

SMOOTH SOLUTIONS FOR THE DYADIC MODEL

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ABSTRACT. We consider the dyadic model, which is a toy model to test issues of well-posedness and blow-up for the Navier–Stokes and Euler equations. We prove well-posedness of positive solutions of the viscous problem in the relevant scaling range which corresponds to Navier–Stokes. Likewise we prove well-posedness for the inviscid problem (in a suitable regularity class) when the parameter corresponds to the strongest transport effect of the non-linearity.

1. INTRODUCTION

We consider the dyadic model introduced in [7, 10] and lately extensively studied in several variants (viscous [8, 4, 5], inviscid [11, 13, 6, 1, 2] and stochastically forced [3]).

The dyadic model has been studied as a *toy model* for the Euler and Navier–Stokes equations as it enjoys the main features of the differential models, such as energy conservation, while having a much simpler mathematical structure. Here we focus on regularity and well-posedness for positive solutions to the viscous (1.1) and to the inviscid problem (1.2).

1.1. **The viscous problem.** Let $\nu > 0$, $\beta > 0$ and consider

$$(1.1) \quad \begin{cases} \dot{X}_n = -\nu\lambda_n^2 X_n + \lambda_{n-1}^\beta X_{n-1}^2 - \lambda_n^\beta X_n X_{n+1}, \\ X_n(0) = x_n, \end{cases} \quad n \geq 1 \quad t \geq 0$$

where $\lambda_0 = 0$, $\lambda_n = \lambda^n$ and $\lambda = 2$. We assume that $x_n \geq 0$ and this implies (see [4]) that the solution remains positive at all times. The parameter β measures the relative strength of the dissipation versus the non-linearity. The range of values $\beta \in (2, \frac{5}{2}]$ is essentially the one corresponding, within the simplification of the model, to the three dimensional Navier-Stokes equations. The range arises from scaling arguments applied to the nonlinear term, we refer to [5] for further details.

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If $\beta \leq 2$ the non linear term is dominated by the dissipative one, in this case Cheskidov [4] proved existence of regular global solutions using classical techniques, while if $\beta > 3$ the non-linearity is too strong and all solutions with large enough initial condition develop a blow-up [4].

The two results above are based on “energy methods” and do not cover the range $\beta \in (2, \frac{5}{2}]$, where it becomes crucial to understand how the structure of the non-linearity drives the dynamics. The method proposed here (which is reminiscent of a technique used in the context of fluid mechanics in [12]) is based on purely dynamical systems techniques.

In order to prove well-posedness of the viscous problem, we identify a minimal condition that implies smoothness of solutions (Proposition 3.3). The main idea then is to show the existence of an invariant region for the vector (X_n, X_{n+1}) by a dynamical argument (Lemma 2.1) which provides the minimal condition. We are led to the following result.

Theorem A. *Let $\beta \in (2, \frac{5}{2}]$, then for every initial condition $(x_n)_{n \geq 1}$ such that*

$$x_n \geq 0 \quad \text{for all } n \geq 1, \quad \text{and} \quad \sum_{n=1}^{\infty} x_n^2 < \infty,$$

there exists a unique solution to problem (1.1), which is smooth, that is

$$\sup_{n \geq 1} (\lambda_n^\gamma X_n(t)) < \infty$$

for all $\gamma > 0$ and $t > 0$.

1.2. The inviscid problem. It turns out that the invariant region provided by Lemma 2.1 is independent of the viscosity. This allows to consider the inviscid problem

$$(1.2) \quad \begin{cases} \dot{X}_n = \lambda_{n-1}^\beta X_{n-1}^2 - \lambda_n^\beta X_n X_{n+1}, \\ X_n(0) = x_n, \end{cases} \quad n \geq 1 \quad t \geq 0.$$

It is known that there are local in time regular solutions (namely, with strong enough decay in n) and that there is a finite time blow-up, that is the quantity

$$\sum_{n=1}^{\infty} (\lambda_n^{\frac{\beta}{3}} X_n(t))^2 \nearrow \infty$$

when t approaches a finite time [10, 7]. Our result gives a different picture, as we prove that the dynamics generated by (1.2) is well-posed in a larger space. The correct interpretation to both results is that the condition above involving the blowing up quantity does not provide the natural space for the solutions of the inviscid problem. Indeed, a $\lambda_n^{-\beta/3}$ decay is borderline for the conservation of energy (which does not hold rigorously for weaker decay, a proof for $\beta \leq 1$ is given in [1]).

To support the physical validity of the solutions we consider, we also prove that the global solution we have found is the unique vanishing viscosity limit. The main result for (1.2) is given in full details as follows.

Theorem B. *Let $\beta = \frac{5}{2}$ and let $x = (x_n)_{n \geq 1}$ with $x_n \geq 0$ for all $n \geq 1$ and*

$$\sup_{n \geq 1} (\lambda_n^\gamma x_n) < \infty,$$

for some $\gamma > \frac{1}{2}$ close enough to $\frac{1}{2}$. Then there is a global in time solution $X = (X_n)_{n \geq 1}$ to (1.2) with initial condition x such that

$$(1.3) \quad \sup_{t \geq 0} (\sup_{n \geq 1} \lambda_n^\gamma X_n(t)) < \infty,$$

which is unique in the class of solutions satisfying the bound (1.3) above.

Moreover, X is the unique vanishing viscosity limit. More precisely, if $X^{[\nu]}$ is the solution to the viscous problem (1.1) with viscosity ν and with initial condition x , then

$$X_n^{[\nu]} \longrightarrow X_n, \quad n \geq 1,$$

as $\nu \rightarrow 0$, uniformly in time on compact sets.

