ON QUANTIZED STOCHASTIC NAVIER-STOKES EQUATIONS

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Abstract. A random perturbation of a deterministic Navier-Stokes equation is considered in the form of an SPDE with Wick type nonlinearity. The nonlinear term of the perturbation can be characterized as the highest stochastic order approximation of the original nonlinear term $u\nabla u$. This perturbation is unbiased in that the expectation of a solution of the perturbed/quantized equation solves the deterministic Navier-Stokes equation. The perturbed equation is solved in the space of generalized stochastic processes using the Cameron-Martin version of the Wiener chaos expansion. The generalized solution can be obtained as a limit or an inverse of solutions to corresponding quantized equations. It is shown that the generalized solution is a Markov process.

1. Introduction

In this paper we will consider a deterministic Navier-Stokes equation

$$\begin{aligned} &\partial_t \mathbf{u}_0 (t,x) = \partial_i \left(a^{ij}(t,x)\,\partial_j \mathbf{u}_0(t,x)\right) - \\ &-u_0^k(t,x)\,\partial_k \mathbf{u}_0(t,x) + \nabla P_0(t,x) + \mathbf{f}(t,x), \\ &\mathbf{u}_0(0,x) = \mathbf{w}(x), \operatorname{div}\, \mathbf{u}_0 = 0, \end{aligned} \tag{1.1}$$

and its stochastic perturbations:

$$\begin{aligned} &\partial_t \mathbf{v}(t,x) = \partial_i \left(a^{ij}(t,x)\,\partial_j \mathbf{v}(t,x)\right) - \\ &v^k(t,x)\,\partial_k \mathbf{v}(t,x) + \nabla P(t,x) + \mathbf{f}(t,x) + \\ &[\sigma^i(t,x)\partial_i \mathbf{v}(t,x) + \mathbf{g}(t,x) - \nabla \tilde{P}(t,x)]\dot{W}_t, \\ &\mathbf{v}(0,x) = \mathbf{w}(x), \operatorname{div}\, \mathbf{v} = 0, \end{aligned} \tag{1.2}$$

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and

$$\begin{aligned}
&\partial_t \mathbf{u}(t,x) = \partial_i \left(a^{ij}(t,x)\,\partial_j \mathbf{u}(t,x)\right) - \\
&u^k(t,x)\,\lozenge\partial_k \mathbf{u}(t,x) + \nabla P(t,x) + \mathbf{f}(t,x) + \\
&[\sigma^i(t,x)\partial_i \mathbf{u}(t,x) + \mathbf{g}(t,x) - \nabla \tilde{P}(t,x)]\dot{W}_t, \\
&\mathbf{u}(0,x) = \mathbf{w}(x), \mathrm{div}\ \mathbf{u} = 0,
\end{aligned} \tag{1.3}$$

where $0 \leq t \leq T, x \in R^d, d \geq 2$, W_t is a cylindrical Wiener process in a separable Hilbert space Y, the coefficients a^{ij}, σ^i and the functions f, g are deterministic, σ^i and g are Y-valued, and $u\lozenge v$ denotes the Wick product. Technically, replacement of problem (1.2) by problem (1.3) amounts to replacement of products of random elements by stochastic convolutions, such as Wick products/Skorokhod integrals ([24], [10], [14] and Section 2.2.1). In the literature on quantum physics, procedures of this type are often called stochastic quantization (see e.g. [21],[9],[3]). Equations subjected to the stochastic quantization procedure are usually referred to as quantized.

The standard properties of the Wick product imply that

$$E\left(\mathbf{u}^k(t,x)\,\lozenge\partial_k \mathbf{u}(t,x)\right) = E\mathbf{u}^k(t,x)\,\partial_k(E\mathbf{u}(t,x)). \tag{1.4}$$

Therefore, the expectation of a solution of (1.3) is a solution of the deterministic Navier Stokes equation (1.1). Thus, equation (1.2) is an unbiased random perturbation of deterministic Navier-Stokes equation (1.1), in that $Eu(t,x) = u_0(t,x)$. This important property does not hold for the stochastic perturbation (1.2) or other standard stochastic perturbations of Navier-Stokes equation.

Stochastic Navier-Stokes equation (1.2) is reasonably well understood and there exists substantial literature on its analytical properties as well as its derivation from the first principles (see e.g. [18], [19] and the references therein). In this paper we will be focusing mostly on equation (1.3).

Burger's equation with Wick product was considered in [5], [7], see also references therein. The analysis in these papers was based on the Itô white noise expansion and S-transform in the Kondratiev's space S_{-1} of generalized stochastic random variables. Wick type SDEs have been introduced in [4], [10] (see [7] as well).

DaPrato and Debussche (see [2]) have investigated stochastic Navier-Stokes equation in R^2 with another version of Wick product in the nonlinear term. The version of the Wick product used in the aforementioned paper was based on an associated invariant distribution. In the setting of [2], the Wick-type nonlinear term differs from the classical one by an additive constant.

It was shown in [19] that under reasonable assumptions stochastic Navier-Stokes equation (1.2) has a square integrable solution. Moreover, this solution can be formally written in the Wiener chaos expansion form:

$$\mathbf{v}\,(t,x)=\sum_{\alpha}\mathbf{v}_{\alpha}\,(t,x)\,\xi_{\alpha},$$

where $\{\xi_\alpha, \alpha\in\mathcal{J}\}$ is the Cameron-Martin basis generated by $\dot{W}_t$, $v_\alpha\,(t,x)=E\left(\mathbf{v}_\alpha\,(t,x)\,\xi_\alpha\right)$, and J is the set of multiindices $\alpha=\{\alpha_k,\ k\geq 1\}$ such that for every k, $\alpha_k\in N$ and $|\alpha|=\sum_k \alpha_k<\infty$.

It was shown in [19] that $v_\alpha\,(t,x):=E\left(\mathbf{v}(t,x)\,\xi_\alpha\right)$ satisfy the propagator equation:

(1.5)

$$\partial_t\mathbf{v}_\alpha\,(t,x)=\partial_i\left(a^{ij}\partial_j\mathbf{v}_\alpha\,(t,x)\right)-$$

$$\sum_p\sum_{0\leq\beta\leq\alpha}c(\alpha,\beta,p)\left(\mathbf{v}_{\beta+p},\nabla\right)\mathbf{v}_{\alpha+p-\beta}\,(t,x)-\nabla P_\alpha\,(t,x)+\mathbf{f}\,(t,x)\,I_{\{|\alpha|=0\}}+$$

$$\sum_k\sqrt{\alpha_k}[\sigma^i\partial_i\mathbf{v}_{\alpha(k)}\,(t,x)-\nabla\tilde{P}_\alpha\,(t,x)+I_{\{|\alpha|=1\}}\mathbf{g}];\mathrm{div}\ \mathbf{v}_\alpha=0,$$

where $\alpha(k)=(\alpha_1,\alpha_2,...\alpha_{k-1},\alpha_k-1,\alpha_{k+1},...)$ and

$$c(\alpha,\beta,p)=\left[\binom{\alpha}{\beta}\binom{\beta+p}{p}\binom{\alpha+p-\beta}{p}\right]^{1/2}.$$

One advantage of the Wiener chaos representation is that it provides convenient formulae for computing statistical moments of the random field $u(t,x)$ (see [18], [19]). For example,

$$\mathbb{E}u^i(t,x)=u_0^i(t,x),$$

$$\mathbb{E}\left(u^i(t,x)u^j\,(t,y)\right)=\sum_{|\alpha|<\infty}u_\alpha^i(t,x)u_\alpha^j(t,y),$$

$$E\left(u^i(s,x)u^j\,(t,y)u^l\,(r,z)\right)=$$
$$\sum_{\alpha,\beta}\sum_{p\leq\alpha\wedge\beta}\left(p!\,(\alpha-p)!\,(\beta-p)!\right)^{-1}u_\alpha^i(s,x)u_\beta^j(t,y)u_{\alpha+\beta-2p}^l\,(r,z)$$

Now, let us consider equation (1.3). Equation (1.3) could be viewed as an approximation of equation (1.2). Indeed, it is a standard fact (see [21], [6], [7]) that

$$\xi_\alpha\Diamond\xi_\beta=\sqrt{(\alpha+\beta)!/\alpha!\beta!}\xi_{\alpha+\beta}\tag{1.6}$$

while

$$\xi_\alpha\xi_\beta=\xi_\alpha\Diamond\xi_\beta+\sum_{\gamma<\alpha+\beta}\kappa_\gamma\xi_\gamma,\tag{1.7}$$

where κ_γ are constants. Therefore, $\xi_\alpha\Diamond\xi_\beta$ is the highest stochastic order approximation of $\xi_\alpha\xi_\beta$ and, respectively, $u\Diamond\nabla u$ is the highest stochastic order approximation of $u\nabla u$.

The coefficients of a formal WCE for a solution of equation (1.3) are given by

$$\partial_t \mathbf{u}_\alpha (t,x) = [\partial_i \left(a^{ij}\partial_j \mathbf{u}_\alpha (t,x)\right) -$$

$$\sum_{0\le\beta\le\alpha} \sqrt{\tbinom{\alpha}{\beta}} \left(\mathbf{u}_{\alpha-\beta}, \nabla\right) \mathbf{u}_\beta (t,x) - \nabla P_\alpha (t,x) + \mathbf{f}(t,x) I_{\{|\alpha|=0\}} \tag{1.8}$$

$$+\sum_k \sqrt{\alpha_k}[\sigma^i \partial_i \mathbf{u}_{\alpha(k)} (t,x) - \nabla \tilde{P}_\alpha (t,x) + I_{\{|\alpha|=1\}}\mathbf{g}]; \text{div } \mathbf{u}_\alpha = 0.$$

Clearly, this system of equations is much simpler than equation (1.5). If $\alpha = 0$, then $u_\alpha(t,x)$ is a solution of deterministic Navier-Stokes equation (1.1). The remaining components are governed by Stokes equations and could be solved sequentially. From the computational point of view this is a substantial advantage. Indeed, the propagator for equation (1.2) is a full nonlinear system while for equation (1.3) is a bi-diagonal system and only the first equation of this system is nonlinear.

It is not clear how, if at all, the quantized Navier-Stokes equation fits into classical Fluid Mechanics. Nevertheless, equation (1.3) is "physical" in that it could be derived from the second Newton law (under appropriate assumptions on the velocity field), much the same way as the classical Navier-Stokes equation (see Appendix II).

The main disadvantage of equation (1.3) is that $\left(u^k (t) \lozenge \partial_k \mathbf{u}(t), u(t)\right)_{L_2} \neq 0$.Therefore, one could not expect that a solution of (1.3) is square integrable. This effect is not specific to stochastic Navier-Stokes equation. In fact, it is common for a large class of stochastic bi-linear PDEs (see e.g. [12], [13]).

In this paper we consider the equation (1.3) in the class of formal Wiener chaos expansions and show that a formal series

$$\mathbf{u}(t,x) = \sum_{|\alpha|<\infty} \mathbf{u}_\alpha (t,x)\, \xi_\alpha \tag{1.9}$$

solves (1.3) if and only if $u_\alpha (t,x)$ are given by equation (1.8). To make this solution square integrable we rescale it by ways of the procedure often called second quantization (see Appendix I). We show that $u(t,x)$ is the limit of square integrable solutions of the rescaled equations.

Convergence of this solution is determined by a system of positive weights $\{r_\alpha\}_{|\alpha|<\infty}$ such that

$$\|\mathbf{u}\|^2_{\mathcal{R}} := \sum_{|\alpha|<\infty} r^2_\alpha E \left\|\mathbf{u}_\alpha (t,x)\right\|^2_{L_2\left((0,T);R^d\right)} < \infty. \tag{1.10}$$

In this paper we establish existence and uniqueness of a generalized solution of equation (1.3) in Sobolev spaces $H^2_2 \cap H^2_p$ for $p > d$. We also demonstrate that the uniqueness holds under the same assumptions that guarantee uniqueness for the deterministic version of this equation. In order to show that $u(t,x)$ belongs to the Kondratiev's space S_{-1} of generalized stochastic random variables, the Catalan numbers (see [22], [8]) are critical for an appropriate choice of the weights r_α in (1.10).

Although ξ_α in (1.9) are not $(\mathcal{F}_t^W)$-adapted, we prove that the generalized solution is $(\mathcal{F}_t^W)$-adapted and Markov. Also, it is established that the generalized solution does not depend on the choice of the basis in $L_2([0,T],Y)$ which is implicitly included in the WCE.

