Approximation of invariant foliations for stochastic dynamical systems *

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Abstract

Invariant foliations are geometric structures for describing and understanding the qualitative behaviors of nonlinear dynamical systems. For stochastic dynamical systems, however, these geometric structures themselves are complicated random sets. Thus it is desirable to have some techniques to approximate random invariant foliations. In this paper, invariant foliations are approximated for dynamical systems with small noisy perturbations, via asymptotic analysis. Namely, random invariant foliations are represented as a perturbation of the deterministic invariant foliations, with deviation errors estimated.

Keywords: Stable and unstable foliations, fiber or leaf, random dynamical systems, fluctuations, asymptotic expansion, SDEs, and SPDEs

1 Introduction and motivation

Invariant foliations, as well as invariant manifolds, provide geometric structures for understanding the qualitative behaviors of nonlinear dynamical systems, and they have been extensively studied for deterministic systems [9, 8, 2].

Invariant manifolds or foliations for finite dimensional stochastic systems or stochastic differential equations (SDEs) were studied in [15, 1, 12, 5]. Recently, the existence of invariant manifolds and invariant foliations for stochastic partial differential equations was investigated in [6, 7, 13, 4] and [11], respectively. In [14], we estimated the impact of small noise on invariant manifolds for nonlinear systems. Note that random center-like invariant manifolds were approximated for some stochastic differential equations (SPDEs) by Wang and Duan [16], and Blomker and Wang [3]. In this paper, we consider a procedure to approximate invariant foliations for nonlinear

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systems perturbed by small noise. We compare invariant foliations for the original deterministic dynamical systems and for the randomly perturbed systems.

We consider the following nonlinear stochastic evolutionary equation with a multiplicative noise, in a separable Hilbert space H with a scalar product $<\cdot,\cdot>$ and the induced norm $\|\cdot\| = \sqrt{<\cdot,\cdot>}$:

$$\frac{dU}{dt} = AU + F(U) + \epsilon\, U \circ \dot{W}, \tag{1}$$

where A is a linear (bounded or unbounded) operator, "$\circ$" is in the sense of Stratonovich stochastic calculus, $W = W(t,\omega)$ is a scalar Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and ϵ is a positive parameter representing the intensity of the noise. This covers some SDEs and SPDEs. Note that the Ito's form of (1) is

$$\frac{dU}{dt} = AU + \epsilon\frac{U}{2} + F(U) + \epsilon\, U\dot{W}.$$

The nonlinearity $F(U)$ satisfies $F(0) = 0$ and $DF(0) = 0$ and is Lipschitz continuous on H

$$\|F(U_1) - F(U_2)\| \le L_F\|U_1 - U_2\|,$$

where L_F is the positive Lipschitz constant and $\|\cdot\|$ the norm in the Hilbert space H. When the nonlinearity $F(U)$ is locally Lipschitz continuous, the approximation result in this paper can be applied to the modified stochastic equation where the nonlinearity is appropriately cut-off and thus obtain approximation information for the local random invariant foliations. The state space H is the Euclidean space $\mathbb{R}^n$ when the above equation is a SDE or a function space if the above equation is a SPDE. When $\epsilon = 0$, Eq. (1) reduces to a deterministic evolutionary equation:

$$\frac{dU}{dt} = AU + F(U). \tag{2}$$

We compare the invariant foliations for the original deterministic system (2) and for the randomly perturbed system (1), and quantify their difference when the noise intensity ϵ is small.

This paper is organized as follows. In section 2, we review some basic concepts of random dynamical systems, and recall the existence result for random invariant foliations. The main result on asymptotic analysis for random invariant foliations is described in section 3, and two illustrative examples are presented in section 4.

2 Random invariant foliation

Following [7], we assume throughout the paper that the linear operator $A : D(A) \to H$ generates a strongly continuous semigroup e^{At} on H, which satisfies the pseudo exponential dichotomy with exponents $\alpha > 0 > \beta$ and a bound $K > 0$, i.e., there exists a continuous projection P^u on H such that

(i) $P^u e^{At} = e^{At} P^u$;

(ii) The restriction $e^{At}|_{R(P^u)}$, $t \geq 0$, is an isomorphism of the range $R(P^u)$ of P^u onto itself, and we define e^{At} for $t < 0$ as the inverse map;

(iii) The following estimates hold

$$\begin{aligned} | e^{At} P^u x | &\leq K e^{\alpha t} | x | \qquad & t \leq 0 \\ | e^{At} P^s x | &\leq K e^{\beta t} | x | \qquad & t \geq 0 \end{aligned} \tag{3}$$

where $P^s = I - P^u$. Denote $H^s = P^s H$, $H^u = P^u H$ and hence $H = H^s \bigoplus H^u$.