The paper is organised as follows. In Section 2 we prove the fundamental invariant region lemma with a dynamical systems technique. The well-posedness of the viscous problem is established in Section 3, while the vanishing viscosity limit and the inviscid problem are analysed in Section 4.

2. THE INVARIANT REGION LEMMA

In this section we prove the key result of the paper. Let $(X_n)_{n \geq 1}$ be a solution to problem (1.1) on a time interval $[0, T]$. In view of Proposition 3.3 below, it is natural to apply the following *change of variables*

$$Y_n = \lambda_n^{\beta-2+\epsilon} X_n,$$

where $\epsilon > 0$ will be chosen suitably in the proof of the lemma below. A straightforward computation shows that $(Y_n)_{n \geq 1}$ solves

$$(2.1) \quad \begin{cases} \dot{Y}_n = -\nu \lambda_n^2 Y_n + \lambda_{n-1}^{2-\epsilon} \lambda^{\beta-2+\epsilon} Y_{n-1}^2 - \lambda_n^{2-\epsilon} \lambda^{2-\beta-\epsilon} Y_n Y_{n+1}, \\ Y_n(0) = y_n, \end{cases}$$

for $n \geq 1$ and $t \in [0, T]$, where clearly $y_n = \lambda_n^{\beta-2+\epsilon} X_n(0)$ for all $n \geq 1$.

For technical reasons we consider a finite dimensional (truncated) version for the equations for Y . For every $N \geq 1$ let $(Y_n^{(N)})_{1 \leq n \leq N}$ be the solution to

$$(2.2) \quad \begin{cases} \dot{Y}_n^{(N)} = -\nu \lambda_n^2 Y_n^{(N)} + \lambda_{n-1}^{2-\epsilon} \lambda^{\beta-2+\epsilon} (Y_{n-1}^{(N)})^2 - \lambda_n^{2-\epsilon} \lambda^{2-\beta-\epsilon} Y_n^{(N)} Y_{n+1}^{(N)}, \\ Y_n(0) = y_n, \end{cases}$$

for $n = 1, \dots, N$, where for the sake of simplicity we have set $Y_0^{(N)} = 0$ and $Y_{N+1}^{(N)} = Y_N^{(N)}$, so to avoid writing the border equations in a different form. Let us now introduce the region A of \mathbf{R}^2 that will be invariant for the vectors $(Y_n^{(N)}, Y_{n+1}^{(N)})$,

$$A := \{(x, y) \in \mathbf{R}^2 : 0 \leq x \leq 1, h(x) < y < g(x)\},$$

where the functions h and g that provide the lower and upper bound of A are defined as

$$g(x) = \min\{mx + \theta, 1\}, \quad h(x) = \begin{cases} 0 & x \leq \delta, \\ c\left(\frac{x-\delta}{1-\delta}\right)^{\lambda^2} & x > \delta. \end{cases}$$

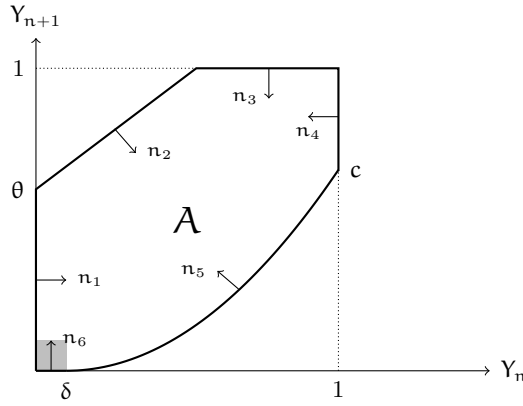


FIGURE 1. The invariant region

Lemma 2.1. *There exist $\delta \in (0, 1)$, $c \in (0, 1)$, $\theta \in (0, 1)$, $m > 0$ and $\epsilon > 0$ such that for every $\beta \in (2, \frac{5}{2}]$ and $\nu \geq 0$ the following statement holds true: if $N \geq 1$ and if $(y_n, y_{n+1}) \in A$ for all $n \leq N$, then $(Y_n^{(N)}(t), Y_{n+1}^{(N)}(t)) \in A$ for all $n = 1, \dots, N$ and $t \geq 0$, where $(Y_n^{(N)})_{1 \leq n \leq N}$ is the solution to (2.2) with initial condition $(y_n)_{1 \leq n \leq N}$.*

Proof. For simplicity we drop the superscript (N) along this proof. Since the pairs $(Y_n, Y_{n+1})_{1 \leq n \leq N}$ satisfy a finite dimensional system of differential equation, it is sufficient to show that the derivative in time of (Y_n, Y_{n+1}) points inward on the border of A when $(Y_n, Y_{n+1}) \in A$ for each $n = 1, \dots, N$ or, equivalently, that the scalar product with the inward normal of the border of A with the vector field

$$\mathfrak{B} = \begin{pmatrix} \dot{Y}_n \\ \dot{Y}_{n+1} \end{pmatrix} = \nu \lambda_n^2 \begin{pmatrix} -Y_n \\ -\lambda^2 Y_{n+1} \end{pmatrix} + \lambda^{\beta-4+2\epsilon} \lambda_n^{2-\epsilon} \begin{pmatrix} Y_{n-1}^2 - \lambda^{6-2\beta-3\epsilon} Y_n Y_{n+1} \\ \lambda^{2-\epsilon} (Y_n^2 - \lambda^{6-2\beta-3\epsilon} Y_{n+1} Y_{n+2}) \end{pmatrix}.$$

is positive when $(Y_n, Y_{n+1}) \in A$ for all $n = 1, \dots, N$. The set A is convex, hence we can consider separately the viscous and the inviscid contribution to \mathfrak{B} .

