Percolation and limit theory for the Poisson lilypond model

Günter Last* and Mathew D. Penrose†‡

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Abstract

The lilypond model on a point process in d-space is a growth-maximal system of non-overlapping balls centred at the points. We establish central limit theorems for the total volume and the number of components of the lilypond model on a sequence of Poisson or binomial point processes on expanding windows. For the lilypond model over a homogeneous Poisson process, we give subexponentially decaying tail bounds for the size of the cluster at the origin. Finally, we consider the enhanced Poisson lilypond model where all the balls are enlarged by a fixed amount (the enhancement parameter), and show that for $d > 1$ the critical value of this parameter, above which the enhanced model percolates, is strictly positive.

Key words and phrases. Poisson process, lilypond model, growth model, stabilization, central limit theorem, continuum percolation.

1 Introduction

Suppose φ is a locally finite set of points of cardinality at least 2 in the space $\mathbb{R}^d$. The *lilypond model* based on φ is the system of balls (or *grains*) $\{B_{\rho(x)}(x) : x \in \varphi\}$ (here $B_r(x) := \{y \in \mathbb{R}^d : |y - x| \leq r\}$ and $|\cdot|$ is Euclidean norm) with the following two properties:

- The *hard-core property*: $\rho(x) + \rho(y) \leq |x - y|$ for all different $x, y \in \varphi$.
- The *smaller grain-neighbour property*: for each $x \in \varphi$ there is at least one $y \in \varphi \setminus \{x\}$ such that $\rho(x) + \rho(y) = |y - x|$ (in which case the points x and y are called *grain-neighbours*) and $\rho(y) \leq \rho(x)$. In this case we call y a *smaller* grain-neighbour of x.

*Institut für Stochastik, Karlsruher Institut für Technologie, 76128 Karlsruhe, Germany. Email: guenter.last@kit.edu

†Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, United Kingdom, Email: m.d.penrose@bath.ac.uk

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In the sequel we shall write $\rho(x, \varphi)$ to denote the dependence of the radii on both φ and $x \in \varphi$. Heveling and Last [9] established existence and uniqueness of the model for all such φ (in fact in greater generality). Define the union set

$$Z(\varphi) := \bigcup_{x \in \varphi} B_{\rho(x,\varphi)}(x). \tag{1.1}$$

In the case of finite φ, the lilypond model may be constructed as follows. All points of φ start growing at the same time and at the same rate. Any given ball ceases its growth as soon as it reaches any other ball. When φ has just a single element x, we define $\rho(x, \varphi) := +\infty$.

The model is of interest since it is a *growth-maximal hard-core* model. No single ball can grow further without overlapping another ball. It also has a *maximin* property: if φ is finite with n elements and the radii $\{\rho(x, \varphi) : x \in \varphi\}$ are listed in ascending order as $\rho_1, \rho_2, \dots \rho_n$, and if $\{\rho'(x) : x \in \varphi\}$ is any other system of radii satisfying the hard-core property, similarly listed in ascending order as $\rho'_1, \dots, \rho'_n$ then $(\rho_1, \dots, \rho_n)$ exceeds $(\rho'_1, \dots, \rho'_n)$ in the lexicographic ordering on $\mathbb{R}^n$.

In this paper we consider the lilypond model on *random* φ. The lilypond model was introduced by Häggström and Meester [8] for the case where φ is a stationary Poisson process Φ in $\mathbb{R}^d$ of intensity one; we call this the *Poisson lilypond model* and set $\Psi = \{(x, \rho(x, \Phi)) : x \in \Phi\}$. They proved that the union set $Z := Z(\Phi)$ does not percolate, i.e. does not have an unbounded connected component. Interestingly, there *does* exist a stationary percolating hard-core system of (non-lilypond) grains on Φ, at least in high dimensions; see [2].

Apart from the one-dimensional case (see Daley, Mallows and Shepp [4]), only a few further probabilistic properties of Ψ are known. Daley, Stoyan and Stoyan [5] give bounds for the *volume fraction*

$$p_Z := \mathbb{P}(0 \in Z) = \mathbb{E}V_d(Z \cap [0, 1]^d) \tag{1.2}$$

of Z, where 0 denotes the origin in $\mathbb{R}^d$ and V_d is Lebesgue measure (volume) on $\mathbb{R}^d$. The latter paper also has some numerical results on the *typical radius* ρ_0 of Ψ. The distribution of ρ_0 is that of $\rho(x, \Phi)$ for a "randomly picked" $x \in \Phi$. Because Φ is Poisson, it is well-known that the distribution of ρ_0 is that of $\rho(0, \Phi^0)$ where $\Phi^0 := \Phi \cup \{0\}$. Because of the hard-core property of Ψ we clearly have

$$p_Z = b_d \mathbb{E}[\rho_0^d], \tag{1.3}$$

where $b_d := V_d(B_1(0))$ is the volume of the unit ball in $\mathbb{R}^d$.

The contributions of the present paper fall into three categories: tail bounds, central limit theorems, and non-percolation under positive enhancement. These may be viewed as extending the percolation theory of the Poisson lilypond model beyond the basic fact that Z does not percolate. We now give an overview of our results, which are proved using notions of *stabilization* developed in Sections 2 and 3.

Given φ as above, and given $x \in \varphi$, define $C'(x, \varphi)$ to be the connected component of $Z(\varphi)$ containing x. We study the 'cluster at the origin' $C'(0, \Phi^0)$, which amounts to studying $C'(x, \Phi)$ for a randomly picked point $x \in \Phi$. Since Z does not percolate, we know that $C'(0, \Phi^0)$ is almost surely bounded. In Section 4 we shall give tail bounds on its size. We consider three different measures of the 'size' of $C'(0, \Phi^0)$, namely volume, metric diameter, and the number of constituent grains. In each case, we shall give upper tail bounds showing that the probability of the size of a cluster exceeding t decays exponentially in a power of t. We also give lower bounds of the same form but with different exponents; it remains open to establish the 'correct' exponent (if any) for the tail decay of the size of the cluster at the origin.

Our central limit theorems are stated with reference to a sequence of expanding windows in $\mathbb{R}^d$. Let $W \subset \mathbb{R}^d$ be a compact convex set containing an open neighborhood of the origin with $V_d(W) = 1$, and set $W_n := n^{1/d}W$, where for $A \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, $a \in \mathbb{R}_+$ we write $x + aA$ for $\{x + ay : y \in A\}$. For $A \subset \mathbb{R}^d$, let $\kappa(A)$ denote the number of connected components of A. In Section 5, we shall derive central limit theorems for (among other things) $V_d(Z(\Phi_n))$, and $\kappa(Z(\Phi_n))$, where we set $\Phi_n := \Phi \cap W_n$. We shall also establish the corresponding de-Poissonized central limit theorems, where instead of Φ_n one considers a point process χ_n consisting of n independent uniformly distributed points in $n^{1/d}W$. Central limit theorems such as these could be of use in establishing confidence intervals for quantities such as the volume fraction, based on simulations.

In Section 6 we consider the *enhanced lilypond model* Z^δ, defined for $\delta > 0$ to be the δ-neighbourhood of Z in $\mathbb{R}^d$. We shall show that for $d \geq 2$, there is a strictly positive critical value of δ such that the enhancement Z^δ percolates almost surely if δ is above this critical value, and does not percolate if δ is below the critical value. This gives us a one-parameter family of random sets in the continuum exhibiting a non-trivial phase transition.

As a final remark here, we compare the Poisson lilypond model to the *random sequential adsorption* (RSA) process with infinite input, in which unit balls arrive at locations given by a homogeneous space-time Poisson process starting at time zero, each ball being irreversibly accepted if it does not overlap any previously accepted ball. Continuing to time infinity, one ends up with a maximal system of balls satisfying the hard-core constraint, as with the lilypond model; see Penrose [12], and Schreiber, Penrose and Yukich [16] for formal definitions, existence and limit theorems for RSA. Noteworthy differences are that for RSA, the radii are all the same, the point process of accepted ball centres is not a spatial Poisson process, and the balls almost surely do not touch.

Notation: We use c, c', c'' and so on to denote positive finite constants whose values are unimportant and may change from line to line. On the other hand, we denote by c_1, c_2 and so on, constants whose values (though still not very important) are fixed and which may reappear in other parts of the paper. For nonempty $A \subset \mathbb{R}^d$ we write $\mathrm{diam}(A)$ for $\sup_{(x,y)\in A\times A} |x - y|$. We write $\mathrm{card}(A)$ for the number of elements of A (possibly ∞).

2 Stabilization

In this section, we establish that there is an almost surely finite random variable $R := R(\Phi)$ such that the radius $\rho(0, \Phi \cup \{0\})$ is unaffected by modifications to the point set Φ outside the ball $B_R(0)$. This is known as a *radius of stabilization* for $\rho(0, \Phi)$. Moreover, we establish tail bounds for R, i.e. bounds on the $\mathbb{P}(R > t)$, which decay exponentially in $t^{d/(d+1)}$. These tail estimates will be crucial in all of our subsequent proofs.

A *point process* is defined as a random variable taking values in the space $\mathbf{N}$ of all locally finite subsets of $\mathbb{R}^d$ equipped with the smallest σ-field $\mathcal{N}$ containing the sets $\{\varphi \in \mathbf{N} : \varphi(B) = k\}$ for all Borel $B \subset \mathbb{R}^d$ and all $k \in \mathbb{N}_0$, where $\varphi(B)$ denotes the number of elements of $\varphi \cap B$. By a (finite) *descending chain* in $\varphi \in \mathbf{N}$ we mean a finite sequence $x_0, \dots, x_n$ ($n \geq 1$) of distinct points of φ for which $|x_{i-1} - x_i| \geq |x_i - x_{i+1}|$ for all $i \in \{1, \dots, n-1\}$. Note that any two points of φ, considered on their own, form a descending chain. Let $\varphi \in \mathbf{N}$ and $x \in \varphi$. If $\varphi(\mathbb{R}^d) \geq 2$ we let $D(x, \varphi)$ denote the *nearest neighbour distance* from x in $\varphi \setminus \{x\}$. That is, we set $D(x, \varphi) := \min\{|x - y| : y \in \varphi \setminus \{x\}\}$. If $\varphi = \{x\}$, set $D(x, \varphi) := +\infty$.

For any $\varphi \in \mathbf{N}$ and $x, y \in \mathbb{R}^d$ we set $\varphi^x := \varphi \cup \{x\}$ and $\varphi^{x,y} = \varphi \cup \{x, y\}$. We construct a closed set $S(y, \varphi) \subset \mathbb{R}^d$ such that if this set is bounded, then the radius $\rho(y, \varphi^y)$ is determined by the restriction of φ to $S(y, \varphi)$ (see (2.2) below). In the trivial case where $\varphi \setminus \{y\} = \emptyset$ we define $S(y, \varphi) := \mathbb{R}^d$. Otherwise, we define

$$S(y, \varphi) := B_{2D(y,\varphi^y)}(y) \cup \bigcup_{(x,r) \in A(y,\varphi)} B_r(x), \tag{2.1}$$

where the set $A(y, \varphi) \subset \varphi \times (0, \infty)$ is defined as follows. A pair (x, r) belongs to $A(\varphi)$ if there is a descending chain $x_0, \dots, x_n$ in φ^0 such that $x_0 = y$, $|x_1 - y| \leq 2D(y, \varphi^y)$, $x_n = x$ and $r = |x - x_{n-1}|$.

The next result tells us essentially that $S(y, \Phi)$ is a *stopping set*; see [1], [17].

Lemma 2.1. *Suppose $y \in \mathbb{R}^d$ and $\varphi, \psi \in \mathbf{N}$ with $\psi \cap S(y, \varphi) = \varphi \cap S(y, \varphi)$. Then $S(y, \psi) = S(y, \varphi)$.*

Proof. Assume $\varphi \setminus \{y\}$ is nonempty (otherwise the result is trivial). Then $D(y, \varphi^y) = D(y, \psi^y)$.

