Effective dynamics
of a coupled microscopic-macroscopic stochastic system *

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Abstract

A conceptual model for microscopic-macroscopic slow-fast stochastic systems is considered. A dynamical reduction procedure is presented in order to extract effective dynamics for this kind of systems. Under appropriate assumptions, the effective system is shown to approximate the original system, in the sense of a probabilistic convergence.

Key Words: Macroscopic-microscopic system, stochastic partial differential equations, averaging principle, effective dynamics, slow-fast scales

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1 Motivation

In modeling complex phenomena in biomedical, geophysical, and chemical systems, we sometimes encounter microscopic-macroscopic stochastic systems. These are systems of coupled stochastic ordinary and partial differential equations (SDEs and SPDEs). The SPDEs describe the macroscopic dynamics while SDEs for the microscopic dynamics. For example, angiogenesis is a vital process in human tissue growth and wound healing. This process involves the growth of new blood vessels from pre-existing vessels where blood cells penetrate into growing tissue, supplying nutrients and oxygen and removing waste products [3]. During the process, blood cells interact with the tissue mass randomly. Here the blood cells may be regarded as "particles" while tissue may be described by a "density" quantity. For another example, in

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fluid flows or oceanic flows, heavy pollutant particles and the mass of a chemical substance may interact in a random fashion. We consider a conceptual microscopic-macroscopic stochastic system where the microscopic component is composed of finite number of "particles" and the macroscopic component is about "densities" evolution of a finite number of substances. Both particles and substances are interacting randomly or are interacting in a random environment.

More specifically, and for simplicity, we assume that there are only two particles whose positions are at $\xi(t)$ and $\eta(t)$, and assume also that there are two substances with densities $u(x,t)$ and $v(x,t)$, respectively. Here both ξ and η satisfy a system of SDEs, and u and v are described by a system of SPDEs. The SDEs and SPDEs are coupled, due to the impact of particles on density evolution. Furthermore, we suppose that ξ and u evolve slowly, but η and v progress much faster. We are interested in deriving an effective model for this coupled stochastic system, hopefully involve only slow variables ξ and u.

First, in §2, we consider a simpler coupled microscopic-macroscopic system when the fast density v is absent and no external noise acting directly on u:

$$\begin{cases} u_t = u_{xx} + f(u,\xi), \quad u(x,0) = u_0(x), \\ \dot{\xi} = b(\xi,\eta) + \sigma_3(\xi)\dot{W}_t, \quad \xi(0) = x_0, \\ \dot{\eta} = \varepsilon^{-1}B(\xi,\eta) + \varepsilon^{-\frac{1}{2}}\sigma_4(\xi,\eta)\dot{W}_t, \quad \eta(0) = y_0, \\ u(0,t) = u(1,t) = 0, \end{cases} \tag{1.1}$$

where $t \in [0,T]$, $x \in [0,1]$, ε is a small positive parameter, and W_t is a standard scalar Brownian motion. The coefficients $f, b, B, \sigma_3, \sigma_4$ all satisfy Lipschitz and boundedness assumptions. In this part, ξ is slow component and η fast component. We derive an effective model involves u and ξ only. This result is summarized in **Theorem 2.1**.

Then in §3, we consider a more complex coupled microscopic-macroscopic stochastic system

$$\begin{cases} u_t = u_{xx} + f(u,v,\xi) + \sigma_1(u)\dot{W}_t^1, \quad u(x,0) = u_0(x), \\ v_t = \frac{1}{\varepsilon}(v_{xx} + g(u,v,\xi)) + \frac{1}{\sqrt{\varepsilon}}\sigma_2(u,v)\dot{W}_t^2, \quad v(x,0) = v_0(x), \\ \dot{\xi} = b(\xi,\eta) + \sigma_3(\xi)\dot{W}_t^3, \quad \xi(0) = x_0, \\ \dot{\eta} = \varepsilon^{-1}B(\xi,\eta) + \varepsilon^{-\frac{1}{2}}\sigma_4(\xi,\eta)\dot{W}_t^3, \quad \eta(0) = y_0, \\ u(0,t) = u(1,t) = 0, \quad v(0,t) = v(1,t) = 0, \end{cases} \tag{1.2}$$

where $t \in [0,T]$, $x \in [0,1]$, ε is a small positive parameter, and $\{W_t^i\}_{t\geq 0}, i = 1,2,3$ are independent scalar Brownian motions. The coefficients $f, g, b, B, \sigma_i's$ satisfy some assumptions. In this setting, ξ and u are slow components while η and v fast components. The first two equations are macroscopic components coupled with the latter two equations for the microscopic components. We derive an effective model involving only u and ξ, and the result is stated in **Theorem 3.1**.

