, M. *Existence of global mild and strong solutions to stochastic hyperbolic evolution equations driven by a spatially homogeneous Wiener process*, J. Evol. Equ. 4, no. 2 (2004) 169–191.

[35] Ondreját, M.; *Uniqueness for stochastic evolution equations in Banach spaces*, Dissertationes Math. (Rozprawy Mat.) **426**, 63 pp., 2004

[36] Ondreját, Martin *Existence of global martingale solutions to stochastic hyperbolic equations driven by a spatially homogeneous Wiener process.* Stoch. Dyn. 6 (2006), no. 1, 23–52.

[37] Peszat, S. and Zabczyk, J.; *Stochastic evolution equations with a spatially homogeneous Wiener process*, Stochastic Process. Appl. 72, no. 2 (1997) 187–204.

[38] Peszat, S.: *The Cauchy Problem for a Non Linear Stochastic Wave Equation in any Dimension*, J. Evol. Equ. **2**, 383-394 (2002)

[39] Peszat, S., Zabczyk, J.; *Non Linear Stochastic Wave and Heat Equations*, Probability Theory Related Fields **116**, No. 3, 421-443, 2000.

[40] Poppenberg, M. *An application of the Nash-Moser theorem to ordinary differential equations in Frchet spaces*, Studia Math. **137**, no. 2, 101–121 (1999)

[41] Pym, J. S., *Inductive and projective limits of normed spaces*, Glasgow Math. J. **9**, 103–105 (1968)

[42] Robertson A. P. and Robertson, Wendy, TOPOLOGICAL VECTOR SPACES. Reprint of the second edition. Cambridge Tracts in Mathematics, 53. Cambridge University Press, Cambridge-New York, 1980.

[43] Rudin Walter; FUNCTIONAL ANALYSIS, Second edition. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991.

[44] Shatah, Jalal; Struwe, Michael; GEOMETRIC WAVE EQUATIONS, Courant Lecture Notes in Mathematics, **2**. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1998.

[45] Shatah, Jalal; *Weak solutions and development of singularities of the* SU(2) σ*-model*, Comm. Pure Appl. Math. **41**, no. 4, 459–469 (1988)

[46] Struwe, Michael; VARIATIONAL METHODS. APPLICATIONS TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS AND HAMILTONIAN SYSTEMS. Fourth edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics **34**, Springer-Verlag, Berlin, 2008.

[47] Reed, M.; ABSTRACT NON LINEAR WAVE EQUATIONS, Lecture Notes in Mathematics 507, Springer-Verlag, 1976.

[48] Segal, I. E. *The global Cauchy problem for a relativistic scalar field with power interaction*, Bull. Soc. Math. France **91**, 129–135 1963

[49] W. A. Strauss, *On weak solutions of semi-linear hyperbolic equations, Anais Acad. Brasil. Ci.* **42**, 645-651 (1970)

[50] Yan, Yi; *When do cylinder* σ*-algebras equal Borel* σ*-algebras in Polish spaces?* Statist. Probab. Lett. **78**, no. 10, 1222–1225 (2008)

[51] Tataru, Daniel *The wave maps equation*, Bull. Amer. Math. Soc. (N.S.) **41**, no. 2, 185–204 (2004)

[52] Zähle, H.; *Weak approximation of SDEs by discrete-time processes.* J. Appl. Math. Stoch. Anal., Art. ID 275747, 15 pages (2008)

[53] Zhou, Yi; *Uniqueness of weak solutions of* $1+1$ *dimensional wave maps.* Math. Z. **232**, no. 4, 707–719 (1999)

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