Suppose $(x, s) \in A(y, \varphi)$. Then there is descending chain $x_0, \dots, x_n$ in φ^y such that $x_0 = y$, $|x_1 - y| \leq 2D(y, \varphi^y)$, $x_n = x$ and $s = |x - x_{n-1}|$. For $1 \leq m \leq n$ we have $x_m \in S(y, \varphi) \cap \varphi$ so that $x_m \in \psi$. Hence, $(x, s) \in A(y, \psi)$, so $A(y, \varphi) \subset A(y, \psi)$.

Conversely, suppose $(x, s) \in A(y, \psi)$. Then there is descending chain $x_0, \dots, x_n$ in ψ^y such that $x_0 = y$, $|x_1 - y| \leq 2D(y, \psi^y)$, $x_n = x$ and $s = |x - x_{n-1}|$.

We claim that $x_m \in \varphi$ for $1 \leq m \leq n$. This is proved by induction on m; if it holds for $1 \leq m \leq k - 1$ then $x_k \in S(y, \varphi) \cap \psi$ so $x_k \in \varphi$. To start the induction note that $|x_1 - y| \leq 2D(y, \varphi^y)$ so $x_1 \in S(y, \varphi) \cap \psi$ so $x_1 \in \varphi$.

By the preceding claim, $(x, s) \in A(y, \varphi)$ so $A(y, \psi) \subset A(y, \varphi)$, and hence $S(y, \psi) \subset S(y, \varphi)$. Therefore $A(y, \psi) = A(y, \varphi)$, so that $S(y, \psi) = S(y, \varphi)$. □

Lemma 2.2. *For any $y \in \mathbb{R}^d$ and $\varphi \in \mathbf{N}$, if the set $S(y, \varphi)$ is bounded, then it satisfies*

$$\rho(y, \varphi^y) = \rho(y, (\varphi^y \cap S(y, \varphi)) \cup \psi), \quad \forall\ \psi \subset \mathbb{R}^d \setminus S(y, \varphi), \psi \in \mathbf{N}. \tag{2.2}$$

Proof. Suppose $S(y, \varphi)$ is bounded and $\psi \in \mathbf{N}$ with $\psi \subset \mathbb{R}^d \setminus S(\varphi)$. Since $S(y, \varphi) = S(y, \varphi^y)$, we can and do assume without loss of generality that $y \in \varphi$. Also, by Lemma 2.1, it suffices to prove the result in the case where $\varphi \subset S(y, \varphi)$, so we assume this too.

Write φ' for $\varphi \cup \psi$, and write $\rho(x)$ for $\rho(x, \varphi)$ and $\rho'(x)$ for $\rho(x, \varphi')$.

Suppose $\rho(y) > \rho'(y)$. Let x_1 be a smaller grain-neighbour of y in φ'. Assume for now that $x_1 \in \varphi$. Then by the hard-core property of the lilypond model on φ, we have $\rho(y) + \rho(x_1) \leq |x_1 - y| = \rho'(y) + \rho'(x_1)$, and hence $\rho(x_1) < \rho'(x_1)$.

Next let x_2 be a smaller grain-neighbour of x_1 in φ. By the hard-core property of the lilypond model on φ', we have $\rho'(x_1)+\rho'(x_2) \leq |x_1-x_2| = \rho(x_1)+\rho(x_2)$, so $\rho'(x_2) < \rho(x_2)$. Let x_3 be a smaller grain-neighbour of x_2 in φ'. Assuming $x_3 \in \varphi$, using once more the hard-core property of the lilypond model on φ yields $\rho(x_3) < \rho'(x_3)$.

Continuing to alternate in this way, we get a sequence of points x_i satisfying

$$\rho(y) > \rho'(y) \geq \rho'(x_1) > \rho(x_1) \geq \rho(x_2) > \rho'(x_2) \geq \rho'(x_3) > \rho(x_3) \geq \cdots$$

and terminating at x_n if for some (odd) n we have $x_n \in \psi$. We see from these inequalities that the possibly terminating sequence $y, x_1, x_2, \ldots$ consists of distinct points. Also, setting $x_0 = y$ and $\rho_i = \rho(x_i)$ and $\rho'_i = \rho'(x_i)$, we have $|x_i - x_{i-1}| = \rho'_i + \rho'_{i-1}$ for odd i and $|x_i - x_{i-1}| = \rho_i + \rho_{i-1}$ for even i, and therefore from the above inequalities $|x_i - x_{i-1}|$ is nonincreasing (in fact, strictly decreasing) in i.

Thus the (possibly terminating) sequence $y, x_1, x_2, \ldots$ forms a descending chain with $|x_1 - y| \leq 2D(y, \varphi^y)$. If the sequence terminates at some point $x_n = z$ with $z \in \psi$, then $z \in S(y, \varphi)$, contradicting the assumption that $\psi \cap S(y, \varphi) = \emptyset$. On the other hand, if the sequence (x_i) does not terminate, then it forms an infinite descending chain in φ, contradicting the assumption that $S(y, \varphi)$ is bounded.

Thus if $\rho(y) > \rho'(y)$ we have derived a contradiction. If $\rho(y) < \rho'(y)$ we argue similarly, this time starting with x_1 a smaller grain-neighbour of y in φ. Again we end up with a contradiction. Thus we must have $\rho(y) = \rho'(y)$. □

Given $\varphi \in \mathbf{N}$, define

$$R(\varphi) := \inf\{r > 0 : S(0, \varphi) \subset B_r(0)\}, \tag{2.3}$$

with the convention $\inf(\emptyset) := +\infty$.

Lemma 2.3. *The function $R : \mathbf{N} \to [0, \infty]$ is Borel-measurable.*

Proof. Let $t > 0$. It suffices to prove $R^{-1}([0, t])$ is measurable, i.e. in $\mathcal{N}$. For $m \in \mathbb{N}$, let $\mathbf{N}_m = \{\varphi \in \mathbf{N} : \varphi(B_t(0)) = m\}$. Then by Lemma 2.1,

$$R^{-1}([0, t]) = \cup_{m=0}^{\infty} \{\varphi \in \mathbf{N}_m : R(\varphi \cap B_t(0)) \leq t\}.$$

However, it is not hard to see that for all $m \in \mathbb{N}$, the function $g : B_t(0)^m \to \{0, 1\}$ given by

$$g(x_1, \ldots, x_m) := \begin{cases} \mathbf{1}_{\{R(\{x_1,\ldots,x_m\}) \leq t\}} & \text{if } x_1, \ldots, x_m \text{ are distinct} \\ 0 & \text{otherwise} \end{cases}$$

is measurable, and hence $\{\varphi \in \mathbf{N}_m : R(\varphi \cap B_t(0)) \leq t\} \in \mathcal{N}$. The result follows. □

By Lemma 2.2, $R(\varphi)$ is a radius of stabilization for $\rho(0, \varphi^0)$, i.e. $\rho(0, \varphi^0)$ is unaffected by modifications to φ outside $B_{R(\varphi)}(0)$. Also, $R(\Phi)$ is known to be almost surely finite [10, 3]. The next result provides another proof of this last fact, and more importantly shows that the tail of the distribution of $R(\Phi)$ decays sub-exponentially.

Lemma 2.4. *There is a constant $c_1 > 0$ such that for any $x \in \mathbb{R}^d$,*

$$\mathbb{P}(R(\Phi) > t) \leq c_1 \exp(-c_1^{-1} t^{d/(d+1)}), \quad t > 0. \tag{2.4}$$

Proof: Let $t \geq 1$. The first step is to bound the distance $D := D(0, \Phi^0)$ from the origin to its nearest neighbour in Φ by a suitable power of t. For the duration of this proof, define

$$\varepsilon := 1/(d+1); \quad K := \max(2, (2b_d e^2)^{1/(d+1)}).$$

Given $t \geq 1$, set $u := K^{-1}t^{\varepsilon}$ and $\ell := \lfloor Kt^{1-\varepsilon} \rfloor$. Let E be the event that there is a descending chain $0, x_1, \ldots, x_\ell$ in Φ^0 such that $K|x_1| \leq t^{\varepsilon}$ (i.e., $|x_1| \leq u$). Then we assert that the event inclusion

$$\{R(\Phi) > t\} \subset \{2KD > t^{\varepsilon}\} \cup E \tag{2.5}$$

holds. To see this, assume on the contrary that $2KD \leq t^{\varepsilon}$ and that E does not occur. Then any descending chain starting at 0 and with its first link of Euclidean length $\leq 2D \leq K^{-1}t^{\varepsilon}$ would have at most $Kt^{1-\varepsilon} - 1$ links, so would end at a point of Euclidean norm at most $u(Kt^{1-\varepsilon} - 1)$. This would imply that $R(\Phi) \leq uKt^{1-\varepsilon} = t$. Hence (2.5) holds.

For any set A, let $(A)_\ell$ denote the set of ℓ-tuples $(a_1, \ldots, a_\ell) \in A^\ell$ such that $a_1, \ldots, a_\ell$ are distinct. Then, using the ℓth order Palm-Mecke formula for the Poisson process (see e.g. Theorem 1.6 of [11]) similarly to Subsection 3.2 of [3], we have that

$$\begin{aligned}
\mathbb{P}(E) &\leq \mathbb{E} \sum_{(x_1,\ldots,x_\ell)\in(\Phi)_\ell} \mathbf{1}\{u \geq |x_1| \geq |x_2 - x_1| \geq \ldots \geq |x_\ell - x_{\ell-1}|\} \\
&= \int \cdots \int \mathbf{1}\{u \geq |x_1| \geq |x_2 - x_1| \geq \ldots \geq |x_\ell - x_{\ell-1}|\} dx_1 \ldots dx_\ell \\
&= \int \cdots \int \mathbf{1}\{u \geq |x_1| \geq |y_2| \geq \ldots \geq |y_\ell|\} dx_1 dy_2 \ldots dy_\ell \\
&= \frac{(b_d u^d)^\ell}{\ell!}.
\end{aligned} \tag{2.6}$$

Hence by Robbins' bound associated with Stirling's formula (see e.g. [6]), and the definition of u and ℓ, and the fact that $K \geq 2$,

$$\mathbb{P}(E) \leq \frac{b_d^\ell u^{\ell d} e^\ell}{\ell^\ell (2\pi)^{1/2}} \leq \left(\frac{b_d t^{\varepsilon d} e}{K^d(Kt^{1-\varepsilon} - 1)}\right)^\ell \leq \left(\frac{2b_d t^{\varepsilon d} e}{K^{d+1}t^{1-\varepsilon}}\right)^\ell.$$

By definition, $\varepsilon d = 1 - \varepsilon$, and $K^{d+1} \geq 2b_d e^2$ so that

$$\mathbb{P}(E) \leq e^{-\ell} \leq \exp(-(K/2)t^{d/(d+1)}).$$

Returning to (2.5) and using the definition of ε again, we see that

$$\mathbb{P}(R(\Phi) > t) \leq \exp(-b_d(2K)^{-d}t^{d/(d+1)}) + \exp(-(K/2)t^{d/(d+1)}), \tag{2.7}$$

and therefore (2.4) holds for suitably chosen c_1. □

Next we extend Lemma 2.4 to the family of binomial point processes

$$\chi_{n,m} := \{n^{1/d}X_i : 1 \leq i \leq m\}, \qquad n, m \in \mathbb{N}. \tag{2.8}$$

where $X_1, X_2, \ldots$ is a sequence of independent random d-vectors uniformly distributed over the convex set W (which was introduced in Section 1).