2. Generalized Random Variables and Processes

2.1. **Wiener Chaos.** To begin with, we shall introduce some basic notation and recall a few fundamental facts of infinite-dimensional stochastic calculus. Let us fix a separable Hilbert spaces Y and $\mathbf{H} = L_2([0,T],Y)$. Let $\{\ell_i, i \geq 1\}$ be a complete orthonormal basis (CONS) in Y and$\{m_i, i \geq 1\}$ be a CONS in $L_2(0,T)$. Denote by $\mathcal{B}$ the class of all CONS in $\mathbf{H}$ of the form $\{e_k = e_k(s) = m_{k_1}(s)\ell_{k_2}\}$ and such that for each k, $\sup_{0\leq s\leq T}|m_k(s)| < \infty$. Obviously, for each k, $\sup_{0\leq s\leq T}|e_k(s)|_Y < \infty$. Let us fix a CONS $b = \{e_k, k \geq 1\} \in \mathcal{B}$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space with a filtration $\mathbb{F}$ of right continuous σ-algebras $(\mathcal{F}_t)_{t\geq 0}$. All the σ-algebras are assumed to be $\mathbf{P}$-completed. Let $W(t)$ be an $\mathbb{F}$-adapted cylindrical Brownian motion in Y, i. e.,

$$W(t) = \sum_{k=1}^{\infty} w_k(t)\ell_k,$$

where $\{w_k(t), k \geq 1\}$ is a sequence of independent standard one-dimensional Brownian motions in $(\Omega, \mathcal{F}, \mathbf{P})$. We write $W(e_k) = \int_0^T e_k(t)dW_t$. For $e_k = e_k(s) = m_{k_1}(s)\ell_{k_2}$,

$$W(e_k) = \int_0^T e_k(t)dW_t = \int_0^T m_{k_1}(t)dw_{k_2}(t),$$

and

$$W(t) = \sum_{k=1}^{\infty}(\int_0^t e_k(s)ds)W(e_k), 0 \leq t \leq T. \tag{2.1}$$

Let $\alpha = \{\alpha_k,\ k \geq 1\}$ be a multiindex, i.e. for every k, $\alpha_k \in N = \{0,1,2,\ldots\}$. We shall consider only such α that $|\alpha| = \sum_k \alpha_k < \infty$, i.e., only a finite number of α_k is non-zero, and we denote by J the set of all such multiindices. For $\alpha, \beta \in J$, we define

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots), \quad \alpha! = \prod_{k\geq 1}\alpha_k!.$$

By ε_k we denote the multi-index α with $\alpha_k = 1$ and $\alpha_j = 0$ for $j \neq k$. Write

$$\alpha(k) = \alpha - \varepsilon_k \tag{2.2}$$

For each multi-index α of length n we relate a set K_α whose elements are positive integers $k_i, i = 1,\ldots,n$, such that each k is represented there by

α_k-copies. To proceed further, we need a description of a multi-index α with $|\alpha| = n > 0$ based on its characteristic set K_α, that is, an ordered n-tuple $K_\alpha = \{k_1, \dots, k_n\}$, where $k_1 \le k_2 \le \dots \le k_n$ characterize the locations and the values of the non-zero elements of α. More precisely, k_1 is the index of the first non-zero element of α, followed by $\max(0, \alpha_{k_1} - 1)$ of entries with the same value. The next entry after that is the index of the second non-zero element of α, followed by $\max(0, \alpha_{k_2} - 1)$ of entries with the same value, and so on.

For an orthonormal basis $\{e_k, k \ge 1\}$ in $L_2([0,T], Y)$ and $\alpha \in I$ with $K_\alpha = \{k_1, \dots, k_n\}$, we denote

$$E_\alpha = \sum_{\sigma \in \mathcal{G}^n} e_{k_{\sigma(1)}} \otimes \dots \otimes e_{k_{\sigma(n)}}, \alpha \in \mathcal{I}, \tag{2.3}$$

where G^n is a permutation group of $\{1, \dots, n\}$. The set

$$\left\{ e_\alpha = \frac{E_\alpha}{\sqrt{\alpha! |\alpha|!}}, \alpha \in \mathcal{I} \right\} \tag{2.4}$$

is a CONS for the symmetric part of $\mathbf{H}^{\otimes n}$.

For $\alpha \in J$, write $\mathcal{H}_\alpha := \prod_{k=1}^\infty H_{\alpha_k}(W(e_k))$, where H_n is the n^{th} Hermite polynomial defined by $H_n(x) = (-1)^N \left(d^n e^{-x^2/2}/dx^n \right) e^{x^2/2}$.

Write $\xi_\alpha = \mathcal{H}_\alpha / \sqrt{a!}$. For $|\alpha| = n$,

$$\xi_\alpha = \sqrt{|\alpha|!} W(e_\alpha), \tag{2.5}$$

where

$$W(e_\alpha) = \int_0^T \int_0^{s_n} \dots \int_0^{s_2} e_\alpha(s_1, \dots, s_n) dW_{s_1} \dots dW_{s_n}. \tag{2.6}$$

If $\alpha = \varepsilon_k$, then

$$W(e_{\varepsilon_k}) = W(e_k) = \int_0^T e_k(t) dW_t \tag{2.7}$$

Theorem 1 (Cameron and Martin [1])**.** *The set $\Xi = \{\xi_\alpha = \xi_\alpha(\mathbf{b}), \alpha \in \mathcal{I}\}$ is an orthonormal basis in $L_2(\Omega, \mathbf{P})$. If $\mathbf{X}$ is a Hilbert space and $\eta \in L_2(\Omega, \mathbf{P}; \mathbf{X})$ and $\eta_\alpha = E(\eta \xi_\alpha)$, then $\eta = \sum_{\alpha \in \mathcal{J}} \eta_\alpha \xi_\alpha$ and $E|\eta|^2 = \sum_{\alpha \in \mathcal{J}} \eta_\alpha^2$.*

The expansion $\eta = \sum_{\alpha \in \mathcal{J}} \eta_\alpha \xi_\alpha$ is often referred to as Wiener chaos expansion.

Remark 1. *The basis $\xi_\alpha, \alpha \in \mathcal{I}$, can be obtained by differentiating stochastic exponent. Let $\mathcal{Z}$ be the set of all real-valued sequences $z = (z_k)$ such that only finite number of z_k is not zero. For $\alpha \in J$, denote $\partial_z^\alpha = \Pi_k \partial^{\alpha_k} / (\partial z_k)^{\alpha_k}$ and let*

$$e_z = e_z(t) = \sum_k z_k e_k(t), 0 \le t \le T,$$

$$p_t(z) = p_t(e_z) = p_t(z, \mathbf{b}) = \exp\left\{\int_0^t e_z(s)dW_s - \frac{1}{2}\int_0^t |e_z(s)|_Y^2 ds\right\}, \tag{2.8}$$
$$p(z) = p_T(z), z \in \mathcal{Z}, 0 \le t \le T.$$

It is a standard fact (see, for example, [16]*) that* $\mathcal{H}_\alpha = \partial_z^\alpha p(z)|_{z=0}, \xi_\alpha = \mathcal{H}_\alpha/\sqrt{\alpha!}$*. Since* $p(z)$ *is analytic, it follows by (2.5),*

$$p(z) = p(z, \mathbf{b}) = \sum_\alpha \frac{\mathcal{H}_\alpha}{\alpha!} z^\alpha = \sum_\alpha z^\alpha \sqrt{\frac{|\alpha|!}{\alpha!}} W(e_\alpha). \tag{2.9}$$

2.2. **Generalized random variables and processes.** Let $\mathbf{b} \in \mathcal{B}$, and $\xi_\alpha = \xi_\alpha(\mathbf{b}), \alpha \in \mathcal{I}$. Let

$$\mathcal{D} = \mathcal{D}(\mathbf{b}) = \left\{ v = \sum_\alpha v_\alpha \xi_\alpha : \ v_\alpha \in \mathbf{R} \text{ and only finite number of } v_\alpha \text{ are not zero} \right\}.$$

Definition 1. *A generalized* $\mathcal{D}$*-random variable with values in a convex topological vector space* E *with Borel* σ*-algebra is a formal series* $u = \sum_\alpha u_\alpha \xi_\alpha$*, where* $u_\alpha \in E, \xi_\alpha = \xi_\alpha(b)$*, and* $\mathbf{b} = \{e_k, k \ge 1\} \in \mathcal{B}$ *is a CONS in* $L_2([0,T], Y)$*.*

Denote the vector space of all generalized $\mathcal{D}$-random variables by $\mathcal{D}' = \mathcal{D}'(\mathbf{b}) = \mathcal{D}'(\mathbf{b}, E)$. The elements of $\mathcal{D}$ are the test random variables for $\mathcal{D}'$. We define the action of a generalized random variable u on the test random variable v by

$$\langle u, v \rangle = \sum_\alpha v_\alpha u_\alpha.$$

For a sequence $u^n \in \mathcal{D}'$ and $u \in \mathcal{D}'$, we say that $u^n \to u$, if for every $v \in \mathcal{D}$,

$$\langle u, v^n \rangle \to \langle u, v \rangle .$$

This implies that $u^n = \sum_\alpha u_\alpha^n \xi_\alpha \to u = \sum_\alpha u_\alpha \xi_\alpha$ if and only if $u_\alpha^n \to u_\alpha$ as $n \to \infty$ for all α.

Remark 2. *Obviously, if* $u = \sum_\alpha u_\alpha \xi_\alpha \in \mathcal{D}'(\mathbf{b}, E)$*,* F *is a vector space and* $f : E \to F$ *is a linear map, then*

$$f(u) = \sum_\alpha f(u_\alpha)\xi_\alpha \in \mathcal{D}'(\mathbf{b}, F).$$

Definition 2. *a) An* E*-valued generalized* $\mathcal{D}$*- process* $u(t)$ *in* $[0,T]$ *is a* $D'(b,E)$*-valued function on* $[0,T]$ *such that for each* $t \in [0,T]$

$$u(t) = \sum_\alpha u_\alpha(t)\xi_\alpha \in \mathcal{D}'(\mathbf{b}, E);$$

and $u_\alpha(t)$ *are deterministic measurable* E*-valued functions on* $[0,T]$*. We denote the linear space of all such processes by* $\mathcal{D}'([0,T], \mathbf{b}, E)$*.*

Remark 3. *If there is no room for confusion, we will often say* $\mathcal{D}$*-process (*$\mathcal{D}$*-random variable) instead of generalized* $\mathcal{D}$*-process (generalized* $\mathcal{D}$*-random variable).*

If E is a normed vector space, we denote

$$L_1(\mathcal{D}'([0,T],\mathbf{b},E))$$
$$= \{u(t) = \sum_\alpha u_\alpha(t)\xi_\alpha \in \mathcal{D}'([0,T],\mathbf{b},E) : \int_0^T |u_\alpha(t)|_E dt < \infty,\ \alpha \in \mathcal{I}\}.$$

For $u(t) = \sum_\alpha u_\alpha(t)\xi_\alpha \in L_1(\mathcal{D}'([0,T],\mathbf{b},E))$ we define $\int_0^t u(s)ds, 0 \le t \le T$, in $\mathcal{D}'([0,T],b,E)$ by

$$\int_0^t u(s)ds = \sum_\alpha \left(\int_0^t u_\alpha(s)ds\right)\xi_\alpha, 0 \le t \le T.$$

If $u(t) = \sum_\alpha u_\alpha(t)\xi_\alpha$ and $u_\alpha(t)$ are differentiable in t, then

$$\frac{d}{dt}u(t) = \dot{u}(t) = \sum_\alpha \dot{u}_\alpha(t)\xi_\alpha.$$

Example 1. *A cylindrical Wiener process $W_t, 0 \le t \le T$, in a Hilbert space Y, and its derivative $dW_t/dt = \dot{W}_t$ are generalized Y-valued stochastic processes. Indeed, by (2.1),*

$$W_t = \sum_k \int_0^t e_k(s)ds\xi_{\varepsilon_k}, 0 \le t \le T,$$

and

$$W_t = \int_0^t \dot{W}_s ds,$$

where

$$\dot{W}_t = \sum_k e_k(t)\xi_{\varepsilon_k}, 0 \le t \le T.$$

2.2.1. *Wick Product and Skorokhod Integral.*

Definition 3. *For ξ_α, ξ_β from Ξ, define the Wick product*

$$\xi_\alpha \Diamond \xi_\beta := \sqrt{\left(\frac{(\alpha+\beta)!}{\alpha!\beta!}\right)}\xi_{\alpha+\beta}. \tag{2.10}$$

In particular, taking in (2.10) $\alpha = k\varepsilon_i$ and $\beta = n\varepsilon_i$ we get

$$H_k(\xi_i)\Diamond H_n(\xi_i) = H_{k+n}(\xi_i). \tag{2.11}$$

For a Hilbert space E and arbitrary $v = \sum_\alpha v_\alpha\xi_\alpha$ and $u = \sum_\alpha u_\alpha\xi_\alpha$ in $\mathcal{D}'(b,E)$, we define their Wick product as a $\mathcal{D}$-generalized real valued random variable given by

$$v\Diamond u = \sum_\alpha \sum_{\beta\le\alpha} (u_\beta, v_{\alpha-\beta})_E \sqrt{\frac{\alpha!}{\beta!(\alpha-\beta)!}}\xi_\alpha \in \mathcal{D}'(\mathbf{b},\mathbf{R}). \tag{2.12}$$

Skorokhod integral assigns to $v \in L_1(\mathcal{D}'([0,T],\mathbf{b},Y))$ a generalized random variable $\delta(v) \in \mathcal{D}'(\mathbf{b},\mathbf{R})$ such that

$$\delta(v) = \int_0^T v(s)dW_s = \sum_\alpha \delta(v)_\alpha \xi_\alpha,$$

where

$$\delta(v)_\alpha = \sum_k \sqrt{\alpha_k} \int_0^T \left(v_{\alpha(k)}(t), e_k(t)\right)_Y dt.$$

and $\alpha(k)$ is given by (2.2). Let $\delta_t(v)$ be a process in $\mathcal{D}'([0,T],\mathbf{b},Y)$ such that

$$\delta_t(v) = \int_0^t v(s)dW_s = \delta\left(v1_{[0,t]}\right) = \sum_\alpha \delta_t(v)_\alpha \xi_\alpha, 0 \le t \le T,$$

where

$$\delta_t(v)_\alpha = \sum_k \sqrt{\alpha_k} \int_0^t \left(v_{\alpha(k)}(s), e_k(s)\right)_Y ds.$$

Since $\dot{W}_t = \sum_k e_k(t)\xi_{\varepsilon_k}$, it follows by (2.12) that

$$v_t \diamondsuit \dot{W}_t = \sum_\alpha \sum_k (v_{\alpha(k)}(t), e_k(t))_Y \sqrt{\alpha_k} \xi_\alpha,$$

and

$$\delta(v) = \int_0^T v_t \diamondsuit \dot{W}_t \, dt \text{ and } \delta_t(v) = \int_0^t v(s) \diamondsuit \dot{W}_s ds.$$

Remark 4. *Skorokhod integral is an extension of the Itô integral*[1].