We start with the viscous part, which we denote by \mathfrak{B}_v (we neglect the multiplicative constant $\nu\lambda_n^2$) and we denote the inward normals as in Figure 1. The scalar product of \mathfrak{B}_v with each $\vec{n}_1, \vec{n}_3, \vec{n}_4, \vec{n}_6$ on the respective pieces of the border of A is clearly positive, as

$$\begin{aligned}\mathfrak{B}_v \cdot \vec{n}_1 &= -Y_n = 0, & \mathfrak{B}_v \cdot \vec{n}_3 &= \lambda^2 Y_{n+1} = \lambda^2, \\ \mathfrak{B}_v \cdot \vec{n}_4 &= Y_n = 1, & \mathfrak{B}_v \cdot \vec{n}_6 &= -\lambda^2 Y_{n+1} = 0,\end{aligned}$$

so we are left with the last two cases, in which, for simplicity, we set $x = Y_n$. First,

$$\mathfrak{B}_v \cdot \vec{n}_2 = -xg'(x) + \lambda^2 Y_{n+1} = -xg'(x) + \lambda^2 g(x) = m(\lambda^2 - 1)x + \theta\lambda^2 > 0,$$

then

$$\mathfrak{B}_v \cdot \vec{n}_5 = xh'(x) - \lambda^2 Y_{n+1} = xh'(x) - \lambda^2 h(x) = \frac{c\delta\lambda^2}{1-\delta} \left(\frac{x-\delta}{1-\delta} \right)^{\lambda^2} \geq 0.$$

We consider now the inviscid term, that we denote by \mathfrak{B}_i (and again we neglect the irrelevant multiplicative factor). Again we set $x = Y_n$ and, for simplicity, $\gamma = 6 - 2\beta - 3\epsilon$. We consider first the easy terms,

$$\begin{aligned}\mathfrak{B}_i \cdot \vec{n}_1 &= Y_{n-1}^2 - \lambda^\gamma x Y_{n+1} = Y_{n-1}^2 \geq 0, \\ \mathfrak{B}_i \cdot \vec{n}_6 &= \lambda^{2-\epsilon}(x^2 - \lambda^\gamma Y_{n+1} Y_{n+2}) = \lambda^{2-\epsilon} x^2 \geq 0.\end{aligned}$$

Next, we consider the piece of the border of A corresponding to \vec{n}_3 . Here $Y_{n+1} = 1$ and $x \leq 1$, moreover since $(Y_{n+1}, Y_{n+2}) \in A$, it follows that $Y_{n+2} \geq c$, hence

$$\mathfrak{B}_i \cdot \vec{n}_3 = \lambda^{2-\epsilon}(\lambda^\gamma Y_{n+1} Y_{n+2} - x^2) \geq \lambda^{2-\epsilon}(\lambda^\gamma c - 1).$$

The term on the right hand side in the formula above is positive if we choose $\lambda^\gamma c = 1$. Likewise on the piece corresponding to \vec{n}_4 we have $x = Y_n = 1$, $Y_{n+1} \geq c$ and $Y_{n-1} \leq 1$, hence

$$\mathfrak{B}_i \cdot \vec{n}_4 = \lambda^\gamma x Y_{n+1} - Y_{n-1}^2 \geq \lambda^\gamma c - 1 \geq 0.$$

We are left with the two challenging inequalities, that we are going to analyse. The first is on the piece of boundary corresponding to \vec{n}_2 , where we have $Y_{n+1} = g(x)$, and, since $(Y_{n+1}, Y_{n+2}) \in A$, $Y_{n+2} \geq h(Y_{n+1}) = h(g(x)) = c \left(\frac{mx + \theta - \delta}{1-\delta} \right)^{\lambda^2}$, if we choose $\theta \geq \delta$. Hence, using the fact that $\lambda^\gamma c = 1$ and that $\gamma \leq 2 - 3\epsilon$,

$$\begin{aligned}(2.3) \quad \mathfrak{B}_i \cdot \vec{n}_2 &= g'(x)(Y_{n-1}^2 - \lambda^\gamma x Y_{n+1}) - \lambda^{2-\epsilon}(x^2 - \lambda^\gamma Y_{n+1} Y_{n+2}) \\ &\geq -\lambda^\gamma x g'(x) g(x) - \lambda^{2-\epsilon}(x^2 - \lambda^\gamma g(x) h(g(x))) \\ &= \lambda^{2-\epsilon}(mx + \theta) \left(\frac{mx + \theta - \delta}{1-\delta} \right)^{\lambda^2} - \lambda^{2-\epsilon} x^2 - \lambda^\gamma mx(mx + \theta) \\ &\geq \lambda^{2-\epsilon}(mx + \theta) \left(\frac{mx + \theta - \delta}{1-\delta} \right)^{\lambda^2} - \lambda^{2-\epsilon} x^2 - \lambda^{2-3\epsilon} mx(mx + \theta).\end{aligned}$$

This last expression depends on x but not on β and it is sufficient to show that it is non-negative for $x \in [0, \frac{1-\theta}{m}]$. This will be done later by a suitable choice of the parameters.