Lemma 2.5. *There is a constant $c_2 > 0$, dependent only on the choice of W, such that for all $n \in \mathbb{N}$ with $n \geq 4$, and all $x \in W_n$, and all $m \in [n/2, 3n/2]$,*

$$\mathbb{P}(R(-x + \chi_{n,m-1}) > t) \leq c_2 \exp(-c_2^{-1} t^{d/(d+1)}), \quad t > 0. \tag{2.9}$$

In proving this, and again later, we shall use the fact that there is a constant $c_0 > 0$, dependent only on W, such that

$$c_0^{-1} r^d \leq V_d(W_n \cap B_r(x)), \qquad n \in \mathbb{N}, x \in W_n, r \in (0, n^{1/d}\mathrm{diam}(W)]. \tag{2.10}$$

To see this, take $u > 0$ such that $B_{2u}(0) \subset W$. For $x \in W$, and $0 < s \leq u$, the convex hull of $\{x\} \cup B_{2u}(0)$ is contained in W, and the intersection of this with $B_s(x)$ contains the intersection of $B_s(x)$ and a cone with apex at x and subtended angle at least $\arcsin(u/\mathrm{diam}(W))$, and hence has volume at least $c^{-1}s^d$ for some constant c independent of x and s. Therefore

$$\inf\{s^{-d} V_d(W \cap B_s(x)) : x \in W, s \in (0, u]\} > 0. \tag{2.11}$$

Since $s^{-d}V_d(W \cap B_s(x))$ is continuous in (s, x), it is bounded away from zero on (s, x) in the compact set $[u, \mathrm{diam}(W)] \times W$, and therefore (2.11) still holds with the range of s extended to $(0, \mathrm{diam}(W)]$, and then (2.10) follows by setting $x = n^{-1/d}y$ and scaling.

Proof of Lemma 2.5. By (2.10), for all $n \geq 4, x \in W_n$, $m \geq n/2$ (so that in particular $m - 1 \geq n/4$) and $r \in (0, n^{1/d}\mathrm{diam}(W)]$, we have

$$\begin{aligned}\mathbb{P}(D(x, \chi_{n,m-1}) \geq r) \leq (1 - c_0^{-1} n^{-1} r^d)^{m-1} &\leq (\exp(-c_0^{-1} n^{-1} r^d))^{m-1} \\ &\leq \exp(-(4c_0)^{-1} r^d),\end{aligned} \tag{2.12}$$

and this holds trivially for $r > n^{1/d}\mathrm{diam}(W)$ as well.

The proof of Lemma 2.5 now mostly follows that of Lemma 2.4. There is a difference in the first term in the right side of (2.7), where we now need to estimate the probability that the nearest point to x in $\chi_{n,m-1}$ is at a distance greater than $(2K)^{-1}t^{\varepsilon}$. For this we can use (2.12).

Instead of the event E featuring in the proof of Lemma 2.4, we now need to consider E', defined to be the event that there is a descending chain $(x, x_1, \ldots, x_\ell)$ in $\chi^x_{n,m-1}$ with $|x_1 - x| \leq u$. Instead of the estimate (2.6), setting $Y_i := n^{1/d}X_i$ we now have

$$\begin{aligned}\mathbb{P}(E') &\leq \mathbb{E} \sum_{(i_1,i_2,\ldots,i_\ell) \in (\{1,2,\ldots,m\})_\ell} \mathbf{1}\{u \geq |Y_{i_1}| \geq |Y_{i_2} - Y_{i_1}| \geq \cdots \geq |Y_{i_\ell} - Y_{i_{\ell-1}}|\} \\ &= \frac{m!}{(m-\ell)!} \int_{(n^{1/d}W)^\ell} \mathbf{1}\{u \geq |x_1| \geq |x_2 - x_1| \geq \cdots \geq |x_\ell - x_{\ell-1}|\} n^{-\ell} dx_1 \ldots dx_\ell \\ &\leq 2^\ell \int \cdots \int \mathbf{1}\{u \geq |x_1| \geq |x_2 - x_1| \geq \ldots \geq |x_\ell - x_{\ell-1}|\} dx_1 \ldots dx_\ell\end{aligned} \tag{2.13}$$

because $m \leq 2n$.

With these changes, we can complete the proof by following the proof of Lemma 2.4; it is easy to modify the argument to allow for the extra factor of 2^ℓ in (2.13) compared to (2.6). ∎

3 External stabilization

In this section we introduce the notion of a *fence*. Loosely speaking, given an annulus in $\mathbb{R}^d$, a configuration $\varphi \in \mathbb{N}$ has a fence if there are enough points in the annulus to guarantee that no lilypond grain centred inside the annulus can penetrate too far outside, and no lilypond grain centred outside the annulus can penetrate too far inside. Combining this with with the notion of a radius of stabilization $R(\varphi)$ as already considered, we shall arrive at a stronger *external* stabilization radius, denoted $R_{\rm ex}(\varphi)$, with similar tail behaviour. Loosely speaking, external stabilization means that changes to φ beyond distance $R_{\rm ex}(\varphi)$ do not affect the grains near the origin, and changes near the origin do not affect the grains beyond distance $R_{\rm ex}(\varphi)$.

Let $x \in \mathbb{R}^d$ and $0 < r < s$. Let $B^o_r(x)$ be the open ball of radius r centred at x, and set $\partial B_s(x) := B_s(x) \setminus B^o_s(x)$, the boundary of $B_s(x)$. Let $(e_1, e_2, e_3, \ldots)$ be an arbitrarily chosen sequence forming a countable dense set in $\partial B_1(0)$. Let $k(s,r)$ be the smallest positive integer k such that there exists an increasing sequence $(j_1, j_2, \ldots, j_k)$ of positive integers, such that

$$\partial B_s(0) \subset \cup_{i=1}^k B^o_r(s e_{j_i}) \tag{3.1}$$

(such a k exists by compactness). Note that

$$k(s,r) = k(s/r, 1), \quad 0 < r < s. \tag{3.2}$$

In other words, $k(s,r)$ depends on (s,r) only through the ratio of s to r.

Let $x \in \mathbb{R}^d$ and $0 < r < s$. Setting $k = k(s,r)$, let $j_1, \ldots, j_k$ be the first sequence of positive integers, according to the lexicographic ordering, such that (3.1) holds. That is, for $0 \le i \le k-1$, having defined $j_1, \ldots, j_i$ let j_{i+1} be the first positive integer j such that $\{j_1 \ldots, j_i, j\}$ can be extended to a set of cardinality k in $\mathbb{N}$ which satisfies (3.1).

For any $x \in \mathbb{R}^d$, if we set $y_i = x + s e_{j_i}$, then the points $y_1, \ldots, y_k$ in $\partial B_s(x)$ satisfy

$$\partial B_s(x) \subset \cup_{i=1}^k B^o_r(y_i).$$

With $y_1, \ldots, y_k$ defined thus, define

$$F(x,s,r) := \cap_{i=1}^k \{\varphi \in \mathbf{N} : \varphi(B^o_r(y_i) \setminus B_s(x)) \ge 2\}. \tag{3.3}$$

In the case $x = 0$ we write simply $F(s,r)$ for $F(0,s,r)$. We think of $F(x,s,r)$ as a set of configurations φ containing a fence in the annulus $B_{s+r}(x) \setminus B_s(x)$. We formalize the fence property as follows:

Lemma 3.1. *Suppose* $0 < r < s < \infty$ *with* $s \ge 2r$, *and let* $x \in \mathbb{R}^d$. *Suppose* $\varphi \in F(x,s,r)$. *Then for any* $z \in \varphi \setminus B^o_s(x)$ *we have* $B_{\rho(z,\varphi)}(z) \cap B_{s-2r}(x) = \emptyset$, *and for any* $y \in \varphi \cap B_s(x)$ *we have* $B_{\rho(y,\varphi)}(y) \subset B^o_{s+2r}(x)$.

Proof. Let $z \in \varphi \setminus B^o_s(x)$. The line segment from z to x includes a point w in $\partial B_s(x)$. Then w lies in at least one of the balls $B^o_r(y_i)$, and since we assume $\varphi \in F(x,s,r)$, there exists $u \in \varphi \cap B^o_r(y_i) \setminus \{z\}$ such that $|u - w| < 2r$. Hence by the triangle inequality, $|z - u| < |z - w| + 2r$. Therefore, since $\rho(z,\varphi) \le |z-u|$, the grain $B_{\rho(z,\varphi)}(z)$ does not intersect $B_{s-2r}(x)$, as asserted.

Let $y \in \varphi \cap B_s(x)$. Take v in $\partial B_s(x)$, such that y lies in the line segment from v to x. Since $\varphi \in F(x,s,r)$, there exists $t \in \varphi \setminus \{y\}$ such that $|t-v| < 2r$. Hence by the triangle inequality, $\rho(y,\varphi) \le |y-t| < |y-v| + 2r$, so that $B_{\rho(y,\varphi)}(y) \subset B^o_{s+2r}(x)$, as asserted. □

Recall that $W \subset \mathbb{R}^d$ is convex and compact with $V_d(W) = 1$, containing an open neighborhood of the origin, and $W_n := n^{1/d} W_n$ for $n \in \mathbb{N}$. For proving results on point processes in W_n, we introduce some further notation. Given $0 < r < s$, given $n \in \mathbb{N}$ and $x \in W_n$, we shall define $F_n(x,s,r)$ similarly to $F(x,s,r)$ but now with the fence involving only regions intersecting W_n. First we define the points $y_i = x + s e_{j_i}$ for $1 \le i \le k(s,r/2)$, similarly to the points y_i in the definition of $F(x,s,r)$, such that

$$\partial B_s(x) \subset \cup_{i=1}^{k(s,r/2)} B^o_{r/2}(y_i). \tag{3.4}$$

Let $z_1, \ldots, z_{k'}$ be those y_i such that $B_{r/2}(y_i) \cap W_n \neq \emptyset$. Note that $k' \le k(s,r/2)$. Set

$$F_n(x,s,r) := \cap_{i=1}^{k'} \{\varphi \in \mathbf{N} : \varphi(B^o_r(z_i)) \ge 2\} \cap \{\varphi \in \mathbf{N} : \varphi \subset W_n\}. \tag{3.5}$$

Lemma 3.2. *Suppose $\varphi \in \mathbf{N}$ and $\psi \in \mathbf{N}$ (possibly with $\psi = \varphi$). Let $r, s, t > 0$ with $s \ge 2r$ and $t \ge s + 4r$, and $x \in \mathbb{R}^d$. Suppose $y \in \varphi \cap B_s(x)$ and $z \in \psi \setminus B_t(x)$ Then:*
(i) If $\varphi \in F(x,s,r)$ and $\psi \in F(x,t,r)$ then $|y-z| > \rho(y,\varphi) + \rho(z,\psi)$.
(ii) If $\varphi \in F_n(x,s,r)$ and $\psi \in F_n(x,t,r)$ then $|y-z| > \rho(y,\varphi) + \rho(z,\psi)$.

Proof. To prove (i), suppose $\varphi \in F(x,s,r)$ and $\psi \in F(x,t,r)$. Then by Lemma 3.1, since $t - s \ge 4r$ we have $B_{\rho(y,\varphi)} \subset B^o_{s+2r}(x)$ and $B_{\rho(z,\psi)} \cap B_{t-2r}(x) = \emptyset$, so that $B_{\rho(y,\varphi)}(y) \cap B_{\rho(z,\psi)}(z) = \emptyset$, establishing part (i).