Proof. Indeed, if $u(t) = \sum_\alpha u_\alpha(t)\xi_\alpha$ is $\mathbb{F}$-adapted Y-valued such that

$$\mathbf{E} \int_0^T |u(t)|_H^2 dt < \infty,$$

then $v = \int_0^T u(t)dW(t) = \sum_\alpha v_\alpha \xi_\alpha$ is square integrable. By Ito formula for the product of $\int_0^t u(s)dW(s)$ and stochastic exponent $p_t(z)$ from Remark 1, we obtain

$$\begin{aligned}\mathbf{E}vp(z) = \mathbf{E} \int_0^T u(t)dW(t)p_T(z) &= \int_0^T \mathbf{E}[p_t(z)(u(t), e_z(t))_Y]dt \\ &= \int_0^T \mathbf{E}[p(z)(u(t), e_z(t))_Y]dt, z \in \mathcal{Z}.\end{aligned}$$

[1]Of course, this statement is well known. However, the proof given here is short and straightforward.

So,

$$\frac{\partial^{|\alpha|}\mathbf{E}vp(z)}{\partial z^{\alpha}}=\sum_k \alpha_k\int_0^T \frac{\partial^{|\alpha(k)|}}{\partial z^{\alpha(k)}}(\mathbf{E}p(z)u(t),e_k(t))_Y]dt \text{ and}$$

$$v_\alpha=(\sqrt{\alpha!})^{-1}\frac{\partial^{|\alpha|}\mathbf{E}vp(z)}{\partial z^{\alpha}}|_{z=0}=\sum_k\int_0^T\sqrt{\alpha_k}(u_{\alpha(k)}(t),e_k(t))_Y]dt.$$

□

3. Quantized Navier-Stokes Equation

For $T>r\geq 0$, let us consider the following Navier-Stokes equation:

$$(3.1)\quad \begin{aligned}&\partial_t\mathbf{u}(t,x)=\partial_i\left(a^{ij}(t,x)\,\partial_j\mathbf{u}(t,x)\right)+b^i(t,x)\partial_i\mathbf{u}(t,x)-\\ &u^k(t,x)\lozenge\partial_k\mathbf{u}(t,x)+\nabla P(t,x)+\mathbf{f}(t,x)+\\ &[\sigma^i(t,x)\partial_i\mathbf{u}(t,x)+\mathbf{g}(t,x)-\nabla\tilde{P}(t,x)]\lozenge\dot{W}_t,\ div\,\mathbf{u}=0,\\ &\mathbf{u}(r,x)=\boldsymbol{w}(x).\end{aligned}$$

The unknowns in the equation (3.1) are the functions $\mathbf{u}=\left(u^l\right)_{1\leq l\leq d}$, P, $\tilde{P}$. It is assumed that $a^{ij},b^i,\mathbf{f}=\left(f^i\right),\mathbf{w}=\left(w^i\right)$ are measurable deterministic functions on $[0,\infty)\times\mathbf{R}^d$, and the matrix $\left(a^{ij}\right)$ is symmetric. Let us assume also that $\sigma^i,\mathbf{g}=\left(g^i\right)$ be Y-valued measurable deterministic functions on $[0,\infty)\times\mathbf{R}^d$.

In addition, we will need the following assumptions.

A1. For all $t\geq 0,x\in\mathbf{R}^d$,

$$K|\lambda|^2\geq a^{ij}\lambda_i\lambda_j\geq\delta|\lambda|^2,$$

where K,δ are fixed strictly positive constants.

A2. For all $t\geq 0,x,\lambda\in\mathbf{R}^d$,

$$\sum_{k=0}^{2}(|\partial^k a^{ij}|+|\partial^k b^i|+|\partial^k\sigma^i|_Y)|\leq K.$$

A3. The functions $\mathbf{f}(t,x),\mathbf{p}(x)$ and $\mathbf{g}(t,x)$ are measurable deterministic, $p>d$, and for all $t>0$,

$$\int_0^t[|\mathbf{f}(r)|_{1,p}^{2p}+|\mathbf{f}(r)|_{1,2}^2+|\mathbf{g}(r)|_{1,p}^{2p}+|\mathbf{g}(r)|_{1,2}^2]dr<\infty,$$

$$|\boldsymbol{w}|_{1,2p}+|\boldsymbol{w}|_{1,2}<\infty.$$

We will seek a solution to (3.1) in the form

$$\mathbf{u}(t)=\sum_\alpha\mathbf{u}_\alpha(t)\xi_\alpha\in\mathcal{D}'([0,T],\mathbf{b},\mathbb{H}_p^2),p\geq 2.$$

In this case, denoting by $\mathcal{P}(\mathbf{v})$ the solenoidal projection of the vector field $\mathbf{v}$, we can rewrite (3.1) in the following equivalent form:

$$\begin{aligned}
&\partial_t \mathbf{u}(t,x) = \mathcal{P}[\partial_i \left(a^{ij}(t,x)\,\partial_j \mathbf{u}(t,x)\right) + b^i(t,x)\partial_i \mathbf{u}(t,x) - \\
&u^k(t,x)\,\partial_k \mathbf{u}(t,x) + \mathbf{f}(t,x)] + \\
&\mathcal{P}[\sigma^i(t,x)\partial_i \mathbf{u}(t,x) + \mathbf{g}(t,x)]\lozenge \dot{W}_t,\ div\, \mathbf{u} = 0, \\
&\mathbf{u}(r,x) = \boldsymbol{w}(x).
\end{aligned} \tag{3.2}$$

Definition 4. *Given* $\mathbf{w} = \left(w^i\right) = \sum_\alpha w_\alpha(x)\xi_\alpha \in \mathcal{D}'(\mathbf{b}, \mathbb{H}_p^2)$ *and* $r < T$, *a generalized* $\mathcal{D}$*-process* $\mathbf{u}(t) = \sum_\alpha u_\alpha(t)\xi_\alpha \in \mathcal{D}'([r,T], \mathbf{b}, \mathbb{H}_p^2)$ *is called* $\mathcal{D}$-$\mathbb{H}_p^2$ *solution of equation (3.1) in* $[r,T]$, *if for each* α, $u_\alpha(t)$ *is strongly continuous in* t,

$$\int_r^T |\mathbf{u}_\alpha(s)|_{3,p}^p\, ds < \infty,$$

and the equality

$$\begin{aligned}
&\mathbf{u}(t) = \mathbf{w} + \textstyle\int_r^t \mathcal{P}[-u^i(r)\lozenge\partial_i \mathbf{u}(r) + \\
&\partial_i(a^{ij}(r)\partial_j \mathbf{u}(r)) + b^i(t,x)\partial_i \mathbf{u}(t,x) + \mathbf{f}(r)]dr + \\
&\textstyle\int_r^t \mathcal{P}[\sigma^k(r)\partial_k \mathbf{u}(r) + \mathbf{g}(r)]\lozenge \dot{W}_r dr.
\end{aligned} \tag{3.3}$$

holds in $\mathcal{D}(\mathbf{b}, \mathbb{H}_p^2(R^d))$ *for every* $r \le t \le T$. *If an* $\mathbb{H}_p^s$*-solution in* $[r,T]$ *is also* $\mathbb{H}_q^s$*-solution in* $[r,T]$, *we call it* $\mathbb{H}_p^s \cap \mathbb{H}_q^s$*-solution in* $[r,T]$.

Applying Remark 2 and definition of the Wick product we obtain the following statement.

Lemma 1. *Assume A1-A3 hold,* $p \ge 2$. *Then* $\mathbf{u}(t) = \sum_\alpha u_\alpha(t)\xi_\alpha \in \mathcal{D}'([r,T], b, \mathbb{H}_p^2)$ *is an* $\mathcal{D}$-$\mathbb{H}_p^2$ *solution in* $[r,T]$ *if and only if for each* $\alpha \in \mathcal{I}, r \le t \le T$,

$$\begin{aligned}
&\mathbf{u}_\alpha(t,x) = \mathbf{w}_\alpha(x) + \textstyle\int_r^t [\partial_i \left(a^{ij}(s,x)\,\partial_j \mathbf{u}_\alpha(s,x)\right) + b^i(s,x)\partial_i \mathbf{u}_\alpha(s,x) - \\
&- \textstyle\sum_{\gamma\le\alpha} \sqrt{\binom{\alpha}{\gamma}} u^k_{\alpha-\gamma}(s,x)\,\partial_k \mathbf{u}_\gamma(s,x) + \nabla P_\alpha(s,x) + \mathbf{f}(s,x)\,1_{\alpha=0}]ds + \\
&+ \textstyle\int_r^t \sum_k \sqrt{\alpha_k}[(\sigma^p(s,x), e_k(s))_Y\,\partial_p \mathbf{u}_{\alpha(k)}(s,x) + \mathbf{1}_{|\alpha|=1}\mathbf{g}(s,x) \\
&-\nabla\left(\tilde{P}_{\alpha(k)}(s,x), e_k(s)\right)_Y]ds,\ \ \text{div}\,\mathbf{u} = 0,
\end{aligned} \tag{3.4}$$

or, equivalently,

$$\mathbf{u}_\alpha(t,x)=\mathbf{w}_\alpha(x)+\int_r^t\mathcal{P}[\partial_i\left(a^{ij}(s,x)\,\partial_j\mathbf{u}_\alpha(s,x)\right)+b^i(s,x)\partial_i\mathbf{u}_\alpha(s,x)-$$

$$+\sum_{\gamma\le\alpha}\sqrt{\binom{\alpha}{\gamma}}u^k_{\alpha-\gamma}(s,x)\,\partial_k\mathbf{u}_\gamma(s,x)+\mathbf{f}(s,x)\,1_{\alpha=0}]ds+$$

$$+\int_r^t\sum_k\sqrt{\alpha_k}\mathcal{P}[(\sigma^p(s,x),e_k(s))_Y\,\partial_p\mathbf{u}_{\alpha(k)}(s,x)+(\mathbf{g}(s,x),e_k(s))_Y\,1_{|\alpha|=1}]ds,$$
$$\text{div }\mathbf{u}_\alpha=0.$$

Remark 5. *a) If $\alpha=0$, the zero term $u_\alpha(t,x)=u_0(t,x)$ of an $\mathcal{D}$-$\mathbb{H}^2_p$ solution in $[r,T]$ satisfies Navier-Stokes equation:*

$$\begin{aligned}&\mathbf{u}_0(t,x)=\mathbf{w}_0(x)+\int_r^t[\partial_i\left(a^{ij}(s,x)\,\partial_j\mathbf{u}_0(s,x)\right)-\\ &-u^k_0(s,x)\,\partial_k\mathbf{u}_0(s,x)+\nabla P_0(s,x)+\mathbf{f}(s,x)]ds,\ \ \text{div }\mathbf{u}_0=0,\end{aligned}\tag{3.5}$$

For the remaining components we have to solve Stokes equations. For example, denote ε_l the multiindex α such that $\alpha_l=1$ and the remaining components are zeros. Then for $\mathbf{u}_{\varepsilon_l}$,

$$\mathbf{u}_{\varepsilon_l}(t,x)=\mathbf{w}_{\varepsilon_i}+\int_r^t[\partial_i\left(a^{ij}(s,x)\,\partial_j\mathbf{u}_{\varepsilon_l}(s,x)\right)-$$

$$-u^k_0(s,x)\,\partial_k\mathbf{u}_{\varepsilon_l}(s,x)-u^k_{\varepsilon_l}(s,x)\,\partial_k\mathbf{u}_0(s,x)+\nabla P_{\varepsilon_l}(s,x)]ds+$$

$$+\int_r^t[(\sigma^p(s,x),e_l(s))_Y\,\partial_p\mathbf{u}_0(t,x)+(\mathbf{g}(s,x),e_l(s))_Y$$
$$-\nabla\left(\tilde{P}_0(s,x),e_l(s)\right)_Y]ds,$$
$$\text{div }\mathbf{u}_{\varepsilon_l}=0,$$

and for $|\alpha|\ge 2$,

$$\mathbf{u}_\alpha(t,x)=\mathbf{w}_\alpha(x)+\int_r^t[\partial_i\left(a^{ij}(s,x)\,\partial_j\mathbf{u}_\alpha(s,x)\right)+b^i(s,x)\partial_i\mathbf{u}_\alpha(s,x)-$$

$$-u^k_0(s,x)\,\partial_k\mathbf{u}_\alpha(s,x)+u^k_\alpha(s,x)\,\partial_k\mathbf{u}_0(s,x)$$

$$-\sum_{\gamma\le\alpha,1\le|\gamma|\le|\alpha|-1}\sqrt{\binom{\alpha}{\gamma}}u^k_{\alpha-\gamma}(s,x)\,\partial_k\mathbf{u}_\gamma(s,x)+\nabla P_\alpha(s,x)]ds+$$

$$+\int_r^t\sum_k\sqrt{\alpha_k}[(\sigma^p(s,x),e_k(s))_Y\,\partial_p\mathbf{u}_{\alpha(k)}(s,x)$$

$$-\nabla\left(\tilde{P}_{\alpha(k)}(s,x),e_k(s)\right)_Y]ds,\ \ \text{div }\mathbf{u}_\alpha=0,$$

b) Since for $|\alpha|\ge 1$, $\mathbf{E}\mathbf{u}_\alpha(t,x)\xi_\alpha=0$, the equation (3.2) (or (3.3)) can be regarded as a random perturbation of the deterministic Navier-Stokes equation (3.5).

c) If A1-A3 hold, then there is $T_1>0$ and a unique $\mathbb{H}^2_2\cap\mathbb{H}^2_{2p}$-solution of (3.5) in $[0,T_1)$.

First, we prove the existence and uniqueness of $\mathcal{D}$-solutions.

Lemma 2. *Assume that A1-A3 hold,* $\mathbf{b} \in \mathcal{B}, \mathbf{w} = \sum_\alpha \mathbf{w}_\alpha \xi_\alpha$,

$$|\mathbf{w}_\alpha|_{1,p} + |\mathbf{w}_\alpha|_{1,2} < \infty \text{ for all } |\alpha| \geq 1,$$
$$|\mathbf{w}_0|_{1,2p} + |\mathbf{w}_0|_{1,2} < \infty.$$

Then for each $r \leq T < T_1$ *there is a unique* $\mathcal{D}$-$\mathbb{H}_p^2 \cap \mathbb{H}_2^2$*-solution of (3.1) in* $[r, T]$.