Prior to this, we consider the second inequality, on the piece corresponding to \bar{n}_5 . Here we have that $Y_{n+1} = h(x)$ and $Y_{n-1} \leq 1$, and, since $(Y_{n+1}, Y_{n+2}) \in A$, $Y_{n+2} \leq g(Y_{n+1}) = g(h(x)) \leq mh(x) + \theta$. Therefore, since $x \left(\frac{x-\delta}{1-\delta}\right)^{\lambda^2} \leq 1$ and $\gamma \geq 1 - 3\epsilon$, hence $\lambda^{-\gamma} \leq \lambda^{3\epsilon-1}$,

$$\begin{aligned}
(2.4) \quad \mathfrak{B}_i \cdot \bar{n}_5 &= \lambda^{2-\epsilon}(x^2 - \lambda^\gamma Y_{n+1} Y_{n+2}) - h'(x)(Y_{n-1}^2 - \lambda^\gamma x Y_{n+1}) \\
&\geq \lambda^{2-\epsilon}(x^2 - \lambda^\gamma h(x)g(h(x))) - h'(x)(1 - \lambda^\gamma x h(x)) \\
&= \lambda^{2-\epsilon} \left[x^2 - \theta \left(\frac{x-\delta}{1-\delta} \right)^{\lambda^2} \right] - m \lambda^{2-\gamma-\epsilon} \left(\frac{x-\delta}{1-\delta} \right)^{2\lambda^2} + \\
&\quad - \frac{\lambda^{2-\gamma}}{1-\delta} \left(\frac{x-\delta}{1-\delta} \right)^{\lambda^2-1} \left[1 - x \left(\frac{x-\delta}{1-\delta} \right)^{\lambda^2} \right] \\
&\geq \lambda^{2-\epsilon} \left[x^2 - \theta \left(\frac{x-\delta}{1-\delta} \right)^{\lambda^2} \right] - m \lambda^{1+2\epsilon} \left(\frac{x-\delta}{1-\delta} \right)^{2\lambda^2} + \\
&\quad - \frac{\lambda^{1+3\epsilon}}{1-\delta} \left(\frac{x-\delta}{1-\delta} \right)^{\lambda^2-1} \left[1 - x \left(\frac{x-\delta}{1-\delta} \right)^{\lambda^2} \right]
\end{aligned}$$

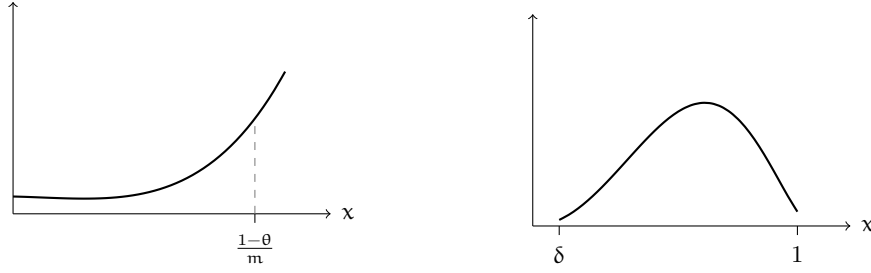
for $x \in [\delta, 1]$. Also this lower bound does not depend on β .

Let ψ_1 and ψ_2 be the right-hand sides of (2.3) and (2.4), respectively, when $\epsilon = 0$, namely,

$$\begin{aligned}
\psi_1(x) &= (mx + \theta) \left(\frac{mx + \theta - \delta}{1-\delta} \right)^{\lambda^2} - x^2 - mx(mx + \theta), \\
\psi_2(x) &= \lambda \left[x^2 - \theta \left(\frac{x-\delta}{1-\delta} \right)^{\lambda^2} \right] - m \left(\frac{x-\delta}{1-\delta} \right)^{2\lambda^2} + \\
&\quad - \frac{1}{1-\delta} \left(\frac{x-\delta}{1-\delta} \right)^{\lambda^2-1} \left[1 - x \left(\frac{x-\delta}{1-\delta} \right)^{\lambda^2} \right].
\end{aligned}$$

It is sufficient to show that both functions have positive minimal values. Continuity then ensures that the same is true for small ϵ . A direct computation shows that both ψ_1 and ψ_2 are positive with the choice $\delta = \frac{1}{10}$, $\theta = \frac{3}{5}$, $m = \frac{3}{4}$. Figure 2 shows a plot of the two functions. \square

Remark 2.2. A cleverer choice of the parameters δ , θ , and m might allow to extend the above result, and in turn the main results of the paper, to larger values of β (although smaller than 3, due to the blow-up results in [4] and [7]).

FIGURE 2. The functions ψ_1 , on the left, and ψ_2 , on the right.

3. UNIQUENESS AND REGULARITY IN THE VISCOUS CASE

Define

$$(3.1) \quad H = \{x = (x_n)_{n \geq 1} \subset \mathbf{R} : \|x\|_H^2 := \sum_{n=1}^{\infty} x_n^2 < \infty\}.$$

Following Cheskidov [4] we introduce weak and Leray–Hopf solutions for (1.1).

Definition 3.1. A weak solution to (1.1) on $[0, T]$ is a sequence of functions $X = (X_n)_{n \geq 1}$ such that $X_n \in C^1([0, T]; \mathbf{R})$ for every $n \geq 1$ and (1.1) is satisfied.

A Leray–Hopf solution is a weak solution X with values in H and such that the energy inequality

$$\|X(t)\|_H^2 + 2\nu \int_s^t \sum_{n=1}^{\infty} (\lambda_n X_n(r))^2 dr \leq \|X(s)\|_H^2,$$

holds for a. e. s and all $t > s$.

The following facts are proved in [4],

- existence of global in time Leray–Hopf solutions for all initial conditions in H ,
- if the initial condition $(x_n)_{n \geq 1}$ is positive, namely $x_n \geq 0$ for all $n \geq 1$, then every weak solution is a Leray–Hopf solution, stays positive for all times and the energy inequality holds for all times,
- if $\beta \leq 2$, there is a unique Leray–Hopf solution which is smooth, for every initial condition in H ,
- if $\beta > 3$, then every positive solution (starting from a large enough initial condition) cannot be smooth for all times.

Our first result is a criterion for uniqueness of positive solutions.