To prove (ii), suppose instead that $\varphi \in F_n(x,s,r)$ and $\psi \in F_n(x,t,r)$. Let u and v be the points on the line segment yz such that $|u-x| = s$ and $|v-x| = t$. Since W_n is convex, u and v are in W_n. Since $u \in \partial B_s(x)$, by (3.4) in the definition of $F_n(x,s,r)$ we have $u \in B_{r/2}(y_i)$ for some i, but then since also $u \in W_n$ we have $y_i = z_j$ for some $j \le k'$. Hence since $\varphi \in F_n(x,s,r)$, by (3.5), there exists $w \in \varphi \cap B^o_r(z_j) \setminus \{y\}$, and by the triangle inequality

$$|w-y| \le |u-y| + |z_j - u| + |w - z_j| \le |u-y| + 3r/2.$$

Similarly, since $v \in W_n \cap \partial B_t(x)$ and $\psi \in F_n(x,t,r)$ we can find $w' \in \psi \setminus \{z\}$ such that $|w'-z| \le |v-z| + 3r/2$. Hence by the hard-core property,

$$\rho(y,\varphi) + \rho(z,\psi) \le |w-y| + |w'-z| \le |u-y| + |v-z| + 3r,$$

whereas since y, u, v, z are collinear and $|u-v| \ge t - s \ge 4r$, we have

$$|y-z| \ge |u-y| + |v-z| + 4r,$$

and part (ii) follows. □

We now give some probability estimates for the point sets $F(x,s,r)$ and $F_n(x,s,r)$.

Lemma 3.3. *There exists $c_3 \in (0,\infty)$ such that if $0 < r < 2r < s$ and $x \in \mathbb{R}^d$, then:*
(i) $\mathbb{P}(\Phi \notin F(x,s,r)) \le k(s,r) c_3 \exp(-c_3^{-1} r^d)$.
(ii) If $m, n \in \mathbb{N}$ and $n \ge 4$ and $m \in [n/2, 3n/2]$ and $x \in W_n$, then $\mathbb{P}(\chi_{n,m} \notin F_n(x,s,r)) \le k(s,r) c_3 \exp(-c_3^{-1} r^d)$.

Proof. (i) With $y_1, \ldots, y_k$ as in the definition (3.3) of $F(x, s, r)$, note that $V_d(B_r^o(y_i) \setminus B_s(x)) \geq (b_d/2)r^d$, so by subadditivity of measure,

$$\mathbb{P}(\Phi \notin F(x, s, r)) \leq k(s, r)(1 + (b_d/2)r^d) \exp(-(b_d/2)r^d), \tag{3.6}$$

and part (i) follows.

For part (ii), first we claim that there is a constant $c > 0$, independent of n and x, such that if $n \geq 4$ and $0 \leq r \leq n^{1/d}\mathrm{diam}(W)$, then with k' and z_i as defined just before (3.5) we have

$$V_d(B_r(z_i) \cap W_n) \geq c^{-1}r^d, \quad 1 \leq i \leq k'. \tag{3.7}$$

Indeed, given $i \leq k'$, if we choose $y \in B_{r/2}(z_i) \cap W_n$, then $B_{r/2}(y) \subset B_r(z_i)$ so the claim follows from (2.10).

By (3.7), subadditivity and the fact that the binomial distribution is stochastically increasing in the success probability, we have

$$\mathbb{P}(\chi_{n,m} \notin F_n(x, s, r)) \leq k(s, r)[(1 - r^d/(nc))^m + m(r^d/(nc))(1 - r^d/(nc))^{m-1}].$$

Hence by the inequality $1 - t \leq e^{-t}$, there is a further constant c' such that for all $n \geq 4$ and $m \in [n/2, 3n/2]$, (so that in particular $m - 1 \geq n/4$ and $m \leq 2n$), for all $x \in W_n$ and $0 < r < s$ with $r \leq n^{1/d}\mathrm{diam}(W)$,

$$\begin{aligned} \mathbb{P}[\chi_{n,m} \notin F_n(x, s, r)] &\leq k(s, r)(1 + mr^d/(cn))(1 - r^d/(cn))^{m-1} \\ &\leq k(s, r)(1 + (2/c)r^d) \exp(-(4c)^{-1}r^d) \\ &\leq k(s, r)c' \exp(-(c')^{-1}r^d). \end{aligned} \tag{3.8}$$

Moreover, if $r > n^{1/d}\mathrm{diam}(W)$ then $s - r > n^{1/d}\mathrm{diam}(W)$ so $k' = 0$, and then trivially (3.8) still holds. This gives us part (ii). □

For $\varphi \in \mathbf{N}$ and $r > 0$, define the set

$$S_r^*(\varphi) := \cup_{x \in \varphi \cap B_{7r}(0) \setminus B_{2r}(0)} S(x, \varphi). \tag{3.9}$$

Lemma 3.4. *Let $r > 0$. Suppose $\varphi \in F(2r, r/2) \cap F(4r, r/2) \cap F(7r, r/2)$, and suppose $S_r^*(\varphi)$ is bounded. Let $\varphi_{\mathrm{in}} \in \mathbf{N}, \varphi_{\mathrm{out}} \in \mathbf{N}$ be such that*

$$\varphi_{\mathrm{in}} \subset B_{2r}(0) \setminus S_r^*(\varphi); \qquad \varphi_{\mathrm{out}} \subset \mathbb{R}^d \setminus (B_{7r}(0) \cup S_r^*(\varphi)).$$

Then

$$\rho(x, \varphi \cup \varphi_{\mathrm{in}}) = \rho(x, \varphi \cup \varphi_{\mathrm{in}} \cup \varphi_{\mathrm{out}}), \qquad x \in \varphi_{\mathrm{in}} \cup (\varphi \cap B_{7r}(0)); \tag{3.10}$$

$$\rho(x, \varphi \cup \varphi_{\mathrm{out}}) = \rho(x, \varphi \cup \varphi_{\mathrm{in}} \cup \varphi_{\mathrm{out}}), \qquad x \in \varphi_{\mathrm{out}} \cup (\varphi \setminus B_{2r}(0)). \tag{3.11}$$

Proof. First note that by Lemma 2.2,

$$\rho(x, \varphi \cup \varphi_{\mathrm{in}}) = \rho(x, \varphi \cup \varphi_{\mathrm{out}}) = \rho(x, \varphi), \qquad x \in \varphi \cap B_{7r}(0) \setminus B_{2r}(0). \tag{3.12}$$

Hence, we can and do consistently define $\rho'(x)$ for all $x \in \varphi \cup \varphi_{\rm in} \cup \varphi_{\rm out}$, by

$$\rho'(x) = \rho(x, \varphi \cup \varphi_{\rm in}), \quad x \in \varphi_{\rm in} \cup (\varphi \cap B_{7r}(0)); \tag{3.13}$$

$$\rho'(x) = \rho(x, \varphi \cup \varphi_{\rm out}), \quad x \in \varphi_{\rm out} \cup (\varphi \setminus B_{2r}(0)). \tag{3.14}$$

Assign grain radius $\rho'(x)$ to each $x \in \varphi \cup \varphi_{\rm in} \cup \varphi_{\rm out}$. We assert that with grain radii assigned in this way, each $x \in \varphi \cup \varphi_{\rm in} \cup \varphi_{\rm out}$ has a smaller grain-neighbour in $\varphi \cup \varphi_{\rm in} \cup \varphi_{\rm out}$,

To verify this assertion, first suppose $x \in B_{4r}(0)$. Then by the defining properties of the lilypond model, there exists $y \in \varphi \cup \varphi_{\rm in}$ such that y is a smaller grain-neighbour of x under the radii $\rho(\cdot, \varphi \cup \varphi_{\rm in})$. Moreover, since $\varphi \in F(4r, r/2) \cap F(7r, r/2)$, if $y \notin B_{7r}(0)$ then by Lemma 3.2, we would have $|x - y| > \rho(x, \varphi \cup \varphi_{\rm in}) + \rho(y, \varphi \cup \varphi_{\rm in})$, contradicting the statement that y is a grain-neighbour of x. Therefore $y \in B_{7r}(0)$, so by (3.13) we have $\rho'(y) = \rho(y, \varphi \cup \varphi_{\rm in})$ (and likewise for x). Therefore y is also a smaller grain-neighbour of x using the radii $\rho'(\cdot)$ as asserted.

Now suppose instead that $x \notin B_{4r}(0)$. Then there exists $y \in \varphi \cup \varphi_{\rm out}$ such that y is a smaller grain-neighbour of x under the radii $\rho(\cdot, \varphi \cup \varphi_{\rm out})$. Moreover, since $\varphi \in F(4r, r/2) \cap F(2r, r/2)$, if $y \in B_{2r}(0)$ then by Lemma 3.2, we would have $|x - y| > \rho(x, \varphi \cup \varphi_{\rm out}) + \rho(y, \varphi \cup \varphi_{\rm out})$ contradicting the statement that y is a grain-neighbour of x. Therefore $y \notin B_{2r}(0)$, so by (3.14) we have $\rho'(y) = \rho(y, \varphi \cup \varphi_{\rm out})$ (and likewise for x). Therefore y is also a smaller grain-neighbour of x using the radii $\rho'(\cdot)$ as asserted.

We shall show that the radii $\rho'(x), x \in \varphi \cup \varphi_{\rm in} \cup \varphi_{\rm out}$, have the hard-core property. This will suffice to give $\rho'(x) = \rho(x, \varphi \cup \varphi_{\rm in} \cup \varphi_{\rm out})$ for all $x \in \varphi \cup \varphi_{\rm in} \cup \varphi_{\rm out}$ as required, because as already mentioned, the lilypond model is the *unique* set of radii satisfying the hard-core and smaller grain-neighbour properties [9].

Let x, y be distinct elements of $\varphi \cup \varphi_{\rm in} \cup \varphi_{\rm out}$, with $|x| \leq |y|$. We consider separately the case with $|y| \leq 7r$, the case with $|x| > 2r$, and the case with $|x| \leq 2r$ and $|y| > 7r$. These three cases cover all possibilities.

In the first case with $|y| \leq 7r$, both x and y are in $\varphi_{\rm in} \cup (\varphi \cap B_{7r}(0))$. By (3.13) and the hard-core property of the lilypond model on $\varphi \cup \varphi_{\rm in}$, we have $\rho'(x) + \rho'(y) \leq |x - y|$.

In the second case with $|x| > 2r$, both x and y are in $\varphi_{\rm out} \cup (\varphi \setminus B_{2r}(0))$. By (3.14) and the hard-core property of the lilypond model on $\varphi \cup \varphi_{\rm out}$, we have $\rho'(x) + \rho'(y) \leq |x - y|$.

Now consider the third case with $|x| \leq 2r$ and $|y| > 7r$. In this case we have $x \in \varphi_{\rm in} \cup (\varphi \cap B_{2r}(0))$ and $y \in \varphi_{\rm out} \cup (\varphi \setminus B_{7r}(0))$, so by (3.13) and (3.14), the assumption that $\varphi \in F(2r, r/2) \cap F(7r, r/2)$, and Lemma 3.2, we have that $\rho'(x) + \rho'(y) < |x - y|$.

Hence the radii $\rho'(x), x \in \varphi \cup \varphi_{\rm in} \cup \varphi_{\rm out}$ have the hard-core property as required. □

We have a similar result to Lemma 3.4 in the case of point processes in W_n.

Lemma 3.5. *Let $r > 0$ and $z \in \mathbb{R}^d$. Suppose $\varphi \in F_n(z, 2r, r/2) \cap F_n(z, 4r, r/2) \cap F_n(z, 7r, r/2)$, with $\varphi(\mathbb{R}^d) \geq 2$. Let $\varphi_{\rm in} \in \mathbf{N}, \varphi_{\rm out} \in \mathbf{N}$ be such that*

$$\varphi_{\rm in} \subset W_n \cap B_{2r}(x) \setminus S_r^*(\varphi); \qquad \varphi_{\rm out} \subset W_n \setminus (B_{7r}(x) \cup S_r^*(\varphi)).$$

Then

$$\rho(x, \varphi \cup \varphi_{\rm in}) = \rho(x, \varphi \cup \varphi_{\rm in} \cup \varphi_{\rm out}), \qquad x \in \varphi_{\rm in} \cup (\varphi \cap B_{7r}(z));$$

$$\rho(x, \varphi \cup \varphi_{\rm out}) = \rho(x, \varphi \cup \varphi_{\rm in} \cup \varphi_{\rm out}), \qquad x \in \varphi_{\rm out} \cup (\varphi \setminus B_{2r}(z)).$$

Proof. The proof is just the same as for Lemma 3.4, only using part (ii) instead of part (i) of Lemma 3.2. □

We now define $\mathbf{U}$ to be the set of all $\varphi \in \mathbf{N}$ such that every point of φ has a *unique smaller grain neighbour* under the lilypond model based on φ, or $\varphi(\mathbb{R}^d) \leq 2$.