2.1 Random dynamical systems

A measurable random dynamical system on Hilbert space $(H, \mathcal{B})$ over a driving system $(\theta(t))_{t \in T}$ with time T is a mapping

$$\varphi : T \times \Omega \times H \to H, (t, \omega, x) \to \varphi(t, \omega, x),$$

with the following properties [1]:

(i) Measurability: φ is $\mathcal{B}(T) \otimes \mathcal{F} \otimes \mathcal{B}$–measurable.

(ii) The mappings $\varphi(t, \omega) = \varphi(t, \omega, \cdot) : H \to H$ form a cocycle over $\theta(\cdot)$, i.e. they satisfy $\varphi(0, \omega) = id_X$ for all $\omega \in \Omega$ and $\varphi(t+s, \omega) = \varphi(t, \theta(s)\omega)\, \varphi(s, \omega)$ for all $s, t \in T$ and $\omega \in \Omega$.

For SDEs and SPDEs [1], we identify $\omega(t) = W(t, \omega)$, and define the driving system $\theta(t)$ is the Wiener shift, i.e., $\theta_t \omega(\tau) = \omega(\tau + t) - \omega(t)$.

To facilitate random dynamical systems study of (1), we convert it into an evolutionary equation with random coefficients, called a random evolutionary equation. To this end, we introduce $z(w)$ as the stationary solution of the following Langevin equation

$$dz + zdt = \epsilon dW.$$

Then $z(w) = \epsilon Z(\omega)$, where $Z(\omega)$ is the stationary solution of $dZ(t) + Z(t)dt = dW(t)$ and it can be expressed as

$$Z(\omega) = \int_{-\infty}^{0} e^{\tau}\, dW(\tau).$$

Moreover,

$$Z(\theta_t \omega) = e^{-t} Z(\omega) + e^{-t} \int_0^t e^{\tau}\, dW(\tau).$$

Define a transform

$$\bar{x} := T(\omega, \bar{X}) = \bar{X} e^{-z(\omega)}$$

with its inverse transform

$$\bar{X} := T^{-1}(\omega, \bar{x}) = \bar{x} e^{z(\omega)}. \tag{4}$$

Denote $U(t, \omega, X)$ as the solution of (1) with initial value X. Introducing

$$u = T(\theta_t \omega, U(t, \omega, X) = e^{-z(\theta_t \omega)} U(t, \omega, X),$$

then the new system state u satisfies the following random evolutionary equation [7]

$$\frac{du}{dt} = Au + z(\theta_t \omega) u + G(\theta_t \omega, u), \quad u(0) = x \in H, \tag{5}$$

where

$$x = T(\omega, X) = e^{-z(\omega)} X$$

and

$$G(\omega, u) := e^{-z(\omega)} F(e^{z(\omega)} u). \tag{6}$$

We often denote the solution of (5) to be $u = \varphi(t, \omega, x)$. The solution mapping of (5), i.e. $(t, \omega, x) \to \varphi(t, \omega, x)$, generates a random dynamical system. Thus (see [7])

$$(t, \omega, X) \to T^{-1}(\theta_t \omega, \varphi(t, \omega, T(\omega, X)) := U(t, \omega, X)$$

is also a random dynamical system. In fact, The relationship between solutions of (1) and (5) is described by

$$U(t, \omega, X) = T^{-1}(\theta_t \omega, \varphi(t, \omega, T(\omega, X))$$
$$\varphi(t, \omega, x) = T(\theta_t \omega, U(t, \omega, T^{-1}(\omega, x)).$$

2.2 Random invariant foliation

The concept of invariant foliation is about quantifying certain sets (called leaves or fibers) in state space H, starting from all points in such a leaf the dynamical orbits have similar asymptotic behaviors. These leaves are thus building blocks for understanding dynamics.