Proposition 3.2 (Uniqueness). Let $X = (X_n)_{n \geq 1}$ be a positive solution to (1.1) on $[0, T]$ such that the quantity

$$(3.2) \quad \sup_{t \in [0, T], n \geq 1} (\lambda_n^{\beta-3} X_n(t))$$

is finite. Then X is the unique weak solution with initial condition $(X_n(0))_{n \geq 1}$.

In particular, if $\beta \leq 3$, there is a unique weak solution for any positive initial condition in H .

Proof. The proof is a minor variation of the idea in [2]. Denote by c_0 the quantity (3.2). Let $Y = (Y_n)_{n \geq 1}$ be another solution with the same initial condition of X and set $Z_n = Y_n - X_n$, $W_n = X_n + Y_n$, then

$$\dot{Z}_n = -\nu \lambda_n^2 Z_n + \lambda_{n-1}^\beta Z_{n-1} W_{n-1} - \frac{1}{2} \lambda_n^\beta (Z_n W_{n+1} + Z_{n+1} W_n).$$

Fix $N \geq 1$ and set $\psi_N(t) = \sum_{n=1}^N \frac{1}{2^n} Z_n^2$, then $\psi_N(0) = 0$ and it is elementary to verify that

$$\frac{d}{dt} \psi_N(t) + 2\nu \sum_{n=1}^N \frac{\lambda_n^2}{2^n} Z_n^2 = -\frac{1}{2} \sum_{n=1}^N \frac{\lambda_n^\beta}{2^n} Z_n^2 W_{n+1} - \frac{\lambda_N^\beta}{2^{N+1}} Z_N Z_{N+1} W_N.$$

In particular (we recall that $\lambda = 2$ and $\lambda_n = \lambda^n$),

$$\begin{aligned} \frac{d}{dt} \psi_N(t) &\leq -\frac{1}{2} \lambda_N^{\beta-1} Z_N Z_{N+1} W_N \\ &= -\frac{1}{2} \lambda_N^{\beta-1} (Y_N^2 Y_{N+1} + X_N^2 X_{N+1} - X_{N+1} Y_N^2 - X_N^2 Y_{N+1}) \\ &\leq \frac{1}{2} \lambda_N^{\beta-1} (X_{N+1} Y_N^2 + X_N^2 Y_{N+1}) \\ &\leq c_0 \lambda_N^2 (X_N^2 + Y_N^2 + Y_{N+1}^2), \end{aligned}$$

and so by integrating in time,

$$\psi_N(t) \leq c_0 \int_0^t \lambda_N^2 (X_N^2 + Y_N^2 + Y_{N+1}^2) ds.$$

Since X and Y are both Leray–Hopf solutions, the right hand side in the above inequality converges to 0 as $N \rightarrow \infty$ and in conclusion $\psi_n(t) = 0$ for all $t \geq 0$ and all $n \geq 1$. \square

3.1. Regularity. Having the key Lemma 2.1 in hand, the missing step for the proof of Theorem A is a regularity criterion. The next result gives a minimal condition of smoothness which is in a way essentially optimal, as shown in Section 3.1.1 below, and which holds for general (positive and non-positive) initial conditions. Set

$$\mathcal{D}^\infty = \{(x_n)_{n \geq 1} : \sup_{n \geq 1} (\lambda_n^\gamma |x_n|) < \infty \text{ for all } \gamma > 0\}.$$

Proposition 3.3. *Let $T > 0$ and let X be a solution to (1.1) on $[0, T]$ such that $X(0) \in \mathcal{D}^\infty$ and*

$$\lim_{n \rightarrow \infty} \left(\sup_{t \in [0, T]} (\lambda_n^{\beta-2} |X_n(t)|) \right) = 0.$$

Then $X(t) \in \mathcal{D}^\infty$ for all $t \in [0, T]$. In particular, the above condition is verified if there is $\epsilon > 0$ such that

$$\sup_{n \geq 1} \left(\sup_{t \in [0, T]} (\lambda_n^{\beta-2+\epsilon} |X_n(t)|) \right) < \infty.$$

Proof. We can assume without loss of generality that $\lambda_n^{\beta-2} |X_n(t)| \leq c_n$ for all $n \geq 1$ and $t \in [0, T]$, with $c_n \downarrow 0$. Since

$$X_n(t) = e^{-\nu \lambda_n^2 t} X_n(0) + \int_0^t e^{-\nu \lambda_n^2 (t-s)} (\lambda_{n-1}^\beta X_{n-1}^2 - \lambda_n^\beta X_n X_{n+1}) ds,$$

we have that

$$|X_n(t)| \leq |X_n(0)| + \lambda^2 c_{n-1} \int_0^t \lambda_n^2 e^{-\nu \lambda_n^2 (t-s)} (|X_{n-1}| + |X_n|) ds,$$

and so for every $\gamma > 0$,

$$G_n \leq \lambda_n^\gamma |X_n(0)| + \frac{\lambda^{2+\gamma}}{\nu} c_{n-1} (G_{n-1} + G_n),$$

where we have set $G_n = \sup_{t \in [0, T]} (\lambda_n^\gamma |X_n(t)|)$. Hence there is n_0 such that for $n \geq n_0$ we have $\lambda^{2+\gamma} \nu^{-1} c_{n-1} \leq \frac{1}{3}$ and so $\sup_{n \geq n_0} G_n < \infty$. The terms G_n for $n \leq n_0$ are bounded due to the assumption. \square