Recall the definition (2.3) of $R(\varphi)$, $\varphi \in \mathbf{N}$. For $x \in \mathbb{R}^d$, and $r > 0$, $n \in \mathbb{N}$ define subsets $E_r(x), U_r(x), G_r(x)$ and $G_{n,r}(x)$ of $\mathbf{N}$ by

$$\begin{aligned} E_r(x) &:= \{\varphi \in \mathbf{N} : R(-y+\varphi) < r,\ y \in \varphi \cap B_{8r}(x)\}; \\ U_r(x) &:= \{\varphi \in \mathbf{N} : \varphi \cap B_{9r}(x) \in \mathbf{U}\}, \end{aligned}$$

and

$$G_r(x) := E_r(x) \cap U_r(x) \cap \cap_{j=1}^{8} F(x, jr, r/2); \tag{3.15}$$

$$G_{n,r}(x) := E_r(x) \cap U_r(x) \cap \cap_{j=1}^{8} F_n(x, jr, r/2). \tag{3.16}$$

For $\varphi \in \mathbf{N}$, we now define our radius of external stabilization $R_{\text{ex}}(\varphi)$ by

$$R_{\text{ex}}(\varphi) := 9 \min\{r \in \mathbb{N} : \varphi \in G_r(0)\}, \tag{3.17}$$

with $\min(\emptyset)$ taken to be $+\infty$. The next result, in which we write B_r for $B_r(0)$, shows that R_{ex} has the external stabilization property.

Lemma 3.6. *Suppose $r > 0$. Then (i) if $\varphi \in G_r$, then for any $\psi \in \mathbf{N}$ with $\psi(B_{8r}) = 0$, we have:*

$$\rho(y, (\varphi \cap B_{8r}) \cup \psi) = \rho(y, (\varphi^0 \cap B_{8r}) \cup \psi), \qquad y \in \psi \cup (\varphi \cap B_{8r} \setminus B_{2r}); \tag{3.18}$$

$$\rho(x, (\varphi \cap B_{8r}) \cup \psi) = \rho(x, \varphi \cap B_{8r}), \qquad x \in \varphi \cap B_{7r}; \tag{3.19}$$

$$\rho(x, (\varphi^0 \cap B_{8r}) \cup \psi) = \rho(x, \varphi^0 \cap B_{8r}), \qquad x \in \varphi^0 \cap B_{7r}. \tag{3.20}$$

Also, (ii) if $n \in \mathbb{N}$, and $x \in W_n$, and $\varphi \in G_{n,r}(x)$, then

$$\rho(y, \varphi) = \rho(y, \varphi^x), \qquad y \in \varphi \setminus B_{2r}(x) \tag{3.21}$$

Proof. (i) Suppose $\varphi \in G_r$. Then $\varphi \in F(2r, r/2) \cap F(4r, r/2) \cap F(7r, r/2)$, and moreover $\varphi \in E_r(0)$ so the set $S_r^*(\varphi)$ defined by (3.9) is contained in $B_{8r} \setminus B_r$. Also $S_r^*(\varphi) = S_r^*(\varphi \cap B_{8r}(0))$ by Lemma 2.1. Therefore we can apply Lemma 3.4 to $\varphi \cap B_{8r}(0)$; taking $\varphi_{\text{out}} = \psi$ and $\varphi_{\text{in}} = \{0\}$, we obtain (3.18) from (3.11) and (3.20) from (3.10), and taking $\varphi_{\text{in}} = \emptyset$ we obtain (3.19) from (3.10), completing the proof of part (i).

Part (ii) is proved by a similar argument, using Lemma 3.5 instead of Lemma 3.4. □

The next result gives us tail bounds on $R_{\text{ex}}(\Phi)$.

Lemma 3.7. *There is a constant $c_4 \in (0, \infty)$ such that for all $r > 0$,*

$$\mathbb{P}(\Phi \notin G_r(0)) \leq c_4 \exp(-c_4^{-1} r^{d/(d+1)}) \tag{3.22}$$

and for $n \geq 4$ and $m \in [n/2, 3n/2]$ and $y \in W_n$,

$$\mathbb{P}(\chi_{n,m} \notin G_{n,r}(y)) \leq c_4 \exp(-c_4^{-1} r^{d/(d+1)}). \tag{3.23}$$

Proof. By Lemma 5.1 of [3], for all $r > 0$ we have $\mathbb{P}(\Phi \in U_r(0)) = 1$. Also, by the Palm-Mecke equation, see e.g. Theorem 9.22 of [11], and (2.4),

$$\begin{aligned}\mathbb{P}(\Phi \notin E_r(0)) &\le \mathbb{E} \sum_{x \in \Phi \cap B_{8r}(0)} \mathbf{1}\{R(-x+\Phi) \ge r\} \\ &= b_d(8r)^d \mathbb{P}(R(\Phi) \ge r) \le b_d(8r)^d c_1 \exp(-c_1^{-1} r^{d/(d+1)}). \end{aligned} \tag{3.24}$$

Also, by Lemma 3.3 (i) and (3.2),

$$\mathbb{P}(\Phi \notin \cap_{j=1}^{8} F(jr, r/2)) \le \sum_{j=1}^{8} k(2j,1) c_3 \exp(-c_3^{-1} r^d)$$

and combined with (3.24) this gives (3.22) for a suitable choice of c_4.

Next, suppose $n \ge 4$ and $n/2 \le m \le 3n/2$, and $y \in W_n$. Then by Lemma 2.5,

$$\begin{aligned}\mathbb{P}(\chi_{n,m} \notin E_r(y)) &\le \frac{m}{n} \int_{B_{6r}(y)} \mathbb{P}(R(-x+\chi_{n,m-1}) > r) dx \\ &\le (3/2) c_2 \exp(-c_2^{-1} r^{d/(d+1)}). \end{aligned} \tag{3.25}$$

Combined with Lemma 3.3 (ii) and (3.2), this gives us (3.23). □

4 Sub-exponential decay

In this section we give tail bounds on the distribution of the size of the component of the Poisson lilypond model containing a typical Poisson point, with 'size' measured either by cardinality, or by metric diameter, or by volume.

For $\varphi \in \mathbf{N}$, define as follows the directed graph $\mathcal{G}(\varphi) = (\varphi, E(\varphi))$ with vertex set φ and edge set $E(\varphi)$. A pair (x, y) is in $E(\varphi)$ if y is a smaller grain-neighbour of x. Let $\mathcal{G}^*(\varphi)$ denote the associated undirected graph. For $x \in \varphi$, let $C(x, \varphi)$ denote the cluster at x, that is, the set of points of φ that are connected to x by a path in the undirected graph $\mathcal{G}^*(\varphi)$. Let $C'(x, \varphi)$ denote the union of lilypond grains centred at points of $C(x, \varphi)$, i.e., the connected component containing x of the set $\cup_{x \in \varphi} B_{\rho(x,\varphi)}(x)$.

Theorem 4.1. *There are strictly positive constants c_5, c_6, c_7 such that*

$$c_5^{-1} \exp(-c_5 r^d) \le \mathbb{P}(\mathrm{diam}(C'(0, \Phi^0)) \ge r) \le c_5 \exp(-c_5^{-1} r^{d/(d+1)}), \quad r > 0; \tag{4.1}$$
$$c_6^{-1} \exp(-c_6 t) \le \mathbb{P}(V_d(C'(0, \Phi^0)) \ge t) \le c_6 \exp(-c_6^{-1} t^{1/(d+1)}), \quad t > 0; \tag{4.2}$$
$$c_7^{-1} \exp(-c_7 n^2) \le \mathbb{P}(\mathrm{card}\, C(0, \Phi^0) \ge n) \le c_7 \exp(-c_7^{-1} n^{d/(d+1)}), \quad n > 0. \tag{4.3}$$

Note that in each of (4.1), (4.2) and (4.3) the power of r, t or n in the exponent is different in the lower bound than in the upper bound. It is an open problem to make these bounds sharper.

Sharper bounds are available in the analogous setting for lattice and continuum percolation. Consider for example the geometric graph on Φ^0, with each pair of points connected by an edge if and only if the distance between them is less than a constant r^*, with r^* chosen to be subcritical. Then results like (4.3) and (4.1) hold with exponents of the form $c \cdot n$, respectively $c \cdot r$, in both the upper and lower bound, though not necessarily both with the same c (see Section 10.1 of [11]). A result like (4.1) holds for subcritical lattice percolation with exponents of the form $c \cdot r$; see (6.10) of [7]. This implies a bound like (4.2) for the Boolean model associated with the subcritical geometric graph just mentioned, with $c \cdot t$ in the exponent. A similar lower bound also holds.

The next lemma will be used in proving Theorem 4.1, and again later. Recall the definitions (3.15) and (3.16) of $G_r(x)$ and $G_{n,r}(x)$ respectively.

Lemma 4.2. *Let $r > 0$ and $x \in \mathbb{R}^d$.*

(i) If $\varphi \in G_r(x)$, then $C(y, \varphi) \subset B_{5r}(x)$ for all $y \in \varphi \cap B_{3r}(x)$, and

$$\bigcup_{y \in \varphi \cap B_{3r}(x)} C'(y, \varphi) \subset B_{6r}(x). \tag{4.4}$$

(ii) If $n \in \mathbf{N}$, and $x \in W_n$, and $\varphi \in G_{n,r}(x)$, then $C(y, \varphi) \subset B_{5r}(x)$.

Proof. (i) First, we assert that each $y \in \varphi \cap B_{5r}(x)$ has a unique smaller grain neighbour in φ. Indeed, y does not have any grain-neighbour (in either φ or $\varphi \cap B_{9r}(x)$) outside $B_{7r}(x)$, by Lemma 3.2 because $\varphi \in F(x, 5r, r/2) \cap F(x, 7r, r/2)$. Also, $\rho(u, \varphi) = \rho(u, \varphi \cap B_{9r}(9x))$ for all $u \in \varphi \cap B_{8r}(x)$ by Lemmas 2.1 and 2.2, because $\varphi \in E_r(x)$. Hence, the unique smaller grain neighbour of y in $\varphi \cap B_{9r}(x)$ (which it has because $\varphi \in G_r(x) \subset U_r(x)$) is also its unique smaller grain neighbour in φ, justifying the assertion.

Hence, in the graph $\mathcal{G}(\varphi)$, each vertex inside $B_{5r}(x)$ has an out-degree of 1. This implies that for any path in $\mathcal{G}^*(\varphi)$ starting inside $B_{3r}(x)$ and ending outside $B_{5r}(x)$ but with all vertices except the last inside $B_{5r}(x)$, if the direction of the edges in $\mathcal{G}(\varphi)$ is taken into consideration the path can reverse its direction at most once. That is, such a path must consist of a directed path in the forward direction (possibly of zero length), followed by a directed path in the reverse direction (also possibly of zero length). Hence, if $y \in \varphi \cap B_{3r}(x)$ and $C(y, \varphi)$ is not contained in $B_{5r}(x)$, then either there is a descending grain-chain in φ starting inside $B_{3r}(x)$ and ending outside $B_{4r}(x)$, or there is a descending grain-chain in φ starting outside $B_{5r}(x)$ and ending inside $B_{4r}(x)$.