2 A stochastic microscopic-macroscopic model

First we consider the following coupled SPDE-SDE system

$$\begin{cases} u_t = u_{xx} + f(u,\xi), \quad u(x,0) = u_0(x), \\ \dot{\xi} = b(\xi,\eta) + \sigma_3(\xi)\dot{W}_t, \quad \xi(0) = x_0, \\ \dot{\eta} = \varepsilon^{-1}B(\xi,\eta) + \varepsilon^{-\frac{1}{2}}\sigma_4(\xi,\eta)\dot{W}_t, \quad \eta(0) = y_0, \\ u(0,t) = u(1,t) = 0, \end{cases} \tag{2.1}$$

for $t \in [0,T]$, $x \in [0,1]$, where ε is a small positive parameter, W is a standard scalar Brownian motion, the coupling term $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the condition: Globally Lipschitz in u and ξ with Lipschitz constant K^f, and $b(\xi,\eta)$, $B(\xi,\eta)$, $\sigma_3(\xi)$, $\sigma_4(\xi,\eta)$ are all globally Lipschitz and bounded.

As in [6], p.268, the slow-fast SDEs for (ξ,η) above have the following averaged effective dynamical description.

We introduce a random process $\eta^\xi(t)$, defined by the stochastic differential equation for fixed $\xi \in \mathbb{R}$,

$$\dot{\eta}^\xi(t) = B(\xi, \eta^\xi(t)) + \sigma_4(\xi, \eta^\xi(t))\dot{W}_t, \qquad \eta^\xi(0) = y_0. \tag{2.2}$$

For any $t \geq 0$ and any $\xi \in \mathbb{R}$, we assume that there exists a function $\bar{b}(\xi)$, such that

$$\mathbb{E}\Big|\frac{1}{T}\int_t^{t+T} b(\xi, \eta^\xi(s))ds - \bar{b}(\xi)\Big| < \chi(T),$$

where the non-negative upper bound function $\chi(T) \to 0$ as $T \to \infty$. Then there is an averaged effective model

$$\dot{\bar{\xi}}(t) \;\; = \;\; \bar{b}(\bar{\xi}(t)) + \sigma_3(\bar{\xi}(t))\dot{W}_t, \quad \bar{\xi}(0) = x_0. \tag{2.3}$$

It follows from [6] that $\sup\limits_{0\leq t\leq T} \mathbb{E}|\xi - \bar{\xi}|^2 \to 0$ and $\lim\limits_{\varepsilon\to 0} \mathbb{P}\big\{\sup\limits_{t\in[0,T]} |\xi - \bar{\xi}| > \delta\big\} = 0$.

Now we consider the following effective system for the original microscopic-macroscopic system (2.1):

$$\begin{cases} \bar{u}_t = \bar{u}_{xx} + f(\bar{u}, \bar{\xi}), & \bar{u}(x,0) = u_0(x), \\ \dot{\bar{\xi}} = \bar{b}(\bar{\xi}) + \sigma_3(\bar{\xi})\dot{W}_t, & \bar{\xi}(0) = x_0. \end{cases} \tag{2.4}$$

We have already known above that ξ converges to $\bar{\xi}$ in probability, uniformly on bounded time intervals. Our goal in this section is to show that u converges to $\bar{u}$ in some probabilistic sense.

Theorem 2.1. *(Effective dynamical reduction I)*
Under the above assumptions on the coefficients, the system (2.4) *is an effective description of the original system* (2.1)*. That is, for any $T > 0$ and $\delta > 0$,*

$$\mathbb{P}\Big\{\sup_{t\in[0,T]} \|u - \bar{u}\|^2 > \delta\Big\} \to 0, \qquad as \quad \varepsilon \to 0.$$

This says that u converges to $\bar{u}$ in probability, uniformly on any finite time intervals.