Let us consider a leaf for random invariant foliation for the above random dynamical system $\varphi(t, \omega, x)$. A leaf passing through a point Φ^0 in the state space H, denoted as $W(\Phi^0, \omega)$, is a random set and is invariant in the following special sense [1, 11]

$$\varphi(t, \omega, W(\Phi^0, \omega)) \subset W(\varphi(t, \omega, \Phi^0), \theta_t \omega) \ \ for \ \ t \geq 0.$$

If we can represent Q as a graph of a C^k (or Lipschitz) mapping, then $Q(\Phi^0, \omega)$ is called a C^k (or Lipschitz) leaf for the random invariant foliation. The existence of random invariant foliation for (5) is shown in [11]. To facilitate our asymptotic analysis in the next section, we recall as follows. We only consider stable leaves, still denoted as $W(\Phi^0,\omega)$. Unstable leaves may be considered similarly.

Define

$$\psi(t) = \tilde{\Phi}(t) - \Phi(t), \tag{7}$$

where $\tilde{\Phi}(t) = \varphi(t,\omega,\tilde{\Phi}^0)$ and $\Phi(t) = \varphi(t,\omega,\Phi^0)$ are two solutions of (5) starting at two initial states $\tilde{\Phi}^0$ and Φ^0, respectively. Also introduce the following Banach space, for each η, $\beta < \eta < \alpha$,

$$\hat{C}_\eta^+ = \{\varphi : [0,\infty) \to H \mid \varphi \ \ is \ \ continuous \ \ and \ \ \sup_{t\in[0,\infty)} \ e^{-\eta t - \int_0^t z(\theta_\tau\omega)d\tau}\|\varphi(t)\| < \infty\}$$

with the norm

$$\|\varphi\|_{\hat{C}_\eta^+} = \sup_{t\in[0,\infty)} \ e^{-\eta t - \int_0^t z(\theta_\tau\omega)d\tau}\|\varphi(t)\|.$$

It is shown in ([11]) that $\tilde{\Phi}^0 \in W(\Phi^0,\omega)$ if and only if there exists a function $\psi(\cdot) \in \hat{C}_\eta^+$ with $\psi(0) = \tilde{\Phi}^0 - \Phi^0$ and

$$\begin{aligned}\psi(t) = \ & e^{At+\int_0^t z(\theta_s\omega)ds}\xi + \int_0^t e^{A(t-s)+\int_s^t z(\theta_r\omega)dr}P^s\Delta G(\theta_s\omega,\psi(t),\Phi(t))ds \\ & + \int_\infty^t e^{A(t-s)+\int_s^t z(\theta_r\omega)dr}P^u\Delta G(\theta_s\omega,\psi(t),\Phi(t))ds\end{aligned} \tag{8}$$

where $\xi = P^s(\tilde{\Phi}^0 - \Phi^0)$, and $\Delta G(\omega,\psi,\Phi) = G(\omega,\psi+\Phi) - G(\omega,\Phi)$. Under the gap condition

$$K = CL\left(\frac{1}{\alpha-\eta} + \frac{1}{\eta-\beta}\right) < 1,$$

there exists an invariant foliation for (5) whose stable leaf is given by

$$W(\Phi^0,\omega) = \left\{\xi + l^s(\xi,\Phi^0,\omega) \mid \xi \in H^s\right\},$$

where $\Phi^0 \in H$, $(\xi,\Phi^0,\omega) \to l^s(\xi,\Phi^0,\omega)$ is measurable and Lipschitz continuous in ξ and

$$\varphi(t,\omega,W(\Phi^0,\omega)) \subset W(\Phi(t,\varphi^0,\omega),\theta_t\omega).$$

Moreover

$$l^s(\xi, \Phi^0, \omega) = P^u\Phi^0 + P^u\psi(0; \xi - P^s\Phi^0, \Phi^0, \omega), \quad \xi \in H^s, \tag{9}$$

where

$$P^u\psi(t, \xi, \Phi^0, \omega) = \int_\infty^t e^{A(t-s)+\int_s^t z(\theta_r\omega)\,dr} P^u(G(\theta_s\omega, \psi(s, \xi, \Phi^0, \omega) + \Phi(s) - G(\theta_s\omega, \Phi(s))\,ds.$$

It also follows from (8) that $\psi(t)$, as defined in (7), satisfies the following equation

$$\frac{d\psi}{dt} = Au + z(\theta_t\omega)\psi + \Delta G(\theta_t\omega, \psi(t), \Phi(t)). \tag{10}$$

3 Asymptotic analysis for random invariant foliation

In this section, we propose an approach to approximate the random invariant foliation by asymptotic analysis for ϵ sufficiently small.