Since $\varphi \in E_r(x)$, we have no descending chain starting inside $B_{3r}(x)$ and ending outside $B_{4r}(x)$ or starting in $B_{6r}(x) \setminus B_{5r}(x)$ and ending inside $B_{4r}(x)$. Moreover, since $\varphi \in F(x, 4r, r/2) \cap F(x, 6r, r/2)$, by Lemma 3.2 there is no edge of $\mathcal{G}(\varphi)$ with one endpoint outside $B_{6r}(x)$ and the other endpoint inside $B_{4r}(x)$. This shows that $C(y, \varphi) \subset B_{5r}(x)$, and since also $\varphi \in F(x, 5r, r/2)$, by Lemma 3.1 we have (4.4).

The proof of (ii) is similar. □

Proof of (4.1). By Lemmas 4.2 and 3.7, we have for $r > 0$ that

$$\mathbb{P}(\{C'(0, \Phi^0) \subset B_{6r}(0)\}^c) \le \mathbb{P}(\Phi \notin G_r(0)) \le c_4 \exp(-c_4^{-1} r^{d/(d+1)}), \tag{4.5}$$

and the upper bound in (4.1) follows.

For the lower bound, observe that for all $r > 0$, we have

$$\mathbb{P}[\mathrm{diam}(C'(0, \Phi^0)) \ge r] \ge \mathbb{P}[\Phi(B_{2r}(0)) = 0] = \exp(-b_d (2r)^d). \quad \square$$

Proof of (4.3). Choosing r so that $2b_d(6r)^d = n$, by (4.5) and a standard Chernoff-type tail estimate for the Poisson distribution (see e.g. Lemma 1.2 of [11]) there are constants c, c' such that for $n \geq 1$ we have

$$\mathbb{P}[\mathrm{card}(C(0,\Phi^0)) \geq n] \leq c \exp(-c^{-1} r^{d/(d+1)}) + \mathbb{P}[\Phi^0(B_{6r}(0)) \geq 2b_d(6r)^d] \leq c' \exp(-(1/c')n^{1/(d+1)})$$

and the upper bound in (4.3) follows.

Let e be the unit vector $(1,0,\ldots,0)$ in $\mathbb{R}^d$. For $i \in \mathbb{N}$, let B_i be the closed ball in $\mathbb{R}^d$, centred on $9^{-i}e$ and having radius 9^{-i-2}. Since $9^{-(i-1)} - 9^{-i} = 8(9^{-i})$, for all $i \geq 2$, $x \in B_i$ and $y \in B_{i-1}$ we have $7(9^{-i}) \leq |x-y| \leq 9^{-(i-1)}$ and also $|x| \leq 10(9^{-i-1})$.

Observe that there is a positive constant c such that for all $n \in \mathbf{N}$ we have

$$\mathbb{P}(\{\Phi(B_9(0)) = n\} \cap \cap_{i=1}^n \{\Phi(B_i) = 1\}) = \left(\prod_{i=1}^n (b_d 9^{-(i+2)d})\right) \exp(-b_d 9^d) \geq c^{-1} \exp(-cn^2). \quad (4.6)$$

If the event inside the left hand side of (4.6) occurs, then labelling the point of $\Phi \cap B_i$ as x_i, each point x_i for $1 \leq i \leq n$ has a smaller grain-neighbour in Φ^0 to its left in the collection $\{0, x_1, x_2, \ldots, x_n\}$. Indeed, every point in this collection to the right of x_i, and also every point of $\Phi \setminus B_9(0)$, is distant more than $2D(x_i, \Phi^0)$ from x_i, so cannot be its smaller grain neighbour.

Therefore, if the event inside the left hand side of (4.6) occurs, then $x_1, \ldots, x_n$ are all in $C(0,\Phi^0)$, so that in this case $\mathrm{card}\, C(0,\Phi^0) \geq n+1$. Hence, the lower bound in (4.3) holds by (4.6). $\square$

Proof of (4.2). Taking r so that $b_d r^d = t$ we have by (4.1) that

$$\mathbb{P}[V_d(C'(0,\Phi^0)) > t] \leq c_5 \exp(-c_5^{-1} r^{d/(d+1)}) = c_5 \exp(-c_5^{-1}(t/b_d)^{1/(d+1)})$$

while for the lower bound, we have

$$\mathbb{P}[V_d(C'(0,\Phi^0)) \geq t] \geq \mathbb{P}[\Phi(B_{2r}(0)) = 0] = \exp(-2^d t). \quad \square$$

5 Central limit theorems

In this section we derive central limit theorems (CLTs) associated with the lilypond model. We consider a sequence of binomial or Poisson point processes with finite total number of points over an expanding sequence of bounded regions in $\mathbb{R}^d$. Thus we consider, for example, $V_d(Z(\Phi_n))$ as defined in Section 1, rather than $V_d(Z(\Phi) \cap W_n)$. Our choice means that we can directly apply results in [14] or [13], although it should be possible to obtain similar results for $V_d(Z(\Phi) \cap W_n)$.

For $n, m \in \mathbf{N}$, recall the definition of $X_1, X_2, \ldots$ and $\chi_{n,m}$ at (2.8). Set $W_n := n^{1/d}W$. We consider the restricted Poisson process $\Phi_n := \Phi \cap W_n$, and also the binomial point process $\chi_n := \chi_{n,n}$. For measurable $g : \mathbb{R}_+ \to \mathbb{R}$, we give CLTs for sums of the form $H_g(\Phi_n)$ and $H_g(\chi_n)$, where for all $\varphi \in \mathbf{N}$ with $2 \leq \varphi(\mathbb{R}^d) < \infty$ we define

$$H_g(\varphi) := \sum_{x \in \varphi} g(\rho(x,\varphi)),$$

and if $\varphi(\mathbb{R}^d) \in \{0,1\}$ we set $H_g(\varphi) := 0$. For example if $g(t) = b_d t^d$, then by the hard-core property $H_g(\Phi_n)$ is the total volume $V_d(Z(\Phi_n))\mathbf{1}_{\{\Phi(W_n)\geq 2\}}$ and $H_g(\chi_n)$ is $V_d(Z(\chi_n))$ for $n \geq 2$.

Theorem 5.1. *Suppose that there exists finite $\beta > 0$ such that $g : \mathbb{R}_+ \to \mathbb{R}$ satisfies the growth bound*

$$\sup_{t\in\mathbb{R}_+} (1+t^\beta)^{-1}|g(t)| < \infty, \tag{5.1}$$

and that g is not Lebesgue-almost everywhere constant. Then there exist constants $0 < \tau_g^2 \leq \sigma_g^2 < \infty$ (dependent on g but independent of the choice of the convex set W) such that as $n \to \infty$ we have $n^{-1}\mathrm{Var}(H_g(\Phi_n)) \to \sigma_g^2$ and $n^{-1}\mathrm{Var}(H_g(\chi_n)) \to \tau_g^2$, and

$$n^{-1/2}(H_g(\Phi_n) - \mathbb{E}H_g(\Phi_n)) \xrightarrow{\mathcal{D}} N(0, \sigma_g^2); \tag{5.2}$$

$$n^{-1/2}(H_g(\chi_n) - \mathbb{E}H_g(\chi_n)) \xrightarrow{\mathcal{D}} N(0, \tau_g^2). \tag{5.3}$$

The next theorem provides a similar CLT for the number of components. Recall that $\kappa(A)$ (for $A \subset \mathbb{R}^d$) denotes the number of connected components of A; for $\varphi \in \mathbf{N}$ with $2 \leq \varphi(\mathbb{R}^d) < \infty$ we define

$$H_\kappa(\varphi) := \kappa(Z(\varphi))$$

which is also the number of components of the graph $\mathcal{G}^*(\varphi)$, and if $\varphi(\mathbb{R}^d) = 1$ we set $H_\kappa(\varphi) := 1$, and if $\varphi(\mathbb{R}^d) = 0$ we set $H_\kappa(\varphi) := 0$.

Theorem 5.2. *There are constants $0 < \tau_\kappa^2 \leq \sigma_\kappa^2 < \infty$ such that $n^{-1}\mathrm{Var}(H_\kappa(\Phi_n)) \to \sigma_\kappa^2$ and $n^{-1}\mathrm{Var}(H_\kappa(\chi_n)) \to \tau_\kappa^2$, and*

$$n^{-1/2}(H_\kappa(\Phi_n) - \mathbb{E}H_\kappa(\Phi_n)) \xrightarrow{\mathcal{D}} N(0, \sigma_\kappa^2); \tag{5.4}$$

$$n^{-1/2}(H_\kappa(\chi_n) - \mathbb{E}H_\kappa(\chi_n)) \xrightarrow{\mathcal{D}} N(0, \tau_\kappa^2). \tag{5.5}$$

We shall prove Theorems 5.1 and 5.2 by using results from [14] which have the advantage of showing that the limiting variance is non-zero. An alternative would be to use results from [13] (an approach based instead on sub-exponential stabilization) which could be used to give a formula for σ^2 in terms of integrated two-point correlation functions, and also to provide Gaussian limits for the random measures associated with the sums $H_g(\Phi_n)$ and $H_g(\chi_n)$; moreover, using Theorem 2.5 of [15], it should be possible to provide Berry-Esseen type error bounds associated with (5.4) converging to zero at rate $O(n^{\varepsilon-1/2})$ for any $\varepsilon > 0$. However, we do not give details of these alternative approaches here.

The approach of [14] is based on a notion of *external* stabilization. Given a real-valued functional $H(\varphi)$, defined on finite $\varphi \in \mathbf{N}$ in a translation-invariant manner (i.e. with $H(x+\varphi) = H(x)$ for all nonempty $\varphi \in \mathbf{N}$ and all $x \in \mathbb{R}^d$), we say that a nonnegative random variable $\tilde{R}$ is a *radius of external stabilization* for H if there is a further random variable $\tilde{\Delta}$ such that

$$H((\Phi^0 \cap B_{\tilde{R}}(0)) \cup \psi) - H((\Phi \cap B_{\tilde{R}}(0)) \cup \psi) = \tilde{\Delta}$$

for all $\psi \in \mathbf{N}$ with $\psi(\mathbb{R}^d) < \infty$ and $\psi(B_{\tilde{R}}(0)) = 0$. If H has a radius of external stabilization $\tilde{R}$ with $\mathbb{P}(\tilde{R} < \infty) = 1$, we say H is *externally stabilizing* and refer to $\tilde{\Delta}$ as the *add-one cost* of H.

To prove Theorems 5.1 and 5.2 we shall use the following result, which is Theorem 2.1 of [14].