Remark 3.4 (Local smooth solutions). The $\lambda_n^{\beta-2}$ decay can be interpreted in terms of local existence and uniqueness of smooth solutions. Indeed, this decay is critical, in the sense that only exponents larger or equal than $\beta - 2$ allow for local smooth solutions (for any general quadratic finite-range interaction non-linearity, without taking the geometry into account). This can be seen in the following way. Set for $\epsilon > 0$

$$(3.3) \quad \mathcal{W}_\epsilon = \{x = (x_n)_{n \geq 1} : \|x\|_{\mathcal{W}_\epsilon} := \sup_{n \geq 1} (\lambda_n^{\beta-2+\epsilon} |x_n|) < \infty\},$$

the result is a standard application of Banach's fixed point theorem to the map

$$\mathcal{F}_n(X)(t) = e^{-\nu \lambda_n^2 t} X_n(0) + \int_0^t e^{-\nu \lambda_n^2 (t-s)} (\lambda_{n-1}^\beta X_{n-1}^2 - \lambda_n^\beta X_n X_{n+1}) ds$$

and the relevant estimate to prove that \mathcal{F} maps a small ball into itself and is a contraction (for a small enough time interval) is

$$\begin{aligned} \int_0^t e^{-\nu \lambda_n^2 (t-s)} \left(\lambda_{n-1}^\beta X_{n-1} Y_{n-1} - \frac{1}{2} \lambda_n^\beta (X_n Y_{n+1} + X_{n+1} Y_n) \right) ds &\leq \\ &\leq \frac{c \lambda}{\nu^{1-\frac{\epsilon}{2}}} \lambda_n^{2-\beta-\epsilon} T^{\frac{\epsilon}{2}} \left(\sup_{t \leq T} \|X\|_{\mathcal{W}_\epsilon} \right) \left(\sup_{t \leq T} \|Y\|_{\mathcal{W}_\epsilon} \right). \end{aligned}$$

The case $\epsilon = 0$ (the critical case!) does not allow for small constants and can be worked out as in [9].

3.1.1. *Stationary solutions and critical regularity.* In this section we show that the condition given in Proposition 3.3 is optimal, by showing that there is a solution to (1.1) such that the quantity $\sup_{t,n} (\lambda_n^{\beta-2} |X_n(t)|)$ is bounded but the solution is not smooth. The example is provided by a time-stationary solution. In order to do this in this section (and only in this section) we shall consider solutions to (1.1) which may have also non-positive components.

We shall call *stationary solution* any sequence $\gamma = (\gamma_n)_{n \geq 1}$ such that

$$(3.4) \quad \nu \lambda_n^2 \gamma_n + \lambda_n^\beta \gamma_n \gamma_{n+1} - \lambda_{n-1}^\beta \gamma_{n-1}^2 = 0, \quad n \geq 1,$$

Proposition 3.5. *Let $\gamma = (\gamma_n)_{n \geq 1}$ be a non-zero stationary solution.*

- *If there is $n_0 \geq 1$ such that $\gamma_{n_0} = 0$, then $\gamma_n = 0$ for all $n \leq n_0$.*
- *Let n_0 be the first index such that $\gamma_{n_0} \neq 0$. Then $\gamma_n < 0$ for all $n > n_0$.*
- *Let n_0 be the first index such that $\gamma_{n_0} \neq 0$. Then there is $c > 0$ such that $\lambda_n^{\beta-2} |\gamma_n| \geq c$, for all $n \geq n_0$.*

Proof. Multiply (3.4) by γ_n and sum up to N to obtain

$$\nu \sum_{n=1}^N \lambda_n^2 \gamma_n^2 + \lambda_N^\beta \gamma_N^2 \gamma_{N+1} = 0, \quad N \geq 1.$$

The first two properties follow from this equality. For the third property, (3.4) implies that

$$\gamma_{n+1} = \frac{\gamma_{n-1}^2}{\lambda_n^\beta \gamma_n} - \nu \lambda_n^{2-\beta} \leq -\nu \lambda_n^{2-\beta},$$

since all γ_n are negative. □

Hence a stationary solutions can decay at most as the critical profile which is borderline in Proposition 3.3. So the existence of a stationary solutions shows that the condition of Proposition 3.3 is optimal. Moreover, if the stationary solution is in H , this provide an example of two weak solutions with the same initial condition (the stationary solution and the Leray-Hopf solution).

We look now for a stationary solution $(\gamma_n)_{n \geq 1}$. Set $u = \lambda^{2\beta-6}$ (notice that $u < 1$ for $\beta < 3$) and $\gamma_n = -\nu \lambda_{n-1}^{2-\beta} a_n$. Then

$$\begin{cases} a_1(a_2 - 1) = 0, \\ a_n a_{n+1} = a_n + u a_{n-1}^2, \quad n \geq 2. \end{cases}$$

One can show that if $u < \frac{1}{3}$, then there are infinitely many stationary solutions such that $0 < c_1 \leq \lambda_n^{\beta-2} |\gamma_n| \leq c_2$. Indeed, consider $a_1 \geq 0$ and $a_2 = 1$ and set

$$A = \frac{1}{2u} \left(1 - \sqrt{\frac{1-3u}{1+u}} \right), \quad B = \frac{1}{2u} \left(1 + \sqrt{\frac{1-3u}{1+u}} \right).$$

It is easy to verify that if $a_{n-1}, a_n \in [A, B]$, then $a_{n+1} \in [A, B]$, and so one needs only to find values of a_1 such that the sequence $(a_n)_{n \geq 1}$ ends up in $[A, B]$.

This requires a few computations which are not relevant for the paper and are omitted.