Consider the stable leaf of the invariant foliation for (5) ($0 < \epsilon \ll 1$), passing through a point $\Phi^0 \in H$,

$$W(\Phi^0, \omega) = \{\xi + l^s(\xi, \Phi^0, \omega)\Big|\xi \in H^s\}. \tag{11}$$

Let the deterministic leaf (i.e. $\epsilon = 0$) be represented as

$$\{\xi + l^{(d)}(\xi)\Big|\xi \in H^s\}, \tag{12}$$

where $l^s(\cdot, \omega) : H^s \to H^u$ and $l^{(d)}(\cdot) : H^s \to H^u$ are Lipschitz mappings. We expand

$$l^s(\xi, \Phi^0, \omega) = l^{(d)}(\xi) + \epsilon l^{(1)}(\xi, \Phi^0, \omega) + \epsilon^2 l^{(2)}(\xi, \Phi^0, \omega) + \cdots + \epsilon^k l^{(k)}(\xi, \Phi^0, \omega) + \cdots \tag{13}$$

Write the solution of (10) in the form

$$\psi(t) = \psi^{(d)}(t) + \epsilon\psi^{(1)}(t) + \cdots + \epsilon^k\psi^{(k)}(t) + \cdots \tag{14}$$

with the initial condition

$$\psi(0) = \xi + l^s(\xi,\omega) - \Phi^0 = \xi - \Phi^0 + l^{(d)}(\xi) + \epsilon l^{(1)}(\xi,\omega) + \cdots . \tag{15}$$

Similarly, write $\Phi(t) = \varphi(t,\omega,\Phi^0)$ as

$$\Phi(t) = \Phi^{(d)}(t) + \epsilon\Phi^{(1)}(t) + \cdots + \epsilon^k \Phi^{(k}(t) + \cdots \tag{16}$$

with initial condition

$$\Phi(0) = \Phi^0 \tag{17}$$

By the Taylor expansion, we obtain

$$e^{z(\theta_t(\omega))} = e^{\epsilon Z(\theta_t(\omega))} = 1 + \epsilon Z(\theta_t(\omega)) + \cdots + \frac{\epsilon^k \left(Z(\theta_t(\omega))\right)^k}{k!} + \cdots \tag{18}$$

and

$$\begin{aligned} & e^{\int_s^t z(\theta_r(\omega))\,dr} = e^{\epsilon \int_s^t Z(\theta_r(\omega))\,dr} \\ & \quad = 1 + \epsilon \int_s^t Z(\theta_r(\omega))\,dr + \cdots + \frac{\epsilon^k \left(\int_s^t Z(\theta_r(\omega))\,dr\right)^k}{k!} + \cdots . \end{aligned} \tag{19}$$

Suppose $F(u)$ is sufficiently smooth with respect to u. With (18), it follows from (6) that

$$\begin{aligned} & G(\theta_t\omega, \psi(t)) = e^{-z(\theta_t(\omega))} F\left(e^{z(\theta_t(\omega))}\psi(t)\right) \\ & = e^{-\epsilon Z(\theta_t(\omega))} F\left(e^{\epsilon Z(\theta_t(\omega))}\psi(t)\right) \\ & = (1 - \epsilon Z(\theta_t(\omega)) + \cdots) F\Big((1 + \epsilon Z(\theta_t(\omega)) + \cdots) \left(\psi^{(d)}(t) + \epsilon\psi^{(1)}(t) + \cdots\right) \Big) \\ & = F(\psi^{(d)}(t)) + \epsilon\left(-Z(\theta_t(\omega))F(\psi^{(d)}(t)) + F_u^{\psi^{(d)}}\left(\psi^{(1)}(t) + Z(\theta_t(\omega))\psi^{(d)}(t)\right)\right) + \cdots , \end{aligned} \tag{20}$$

where $F_u^{\psi^{(d)}(t)}$ represents the first order Fréchet derivative [10] of the function $F(u)$ with respect to u and evaluated at $\psi^{(d)}(t)$. In Euclidean space, the Fréchet derivative reduces to the classical derivative.