Proof. Recall the Gronwall's inequality in differential form: Let $z : [0,T] \to \mathbb{R}$ satisfy the differential inequality

$$\frac{dz}{dt} \leq g(t)z + h(t).$$

Then

$$z(t) \leq z(0)\exp\Big(\int_0^t g(r)\,dr\Big) + \int_0^t \exp\Big(\int_s^t g(r)\,dr\Big)h(s)\,ds.$$

Denoting $U = u - \bar{u}$, then

$$U_t \;\; = \;\; U_{xx} + f(u,\xi) - f(\bar{u}, \bar{\xi}), \tag{2.5}$$

Multiply each side of the equation above by $2U$ and taking integral, by Young's inequality and the global Lipschitz condition on f, we get

$$\begin{aligned}\frac{d}{dt}\|U\|^2 &= -2\|U_x\|^2 + \|U\|^2 + K^f\|U\|^2 + K^f|\xi - \bar{\xi}|^2 \\ &\leq (1+K^f)\|U\|^2 + K^f|\xi - \bar{\xi}|^2.\end{aligned}$$

Taking expectation and by the Gronwall's inequality, we obtain

$$\begin{aligned}\|U\|^2 &\leq \int_0^t e^{-(1+K^f)(s-t)} K^f |\xi - \bar{\xi}|^2\, ds \\ &\leq e^{(1+K^f)T} K^f \int_0^T |\xi - \bar{\xi}|^2\, ds.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{P}\Big\{\sup_{t\in[0,T]} \|U\|^2 > \delta\Big\} &\leq \mathbb{P}\Big\{\sup_{t\in[0,T]} e^{(1+K^f)T} K^f \int_0^T |\xi - \bar{\xi}|^2\, ds > \delta\Big\} \\ &\leq \mathbb{P}\Big\{\int_0^T \sup_{s\in[0,T]} |\xi - \bar{\xi}|^2\, ds = T \sup_{t\in[0,T]} |\xi - \bar{\xi}|^2 > \delta/(e^{(1+K^f)T} K^f)\Big\} \\ &= \mathbb{P}\Big\{\sup_{t\in[0,T]} |\xi - \bar{\xi}|^2 > \delta/(T e^{(1+K^f)T} K^f)\Big\}.\end{aligned}$$

By the result $\lim_{\varepsilon\to 0} \mathbb{P}\{\sup_{t\in[0,T]} |\xi - \bar{\xi}| > \delta\} = 0$ in [6], we finally have $\mathbb{P}\Big\{\sup_{t\in[0,T]} \|U\|^2 > \delta\Big\} \to 0$ as $\varepsilon \to 0$. This completes the proof. □

3 A more complex stochastic microscopic-macroscopic model

In this section we consider teh following more complicated slow-fast microscopic-macroscopic stochastic system

$$\begin{cases} u_t = u_{xx} + f(u, v, \xi) + \sigma_1(u)\dot{W}_t^1, \quad u(x, 0) = u_0(x), \\ v_t = \frac{1}{\varepsilon}(v_{xx} + g(u, v, \xi)) + \frac{1}{\sqrt{\varepsilon}}\sigma_2(u, v)\dot{W}_t^2, \quad v(x, 0) = v_0(x), \\ \dot{\xi} = b(\xi, \eta) + \sigma_3(\xi)\dot{W}_t^3, \quad \xi(0) = x_0, \\ \dot{\eta} = \varepsilon^{-1}B(\xi, \eta) + \varepsilon^{-\frac{1}{2}}\sigma_4(\xi, \eta)\dot{W}_t^3, \quad \eta(0) = y_0, \\ u(0, t) = u(1, t) = 0, \quad v(0, t) = v(1, t) = 0, \end{cases} \tag{3.1}$$

for $t \in [0, T]$, $x \in [0, 1]$, where ε is a small positive parameter, and $\{W_t^i\}_{t\geq 0}, i = 1, 2, 3$ are independent scalar Brownian motions. For the coefficients we have the following assumptions:
H1: The drift coefficients $f(u, v, \xi)$:$\mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, diffusion coefficients $\sigma_1(u)$:$\mathbb{R} \to \mathbb{R}$ are Lipschitz continuous with respect to all three variables and hence also linear growth, i.e. there exist constant K_f, K_{σ_1} such that for any u_1, u, v_1, $v \in H$ and ξ_1, $\xi \in \mathbb{R}$,

$$\begin{aligned}|f(u_1, v_1, \xi_1) - f(u, v, \xi)|^2 &\leq K_f(|u_1 - u|^2 + |v_1 - v|^2 + |\xi_1 - \xi|^2), \\ |f(u, v, \xi)|^2 &\leq K_f(1 + |u|^2 + |v|^2 + |\xi|^2), \\ |\sigma_1(u_1) - \sigma_1(u)|^2 &\leq K_{\sigma_1}|u_1 - u|^2, \\ |\sigma_1(u)|^2 &\leq K_{\sigma_1}(1 + |u|^2).\end{aligned} \tag{3.2}$$