Proof. For $\alpha = 0$, the equation (3.5) is Navier-Stokes equation. According to Theorem 3 in [18], there is $T_1 > 0$ and a unique $\mathbb{H}_{2p}^1 \cap \mathbb{H}_2^1$-solution $\mathbf{u}_0(t), t < T_1$, to (3.5) such that

$$\sup_{0 \leq s \leq t} |\mathbf{u}_0(s)|_{1,l}^l + \int_r^t |\partial^2 \mathbf{u}_0(r)|_{0,l}^l dr < \infty,$$

where $l = 2, 2p$. By Sobolev embedding theorem, for all $r \leq t < T$,

$$\sup_{x, r \leq t} |\mathbf{u}_0(r, x)| + \int_r^t \sup_x |\partial \mathbf{u}_0(r, x)|^{2p} dr < \infty.$$

Therefore,

$$\int_r^t [|u_0^k\,(s,x)\,\partial_k \mathbf{u}_0(s,x)|^l + |\partial_j u_0^k\,(s,x)\,\partial_k \mathbf{u}_0(s,x)|^l +$$
$$|u_0^k\,(s,x)\,\partial_{kj}^2 \mathbf{u}_0(s,x)|^l] dsdx < \infty,$$

$l = 2, p$. By Proposition 4.7 in [17],

$$\sup_{r \leq t} |\mathbf{u}_0(r)|_{2,l}^l + \int_r^t |\partial^3 \mathbf{u}_0(r)|_l^l dr < \infty, \tag{3.6}$$

$l = 2, p$, and by Sobolev embedding theorem, for all $t < T_1$,

$$\sup_{x, r \leq t} \sum_{k=0}^{1} |\partial^k \mathbf{u}_0(r, x)| + \int_r^t \sup_x |\partial^2 \mathbf{u}_0(r, x)|^p dr < \infty.$$

Assume that for $|\alpha| \leq n$, the estimate

$$\sup_{r \leq t} |\mathbf{u}_\alpha(r)|_{2,l}^l + \int_r^t |\partial^3 \mathbf{u}_\alpha(r)|_l^l dr < \infty, \tag{3.7}$$

$l = 2, p$, holds. By Sobolev embedding theorem, it implies that

$$\sup_{x, r \leq t} \sum_{k=0}^{1} |\partial^k \mathbf{u}_\alpha(r, x)| + \int_r^t \sup_x |\partial^2 \mathbf{u}_\alpha(r, x)|^p dr < \infty,$$

if $|\alpha| \leq n, t < T_1$. Then for $|\alpha| = n+1$,

$$\mathbf{u}_\alpha(t,x) = \mathbf{w}_\alpha(x) + \int_r^t [\partial_i\left(a^{ij}(s,x)\,\partial_j \mathbf{u}_\alpha(s,x)\right) -$$

$$-u_0^k(s,x)\,\partial_k \mathbf{u}_\alpha(s,x) + u_\alpha^k(s,x)\,\partial_k \mathbf{u}_0(s,x)$$

$$-\sum_{\gamma\leq\alpha, 1\leq|\gamma|\leq|\alpha|-1}\sqrt{\binom{\alpha}{\gamma}}u^i_{\alpha-\gamma}(s,x)\,\partial_i \mathbf{u}_\gamma(s,x) + \nabla P_\alpha(s,x)]ds+$$

$$+\int_r^t \sum_k \sqrt{\alpha_k}[(\sigma^p(s,x), e_k(s))_Y\,\partial_p \mathbf{u}_{\alpha(k)}(t,x)$$

$$-\nabla\left(\tilde{P}_{\alpha(k)}(t,x), e_k(s)\right)_Y]ds, \text{ div } \mathbf{u}_\alpha = 0,$$

and by Proposition 4.7 in [17], (3.7) holds for $|\alpha| = n+1$. □

Because of the uniqueness, the $\mathcal{D}$-solution has a restarting property. More specifically, the following statement holds:

Corollary 1. *Assume that A1-A3 hold,* $b \in B, w = \sum_\alpha w_\alpha \xi_\alpha$,

$$|\mathbf{w}_\alpha|_{1,p} + |\mathbf{w}_\alpha|_{1,2} < \infty \textit{ for all } |\alpha| \geq 1,$$
$$|\mathbf{w}_0|_{1,2p} + |\mathbf{w}_0|_{1,2} < \infty.$$

Let $u^{r,\mathbf{w}}(t)$ *be the solution to (3.1)in* $[r,T], T < T_1$, *and* $r \leq r' \leq t \leq T$. *Then*

$$\mathbf{u}^{r,\mathbf{w}}(t) = \mathbf{u}^{r',\mathbf{u}(r')}(t).$$

Proof. Indeed for $u(t) = u^{r,\mathbf{w}}(t)$, and $r \leq r' \leq t \leq T$, we have

$$\mathbf{u}(t) = \mathbf{u}(r') + \int_{r'}^t \{\partial_i\left(a^{ij}(s)\,\partial_j \mathbf{u}(s)\right) + b^i(s)\partial_i \mathbf{u}(s) -$$

$$-u^k(s)\,\Diamond\partial_k \mathbf{u}(s) + \nabla P(s) + \mathbf{f}(s)$$

$$+[\sigma^i(s)\partial_i \mathbf{u}(s) + \mathbf{g}(s) - \nabla\tilde{P}(s)]\Diamond\dot{W}_s\}ds, \text{ div } \mathbf{u} = 0,$$

and the statement follows by Lemma 2. □

3.1. **Quantization and approximation of the generalized solution.**

To begin with, we will derive more precise estimates for $\mathcal{D}$-$\mathbb{H}^2_p$ solutions of equation (3.1). One could hardly expect that the $\mathcal{D}$-$\mathbb{H}^2_p$ solution of quantized Navier-Stokes equation has finite variance, i.e. $\sum_\alpha |\mathbf{u}_\alpha(t)|^2 < \infty$. However, in this subsection we will show that the solution could be obtained as the limit of square integrable solutions of the equations rescaled in a special way that originates in quantum physics and is usually referred to as second quantization (see [21], and Appendix I).

Lemma 3. *Assume A1-A3 hold and* $\sup_k \int_0^T |e_k(s)|^p ds < \infty$. *Let* $u(t) = \sum_\alpha u_\alpha(t)\xi_\alpha \in \mathcal{D}'([0,T], b, \mathbb{H}_p^2 \cap \mathbb{H}_2^2)$ *be* $\mathbb{H}_p^2 \cap \mathbb{H}_2^2$*-solution of equation (3.1) in* $[0,T]$. *Denote*

$$\tilde{L}_\alpha = \sup_{t \le T} |\mathbf{u}_\alpha(t)|_{2,p} + \sup_{t \le T} |\mathbf{u}_\alpha(t)|_{2,2}, \alpha \in \mathcal{I}.$$

Then there is a constant B_0 *such that*

$$\tilde{L}_\alpha \le \sqrt{\alpha!} C_{|\alpha|-1} \binom{|\alpha|}{\alpha} B_0^{|\alpha|-1} K^{|\alpha|}, |\alpha| \ge 2,$$

where $K = 1 + \sup_i \tilde{L}_{\varepsilon_i}$, *and*

$$C_{|\alpha|-1} = \frac{1}{|\alpha|-1} \binom{2(|\alpha|-1)}{|\alpha|-1}, |\alpha| \ge 2$$

are the Catalan numbers (see e.g. [22], [8] *). Moreover, there is a number* $q > 2$ *so that*

$$\sum_\alpha \frac{(2\mathbf{N})^{-q\alpha} \tilde{L}_\alpha^2}{\alpha!} < \infty,$$

i.e., the solution $\mathbf{u}$ *belongs to the Kondratiev space of generalized random functions* $S_{-1,-q}(\mathbb{H}_p^2 \cap \mathbb{H}_2^2)$.

Proof. By Proposition 4.7 in [17], $\tilde{L}_0 + \sup_i \tilde{L}_{\varepsilon_i} < \infty$, and there is a constant C independent of $|\alpha| \ge 2$ such that

$$\begin{aligned}
&\sup_{r \le T} |\mathbf{u}_\alpha(r)|_{2,l}^l + \int_0^t |\partial^2 \mathbf{u}_\alpha(r)|_{1,l}^l dr \\
&\le C \int_0^t | \sum_{\gamma \le \alpha, 1 \le |\gamma| \le |\alpha|-1} \sqrt{\binom{\alpha}{\gamma}} u_{\alpha-\gamma}^k (s,x) \, \partial_k \mathbf{u}_\gamma(s,x) + \\
&+ \sum_k \sqrt{\alpha_k} \, (\sigma^p(s,x), e_k(s))_Y \, \partial_p \mathbf{u}_{\alpha(k)} (s,x) \, |_{1,l}^l ds,
\end{aligned}$$

$l = 2, p$. So, there is a constant B_0 so that for $|\alpha| = n \ge 2, \hat{L}_\alpha = (\alpha!)^{-1/2} \tilde{L}_\alpha, \hat{L}_{\varepsilon_i} = \tilde{L}_{\varepsilon_i}$ we have

$$\hat{L}_\alpha \le B_0 \left(\sum_{\gamma \le \alpha, 1 \le |\gamma| \le |\alpha|-1} \hat{L}_{\alpha-\gamma} \hat{L}_\gamma + 1_{\sigma \ne 0} \sum_k \hat{L}_{\alpha(k)} 1_{\alpha_k \ne 0} \right)$$

and denoting $L_\alpha = \hat{L}_\alpha$ if $|\alpha| > 1, L_\alpha = 1 + \hat{L}_\alpha$ if $|\alpha| = 1$ we have

$$L_\alpha \le B_0 \sum_{\gamma \le \alpha, 1 \le |\gamma| \le |\alpha|-1} L_{\alpha-\gamma} L_\gamma$$

and by [8][2] for $|\alpha| \geq 2$

$$\begin{aligned} L_\alpha &\leq C_{|\alpha|-1} B_0^{|\alpha|-1} \binom{|\alpha|}{\alpha} \prod_i (1+\tilde{L}_{\varepsilon_i})^{\alpha_i} \\ &\leq C_{|\alpha|-1} \binom{|\alpha|}{\alpha} B_0^{|\alpha|-1} K^{|\alpha|}. \end{aligned}$$

So,

$$\begin{aligned} \tilde{L}_\alpha &\leq \sqrt{\alpha!} C_{|\alpha|-1} \binom{|\alpha|}{\alpha} B_0^{|\alpha|-1} K^{|\alpha|}, \\ \tilde{L}_\alpha^2 &\leq \alpha! C_{|\alpha|-1}^2 \binom{|\alpha|}{\alpha} (2\mathbf{N})^\alpha B_0^{2(|\alpha|-1)} K^{2|\alpha|} \end{aligned}$$

and

$$\frac{r^\alpha \tilde{L}_\alpha^2}{\alpha!} \leq C_{|\alpha|-1}^2 \binom{|\alpha|}{\alpha} (2\mathbf{N}r)^\alpha B_0^{2(|\alpha|-1)} K^{2|\alpha|}.$$

Therefore with $r = (r_i), r_i = (2i)^{-q}, q > 2$,

$$\begin{aligned} \sum_{|\alpha|=n} \frac{r^\alpha \tilde{L}_\alpha^2}{\alpha!} &= C_{n-1}^2 B_0^{2(n-1)} K^{2n} \sum_{|\alpha|=n} \binom{|\alpha|}{\alpha} (2\mathbf{N}r)^\alpha \\ &\leq C_{n-1}^2 B_0^{2(n-1)} K^{2n} 2^n 2^{-qn} \left(\sum_{i=1}^\infty i^{q-1} \right)^n. \end{aligned}$$

For large n, the Catalan numbers

$$C_{n-1} \approx \frac{4^{n-1}}{\sqrt{\pi}(n-1)^{3/2}}$$

and there is a number $q > 2$ such that

$$\sum_{n=0}^\infty \sum_{|\alpha|=n} \frac{r^\alpha \tilde{L}_\alpha^2}{\alpha!} < \infty.$$

So, the solution $\mathbf{u}(t)$ belongs to Kondratiev's space $S_{-1,-q}(\mathbb{H}_p^2 \cap \mathbb{H}_2^2)$. □

For $\varepsilon > 0$ define a self-adjoint positive operator D_ε on H such that $D_\varepsilon e_k = 2^{-\varepsilon k} e_k$ and a sequence of positive numbers $\kappa_{\varepsilon,n} = e^{-\varepsilon e^n}$. Let $C_\varepsilon = \sum_{n=0}^\infty \kappa_{\varepsilon,n} D_\varepsilon^{\otimes n}$. Then

$$C_\varepsilon e_\alpha = \kappa_{\varepsilon,|\alpha|} D_\varepsilon^{\otimes n} e_\alpha = \kappa_{\varepsilon,|\alpha|} \left(\Pi_k 2^{-\varepsilon k \alpha_k} \right) e_\alpha = \kappa_{\varepsilon,|\alpha|} \left(2^{-\varepsilon \mathbf{N}} \right)^\alpha e_\alpha.$$

Let $\Gamma(C_\varepsilon)$ be the second quantization operator related for the operator C_ε (see Appendix I).

[2]Kaligotla and Lototsky [8] used Catalan numbers for quantization of stochastic Burgers equation.