Substituting (13), (20) and (14) into (5), and equating the terms with the same power of ϵ, we get

$$\begin{cases} \frac{d\psi^{(d)}(t)}{dt} = A\psi^{(d)}(t) + F(\psi^{(d)}(t) + \Phi^{(d)}(t) - F(\Phi^{(d)}(t), \\ \psi^{(d)}(0) = \xi + l^{(d)}(\xi) - \Phi^0, \end{cases} \tag{21}$$

and

$$\begin{cases} \frac{d\psi^{(1)}(t)}{dt} = \left[A + F_u^{\psi^{(d)}+\Phi^{(d)}}\right]\psi^{(1)}(t) + \tilde{\lambda}, \\ \psi^{(1)}(0) = l^{(1)}(\xi,\omega), \end{cases} \tag{22}$$

where

$$\begin{aligned} \tilde{\lambda} = {} & Z(\theta_t(\omega))\left[\psi^{(d)}(t) + F(\Phi^{(d)}(t)) - F(\psi^{(d)}(t) + \Phi^{(d)}(t)) - F_\psi^{\psi^{(d)}(t)}\psi^{(d)}(t)\right] \\ & + F_u^{\psi^{(d)}+\Phi^{(d)}}\left(\Phi^{(1)}(t) + Z(\theta_t\omega)(\psi^{(d)}(t) + \Phi^{(d)}(t))\right) \\ & - F_u^{\Phi^{(d)}}\left(\Phi^{(1)}(t) + Z(\theta_t\omega)\Phi^{(d)}(t)\right). \end{aligned} \tag{23}$$

Solve for $\psi^{(d)}(t)$ and $\psi^{(1)}(t)$,

$$\psi^{(d)}(t) = e^{At}\psi^{(d)}(0) + \int_0^t e^{A(t-s)}\left(F(\psi^{(d)}(t) + \Phi^{(d)}(t) - F(\Phi^{(d)}(t)\right) ds, \tag{24}$$

$$\psi^{(1)}(t) = e^{At+\int_0^t F_\psi^{\psi^{(d)}(s)}\,ds}\left(h^{(1)}(\xi,\omega) - \int_0^t e^{-As+\int_s^0 F_\psi^{\psi^{(d)}(r)}\,dr}\tilde{\lambda}\,ds\right). \tag{25}$$

Similarly, we have

$$\begin{cases} \frac{d\Phi^{(d)}(t)}{dt} = A\Phi^{(d)}(t) + F(\Phi^{(d)}(t)), \\ \Phi^{(d)}(0) = \Phi^0, \end{cases} \tag{26}$$

and

$$\begin{cases} \frac{d\Phi^{(1)}(t)}{dt} = \left[A + F_u^{\Phi^{(d)}}\right]\Phi^{(1)}(t) + \tilde{B} \\ \Phi^{(1)}(0) = 0, \end{cases} \tag{27}$$

where

$$\tilde{B} = -Z(\theta_t(\omega)) \left[-\Phi^{(d)}(t) + F(\Phi^{(d)}(t)) - F_u^{\Phi^{(d)}} \Phi^{(d)}(t) \right]. \tag{28}$$

Moreover, solve for $\Phi^{(d)}(t)$ and $\Phi^{(1)}(t)$,

$$\Phi^{(d)}(t) = e^{At}\Phi^{(d)}(0) + \int_0^t e^{A(t-s)} F(\Phi^{(1)}(s))\, ds, \tag{29}$$

$$\Phi^{(1)}(t) = e^{At + \int_0^t F_u^{\Phi^{(d)}(s)}\, ds} \left(l^{(1)}(\xi, \omega) + \int_0^t e^{-As + F_u^{\Phi^{(d)}}} \tilde{B}\, ds \right). \tag{30}$$

With (18), (19) and (20), the right hand side of (9) can be written as

$$P^u \Phi^0 + \int_0^\infty e^{-As + \int_s^t z(\theta_r(\omega))\, dr} P^u G(\theta_s \omega, u(s))\, ds = I_0 + \epsilon I_1 + R_2, \tag{31}$$

where R_2 represents the remainder term and the other two terms are,

$$I_0 = P^u \Phi^0 + \int_\infty^0 e^{-As} P^u \left[F(\Phi_0(s) + \Psi^{(d)}(s)) - F(\Phi_0(s)) \right] ds,$$
$$I_1 = \int_\infty^0 e^{-As} \left\{ \left(\int_s^t Z(\theta_r(\omega))\, dr - Z(\theta_s(\omega)) \right) \tilde{C} \right\} ds,$$

with

$$\tilde{C} = F(\psi^{(d)}(s) + \Phi^{(d)}(s)) - F(\Phi^{(d)})(s) + F_u^{\psi^{(d)} + \Phi^{(d)}} \left(\psi^{(1)}(s) + \Phi^{(1)}(s) + Z(\theta_s \omega)(\psi^{(d)} + \Phi^{(d)}) \right). \tag{32}$$