In addition, f is bounded, i.e. exists C_f, such that

$$|f(u,v,\xi)| \leq C_f. \tag{3.3}$$

H2: There exist constant K_g, K_{σ_2}, such that for any u_1, u, v_1, v, ξ_1, ξ,

$$\begin{aligned} |g(u_1,v_1,\xi_1) - g(u,v,\xi)|^2 &\leq K_g(|u_1-u|^2 + |v_1-v|^2 + |\xi_1-\xi|^2), \\ |g(u,v,\xi)|^2 &\leq K_g(1+|u|^2+|v|^2+|\xi|^2), \\ |\sigma_2(u_1,v_1) - \sigma_2(u,v)|^2 &\leq K_{\sigma_2}(|u_1-u|^2 + |v_1-v|^2), \\ |\sigma_2(u)|^2 &\leq K_{\sigma_2}(1+|u|^2). \end{aligned} \tag{3.4}$$

Moreover, there exist constants $\alpha > 0$ and C_{σ_2}, such that

$$\begin{aligned} v \cdot g(u,v,\xi) &\leq \alpha |v|^2, \\ \sigma_2(u,v) &\leq C_{\sigma_2}. \end{aligned} \tag{3.5}$$

H3: There exist K_b, C_b, K_{σ_3}, C_{σ_3} such that for any ξ, ξ_1, η, η_1,

$$\begin{aligned} |b(\xi,\eta) - b(\xi_1,\eta_1)|^2 &\leq K_b(|\xi-\xi_1|^2 + |\eta-\eta_1|^2), \\ |b(\xi,\eta)|^2 &\leq K_b(1+|\xi|^2+|\eta|^2), \\ |\sigma_3(\xi) - \sigma_3(\xi_1)|^2 &\leq K_{\sigma_3}|\xi-\xi_1|^2, \\ |\sigma_3(\xi)|^2 &\leq K_{\sigma_3}(1+|\xi|^2), \\ |b(\xi,\eta)| &\leq C_b, \\ |\sigma_3(\xi)| &\leq C_{\sigma_3}. \end{aligned} \tag{3.6}$$

Furthermore, there exists a constant $\beta > 0$, such that

$$\xi \cdot b(\xi,\eta) \leq \beta(1+|\xi|^2). \tag{3.7}$$

H4: There exist K_B, C_B, K_{σ_4}, C_{σ_4} such that for any ξ, ξ_1, η, η_1,

$$\begin{aligned} |B(\xi,\eta) - B(\xi_1,\eta_1)|^2 &\leq K_B(|\xi-\xi_1|^2 + |\eta-\eta_1|^2), \\ |B(\xi,\eta)|^2 &\leq K_B(1+|\xi|^2+|\eta|^2), \\ |\sigma_4(\xi,\eta) - \sigma_4(\xi_1,\eta_1)|^2 &\leq K_{\sigma_4}(|\xi-\xi_1|^2 + |\eta-\eta_1|^2), \\ |\sigma_4(\xi,\eta)|^2 &\leq K_{\sigma_4}(1+|\xi|^2+|\eta|^2), \\ |B(\xi,\eta)| &\leq C_B, \\ |\sigma_4(\xi,\eta)| &\leq C_{\sigma_4}. \end{aligned} \tag{3.8}$$

H5: $2\lambda_1 + 2\alpha - K_{\sigma_2} > 0$, where λ_1 is the first eigenvalue of the operator $-\Delta$.

Let H be the Hilbert space $L^2(D)$, equipped with inner products $(\cdot,\cdot)_H$, and norm $\|\cdot\| = (\cdot,\cdot)_H^{\frac{1}{2}}$. Define the operator $A = \Delta$ with zero Dirichlet boundary condition. Let $\{e_k(x)\}_{k\geq 1}$ be the complete orthogonal system of eigenfunctions in H such that, for $k = 1, 2\cdots$,

$$-\Delta e_k = \lambda_k e_k, \ \ e_k|_{\partial D} = 0, \tag{3.9}$$

with $0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_k \leq \cdots$. It is well known that the semigroup $\{G_t\}_{t\geq 0}$ generated by Δ can be defined by,

$$(G_t h)(\varsigma) = \int_D G(\varsigma,\zeta,t)h(\zeta)d\zeta,$$

for any $h(\varsigma) \in H$, where $G(\varsigma, \zeta, t) = \sum_{k=1}^{\infty} e^{-\alpha_k t} e_k(\varsigma) e_k(\zeta)$. It is clear that $\|G_t h\| \leq \|h\|$, thus $\{G_t\}_{t\geq 0}$ is a contraction semigroup. Let V be the Sobolev space H_0^1 of order 1 with Dirichlet boundary conditions, which is densely and continuously injected in the Hilbert space H. V, H and $V^\star$ satisfies a Gelfand triple