Proposition 1. *Assume A1-A3 hold and* $\sup_k \int_0^T |e_k(s)|^p ds < \infty$. *Let* $\mathbf{u}(t) = \sum_\alpha \mathbf{u}_\alpha(t)\xi_\alpha \in \mathcal{D}'([0,T], \mathbf{b}, \mathbb{H}_p^2 \cap \mathbb{H}_2^2)$ *be a generalized* $\mathbb{H}_p^2 \cap \mathbb{H}_2^2$*-solution of equation (3.1) in* $[0,T]$ *and*

$$\mathbf{u}_\varepsilon(t) = \Gamma(C_\varepsilon)\mathbf{u}(t) = \sum_{n=0}^{\infty} \kappa_{\varepsilon,n} \sum_{|\alpha|=n} \mathbf{u}_\alpha(t)(2^{-\varepsilon \mathbf{N}})^\alpha \xi_\alpha,$$

Then $\mathbf{u}_\varepsilon(t)$ *is* $\mathbb{H}_2^2$*-valued square integrable process satisfying the equation*

$$\begin{aligned} &\partial_t \mathbf{u}_\varepsilon(t,x) = \partial_i\left(a^{ij}(t,x)\partial_j \mathbf{u}_\varepsilon(t,x)\right) + b^i(t,x)\partial_i \mathbf{u}_\varepsilon(t,x) - \\ &\Gamma(C_\varepsilon)\Gamma(C_\varepsilon)^{-1} u_\varepsilon^k(t,x) \Diamond \Gamma(C_\varepsilon)^{-1}\partial_k \mathbf{u}_\varepsilon(t,x) + \nabla P_\varepsilon(t,x) + \mathbf{f}(t,x) + \\ &\Gamma(C_\varepsilon)[\sigma^i(t,x)\Gamma(C_\varepsilon)^{-1}\partial_i \mathbf{u}_\varepsilon(t,x) + \mathbf{g}(t,x) - \\ &\Gamma(C_\varepsilon)^{-1}\nabla \tilde{P}_\varepsilon(t,x)] \Diamond \Gamma(C_\varepsilon)^{-1}\dot{W}_t^\varepsilon, \\ &div\, \mathbf{u}_\varepsilon = 0, \mathbf{u}_\varepsilon(0,x) = \mathbf{w}(x). \end{aligned} \tag{3.8}$$

For the coefficients of $\mathbf{u}_\varepsilon(t) = \sum_\alpha \mathbf{u}_{\varepsilon,\alpha}(t)\xi_\alpha$ *we have*

(3.9)

$$\begin{aligned} &\mathbf{u}_{\varepsilon,\alpha}(t,x) = 1_{\alpha=0}\mathbf{w}(x) + \int_0^t [\partial_i\left(a^{ij}(s,x)\partial_j \mathbf{u}_{\varepsilon,\alpha}(s,x)\right) + b^i(s,x)\partial_i \mathbf{u}_{\varepsilon,\alpha}(s,x) - \\ &\sum_{\gamma\le\alpha} \sqrt{\tbinom{\alpha}{\gamma}} e^{-\varepsilon e^{|\alpha|} + \varepsilon e^{|\gamma|} + \varepsilon e^{|\alpha-\gamma|}} u_{\varepsilon,\alpha-\gamma}^k(s,x)\partial_k \mathbf{u}_{\varepsilon,\gamma}(s,x) + \\ &\nabla P_{\varepsilon,\alpha}(s,x) + \mathbf{f}(s,x)1_{\alpha=0}]ds + \\ &\int_0^t \sum_k \sqrt{\alpha_k} e^{-\varepsilon e^{|\alpha|} + \varepsilon e^{|\alpha|-1}} 2^{-\varepsilon k \alpha_k} [(\sigma^p(s,x), e_k(s))_Y \partial_p \mathbf{u}_{\varepsilon,\alpha(k)}(s,x) + \\ &\mathbf{1}_{|\alpha|=1}\mathbf{g}(s,x) - \nabla\left(\tilde{P}_{\varepsilon,\alpha(k)}(s,x), e_k(s)\right)_Y]ds, \ \text{div}\ \mathbf{u}_{\varepsilon,\alpha} = 0. \end{aligned}$$

Moreover, $\mathbf{u}_\varepsilon(t) \in \cap_{\rho,q'} S_{\rho,q'}(\mathbb{H}_p^2 \cap \mathbb{H}_2^2), t \in [0,T], \mathbf{u}(t) = \Gamma(C_\varepsilon)^{-1}\mathbf{u}_\varepsilon(t)$ *and* $\mathbf{u}_\varepsilon(t) \to \mathbf{u}(t)$ *as* $\varepsilon \to 0$ *in* $S_{-1,-q}(\mathbb{H}_p^2 \cap \mathbb{H}_2^2)$, *where* q *is a number in Lemma 3.*

Proof. Let

$$\tilde{L}_\alpha = \sup_{t\le T} |\mathbf{u}_\alpha(t)|_{2,p} + \sup_{t\le T} |\mathbf{u}_\alpha(t)|_{2,2},$$
$$\tilde{L}_{\varepsilon,\alpha} = \sup_{t\le T} |\mathbf{u}_{\varepsilon,\alpha}(t)|_{2,p} + \sup_{t\le T} |\mathbf{u}_{\varepsilon,\alpha}(t)|_{2,2}.$$

By Lemma 3, there is a number $q > 2$ such that

$$\sum_\alpha \frac{(2\mathbf{N})^{-q\alpha}\tilde{L}_\alpha^2}{\alpha!} < \infty.$$

For every $\rho \geq 0, q' \geq 0$, there is a constant $C_0 = C_0(\rho, q')$ so that for any α,

$$(\alpha!)^{\rho}(2\mathbf{N})^{q'\alpha}e^{-\varepsilon e^{|\alpha|}}\left(2^{-\varepsilon\mathbf{N}}\right)^{\alpha} \leq (|\alpha|!)^{\rho}e^{-\varepsilon e^{|\alpha|}}2^{q'|\alpha|}\Pi_i(i^{q'}2^{-\varepsilon i})^{\alpha_i}$$
$$\leq (|\alpha|!)^{\rho}e^{-\varepsilon e^{|\alpha|}}2^{q'|\alpha|}(\sum_i i^{q'}2^{-\varepsilon i})^{|\alpha|} \leq C_0 < \infty.$$

So,

$$\sum_\alpha (\alpha!)^{\rho}(2\mathbf{N})^{q'\alpha}L^2_{\varepsilon,\alpha} = \sum_\alpha (\alpha!)^{\rho}(2\mathbf{N})^{q'\alpha}e^{-\varepsilon e^{|\alpha|}}\left(2^{-\varepsilon\mathbf{N}}\right)^{\alpha}L^2_\alpha$$
$$\leq C_0(\rho+1, q'+q)\sum_\alpha \frac{(2\mathbf{N})^{-q\alpha}L^2_\alpha}{\alpha!}$$
$$< \infty.$$

So, $\mathbf{u}_\varepsilon(t) \in \cap_{\rho,q'}S_{\rho,q'}(\mathbb{H}^2_p \cap \mathbb{H}^2_2), t \in [0,T]$. In particular,

$$\mathbf{E}|\mathbf{u}_\varepsilon(t)|^2_{2,2} \leq \sum_\alpha |\mathbf{u}_{\varepsilon,\alpha}(t)|^2_{2,2} < \infty, t \in [0,T].$$

Therefore $\mathbf{u}_\varepsilon(t)$ is $\mathbb{H}^2_2$-valued square integrable process and (3.8), (3.9) follow by Remark 7. Obviously, $\mathbf{u}(t) = \Gamma(C_\varepsilon)^{-1}\mathbf{u}_\varepsilon(t)$. Also,

$$\mathbf{u}_\varepsilon(t,x) - \mathbf{u}(t,x) = \sum_\alpha \mathbf{u}_{\varepsilon,\alpha}(t,x)\xi_\alpha - \sum_\alpha \mathbf{u}_\alpha(t,x)\xi_\alpha$$
$$= \sum_\alpha \kappa_{\varepsilon,|\alpha|}\left(2^{-\varepsilon\mathbf{N}}\right)^{\alpha}\mathbf{u}_\alpha(t,x)\xi_\alpha - \sum_\alpha \mathbf{u}_\alpha(t,x)\xi_\alpha$$
$$= \sum_\alpha [1 - \kappa_{\varepsilon,|\alpha|}\left(2^{-\varepsilon\mathbf{N}}\right)^{\alpha}]\mathbf{u}_\alpha(t,x)\xi_\alpha,$$

and, obviously,

$$|\mathbf{u}_\varepsilon(t) - \mathbf{u}(t)|^2_{\mathcal{S}_{-1,-q}(\mathbb{H}^2_p\cap\mathbb{H}^2_2)} \leq \sum_\alpha |1 - \kappa_{\varepsilon,|\alpha|}\left(2^{-\varepsilon\mathbf{N}}\right)^{\alpha}|^2\frac{(2\mathbf{N})^{-q\alpha}\tilde{L}^2_\alpha}{\alpha!} \to 0$$

as $\varepsilon \to 0$ by Lebesgue's dominated convergence theorem. □

4. Non-anticipation and Markov Property, Independence of Basis

In this Section we will show that a generalized $\mathcal{D}$-$\mathbb{H}^2_p \cap \mathbb{H}^2_2$-solution of equation (3.1) has the following properties: it is adapted with respect to the filtration $(\mathcal{F}^W_t)$ generated by the Wiener process W_t; it is independent of the choice of the basis $\mathbf{b}$, and it is a generalized Markov process.

4.1. **Equivalent characterization of $\mathcal{D}$-generalized solution.** A more convenient equivalent characterization of $\mathcal{D}$-generalized solution to (1.2) is based on an equivalent description of $\mathcal{D}(\mathbf{b})$. It allows to introduce the notion of an adapted solution and extend $\mathcal{D}(\mathbf{b})$ to a space of test functions that is independent of $\mathbf{b} \in \mathcal{B}$.

4.1.1. *Equivalent description of test function space.* Often it is convenient to use the exponents $p(z), z \in \mathcal{Z}$, defined in Remark 1 to describe the test function space $\mathcal{D}(\mathbf{b})$.

According to (2.9),

$$p(z) = p(e_z) = p(z, \mathbf{b}) = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} z^{\alpha} \sqrt{\frac{|\alpha|!}{\alpha!}} W(e_{\alpha}).$$

Recall that for $|\alpha| = n$,

$$W(e_{\alpha}) = \int_0^T \int_0^{s_n} \dots \int_0^{s_2} e_{\alpha}(s_1, \dots, s_n) dW_{s_1} \dots dW_{s_n},$$

and $\xi_{\alpha} = \sqrt{|\alpha|!} W(e_{\alpha})$. Denote

$$p_n(z) = p_n(e_z) = \sum_{|\alpha|=n} \frac{z^{\alpha}}{\sqrt{\alpha!}} \xi_{\alpha}$$

$$= \int_0^T \int_0^{s_{n-1}} \dots \int_0^{s_2} e_z(s_1) \dots e_z(s_n) dW_{s_1} \dots dW_{s_n}, n \geq 2, \tag{4.1}$$

$$p_1(z) = p_1(e_z) = \int_0^T e_z(s_1) dW_{s_1}, p_0(z) = p_0(e_z) = 1.$$

Proposition 2. *For $\mathbf{b} \in \mathcal{B}$, let $\mathcal{V} = \mathcal{V}(\mathbf{b})$ be the linear space of random variables that consists of all finite linear combinations of $p_n(z), z \in \mathcal{Z}, n \geq 0$. Then $\mathcal{V}(\mathbf{b}) = \mathcal{D}(\mathbf{b})$.*

Proof. Obviously, $\mathcal{V} = \mathcal{V}(\mathbf{b}) \subseteq \mathcal{D} = \mathcal{D}(\mathbf{b})$. For $\alpha \in \mathcal{I}$, denote $\kappa(\alpha) = \max\{k : \alpha_k \neq 0\}$. Fix N, n and $\alpha = (\alpha_k) \in \mathcal{I}$ such that $|\alpha| = N, \kappa(\alpha) = n$. Consider a finite dimensional Hilbert space

$$G = \left\{ \sum_{|\alpha|=N, \kappa(\alpha) \leq n} v_{\alpha} \xi_{\alpha} : v_{\alpha} \in \mathbf{R} \right\}.$$

with inner product

$$(\sum_a v_{\alpha} \xi_{\alpha}, \sum_a v'_{\alpha} \xi_{\alpha})_G = \sum_{\alpha} v_{\alpha} v'_{\alpha}.$$

Consider a vector subspace $\tilde{G} \subseteq G$ generated by $p_N(z), z = (z_1, \dots, z_n, 0, \dots) \in \mathcal{Z}$. It is enough to show that $\tilde{G} = G$. Indeed, the subspace $\tilde{G}$ is finite-dimensional and obviously closed. Assume there is a vector $\sum_{\alpha} v_{\alpha} \xi_{\alpha} \in G$ which is orthogonal to $\tilde{G}$. So, for all $z = (z_1, \dots, z_n, 0, \dots) \in \mathcal{Z}$,

$$(\sum_{\alpha} v_{\alpha} \xi_{\alpha}, p_N(z))_G = \sum_{\alpha} v_{\alpha} \frac{z^{\alpha}}{\sqrt{\alpha!}} = 0$$

which implies that all $v_{\alpha} = 0$. Therefore $\tilde{G} = G$.This completes the proof. □

Due to Proposition 2, we can characterize convergence in $\mathcal{D}' = \mathcal{D}'(\mathbf{b})$ by test functions of the form $p_m(z)$. Indeed, for $z \in \mathcal{Z}$, $v \in \mathcal{D}$, and $m \geq 0$, we have

$$\langle p_m(z), v\rangle = \sum_{|\alpha|=m} v_\alpha \frac{z^\alpha}{\sqrt{\alpha!}}. \tag{4.2}$$

Therefore we have the following necessary and sufficient condition:

Corollary 2. *A sequence $v^n \to v$ in $\mathcal{D}'$ if and only if for all $z \in \mathcal{Z}$ and all $m \geq 0$*

$$\langle p_m(z), v^n\rangle \to \langle p_m(z), v\rangle.$$

4.1.2. *Action of a Skorokhod integral on $p_M(z)$.* Consider $v(t) = \sum_\alpha v_\alpha(t)\xi_\alpha \in \mathcal{D}'(\mathbf{b}; [0,T], Y)$ such that for all α, k,

$$\int_0^T |(v_\alpha(s), e_k(s))_Y| ds < \infty.$$

Recall that the Skorokhod integral assigns to such v a generalized random process

$$\delta_t(v) = \int_0^t v(s)dW_s = \delta\left(v1_{[0,t]}\right) = \sum_\alpha \delta_t(v)_\alpha \xi_\alpha, 0 \leq t \leq T,$$

with

$$\delta_t(v)_\alpha = \sum_k \sqrt{\alpha_k} \int_0^t \left(v_{\alpha(k)}(s), e_k(s)\right)_Y ds.$$