Substituting (13) and (31) into (9), and matching the powers in ϵ, we get

$$l^{(d)}(\xi) = I_0 = P^u \Phi^0 + \int_\infty^0 e^{-As} P^u \left[F(\Phi_0(s) + \Psi^{(d)}(s)) - F(\Phi_0(s)) \right] ds, \tag{33}$$

and

$$l^{(1)}(\xi, \Phi^0, \omega) = \int_\infty^0 e^{-As} \left\{ \left(\int_s^t Z(\theta_r(\omega))\, dr - Z(\theta_s(\omega)) \right) \tilde{C} \right\} ds.$$

As a summary, we obtain the following result about approximating invariant foliation for the random evolutionary equation (5), including some random ordinary or partial differential equations.

Theorem 1 (Approximate invariant foliation for random evolutionary equations)**.** *Let $W(\Phi^0,\omega) = \{\xi + l^s(\xi,\omega,\Phi^0)\Big|\xi \in H^s\}$ represent a stable leaf, passing through a point Φ^0, of the invariant foliation for the random evolutionary equation $\frac{du}{dt} = Au + z(\theta_t\omega)u + G(\theta_t\omega, u)$. Assume that*

(i) $F(u)$ is twice continuously Fréchet differentiable with respect to u;

(ii) For some η ($\alpha > \eta > \beta$), the following gap condition is satisfied

$$K\, L_F \left(\frac{1}{\eta-\beta} + \frac{1}{\alpha-\eta}\right) < 1. \tag{34}$$

Then for ϵ sufficiently small, the leaf of the random invariant foliation can be approximated as

$$W(\Phi^0,\omega) = \{\xi + l^{(d)}(\xi) + \epsilon l^{(1)}(\xi,\Phi^0,\omega) + R_2 \;\Big|\; \xi \in H^s\},$$

where $\|R_2\| \le C(\omega)\epsilon^2$ with $C(\omega) < \infty$, a.s.,

$$l^{(d)}(\xi) = P^u\Phi^0 + \int_\infty^0 e^{-As}P^u\left[F(\Phi_0(s) + \Psi^{(d)}(s)) - F(\Phi_0(s))\right] ds, \tag{35}$$

and

$$l^{(1)}(\xi,\Phi^0,\omega) = \int_\infty^0 e^{-As}\left\{\left(\int_s^t Z(\theta_r(\omega))\,dr - Z(\theta_s(\omega))\right) P^uF(u_0) \right.$$
$$\left. \times \left(F(\psi^{(d)}(s)+\Phi^{(d)}(s)) - F(\Phi^{(d)})(s) + F_u^{\psi^{(d)}+\Phi^{(d)}}\left(\psi^{(1)}(s)+\Phi^{(1)}(s)+Z(\theta_s\omega)(\psi^{(d)}+\Phi^{(d)})\right)\right)\right\} ds. \tag{36}$$

4 Examples

Let us look at two examples.

Example 1

Consider a SDE system

$$\begin{cases} \dot{X} = -X + \epsilon X \circ \dot{W}, \\ \dot{Y} = Y + X^2 + \epsilon Y \circ \dot{W}, \end{cases} \tag{37}$$

where $\epsilon > 0$ and W is a scalar Brownian motion. In this example, $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $H = \mathbb{R}^2$ (a finite dimensional Hilbert space), $H^s = \left\{ \begin{pmatrix} \bar{x} \\ 0 \end{pmatrix} \Big| \bar{x} \in \mathbb{R} \right\}$, and $H^u = \left\{ \begin{pmatrix} 0 \\ \bar{y} \end{pmatrix} \Big| \bar{y} \in \mathbb{R} \right\}$. The transformed differential equations with random coefficients are

$$\begin{cases} \dot{x} = -x + \epsilon Z(\theta_t\omega)x, \\ \dot{y} = y + \epsilon Z(\theta_t\omega)y + e^{\epsilon Z(\theta_t\omega)}x^2, \end{cases} \tag{38}$$

where $Z(\omega)$ is the stationary solution of $dZ + Zdt = dW$, i.e. $Z(\omega) = \int_{-\infty}^{0} e^{\tau} dW(\tau)$ and $Z(\theta_t\omega) = e^{-t}Z(\omega) + e^{-t}\int_0^t e^{\tau} dW(\tau)$.