3.2. Proof of Theorem A. We have now all ingredients for the proof of the main theorem concerning the viscous case.

Proof of Theorem A. Let $x \in H$ be positive and let $X = (X_n)_{n \geq 1}$ be the unique weak solution starting at x . To prove the theorem, it is sufficient to show the following two claims,

1. for some $\epsilon > 0$ and for every $t_0 > 0$ the quantity $\sup_{n \geq 1} (\lambda^{\beta-2+\epsilon} X_n(t_0))$ is finite and

$$\sup_{t \geq t_0} \left(\sup_{n \geq 1} (\lambda^{\beta-2+\epsilon} X_n(t)) \right) \leq \frac{1}{\delta} \sup_{n \geq 1} (\lambda^{\beta-2+\epsilon} X_n(t_0))$$

for every $n \geq 1$ and $t \geq t_0$, where δ is the constant in Lemma 2.1.

2. if $\sup_{n \geq 1} (\lambda^{\beta-2+\epsilon} X_n(t_0))$ is finite, then there exists $t'_0 > t_0$ such that X_n is smooth in $(t_0, t'_0]$.

Indeed, if for $t_0 > 0$ the first claim holds true, then the second claim applies and the solution satisfies the assumptions of Proposition 3.3 for any initial time $t > t_0$ sufficiently small. Hence X is smooth for $t > t_0$ and since by the first claim t_0 can be chosen arbitrarily close to 0, the theorem is proved.

We prove the first claim. By the energy inequality,

$$\sum_{n=1}^{\infty} \int_0^t (\lambda_n X_n(s))^2 ds < \infty,$$

hence $\sup_{n \geq 1} (\lambda_n X_n(t)) < \infty$ for a. e. $t > 0$. Let $t_0 > 0$ be one of these times and set $K_0 = \sup_{n \geq 1} \lambda_n^{\beta-2+\epsilon} X_n(t_0)$, where ϵ is the parameter which has been set in the proof of Lemma 2.1. Let

$$\bar{Y}_n(t) = \frac{\delta}{K_0} \lambda_n^{\beta-2+\epsilon} X_n\left(\frac{\delta}{K_0} t\right), \quad n \geq 1, t \geq t_0,$$

where δ is the constant from Lemma 2.1. It turns out that $(\bar{Y}_n)_{n \geq 1}$ is solution to (2.1) but with viscosity $\bar{\nu} = \frac{\delta}{K_0} \nu$. Uniqueness of $(X_n)_{n \geq 1}$ clearly ensures uniqueness of $(\bar{Y}_n)_{n \geq 1}$ for equation (2.1) and so it is standard to show that the solutions $(\bar{Y}_n^{(N)})_{n \geq 1}$ of (2.2) (with viscosity $\bar{\nu}$) converge to $(\bar{Y}_n)_{n \geq 1}$. Clearly $\sup_{n \leq N} \bar{Y}_n^{(N)}(t_0) \leq \delta$ for all $N \geq 1$, therefore Lemma 2.1 ensures that $\bar{Y}_n^{(N)}(t) \leq 1$ and in turns $\lambda^{\beta-2+\epsilon} X_n(t) \leq \frac{K_0}{\delta}$ for all $n \geq 1$ and $t \geq t_0$. The proof of the first claim is complete.

We finally prove the second claim. Let $V_n = X_n e^{\nu \lambda_n (t-t_0)}$, then to prove smoothness of X in a small interval, it is sufficient to show that V is bounded

(uniformly in n) in the same interval. A direct computation shows that

$$\begin{aligned}\dot{V}_n &= -\nu(\lambda_n^2 - \lambda_n)V_n + \lambda_{n-1}^\beta V_{n-1}^2 - \lambda_n^\beta e^{-\nu\lambda_{n+1}(t-t_0)} V_n V_{n+1} \\ &\leq -\frac{\nu}{2}\lambda_n^2 V_n + \lambda_{n-1}^\beta V_{n-1}^2,\end{aligned}$$

so by comparison for ordinary differential equations we have that $V_n(t) \leq \tilde{V}_n(t)$ for all $t \geq t_0$ for which \tilde{V} is finite, where \tilde{V} is the solution to

$$\dot{\tilde{V}}_n = -\frac{\nu}{2}\lambda_n^2 \tilde{V}_n + \lambda_{n-1}^\beta \tilde{V}_{n-1}^2,$$

with initial condition $\tilde{V}_n(t_0) = V_n(t_0)$. Since by assumption the quantity

$$\sup_n (\lambda_n^{\beta-2+\epsilon} \tilde{V}_n(t_0)) = \sup_n (\lambda_n^{\beta-2+\epsilon} V_n(t_0))$$

is bounded, it follows that $\tilde{V}(t_0) \in \mathcal{W}_\epsilon$, where \mathcal{W}_ϵ has been defined in (3.3). Following the same lines of Remark 3.4, one can apply Banach's fixed point theorem to \tilde{V} in the space \mathcal{W}_ϵ to show existence of a solution in a small time interval. \square

4. THE INVISCID LIMIT

Following [2], we give the following definitions of solution.

Definition 4.1. A solution on $[0, T)$ (global if $T = \infty$) of (1.2) is a sequence $X = (X_n)_{n \geq 1}$ of functions such that $X_n \in C^1([0, T); \mathbf{R})$ for all $n \geq 1$ and (1.2) is satisfied.

A *Leray–Hopf* solution is a weak solution such that $X(t) \in H$ (where H is defined in (3.1)) and the energy inequality

$$\|X(t)\|_H \leq \|X(s)\|_H$$

holds for all $s \geq 0$ and $t \geq s$.

We give a short summary of known facts on solutions to (1.2).