Remark 6. *For $p_M(z) \in \mathcal{D}(\mathbf{b}), z \in Z, M \geq 1,$*

$$\langle p_M(z), \delta_t(v)\rangle = \int_0^t (\langle p_{M-1}(z), v(s)\rangle, e_z(s))_Y \ ds.$$

Indeed,

$$\begin{aligned}
\langle p_M(z), \delta_t(v)\rangle &= \sum_{|\alpha|=M} \delta_t(v)_\alpha \frac{z^\alpha}{\sqrt{\alpha!}} \\
&= \sum_{|\alpha|=M} \sum_k \sqrt{\alpha_k} \int_0^t \left(v_{\alpha(k)}(s), e_k(s)\right)_Y ds \frac{z^\alpha}{\sqrt{\alpha!}} \\
&= \sum_k \sum_{|\alpha|=M} \int_0^t \left(v_{\alpha(k)}(s), e_k(s)\right)_Y \frac{z^{\alpha(k)} z_k}{\sqrt{\alpha(k)!}} ds \\
&= \sum_k \int_0^t (\langle p_{M-1}(z), v(s)\rangle, e_k(s))_Y z_k ds \\
&= \int_0^t (\langle p_{M-1}(z), v(s)\rangle, e_z(s))_Y \ ds.
\end{aligned}$$

4.1.3. *Action of a Wick product on $p_M(z)$.* Recall that for a Hilbert space E and arbitrary $v=\sum_\alpha v_\alpha\xi_\alpha$ and $u=\sum_\alpha u_\alpha\xi_\alpha$ in $\mathcal{D}'(\mathbf{b},E)$, we define

$$v\Diamond u=\sum_\alpha\sum_{\beta\le\alpha}(u_\beta,v_{\alpha-\beta})_E\sqrt{\frac{\alpha!}{\beta!(\alpha-\beta)!}}\xi_\alpha\in\mathcal{D}'(\mathbf{b},\mathbf{R}).$$

In particular,

$$\xi_\alpha\Diamond\xi_\beta=\xi_{\alpha+\beta}\sqrt{\frac{(\alpha+\beta)!}{\beta!\alpha!}}.$$

The following statement holds.

Lemma 4. *For a Hilbert space E, arbitrary elements $v=\sum_\alpha v_\alpha\xi_\alpha$ and $u=\sum_\alpha u_\alpha\xi_\alpha$ from $\mathcal{D}'(\mathbf{b},E)$, and $z\in\mathcal{Z}, M\ge 0$,*

$$\langle p_M(z),v\Diamond u\rangle=\sum_{K+L=M}\langle p_K(z),v\rangle\langle p_L(z),u\rangle\,;$$

In particular,

$$\langle 1,v\Diamond u\rangle=(\langle 1,v\rangle,\langle 1,u\rangle)_E$$

(the expected value of $v\Diamond u$ is the product of expected values).

Proof. According to (4.2),

$$\begin{aligned}\langle p_M(z),v\Diamond u\rangle&=\sum_{|\alpha|=M}\frac{(v\Diamond u)_\alpha}{\sqrt{\alpha!}}z^\alpha\\&=\sum_{|\alpha|=M}\sum_{\beta\le\alpha}(u_\beta,v_{\alpha-\beta})_E\sqrt{\frac{1}{\beta!(\alpha-\beta)!}}z^\alpha\\&=\sum_{|\alpha|=M}\sum_{\beta\le\alpha}(\frac{z^\beta}{\sqrt{\beta!}}u_\beta,\frac{z^{\alpha-\beta}}{\sqrt{(\alpha-\beta)!}}v_{\alpha-\beta})_E\\&=\left(\sum_{K+L=M}\sum_{|\beta|=K}\frac{z^\beta}{\sqrt{\beta!}}u_\beta,\sum_{|\gamma|=L}\frac{z^\gamma}{\sqrt{\gamma!}}u_\gamma\right)_E\\&=(\langle p(z),v\rangle,\langle p(z),u\rangle)_E\,.\end{aligned}$$

□

4.1.4. *An equivalent characterization of the solution.* Now, we will characterize the solution of equation (3.1) by its action on test functions $p_M(z), z\in\mathcal{Z}, M\ge 0$. The following statement holds.

Proposition 3. *Assume A1-A3 hold, $p\ge 2$,$w=\sum_\alpha w_\alpha\xi_\alpha\in\mathcal{D}'(\mathbf{b},\mathbb{H}_p^2)$ and $u(t)=\sum_\alpha u_\alpha(t)\xi_\alpha\in\mathcal{D}'([r,T],\mathbf{b},\mathbb{H}_p^2)$.Then $u(t)$ is $\mathcal{D}$-$\mathbb{H}_p^2$ solution of (3.1) in $[r,T]$ if and only if for all $z\in\mathcal{Z}$ and $M\ge 0$,*

$$\mathbf{u}^{M,z}(t,x)=\langle\mathbf{u}(t,x),p_M(z)\rangle=\sum_{|\alpha|=M}\frac{\mathbf{u}_\alpha(t,x)z^\alpha}{\sqrt{\alpha!}}$$

is an $\mathbb{H}_p^2$*-solution of equation*

(4.3)

$$\mathbf{u}^{M,z}(t,x)=\mathbf{w}^{M,z}(x)+\int_r^t[\partial_i\left(a^{ij}(s,x)\partial_j\mathbf{u}^{M,z}(s,x)\right)+$$

$$b^i(s,x)\partial_i\mathbf{u}^{M,z}(s,x)-\sum_{K+L=M}u^{k,K,,z}(s,x)\partial_k\mathbf{u}^{L,z}(s,x)+$$

$$\nabla P^{M,z}(s,x)+1_{M=0}\mathbf{f}(s,x)]ds+\int_r^t[(\sigma^p(s,x),e_z(s))_Y\partial_p\mathbf{u}^{M-1,z}(s,x)+$$

$$1_{M=1}(\mathbf{g}(s,x),e_z(s))_Y-\nabla\left(\tilde{P}^{M,z}(s,x),e_z(s)\right)_Y]ds,\ \ \operatorname{div}\mathbf{u}^{M,z}=0,$$

where $M\geq 0$, $w^{M,z}(x)=\langle\mathbf{w}(x),p_M(z)\rangle=\sum_{|\alpha|=M}w_\alpha(x)z^\alpha/\sqrt{\alpha!}$, *and* $u^{-1,z}(t,x)=0$.

If $M\geq 1$, *equation (4.3) is Stokes equation; if* $M=0$, *it is Navier-Stokes equation.*

Proof. If $u(t)=\sum_\alpha u_\alpha(t)\xi_\alpha\in\mathcal{D}'([r,T],b,\mathbb{H}_p^2)$ is a $\mathcal{D}$-$\mathbb{H}_p^2$ solution, then we obtain (4.3) by multiplying both sides of (3.4) by $z^\alpha/\sqrt{\alpha!}$ and adding. On the other hand, $u_\alpha(t,x)=\partial_z^\alpha u^{M,z}(t,x)$ and if (4.3) holds, then using Proposition 4.7 in [17] and differentiating (4.3) in z we obtain (3.4) by induction. The statement follows by Lemma 1. □

4.2. **Adapted and independent of basis generalized processes.** Denote $L_\infty([0,T],Y)$ the space of measurable Y-valued bounded functions on $[0,T]$. For $h\in L_\infty([0,T],Y)$, $M\geq 0$, we denote

$$p_{M,t}(h)=\int_0^t\int_0^{s_M}\dots\int_0^{s_2}h(s_1)\dots h(s_M)dW_{s_1}\dots dW_{s_M},0\leq t\leq T.$$

By (4.1), $p_M(z)=p_M(e_z)=p_{M,T}(e_z),z\in\mathcal{Z}$.

Lemma 5. *(i) If* $\{m_k,k\geq 1\}$ *is a CONS in* $L_2(0,T)$, *and* $\{\ell_k,k\geq 1\}$ *is a CONS in* Y,$h\in L_\infty([0,T],Y)$, *then for each* $n,n'\geq 1$, *there is* $z\in\mathcal{Z}$ *such that*

$$h_{n,n'}(t)=\sum_{i=1}^{n'}\sum_{k=1}^{n}\int_0^T(h(s),\ell_k)m_i(s)ds\ell_k m_i(t)=e_z(t),$$

$0\leq t\leq T$. *Obviously,* $p_{M,T}(h_{n,n'})\in\mathcal{D}(\mathbf{b})$, $\mathbf{b}=\{e_k=m_{i_k}\ell_{j_k},k\geq 1\}$,

$$h_{n,n'}\to h\ \text{in}\ L_2([0,T],Y),$$
$$p_{M,T}(h_{n,n'})\to p_{M,T}(h)\ \text{in}\ L_2(\Omega,\mathbf{P}),$$

as $n,n'\to\infty$.

(ii) Assume (m_i) *is trigonometric basis or unconditional* $L_p([0,T])$*-basis (for example, Haar basis, see* [15]*),* $h\in L_\infty([0,T],Y)$. *Then there is a sequence* $z(n)\in\mathcal{Z}$ *such that* $e_{z(n)}\to h$ *in* $L_p([0,T],Y)$ *for all* $p\geq 2$, *as* $n\to\infty$.

Proof. We prove the second part of the statement. Let

$$h_n(s) = \sum_{k=1}^{n} (h(s), \ell_k)_Y \ell_k, n \geq 1.$$

Then $|h_n(s)|_Y \leq |h(s)|_Y$ and for all $p \geq 2$,

$$\int_0^T |h_n(s) - h(s)|_Y^p ds \to 0$$

as $n \to \infty$. If (m_i) is trigonometric basis or unconditional $L_p([0,T])$-basis (for example, Haar basis), then for each n

$$\int_0^T |h_{n,n'}(s) - h_n(s)|^p ds \to 0$$

as $n' \to \infty$. So, there is a subsequence l_n such that

$$\int_0^T |h_{n,l_n}(s) - h(s)|^p ds \to 0$$

as $n \to \infty$, and 2. follows according to part 1. of this remark. □

Let $\mathcal{T}$ be the space of all linear combinations of $p_{M,T}(h), h \in L_\infty([0,T],Y)$, $M \geq 0$. Obviously, $\cup_{\mathbf{b}\in\mathcal{B}}\mathcal{D}(\mathbf{b}) \subseteq \mathcal{T}$, and $\mathcal{T}$ does not depend on any particular $\mathbf{b} \in \mathcal{B}$.

We say that $h_n \to h$ in $L_\infty([0,T],Y)$ if $h_n \to h$ in $L_p([0,T],Y)$ for all $p \geq 2$. Denote $\mathcal{T}'(\mathbf{b}) = \mathcal{T}'(\mathbf{b},E)$ the set of all $v \in \mathcal{D}'(\mathbf{b},E)$ such that for each $h \in L_\infty([0,T],Y)$ and any sequence $e_{z(n)} \to h$ in $L_\infty([0,T],Y)$, the limit $\lim_{n\to\infty} \langle p_{M,T}(z(n)), v\rangle$ exists for all $M \geq 0$, and does not depend on a particular sequence $z(n)$ such that $e_{z(n)} \to h$ in $L_\infty([0,T],Y)$. We define

$$\langle p_{M,T}(h), v\rangle = \lim_{n\to\infty} \langle p_M(z(n)), v\rangle .$$

Let $\mathcal{T}'(\mathbf{b};[0,T]) = \mathcal{T}'(\mathbf{b};[0,T],E)$ be the space of all $v \in \mathcal{D}'(\mathbf{b};[0,T],E)$ such that $v(t) \in \mathcal{T}'(\mathbf{b},E), 0 \leq t \leq T$.

Definition 5. *a) If we have two different CONS* $\mathbf{b} = \{e_k : k \geq 1\}$,$\mathbf{b}' = \{e'_k : k \geq 1\}$ *and* $v \in \mathcal{T}'(\mathbf{b})$, $u \in \mathcal{T}'(\mathbf{b}')$, *then we say* $v = u$, *if for all* $h \in L_\infty([0,T],Y)$, $M \geq 0$,

$$\langle p_{M,T}(h), v\rangle = \langle p_{M,T}(h), u\rangle . \tag{4.4}$$

Let $\mathcal{T}' = \mathcal{T}'(E) = \cup_{\mathbf{b}\in\mathcal{B}}\mathcal{T}'(\mathbf{b},E)$ *be factorized by the equivalence relation (4.4). If* $u \in \mathcal{T}'(E)$ *we say that* u *is a generalized E-valued random variable. We denote* $\mathcal{T}'([0,T]) = \mathcal{T}'([0,T],E) = \cup_{\mathbf{b}\in\mathcal{B}}\mathcal{T}'(\mathbf{b};[0,T],E)$ *factorized by the equivalence relation*

$$\langle p_{M,T}(h), v(t)\rangle = \langle p_{M,T}(h), u(t)\rangle , 0 \leq t \leq T.$$

If $u \in \mathcal{T}'([0,T],E)$, *we say* u *is a generalized E-valued stochastic process.*

b) If $v \in \mathcal{D}'(\mathbf{b};[0,T],E), \mathbf{b} = \{e_k : k \geq 1\} \in \mathcal{B}$, *we say* $u \in \mathcal{T}'([0,T],E)$ *is an extension of* v, *if for every* $z \in \mathcal{Z}, M \geq 0$,

$$\langle p_{M,T}(e_z), v(t)\rangle = \langle p_{M,T}(e_z), u(t)\rangle , 0 \leq t \leq T,$$

where $e_z = \sum_k z_k e_k$.