The stable leaf of the invariant foliation for (38), passing through a point (x_0, y_0), can be approximated as

$$W(x_0, y_0, \omega) =$$
$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \Big| y - y_0 = -\frac{x^2 - x_0^2}{3} + \epsilon Z(\omega)\left(y_0 + \frac{2}{3}x_0^2\right) - \epsilon\frac{x^2 - x_0^2}{3}\left(\int_0^{\infty} e^{-3\tau}\, dW_\tau\right) + O(\epsilon^2) \;\; x \in \mathbb{R} \right\}.$$

Figure 1 compares a random stable leaf (two samples are shown here) for (38) with the deterministic stable leaf, passing through the point $x_0 = 0$ and $y_0 = 0$.

Example 2

Consider the following SPDE

$$\begin{cases} U_t = (U_{xx} + 10U) - U^3 + \epsilon U \circ \dot{W}, x \in [0, 1], \\ U(0, t) = U(1, t) = 0, \end{cases}$$

where $\epsilon > 0$ and W is a scalar Brownian motion. In this example, $A = \Delta + 10$, $H = L^2(0, 1)$, $D(A) = H_0^2(0, 1)$, $F(u) = u^3$. Note that the eigenvalues of A are $\lambda_n = 10 - (n\pi)^2$, and the corresponding normalized eigenfunctions are $e_n = \sqrt{2}\sin(n\pi x)$, $n = 1, 2 \cdots$. Here $H^s = Span\{e_1\}$ and $H^u = Span\ \{e_2, e_3, \cdots, e_n, \cdots\}$.

The transformed random partial differential equation is

$$\begin{cases} u_t = (u_{xx} + 10u) + Z(\theta_t\omega)u - e^{\epsilon 2Z(\theta_t\omega)}u^3, x \in [0, 1], \\ u(0, t) = u(1, t) = 0. \end{cases}$$

Unlike Example 1, here we can not express $l^{(1)}(\xi, \Phi^0, \omega)$ analytically, but only estimate it via (36).

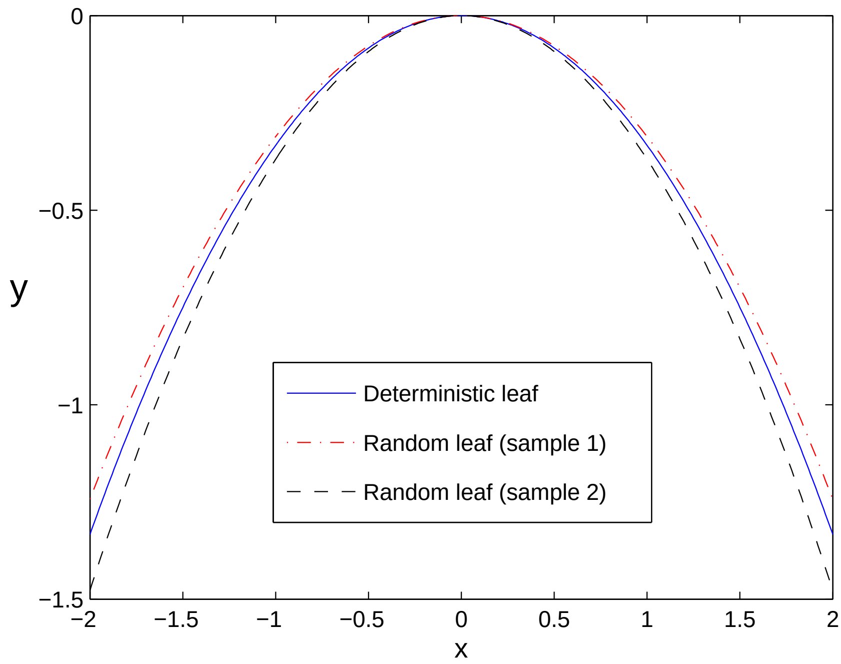

Figure 1: The random stable leave passing through the point $(0, 0)$ for Example 1. Two samples of the random stable leaf are shown here, together with the stable leaf for the corresponding deterministic system ($\epsilon = 0$)

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