Theorem 5.3. *Suppose H is externally stabilizing with add-one cost $\tilde{\Delta}$, and satisfies the moments condition*

$$\sup_{n,m\in\mathbb{N}:n\geq 4, m\in[n/2,3n/2]} \sup_{x\in W_n} \mathbb{E}(|H(\chi_{n,m}^x) - H(\chi_{n,m})|^4) < \infty \tag{5.6}$$

along with (for some $\beta \in (0,\infty)$) the growth bound

$$|H(\varphi)| \leq \beta(\mathrm{diam}(\varphi) + \varphi(\mathbb{R}^d))^\beta, \quad \varphi \in \mathbf{N},\ 0 < \varphi(\mathbb{R}^d) < \infty. \tag{5.7}$$

Then there are constants $0 \leq \tau^2 \leq \sigma^2 < \infty$ such that $n^{-1}\mathrm{Var}(H(\Phi_n)) \to \sigma^2$ and $n^{-1}\mathrm{Var}(H(\chi_n)) \to \tau^2$, and

$$n^{-1/2}(H(\Phi_n) - \mathbb{E}H(\Phi_n)) \xrightarrow{\mathcal{D}} N(0,\sigma^2);$$
$$n^{-1/2}(H(\chi_n) - \mathbb{E}H(\chi_n)) \xrightarrow{\mathcal{D}} N(0,\tau^2).$$

Moreover, if $\tilde{\Delta}$ has a non-degenerate distribution then $\tau^2 > 0$

Proof of Theorem 5.1. We shall use Theorem 5.3. Recall the definition 3.17 of $R_{\mathrm{ex}}(\varphi)$. For $\varphi \in \mathbf{N}$ with $R_{\mathrm{ex}}(\varphi) = 9r < \infty$, set

$$\Delta_g(\varphi) := \left(\sum_{x\in\varphi^0\cap B_{2r}(0)} g(\rho(x,\varphi^0))\right) - \sum_{x\in\varphi\cap B_{2r}(0)} g(\rho(x,\varphi)).$$

By Lemma 3.6, if $\varphi \in \mathbf{N}$ with $R_{\mathrm{ex}}(\varphi) < \infty$ and $\psi \in \mathbf{N}$ with $\psi(\mathbb{R}^d) < \infty$ and $\psi(B_{R_{\mathrm{ex}}(\varphi)}(0)) = 0$, then

$$H_g((\varphi^0 \cap B_{R_{\mathrm{ex}}(\varphi)}(0)) \cup \psi) - H_g(\varphi \cap B_{R_{\mathrm{ex}}(\varphi)}(0)) = \Delta_g(\varphi).$$

Hence, since $R_{\mathrm{ex}}(\Phi) < \infty$ almost surely by Lemma 3.7, H_g is externally stabilizing. Using (5.1), it is easy to see that $H = H_g$ also satisfies the growth bound (5.7). We need to check that H_g satisfies the moments condition (5.6), and that the add-one cost $\Delta_g(\Phi)$ is non-degenerate. We demonstrate these in Lemmas 5.4 and 5.5 below. Given these, we can apply Theorem 5.3 to get the result. □

Lemma 5.4. *Suppose $g : \mathbb{R}_+ \to \mathbb{R}$ satisfies the growth bound (5.1). Then the functional $H = H_g$ satisfies the moments condition (5.6).*

Proof. Since $\chi_{n,m} \subset W_n$, by Lemma 3.6 (ii), and Lemma 3.7, for all $r > 0$, $n \geq 4$, $x \in W_n$ and $m \in [n/2, 3n/2]$ we have

$$\mathbb{P}\left[H_g(\chi_{n,m}^x) - H_g(\chi_{n,m}) \neq \left(\sum_{z\in\chi_{n,m}^x\cap B_{2r}(x)} g(\rho(z,\varphi^x))\right) - \sum_{y\in\chi_{n,m}\cap B_{2r}(x)} g(\rho(y,\varphi))\right]$$
$$\leq c_4 \exp(-c_4^{-1} r^{d/(d+1)}). \tag{5.8}$$

Now observe that there is a constant c_8 such that by the assumed growth bound (5.1) on g and the bound $\rho(x,\varphi) \leq D(x,\varphi)$, if $2 \leq \varphi(B_{2r}(x)) \leq 4b_d(2r)^d$, then

$$\left|\left(\sum_{z\in\varphi^x\cap B_{2r}(x)} g(\rho(z,\varphi^x))\right) - \sum_{y\in\varphi\cap B_{2r}(x)} g(\rho(y,\varphi))\right| \leq c_8(1 + r^{d+\beta}). \tag{5.9}$$

By (2.10), if $n \geq 4$ and $m \in [n/2, 3n/2]$ and $x \in W_n$ and $0 < r \leq n^{1/d}\mathrm{diam}(W)$, then

$$\mathbb{E}\chi_{n,m}(B_{2r}(x)) = (m/n)V_d(B_{2r}(x) \cap W_n) \in \left[(2c_0)^{-1}(2r)^d, 2b_d(2r)^d\right]. \tag{5.10}$$

In particular, if $(8c_0)^{1/d} \leq r \leq n^{1/d}\mathrm{diam}(W)$, then $\mathbb{E}\chi_{n,m}(B_{2r}(x)) \geq 2$. By a standard Chernoff-type tail estimate for the binomial distribution (see e.g. Lemma 1.1 of [11]), and (5.10), there is a constant $c_9 \in (0, \infty)$ such that if $(8c_0)^{1/d} \leq r \leq n^{1/d}\mathrm{diam}(W)$, then

$$1 - \mathbb{P}[2 \leq \chi_{n,m}(B_{2r}(x)) \leq 4b_d(2r)^d] \leq 2\exp(-c_9^{-1}r^d). \tag{5.11}$$

Moreover, if $r > n^{1/d}\mathrm{diam}(W)$ then since $V_d(W) = 1$, the Bieberbach (isodiametric) inequality (see e.g. [11]) yields

$$4b_d(2r)^d \geq 2^{2d+2}nb_d(\mathrm{diam}(W)/2)^d \geq 2^{2d+2}n,$$

so if $2 \leq m \leq 3n/2$ then trivially (5.11) holds in this case too. Combining (5.8), (5.9) and (5.11) gives us, for all $n \geq 4$, $m \in [n/2, 3n/2]$, $x \in W_n$ and $r \geq (8c_0)^{1/d}$, the tail bound

$$\mathbb{P}[|H_g(\chi_{n,m}^x) - H_g(\chi_{n,m})| > c_8(1 + r^{d+\beta})] \leq c_4 \exp(-c_4^{-1}r^{d/(d+1)}) + 2\exp(-c_9 r^d),$$

which suffices to give us the moments bound (5.6) for H_g. □

Lemma 5.5. *The random variable $\Delta_g(\Phi)$ has a nondegenerate distribution.*

Proof. By assumption g is not almost everywhere constant, so by the Lebesgue density theorem (as stated in e.g. [11]) there exist numbers $a > b > 0$ and $t_1 > 0$, $t_2 > 0$ and $\varepsilon \in (0, \min(t_1, t_2)/10)$ such that $\int_{t_1}^{t_1+\varepsilon} \mathbf{1}\{g(t) \geq a\}dt > 0$, and $\int_{t_2}^{t_2+\varepsilon} \mathbf{1}\{g(t) \leq b\}dt > 0$. Choose $r > \max(t_1, t_2)$ such that $\mathbb{P}(\Phi \in F(99r, r)) > 0$ (this is possible by (3.6)).

Enumerate the points of Φ as $X_1, X_2, \ldots$ with $|X_1| < |X_2| < \cdots$. Set $D := |X_1 - X_2|$ and $R_0 := \inf\{|x| : x \in B_{D/2}(X_1)\}$. For $i = 1, 2$ define the event

$$E_i := \{|D| \leq \varepsilon\} \cap \{R_0 \in (t_i, t_i + \varepsilon)\} \cap \{|X_3| > 99r\} \cap \{\Phi \in F(99r, r)\}$$

Then $\mathbb{P}(E_1) > 0$ and $\mathbb{P}(E_2) > 0$. Also, given E_i occurs the value of $\rho(0, \Phi^0)$ is equal to R_0 and $H_g(\Phi^0 \cap B_s(0)) - H_g(\Phi \cap B_s(0)) = g(R_0)$ for any $s > 100r$, so (provided also $R_{\mathrm{ex}}(\Phi) < \infty$) the value of $\Delta_g(\Phi)$ is equal to $g(R_0)$. Moreover, given E_i, the distribution of R_0 is absolutely continuous on $(t_i, t_i + \varepsilon)$ with a strictly positive density. Therefore $\mathbb{P}[\Delta_g(\Phi) \geq a|E_1]$ and $\mathbb{P}[\Delta_g(\Phi) \leq b|E_2]$ are both strictly positive. Hence $\mathbb{P}(\Delta_g(\Phi) \geq a) > 0$ and $\mathbb{P}(\Delta_g(\Phi) \leq b) > 0$. Thus $\Delta_g(\Phi)$ has a nondegenerate distribution as required. □

We now proceed towards a proof of Theorem 5.2. We wish to show the the functional H_κ has an almost surely finite radius of external stabilization. In fact we shall again use R_{ex}, as defined at (3.17).

Suppose $R_{\mathrm{ex}}(\varphi) = 9r < \infty$. Let $\varphi^* := \cup_{x \in \varphi \cap B_{3r}(0)} C(x, \varphi)$. Let $N(\varphi)$ be the number of components of the subgraph of $\mathcal{G}(\varphi)$ induced by the set of vertices φ^*. Let $N_0(\varphi)$ be the number of components of the subgraph of $\mathcal{G}(\varphi^0)$ induced by $\varphi^* \cup \{0\}$. Let

$$\Delta_\kappa(\varphi) := N_0(\varphi) - N(\varphi).$$

The following lemma says that R_{ex} serves as a radius of external stabilization for the functional H_κ:

Lemma 5.6. *(i) If $\varphi \in \mathbf{N}$ with $R_{\text{ex}}(\varphi) < \infty$ and $\psi \in \mathbf{N}$ with $\psi(\mathbb{R}^d) < \infty$ and $\psi(B_{R_{\text{ex}}(\varphi)}(0)) = 0$, then*

$$H_\kappa((\varphi^0 \cap B_{R_{\text{ex}}(\varphi)}(0)) \cup \psi) - H_\kappa((\varphi \cap B_{R_{\text{ex}}(\varphi)}(0)) \cup \psi) = \Delta_\kappa(\varphi). \tag{5.12}$$

(ii) Let $n \in \mathbb{N}$. Suppose $x \in W_n$, and $\varphi \in G_{n,r}(x)$. Then

$$H_\kappa(\varphi^x) - H_\kappa(\varphi) = H_\kappa(\varphi^x \cap B_{7r}(x)) - H_\kappa(\varphi \cap B_{7r}(x)).$$

Proof. Suppose $R_{\text{ex}}(\varphi) = 9r < \infty$. Then by Lemma 3.6, the radii of lilypond grains centred outside $B_{2r}(0)$ are unchanged when a point is inserted at 0, while the radii of grains centred inside $B_{7r}(0)$ are unaffected by changes to φ outside $B_{9r}(0)$.

Set $\varphi_1 := (\varphi \cap B_{9r}(0)) \cup \psi$. Let $x \in \varphi^*$ and $y \in \varphi_1 \setminus \varphi^*$. Then we claim that $\{x, y\} \notin \mathcal{G}^*(\varphi_1)$ and $\{x, y\} \notin \mathcal{G}^*(\varphi_1^0)$. Indeed, by Lemma 4.2 we have $x \in B_{5r}(0)$, so for $y \notin B_{7r}(0)$, the claim follows from Lemma 3.2 and the fact that $\varphi \in F(7r, r/2) \cap F(5r, r/2)$ (so also $\varphi^0 \in F(7r, r/2) \cap F(5r, r/2)$). In the case $y \in B_{7r}(0)$, the claim follows from the definition of φ^*, along with the fact that $\rho(y, \varphi_1) = \rho(y, \varphi)$ and $\rho(x, \varphi_1) = \rho(x, \varphi)$. Similarly, $\{0, y\} \notin \mathcal{G}^*(\varphi_1^0)$.

Therefore the components of $\mathcal{G}^*(\varphi_1)$ induced by all $y \in \varphi_1 \setminus \varphi^*$ do not meet the components containing all $x \in \varphi^*$ and this still holds after the addition of 0. So the contribution of these components to $H_\kappa(\varphi_1^0) - H_\kappa(\varphi_1)$ is zero and (5.12) follows.