- There is at least one global in time Leray–Hopf solutions for all initial conditions in H (see [6], the proof is given for $\beta = \frac{5}{2}$ but the extension to all β is straightforward).
- There is a unique local in time solution for “regular” enough initial conditions [7].
- If the initial condition $(x_n)_{n \geq 1}$ is *positive*, then every weak solution is a Leray–Hopf solution and stays positive for all times [2].
- If $\beta \leq 1$, there is a unique Leray–Hopf solution for every positive initial condition [2].
- No positive solution can be smooth for all times. In [6] they prove that, if $\beta = \frac{5}{2}$, then the quantity $\lambda_n^{5/6} X_n(t)$ cannot be bounded for all times.

We first start by giving a uniqueness criterion, based again on the idea in [2].

Lemma 4.2 (Uniqueness). *Given $T > 0$, let $X = (X_n)_{n \geq 1}$ be a positive solution to (1.2) on $[0, T]$.*

■ *If the quantity*

$$(4.1) \quad \sup_{t \in [0, T], n \geq 1} (\lambda_n^{\beta-1} X_n(t))$$

is finite, then X is the unique solution with initial condition $(X_n(0))_{n \geq 1}$ in the class of Leray–Hopf solutions.

■ *If for some $\epsilon > 0$ the quantity*

$$(4.2) \quad \sup_{t \in [0, T]} \sup_{n \geq 1} (\lambda_n^{\frac{1}{3}(\beta-1)+\epsilon} X_n(t))$$

is finite, then X is the unique solution with initial condition $(X_n(0))_{n \geq 1}$ in the class of solutions satisfying (4.2).

Proof. We follow the same lines (with the same notation) of the proof of Proposition 3.2. Denote by c_0 the quantity (4.1). Let $Y = (Y_n)_{n \geq 1}$ be another solution with the same initial condition of X . Then for $N \geq 1$,

$$\begin{aligned} \frac{d}{dt} \psi_N(t) &\leq -\frac{1}{2} \lambda_N^{\beta-1} Z_N Z_{N+1} W_N \\ &\leq \frac{1}{2} \lambda_N^{\beta-1} (X_{N+1} Y_N^2 + X_N^2 Y_{N+1}) \\ &\leq c_0 (X_N^2 + Y_N^2 + Y_{N+1}^2), \end{aligned}$$

and so by integrating in time,

$$\psi_N(t) \leq c_0 \int_0^t (X_N^2 + Y_N^2 + Y_{N+1}^2) ds.$$

Since X and Y are both Leray–Hopf solutions, the right hand side in the above inequality converges to 0 as $N \rightarrow \infty$ and in conclusion $\psi_n(t) = 0$ for all $t \geq 0$ and all $n \geq 1$.

For the second statement, let X, Y two solutions in the class, that is with (4.2) finite for both X and Y . As in the proof of the previous claim,

$$\frac{d}{dt} \psi_N(t) \leq \frac{1}{2} \lambda_N^{\beta-1} (X_{N+1} Y_N^2 + X_N^2 Y_{N+1}) \leq c \lambda_N^{-3\epsilon},$$

and so $\psi_N(t) \leq \lambda_N^{-3\epsilon} t$, which implies that $X = Y$. □

Proof of Theorem B. Assume that $\beta = \frac{5}{2}$ and that $\sup_n \lambda_n^\gamma x_n < \infty$ for some $\gamma > \beta - 2 = \frac{1}{2}$. First, notice that the second statement of the previous lemma ensures that there is at most one solution satisfying (1.3). So to show that the inviscid dynamics is bounded in the scaling λ_n^γ , we proceed by showing that the viscous dynamics is convergent as $\nu \rightarrow 0$. This shows both statements of the theorem at once.

Given $\nu > 0$, let $(X_n^{[\nu]})_{n \geq 1}$ be the solution to the viscous problem (1.1). Again by uniqueness, it is sufficient to work on a finite interval of time $[0, T]$, with $T > 0$. So we fix $T > 0$. We know by Theorem A (possibly taking, without loss of generality, a smaller value of γ) that

$$C_0 := \sup_{\nu > 0} \sup_{t \in [0, T]} \sup_{n \geq 1} (\lambda_n^\gamma X_n^{[\nu_k]}(t)) < \infty,$$

where C_0 depends only on the initial condition. Moreover for every $n \geq 1$ and $\nu \leq 1$,

$$|\dot{X}_n^{[\nu]}| \leq \nu \lambda_n^2 X_n^{[\nu]} + \lambda_{n-1}^\beta (X_{n-1}^{[\nu]})^2 + \lambda_n^\beta X_n^{[\nu]} X_{n+1}^{[\nu]} \leq c_n(C_0)$$

where c_n is a number independent of $\nu \leq 1$ (although it does depend on n). Hence by the Ascoli–Arzelà theorem for each n the family $\{X_n^{[\nu]} : \nu \in (0, 1]\}$ is compact in $C([0, T]; \mathbf{R})$. By a diagonal procedure, we can find a common sequence $(\nu_k)_{k \in \mathbf{N}}$ and a limit point $(X_n^{[0]})_{n \geq 1}$ such that $X_n^{[\nu_k]} \rightarrow X_n^{[0]}$ uniformly on $[0, T]$ for every $n \geq 1$. Clearly any limit point is positive, satisfies the equations (1.2) and the bound (1.3), hence by the previous lemma there is only one limit point and $X_n^{[\nu]} \rightarrow X_n^{[0]}$ uniformly as $\nu \downarrow 0$. \square

Remark 4.3. Clearly the family $(X^{[\nu]})_{\nu \leq 1}$ has limit points also when $\beta \neq \frac{5}{2}$. Moreover all limit points are bounded in the scaling $\lambda_n^{\beta-2}$ if $\beta \in (2, \frac{5}{2}]$ by virtue of Lemma 2.1. The main limitation is that the uniqueness lemma does not apply.

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