Now, we introduce the notion of an adapted generalized process. Let $\mathcal{F}_t^W = \sigma(W_s, s \le t)$, $0 \le t \le T$, $\mathbb{F}^W = (\mathcal{F}_t^W)$. We say $v \in \mathcal{T}'(\mathbf{b}; [0,T], E)$ is $\mathbb{F}$-adapted if for each $h \in L_\infty([0,T], Y)$, $M \ge 0$, and $t \le T$,

$$\langle p_{M,T}(h), v(t) \rangle = \langle p_{M,t}(h), v(t) \rangle .$$

Example 2. *Let W_t be a cylindrical Wiener process in a Hilbert space Y and $\dot{W}_t = \frac{d}{dt} W_t$. Then (see Example 1 as well) W_t and $\dot{W}_t$ are generalized Y-valued adapted stochastic processes. For any $h \in L_\infty([0,T], Y)$,*

$$\langle W_t, p_{M,T}(h) \rangle = \int_0^t h(s) ds = \langle W_t, p_{M,t}(h) \rangle ,$$
$$\left\langle \dot{W}_t, p_{M,T}(h) \right\rangle = h(t) = \left\langle \dot{W}_t, p_{M,t}(h) \right\rangle$$

if $M = 1$, and $\langle W_t, p_T^M(h) \rangle = \left\langle \dot{W}_t, p_T^M(h) \right\rangle = 0$ otherwise.

4.3. **Independence of basis, non-anticipating and Markov property of the solution.** Proposition 3 suggests the following definition of a generalized solution to (3.1).

Definition 6. *Given $w = \sum_\alpha w_\alpha \xi_\alpha \in \mathcal{T}'(\mathbf{b}, \mathbb{H}_p^2), T \ge r$, a generalized process $\mathbf{u}(t) = \sum_\alpha \mathbf{u}_\alpha(t) \xi_\alpha \in \mathcal{T}'([r,T], b, \mathbb{H}_p^2)$ is called $\mathbb{H}_p^2$-solution of equation (3.1) in $[r,T]$, if for each $h \in L_\infty([0,T], Y), M \ge 0$, the function $\mathbf{u}^{M,h}(t,x) = \langle p_{M,T}(h), \mathbf{u}(t,x) \rangle$ is an $\mathbb{H}_p^2$-solution in $[r,T]$ of Stokes ($M \ge 1$) or Navier-Stokes ($M = 0$) equation*

(4.5)

$$\mathbf{u}^{M,h}(t,x) = \mathbf{w}^{M,h}(x) + \int_r^t [\partial_i \left(a^{ij}(s,x) \partial_j \mathbf{u}^{M,h}(s,x) \right) +$$
$$b^i(s,x) \partial_i \mathbf{u}^{M,h}(s,x) - \sum_{K+L=M} u^{k,K,,h}(s,x) \partial_k \mathbf{u}^{L,h}(s,x) +$$
$$\nabla P^{M,h}(s,x) + 1_{M=0} \mathbf{f}(s,x)] ds + \int_r^t [(\sigma^p(s,x), h(s))_Y \partial_p \mathbf{u}^{M-1,h}(s,x) +$$
$$1_{M=1} (\mathbf{g}(s,x), h(s))_Y - \nabla \left(\tilde{P}^{M,h}(s,x), h(s) \right)_Y] ds, \ \text{div } \mathbf{u}^{M,h} = 0,$$

where $M \ge 0$, $\mathbf{w}^{M,h}(x) = \langle p_{M,T}(h), \mathbf{w}(x) \rangle$, and $u^{-1,h}(t,x) = 0$.

Obviously, a generalized solution is a $\mathcal{D}$-solution. Now we are in a position to prove the main result:

Theorem 2. *Assume that A1-A3 hold, $p > d$. Then for each $T < T_1$ there is a unique $\mathbb{H}_p^2 \cap \mathbb{H}_2^2$-solution of (3.1) in $[0,T]$. Moreover, the solution is $\mathbb{F}^W = (\mathcal{F}_t^W)$-adapted and it extends all $\mathcal{D}(\mathbf{b})$-solutions, $\mathbf{b} \in \mathcal{B}$.*

Proof. Fix $T < T_1$ and choose a special CONS $\{m_k, k \ge 1\}$ in $L_2(0,T)$ such that for each $h \in L_\infty([0,T], Y)$ there is a sequence $z(N) \in \mathcal{Z}$ (see Lemma 5) for which $e_{z(N)} \to h$ in $L_p([0,T], Y)$, for all $p \ge 2$, as $N \to \infty$ (for example,

(m_k) is trigonometric basis or unconditional $L_p([0,T])$-basis (Haar basis), see [15]). According to Lemma 2, there is a unique $\mathcal{D}$-$\mathbb{H}_p^2 \cap \mathbb{H}_2^2$-solution $\mathbf{u}(t,x) = \sum_\alpha \mathbf{u}_\alpha(t,x)\xi_\alpha$ of (3.1) in $[0,T]$. The coefficient functions $\mathbf{u}_\alpha(t,x)$ satisfy (3.4) and by Proposition 3, (4.3) holds for all $M \geq 0, z \in \mathcal{Z}$. Fix $h \in L_\infty([0,T],Y)$ and consider an arbitrary $e_{z(N)} \to h$ in $L_p([0,T],Y)$, for all $p \geq 2$, as $N \to \infty$.

Then for

$$\mathbf{u}^{M,z(N)}(t,x) = \langle \mathbf{u}(t,x), p_M(z(N))\rangle = \sum_{|\alpha|=M} \mathbf{u}_\alpha(t,x) z(N)^\alpha/\sqrt{\alpha!},$$

we have

$$\mathbf{u}^{M,z(N)}(t,x) = \mathbf{w}^{M,z(N)}(x) + \int_r^t [\partial_i \left(a^{ij}(s,x)\,\partial_j \mathbf{u}^{M,z(N)}(s,x)\right) +$$

$$b^i(s,x)\partial_i \mathbf{u}^{M,z(N)}(s,x) - + \textstyle\sum_{K+L=M} u^{k,K,,z(N)}(s,x)\,\partial_k \mathbf{u}^{L,z(N)}(s,x) +$$

$$\nabla P^{M,z(N)}(s,x) + 1_{M=0}\mathbf{f}(s,x)]ds +$$

$$\int_r^t [\left(\sigma^p(s,x), e_{z(N)}(s)\right)_Y \partial_p \mathbf{u}^{M-1,z(N)}(s,x) + 1_{M=1}(\mathbf{g}(s,x), e_{z(N)}(s))_Y -$$

$$\nabla\left(\tilde{P}^{M,z}(s,x), e_{z(N)}(s)\right)_Y]ds, \ \ \text{div}\ \mathbf{u}^{M,z(N)} = 0.$$

By induction in M, Theorem 3 in [18] and Proposition 4.7 in [17], there is a unique $\mathbb{H}_p^2 \cap \mathbb{H}_p^2$-solution $u^M(t,x)$ of equation

$$\mathbf{u}^M(t,x) = \mathbf{w}^{M,h}(x) + \int_r^t [\partial_i \left(a^{ij}(s,x)\,\partial_j \mathbf{u}^M(s,x)\right) + b^i(s,x)\partial_i \mathbf{u}^M(s,x) -$$

$$+ \textstyle\sum_{K+L=M} u^{k,K}(s,x)\,\partial_k \mathbf{u}^L(s,x) + \nabla P^M(s,x) + 1_{M=0}\mathbf{f}(s,x)]ds +$$

$$+ \int_r^t [(\sigma^p(s,x), h(s))_Y \partial_p \mathbf{u}^{M-1}(s,x) + 1_{M=1}(\mathbf{g}(s,x), h(s))_Y$$

$$-\nabla\left(\tilde{P}^M(s,x), h(s)\right)_Y]ds, \ \ \text{div}\ \mathbf{u}^M = 0,$$

where $\mathbf{u}^{-1}(t,x) = 0$.

According to Proposition 4.7 in [17],

$$\sup_{t\leq T} |\mathbf{u}^{M,z(N)}(t) - \mathbf{u}^M(t)|_{2,l}^l + \int_r^T |\partial^2 \mathbf{u}^{M,z(N)}(t) - \partial^2 \mathbf{u}^M(t)|_{1,l}^l dt$$

$$\leq C[\int_r^T |\sum_{\substack{K+L=M,\\ 1\leq K,L\leq M-1}} (u^{k,K,,z(N)}(s)\,\partial_k \mathbf{u}^{L,z(N)}(s) - u^{k,K}(s)\,\partial_k \mathbf{u}^L(s)|_{1,l}^l ds$$

$$+ \int_0^T |\,(\sigma^p(s), h(s))_Y \partial_p \mathbf{u}^{M-1,z(N)}(s) - \left(\sigma^p(s), e_{z(N)}(s)\right)_Y \partial_p \mathbf{u}^{M-1}(s)|_{1,l}^l ds$$

$$+ 1_{M=0} \int_r^T |(\mathbf{g}(s), h(s))_Y - (\mathbf{g}(s), e_{z(N)}(s))_Y|_{1,l}^l ds].$$

Using induction in M and embedding theorems we obtain that

$$\sup_{t\leq T}|\mathbf{u}^{M,z(N)}(t)-\mathbf{u}^{M}(t)|_{2,l}^{l}+\int_{r}^{T}|\partial^{2}\mathbf{u}^{M,z(N)}(t)-\partial^{2}\mathbf{u}^{M}(t)|_{1,l}^{l}dt\rightarrow 0$$

as $N\rightarrow\infty$ for $M\geq 0$. Since $h\in L_{\infty}([0,T],Y)$ is arbitrary, $\mathbf{u}(t)=\sum_{\alpha}\mathbf{u}_{\alpha}(t)\xi_{\alpha}\in\mathcal{T}'([0,T],\mathbf{b},\mathbb{H}_{p}^{2})$ is $\mathbb{H}_{p}^{2}$-solution of equation (3.1) in $[0,T]$, and

$$\mathbf{u}^{M,h}(t,x)=\langle p_{M,T}(h),\mathbf{u}(t,x)\rangle\,,h\in L_{\infty}([0,T],Y)$$

satisfies (4.5). Since for each CONS $\mathbf{b}'=(e_{k}')\in\mathcal{B}$ any linear combination of e_{k}' belongs to $L_{\infty}([0,T],Y)$, the generalized solution $\mathbf{u}(t)$ extends any $\mathcal{D}$-solution.

Now we will prove that the unique generalized $\mathbb{H}_{p}^{2}\cap\mathbb{H}_{2}^{2}$-solution of (3.1) in $[0,T]$ is $\mathbb{F}^{W}$-adapted. We fix $t^{*}\in(0,T),r\leq t^{*}$ and consider a special basis $\bar{m}_{i}(t)$ in $L_{2}((0,T))$ so that each $\bar{m}_{i}$ is supported either in $[0,t^{*}]$ or in $[t^{*},T]$ and such that for each $h\in L_{\infty}([0,T],Y)$ there is a sequence $z(N)\in\mathcal{Z}$ (see Lemma 5) for which $e_{z(N)}\rightarrow h$ in $L_{p}\left([0,T],Y\right)$, for all $p\geq 2$, as $N\rightarrow\infty$ (for example, $(\bar{m}_{k})$ is a combination of two trigonometric or unconditional $L_{p}\left([0,T]\right)$-basis (Haar basis) on $(0,t^{*})$ and (t^{*},T)). Denote by $\bar{\xi}_{\alpha},\ \alpha\in\mathcal{J}$, the corresponding orthonormal basis in $L_{2}(\mathbb{F}_{T}^{W})$. Then

$$\mathbf{u}(t)=\sum_{\alpha}\bar{\mathbf{u}}_{\alpha}(t)\bar{\xi}_{\alpha}$$

(with $\bar{u}_{\alpha}(t)$ satisfying the system (3.4)) is a generalized $\mathbb{H}_{p}^{2}\cap\mathbb{H}_{2}^{2}$-solution of (3.1) in $[0,T]$.