Part (ii) is proved by much the same argument as for Part (i), using Lemma 3.5 instead of Lemma 3.4 and Lemma 3.2 (ii) instead of Lemma 3.2 (i). □

Proof of Theorem 5.2. By Lemmas 3.7 and 5.6, H_κ is externally stabilizing. It is easy to see that H_κ satisfies the growth bound (5.7) with $\beta = 1$. We need to check that H_κ satisfies the moments condition (5.6), and that the limiting add-one cost $\Delta_\kappa(\Phi)$ is non-degenerate. We demonstrate these in Lemmas 5.7 and 5.8 below. Given these, we can apply Theorem 5.3 to get the result. □

Lemma 5.7. *The functional $H = H_\kappa$ satisfies the moments condition (5.6).*

Proof. By Lemmas 5.6 (ii) and 3.7, for $n, m \in \mathbb{N}$ with $n \geq 4$ and $n/2 \leq m \leq 3n/2$, and $x \in W_n$ and $r > 0$,

$$\mathbb{P}[H_\kappa(\chi_{n,m}^x) - H_\kappa(\chi_{n,m}) \neq H_\kappa(\chi_{n,m}^x \cap B_{7r}(x)) - H_\kappa(\chi_{n,m} \cap B_{7r}(x))] \leq c_4 \exp(-c_4^{-1} r^{d/(d+1)}).$$

Therefore, for all $\ell \in \mathbb{N}$,

$$\mathbb{P}(|H_\kappa(\chi_{n,m}^x) - H_\kappa(\chi_{n,m})| > \ell) \leq c \exp(-c^{-1} r^{d/(d+1)}) + \mathbb{P}(\chi_{n,m}^x(B_{7r}(x)) > \ell). \tag{5.13}$$

Now take $r := r(\ell) := (1/7)(\ell/3b_d)^{1/d}$. Then $\mathbb{E}\chi_{n,m}(B_{7r}) \leq b_d(3/2)(7r)^d = \ell/2$. Hence the last probability in (5.13) decays exponentially in ℓ by standard Chernoff tail estimates for the binomial distribution (see e.g. Lemma 1.1 of [11]), so that overall there is a constant c such that the right hand side of (5.13) is bounded by $c \exp(-c^{-1} \ell^{1/(d+1)})$, for all $\ell \leq n$, independently of n and m. Moreover, the left side of (5.13) is zero for $\ell > n$. This gives the uniform bound (5.6) on fourth moments for H_κ.

Lemma 5.8. *The limiting add-one cost $\Delta_\kappa(\Phi)$ has a non-degenerate distribution.*

Proof. Let $e \in \mathbb{R}^d$ be a unit vector. Enumerate the points of Φ as $X_1, X_2, \dots$ with $|X_1| < |X_2| < \cdots$. Given $r \geq 1$, define events $E_{0,r}$ and $E_{1,r}$ as follows:

$$E_{0,r} := \{\Phi(B_1(5e)) = 2\} \cap \{|X_3| > 99r\} \cap \{\Phi \in F(99r, r)\};$$

$$E_{1,r} := \{\Phi(B_1(0)) = 1\} \cap \{\Phi(B_1(5e)) = 2\} \cap \{|X_4| > 99r\} \cap \{\Phi \in F(99r, r)\}.$$

Then by (3.6), for large enough r we have $\mathbb{P}(E_{0,r}) > 0$ and $\mathbb{P}(E_{1,r}) > 0$. But if $E_{0,r}$ occurs and $R_{\text{ex}}(\Phi) < \infty$ then $\Delta_\kappa(\Phi) = 0$, while if $E_{1,r}$ occurs and $R_{\text{ex}}(\Phi) < \infty$ then $\Delta_\kappa(\Phi) = 1$. This gives us the result. □

6 Percolation theory for the enhanced model

One interpretation (in $d = 2$) of the absence of percolation in the Poisson lilypond model, is that a frog is unable to travel infinitely far by a continuous path along the lily pads. For the sake of greater realism, it is natural to ask what happens if the frog is allowed to jump. More mathematically, for $\delta > 0$ we consider the *enhanced union set* Z^δ, where for any $A \subset \mathbb{R}^d$ we set $A^\delta := \cup_{x \in A} B_\delta(x)$ so that in particular

$$Z^\delta := \bigcup_{x \in \Phi} B_{\rho(x,\Phi)+\delta}(x). \tag{6.1}$$

We investigate the connectivity properties of Z^δ (the parameter δ represents half the distance which the frog is able to jump). In particular, we are concerned with the probability that Z^δ has an unbounded connected component. Given δ, the event that this occurs is invariant under translations of the Poisson process Φ, so by the ergodic property of this Poisson process (see Proposition 2.6 of [10]), or alternatively by an argument using the Kolmogorov zero-one law, this probability is either zero or 1, and it is one if and only if

$$\mathbb{P}(E_\infty(\delta)) > 0$$

where $E_\infty(\delta)$ denotes the event that there is an infinite component of Z^δ containing the origin. Accordingly, we define the *critical enhancement* $\delta_c := \delta_c(d)$ by

$$\delta_c := \inf\{\delta > 0 : \mathbb{P}(E_\infty(\delta)) > 0\}.$$

Our main result in this section says that if the range of our jumping frog is sufficiently small then it is still unable to travel infinitely far.

Theorem 6.1. *If $d = 1$ then $\delta_c = \infty$. If $d \geq 2$, then $0 < \delta_c < \infty$.*

For $d = 1$, it is easy to see that $\delta_c = \infty$. For $d \geq 2$, the fact that δ_c is finite is immediate from the basic fact in continuum percolation, that for sufficiently large r the union of balls $\cup_{x\in\Phi} B_r(x)$ percolates (see for example [7] or [10]). Thus, to prove Theorem 6.1 we need only to consider the case with $d \geq 2$ and show $\delta_c > 0$.

Let $D_r(x, \varphi)$ be the minimum of all non-zero pairwise distances between lilypond balls centred in $B_{8r}(x)$. That is, let

$$D_r(x, \varphi) = \min\{|u - v| - \rho(u, \varphi) - \rho(v, \varphi) : u, v \in \varphi \cap B_{8r}(x), |u - v| > \rho(u, \varphi) + \rho(v, \varphi)\}$$

and set $D_r(x, \varphi) = +\infty$ if there are no such pairs (u, v). Note that $D_r(x, \varphi)$ is strictly positive since, if finite, it is the minimum of a finite set of strictly positive numbers.

Lemma 6.2. *Given $x \in \mathbb{R}^d$ and $r > 0$, the event*

$$\{\Phi \in G_r(x)\} \cap \{D_r(x, \Phi) > \delta\}$$

is measurable with respect to $\sigma(\Phi \cap B_{9r}(x))$.

Proof. Let $\varphi \in \mathbf{N}$. By Lemma 2.1, $S(y, \varphi) \subset B_r(y)$ if and only if $S(y, \varphi \cap B_r(y)) \subset B_r(y)$. Hence, $\varphi \in G_r(x)$ if and only if $\varphi \cap B_{9r}(x) \in G_r(x)$. If this is the case, then by Lemmas 2.1 and 2.2, we also have $D_r(x, \varphi) = D_r(x, \varphi \cap B_{9r}(x))$. Hence the displayed event is identical to the event

$$\{(\Phi \cap B_{9r}(x)) \in G_r(x)\} \cap \{D_r(x, \Phi \cap B_{9r}(x)) > \delta\}$$

which is measurable with respect to $\sigma(\Phi \cap B_{9r}(x))$. □

A key claim is the following:

Lemma 6.3. *Let $\delta \in (0, 1/2)$ and let $r > 2$. Let $x \in \mathbb{R}^d$. If there is a continuous path in Z^δ from $\mathbb{R}^d \setminus B_{7r}(x)$ to $B_r(x)$, then the event $\{\Phi \notin G_r(x)\} \cup \{D_r(x, \Phi) \leq 2\delta\}$ occurs.*

Proof. Suppose that $\{\Phi \notin G_r(x)\} \cup \{D_r(x, \Phi) \leq 2\delta\}$ does not occur. Let T denote the union of components of Z which intersect with $B_{2r}(x)$. Any component of Z with all its Poisson points outside $B_{3r}(x)$ is contained in $\mathbb{R}^d \setminus B_{2r}(x)$, by Lemma 3.1 because $\Phi \in F(x, 3r, r/2)$. Hence by Lemma 4.2,

$$T \subset \cup_{y \in \Phi \cap B_{3r}(x)} C'(y, \Phi) \subset B_{6r}(x). \tag{6.2}$$

Consider all lilypond balls centred at Poisson points outside T. Those centred inside $B_{8r}(x)$ are at distance more than 2δ from T, because $D_r(x, \Phi) > 2\delta$ by assumption. Those centred outside $B_{8r}(x)$ do not intersect $B_{7r}(x)$ because of the assumption that $\Phi \in F(x, 8r, r/2)$ and Lemma 3.1, and so are also distant more than 2δ from T, since (6.2) holds and $r > 2\delta$.

Thus, the set T is distant more than 2δ from $Z \setminus T$, and hence T^δ is disconnected from the rest of Z^δ. Finally, by definition of T, $T^\delta \subset B_{7r}(x)$ and $Z^\delta \setminus T^\delta$ is disjoint from $B_r(x)$, both because $\delta < r$, so that there is no continuous path in Z^δ from $B_r(x)$ to $\mathbb{R}^d \setminus B_{7r}(x)$.
□

Proof of Theorem 6.1. Recall the definitions of $R(\varphi)$ and $G_r(x)$ at (2.3) and (3.15) respectively. By Lemma 3.7, $\mathbb{P}(\Phi \notin G_r(0)) \to 0$ as $r \to \infty$. Also, $D_r(x)$ is a strictly positive random variable, Hence, given $\varepsilon > 0$, we can choose $r > 2$ large enough and $\delta \in (0, 1/2)$ small enough such that

$$\mathbb{P}(\{\Phi \notin G_r(0)\} \cup \{D_r(0) \leq 2\delta\}) < \varepsilon. \tag{6.3}$$

Note that $\mathbb{P}(\{\Phi \notin G_r(x)\} \cup \{D_r(x) \leq 2\delta\})$ is the same for all x.

Now divide $\mathbb{R}^d$ into boxes (cubes) of side $2rd^{-1/2}$, labelled $Q_z, z \in \mathbb{Z}^d$, by setting

$$Q_z = 2rd^{-1/2}z + [-rd^{-1/2}, rd^{-1/2}]^d.$$

Define the random field $(Y_z, z \in \mathbb{Z}^d)$ by

$$Y_z := 1 - \mathbf{1}_{\{\Phi \notin G_r(2rd^{-1/2}z)\} \cup \{D_r(2rd^{-1/2}z, \Phi) \leq \delta\}}.$$

Since $Q_z \subset B_r(2rd^{-1/2}z)$, by Lemma 6.3 if there is a continuous path in Z^δ from Q_z to $\mathbb{R}^d \setminus B_{6r}(2rd^{-1/d}z)$, then $Y_z = 0$ almost surely.

If there is an infinite component in Z^δ, then there must be an unbounded continuous path in Z^δ, and by taking successive boxes along the path, there is an infinite sequence $(z_1, z_2, z_3, \ldots)$ of distinct elements with each $z_i \in \mathbb{Z}^d$ and $\|z_i - z_{i+1}\|_\infty \leq 1$ for each i, such that $Y_{z_i} = 0$ for all i.

Given $z \in \mathbb{Z}^d$, by Lemma 6.2 the random variable Y_z is measurable with respect to $\sigma(\Phi \cap B_{9r}(2rd^{-1/2}z))$. Hence the random field $(Y_z, z \in \mathbb{Z}^d)$ is independent of $Y_{z'}$ for all all sites z' with $2rd^{-1/2}|z' - z| > 18r$, i.e. with $|z' - z| > 9d^{1/2}$. Thus Y_z is independent of $Y_{z'}$ whenever the graph distance between z and z' exceeds $9d$. In fact, $(Y_z, z \in \mathbb{Z}^d)$ is a $9d$-dependent random field in the sense of [7].

Let p_c be the critical probability for site percolation on the lattice with vertex set $\mathbb{Z}^d$ and edges between each pair (z, z') with $\|z - z'\|_\infty = 1$; it is well known [7] that $p_c > 0$. By ([7], Theorem (7.65)), and (6.3), we can choose r to be so large and δ to be so small that the random field $(Y_z, z \in \mathbb{Z}^d)$ stochastically dominates a random field $(Y'_z, z \in \mathbb{Z}^d)$ consisting of independent Bernoulli random variables with $\mathbb{P}(Y'_z = 0) = p_c/2$ for each z. Thus, with this choice of δ there is almost surely no infinite path through the lattice of sites with $Y_z = 0$, and hence no infinite component in Z^δ; hence $\delta_c > 0$ as asserted. □

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