Let $\mathcal{J}'=\{\alpha\in\mathcal{J}:\alpha$ has a non zero component corresponding to $\bar{m}_{k}$ whose support is in $(t^{*},T)\}$. It follows by induction on $|\alpha|=n$ that for $t\in(0,t^{*})$, $\bar{u}_{\alpha}(t)=0$ if $\alpha\in\mathcal{J}'$. As a result,

$$\mathbf{u}(t)=\sum_{\alpha\in\mathcal{J}}\bar{\mathbf{u}}_{\alpha}(t)\bar{\xi}_{\alpha}=\sum_{\alpha\notin\mathcal{J}'}\bar{\mathbf{u}}_{\alpha}(t)\bar{\xi}_{\alpha},\ t\in[r,t^{*}].$$

Obviously, $\bar{\xi}_{\alpha}$ are $\mathbb{F}_{t^{*}}^{W}$-measurable for $\alpha\notin J'$. Also, for any $z\in Z,M\geq 0$,

$$p_{M}(z)=\sum_{\alpha}\frac{z^{\alpha}}{\sqrt{\alpha!}}\bar{\xi}_{\alpha}$$

and for $t\leq t^{*}$,

$$\begin{aligned}\langle p_{M}(z),\mathbf{u}(t)\rangle&=\sum_{\alpha\notin\mathcal{J}'}\bar{\mathbf{u}}_{\alpha}(t)\frac{z^{\alpha}}{\sqrt{\alpha!}}=\langle p_{M,t^{*}}(z),\mathbf{u}(t)\rangle\\&=\left\langle p_{M}(e_{z}1_{(0,t^{*})}),\mathbf{u}(t)\right\rangle\end{aligned}$$

(note that $e_{z}=\sum_{k}z_{k}e_{k},e_{z}1_{(0,t^{*})}=\sum_{k\notin G}z_{k}e_{k}$, where G is the set of all k such that $\bar{m}_{j_{k}}$ in $e_{k}=\bar{m}_{j_{k}}l_{j_{k}}$ has its support in (t^{*},T)). □

The solution above has the restarting property as well. By the same arguments as in Corollary 1 we have

Corollary 3. *Let* $w = \sum_\alpha w_\alpha \xi_\alpha \in \mathcal{T}'(b, \mathbb{H}_2^2 \cap \mathbb{H}_2^2)$,

$$|\mathbf{w}_\alpha|_{1,p} + |\mathbf{w}_\alpha|_{1,2} < \infty \text{ for all } |\alpha| \geq 1,$$
$$|\mathbf{w}_0|_{1,2p} + |\mathbf{w}_0|_{1,2} < \infty.$$

Assume that A1-A3 hold, $p > d$. *Let* $u^{r,\mathbf{w}}(t)$ *be* $\mathbb{H}_p^2 \cap \mathbb{H}_2^2$*-solution to (3.1)in* $[r, T], T < T_1$, *and* $r \leq r' \leq t \leq T$. *Then*

$$\mathbf{u}^{r,\mathbf{w}}(t) = \mathbf{u}^{r',\mathbf{u}(r')}(t). \tag{4.6}$$

Corollary 4. *(Markov Property) Assume that the assumptions of Corollary 3 hold true, and, in addition,* w *is* $\mathbb{F}_r^W-$*measurable, then* $u^{r,\mathbf{w}}(t)$ *is* $(\mathbb{F}_t^W)$*-adapted. This together with (4.6) can be interpreted as Markov property.*

5. Appendix I. Quantization and second quantization of Wiener Chaos

Consider a generalized random variable $u = \sum_\alpha u_\alpha \xi_\alpha = \sum_\alpha u_\alpha \sqrt{|\alpha|!} W(e_\alpha) \in \mathcal{D}'(\mathbf{b}), \mathbf{b} = \{e_k, k \geq 1\} \in B$, where

$$W(e_\alpha) = \int_0^T \int_0^{s_n} \ldots \int_0^{s_2} e_\alpha(s_1, \ldots, s_n) dW_{s_1} \ldots dW_{s_n},$$

and $n = |\alpha|$. Since

$$e_\alpha = \sum_{\sigma \in \mathcal{P}^n} e_{k_{\sigma(1)}} \otimes \ldots e_{k_{\sigma(n)}} / \sqrt{\alpha!|\alpha|!},$$

(P^n is the permutation group of the set $\{1, \ldots, n\}$) is a CONS of the symmetric part $H^{\hat{\otimes} n}$ of $H^{\otimes n}$ (recall $H = L_2(0, T]) \times Y$), we can interpret

$$u = \sum_\alpha u_\alpha \xi_\alpha = \sum_\alpha u_\alpha \sqrt{|\alpha|!} W(e_\alpha)$$

as a result of the noise W acting on

$$\hat{u} = \sum_\alpha u_\alpha \sqrt{|\alpha|!} e_\alpha \in \mathcal{H} = \sum_{n=0}^\infty \mathbf{H}^{\hat{\otimes} n}$$

(here $H^{\hat{\otimes} 0} = R$):

$$W(\hat{u}) = \sum_\alpha u_\alpha \sqrt{|\alpha|!} W(e_\alpha).$$

Let $A = (A_n)_{n \geq 0.}$ be a self-adjoint positive operator on H (see above) such that $A_n e_\alpha = \lambda(\alpha) e_\alpha$, where $|\alpha| = n$ and $\lambda(\alpha)$, $\alpha \in I$, are positive numbers.

Definition 7. *Quantization operator* $\Gamma(A) : \mathcal{D}'(\mathbf{b}) \to \mathcal{D}'(\mathbf{b})$ *is defined by*

$$\Gamma(A)u = \Gamma(A)W(\hat{u}) = W(A\hat{u}) = \sum_\alpha u_\alpha \sqrt{|\alpha|!} W(Ae_\alpha)$$
$$= \sum_\alpha u_\alpha \sqrt{\alpha!} \lambda(\alpha) W(e_\alpha) = \sum_\alpha u_\alpha \lambda(\alpha) \xi_\alpha.$$

Since $\lambda(\alpha) > 0$, we can define

$$\Gamma(A)^{-1}u = \Gamma(A^{-1})u = \sum_{\alpha} u_{\alpha}\lambda(\alpha)^{-1}\xi_{\alpha}.$$

Example 3. *1. (second quantization in space-time) Consider a self-adjoint positive operator B on H such that $Be_k = \lambda_k e_k$. Let $A = (B^{\otimes n})$. Then*

$$Ae_{\alpha} = B^{\otimes n}e_{\alpha} = \lambda^{\alpha}e_{\alpha},\ |\alpha| = n,$$

where $\lambda = (\lambda_k)$ and $\lambda^{\alpha} = \Pi_k \lambda_k^{\alpha_k}$. We have

$$\Gamma(A)u = \sum_{\alpha} u_{\alpha}\lambda^{\alpha}\xi_{\alpha}.$$

2. (second quantization in space) Consider a self-adjoint positive operator B on Y such that the sequence of its eigenvectors $(\ell_p)_{p\geq 1}$ ($B\ell_p = \lambda_p\ell_p, \lambda_p > 0$) is a CONS in Y. Let $b = \{e_k, k \geq 1\}$, where $e_k(s) = m_{i_k}(s)\ell_{j_k}$. We extend B to H by

$$Be_k = B(m_{i_k}\ell_{j_k}) = m_{i_k}B\ell_{j_k} = \lambda_{j_k}e_k$$

and quantize in space-time using $A = (B^{\otimes n})$. For $u = \sum_{\alpha} u_{\alpha}\xi_{\alpha}$ we have

$$\Gamma(A)u = \sum_{\alpha} u_{\alpha}\lambda^{\alpha}\xi_{\alpha},$$

where $\lambda^{\alpha} = \Pi_k \lambda_{j_k}^{\alpha_k}$.

3. Consider a self-adjoint positive operator B on H such that $Be_k = \lambda_k e_k$ and a sequence of positive numbers q_n. Let $A = \sum_{n=0}^{\infty} q_n B^{\otimes n}$. Then

$$Ae_{\alpha} = q_n B^{\otimes n}e_{\alpha} = q_n\lambda^{\alpha}e_{\alpha},\ |\alpha| = n,$$

where $\lambda = (\lambda_k)$ and $\lambda^{\alpha} = \Pi_k \lambda_k^{\alpha_k}$. We have

$$\Gamma(A)u = \sum_{n=0}^{\infty} q_n \sum_{|\alpha|=n} u_{\alpha}\lambda^{\alpha}\xi_{\alpha}.$$

For the Wick product we have the following obvious statement.

Remark 7. *1. Assume $A = (A_n)$ is a self-adjoint positive operator on H such that $Ae_{\alpha} = A_n e_{\alpha} = \lambda(\alpha)e_{\alpha}, |\alpha| = n$ and $\lambda(\alpha)$,$\alpha \in I$, are positive numbers, $u, v \in S(b)$. Then, denoting $\Gamma(A)u = u^A, \Gamma(A)v = v^A$, we have*

$$\begin{aligned}\Gamma(A)(u\Diamond v) &= \Gamma(A)\left(\Gamma(A)^{-1}u^A\Diamond\Gamma(A)^{-1}v^A\right)\\ &= \sum_{\alpha} c_{\alpha}\xi_{\alpha},\end{aligned}$$

where

$$c_{\alpha} = \sum_{\beta\leq\alpha} \frac{\lambda(\alpha)}{\lambda(\beta)\lambda(\alpha-\beta)} u_{\beta}^A v_{\alpha-\beta}^A \sqrt{\frac{\alpha!}{\beta!(\alpha-\beta)!}}.$$

In particular, if $\Gamma(A)$ *is the second quantization in space-time, then*

$$\frac{\lambda(\alpha)}{\lambda(\beta)\lambda(\alpha-\beta)}=\frac{\lambda^{\alpha}}{\lambda^{\beta}\lambda^{\alpha-\beta}}=1$$

and

$$\Gamma(A)(u\Diamond v)=u^{A}\Diamond v^{A}.$$

2. For the Skorokhod stochastic integral, we have

$$\begin{aligned}\Gamma(A)\left(\delta(u)\right)&=\Gamma(A)\int_0^T u_s\Diamond\dot{W}_s ds\\&=\int_0^T\Gamma(A)\left(\Gamma(A)^{-1}u_s^A\Diamond\Gamma(A)^{-1}\dot{W}_s^A\right)ds\\&=\int_0^T\Gamma(A)\left(\Gamma(A)^{-1}u_s^A\Diamond W_s\right)ds;\end{aligned}$$

for the coefficients

$$\begin{aligned}\left(\Gamma(A)\delta(u)\right)_{\alpha}&=\sum_k\int_0^T(u^A_{\alpha(k)}(t),\lambda(\varepsilon_k)e_k(t))_Y dt\sqrt{\alpha_k}\frac{\lambda(\alpha)}{\lambda(\alpha(k))\lambda(\varepsilon_k)}\xi_{\alpha}\\&=\sum_k\int_0^T(u^A_{\alpha(k)}(t),e_k(t))_Y dt\sqrt{\alpha_k}\frac{\lambda(\alpha)}{\lambda(\alpha(k))}\xi_{\alpha}.\end{aligned}$$

Also,

$$\left(\Gamma(A)(u(t)\Diamond\dot{W}_t)\right)_{\alpha}=\sum_k(v^A_{\alpha(k)}(t),e_k(t))_Y\sqrt{\alpha_k}\frac{\lambda(\alpha)}{\lambda(\alpha(k))}\xi_{\alpha}.$$

Indeed,

$$\Gamma(A)v_t\Diamond\dot{W}_t=\sum_{\alpha}\sum_k(v_{\alpha(k)}(t),e_k(t))_Y\sqrt{\alpha_k}\lambda(\alpha)\xi_{\alpha}$$

$$=\sum_{\alpha}\sum_k(\lambda(\alpha(k))v_{\alpha(k)}(t),\lambda(\varepsilon_k)e_k(t))_Y\sqrt{\alpha_k}\frac{\lambda(\alpha)}{\lambda(\alpha(k))\lambda(\varepsilon_k)}\xi_{\alpha}.$$

6. Appendix II. Derivation of quantized Navier-Stokes equation

To simplify discussion, we will consider a velocity field which depends only on one standard Gaussian random variable $\eta\sim N(0,1)$, rather than a trajectory of the Wiener process W_t. An interested reader would have little difficulties extending the arguments below to the setting with Wiener process.

Consider a velocity field

$$\mathbf{u}(t,x)=\sum_{h=0}^{\infty}\mathbf{u}_n(t,x)\,\xi_n(\eta).$$

Note that in our setting the Cameron-Martin expansion (see Theorem 1) is indexed by integers rather than multi-indexes. Assume that for every n, u_n is analytic in x in that it could be written as

$$\mathbf{u}_n(t,x) = \sum_{\gamma\in\mathbf{N}^d} \mathbf{c}_{n,\gamma}(t)\, x^{\gamma}.$$

Let $Z = (Z_1, \dots, Z_d)$ be a $\mathcal{F}^{\eta}$-measurable. Then by substituting Z into u we get

$$\mathbf{u}(t,Z) := \sum_n \left(\sum_\gamma \mathbf{c}_{n,\gamma}(t)\, Z^{\gamma} \right) \xi_n(\eta). \tag{6.1}$$

Now, let us introduce the Wick-powers of Z :

$$Z^{\Diamond\gamma} := Z_1^{\Diamond\gamma_1} \Diamond ... \Diamond Z_d^{\gamma_d}, \gamma = (\gamma_1, \dots, \gamma_d) \in \mathbf{N}^d.$$

Next we will replace the standard algebra in (6.1) by the Wick algebra:

$$\mathbf{u}_n^{\Diamond}(t,Z) := \sum_\gamma \mathbf{c}_{n,\gamma}(t)\, Z^{\Diamond\gamma}$$

Consider now the following random field

$$\mathbf{u}^{\Diamond}(t,Z) := \sum_{n\geq 0} \mathbf{u}_n^{\Diamond}(t,Z) \Diamond \xi_n(\eta)$$

Remark 8. *Note that Wick algebra on nonrandom elements reduces to the standard deterministic algebra.*

Let $X_t = (X_t^1, \dots, X_t^d)$ be a solution of the following dynamic equation

$$\dot{X}_t = \mathbf{u}^{\Diamond}(t, X_t).$$

Then by the Wick chain rule

$$\begin{aligned} \ddot{X}_t &= \frac{d}{dt}\mathbf{u}^{\Diamond}(t,X_t) = \partial_t \mathbf{u}^{\Diamond}(t,X_t) + \nabla \mathbf{u}^{\Diamond}(t,X_t) \Diamond \dot{X}_t, \\ &= \partial_t \mathbf{u}^{\Diamond}(t,X_t) + \mathbf{u}^{\Diamond}(t,X_t) \nabla \Diamond \mathbf{u}^{\Diamond}(t,X_t). \end{aligned}$$

If $\mathbf{F} = \mathbf{F}(t,x)$ is an acting force, this yields quantized (Wick) Euler equation

$$\partial_t \mathbf{u}^{\Diamond}(t,x) = -\mathbf{u}^{\Diamond}(t,x) \nabla \Diamond \mathbf{u}^{\Diamond}(t,x) + \mathbf{F}(t,x)$$

If there is no randomness, due to Remark 8, this equation reduces to the standard Euler equation:

$$\partial_t \mathbf{u}(t,x) = -\mathbf{u}(t,x) \nabla \mathbf{u}(t,x) + \mathbf{F}(t,x).$$

Now, by taking $\mathbf{F} = \Delta\mathbf{u} - \nabla P$, where P stands for pressure, we get the quantized Navier-Stokes equation

$$\partial_t \mathbf{u}^{\Diamond}(t,x) = \Delta\mathbf{u} - \mathbf{u}^{\Diamond}(t,x) \nabla \Diamond \mathbf{u}^{\Diamond}(t,x) - \nabla P + \mathbf{F}(t,x).$$

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