$$V \subset H \subset V^\star,$$

and

$$\Delta : V \to V^\star.$$

With the Poincare inequality, we have

$$\langle \Delta v, v \rangle = -\|\nabla v\|^2 \leq -\lambda \|v\|^2, \tag{3.10}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairs of $(V^\star, V)$.
Under the assumptions, the macroscopic fast equation has a unique stationary solution, with distribution μ^u independent of ε, and the average is

$$\bar{f}(u, \xi) \quad = \quad \int_H f(u, v, \xi)\, \mu^u(dv), \quad u \in H, \quad \xi \in \mathbb{R}. \tag{3.11}$$

Then we deal with the following macroscopic effective system

$$\bar{u}_t \quad = \quad \bar{u}_{xx} + \bar{f}(\bar{u}, \bar{\xi}) + \sigma_1(\bar{u}) \dot{W}_t^1, \quad \bar{u}(x, 0) = u_0(x). \tag{3.12}$$

Moreover, an averaged microscopic effective model for ξ is defined as in the last section.

Now we consider the following effective system for the original microscopic-macroscopic system (3.1):

$$\begin{cases} \bar{u}_t = \bar{u}_{xx} + \bar{f}(\bar{u}, \bar{\xi}) + \sigma_1(\bar{u}) \dot{W}_t^1, \quad \bar{u}(x, 0) = u_0(x), \\ \dot{\bar{\xi}} = \bar{b}(\bar{\xi}) + \sigma_3(\bar{\xi}) \dot{W}_t^3, \quad \bar{\xi}(0) = x_0. \end{cases} \tag{3.13}$$

We have already known above that ξ converges to $\bar{\xi}$ in probability, uniformly on bounded time intervals. Our goal in this section is to show that u converges to $\bar{u}$ in some probabilistic sense.

The well-posedness for both systems (3.1) and (3.13) is verified as in [5].

Definition 3.1. *(Mild solution). For fixed ξ, an $H \times H-$ valued predictable process $(u(t), v(t))$ is called a* mild solution *of the first two components of Eq.* (3.1) *if for any $t \in [0, T]$,*

$$\begin{cases} u(t) = G_t u_0 + \int_0^t G_{t-s} f(u(s), v(s), \xi) ds + \int_0^t G_{t-s} \sigma_1(u(s)) dW_s^1, \\ v(t) = G_t^\varepsilon v_0 + \frac{1}{\varepsilon} \int_0^t G_{t-s}^\varepsilon g(v(s), v(s), \xi) ds + \frac{1}{\sqrt{\varepsilon}} \int_0^t G_{t-s}^\varepsilon \sigma_2(u(s), v(s)) dW_s^2, \end{cases} \tag{3.14}$$

where $\{G_t^\epsilon\}_{t\geq 0}$ denote the semigroup generated by differential operator $\frac{\Delta}{\epsilon}$.

Definition 3.2. *(Strong solution). For fixed ξ, a $V \times V-$ valued predictable process $(u(t), v(t))$ is called a strong solution of the first two components of Eq.* (3.1) *if, for any $\varphi \in V$,*

$$\begin{cases} \big(u(t), \varphi\big)_H = \big(u_0, \varphi\big)_H + \int_0^t \langle \Delta u(s)^\epsilon, \varphi \rangle ds + \int_0^t \big(f(u(s), v(s), \xi), \varphi\big)_H ds \\ \qquad\qquad\quad + \int_0^t \big(\sigma_1(u(s)), \varphi\big)_H dW_s^1, \\ \big(v(t), \varphi\big)_H = \big(v_0, \varphi\big)_H + \frac{1}{\varepsilon} \int_0^t \langle \Delta v(s), \varphi \rangle ds + \frac{1}{\epsilon} \int_0^t \big(g(u(s), v(s), \xi), \varphi\big)_H ds \\ \qquad\qquad\quad + \frac{1}{\sqrt{\epsilon}} \int_0^t \big(\sigma_2(u(s), v(s)), \varphi\big)_H dW_s^2, \end{cases} \tag{3.15}$$

hold for any $t \in [0, T]$ a.s..

Under the assumptions we listed, for any fixed $u_0 \in H$ and any $v_0 \in H$, the first two equations of the Eq. (3.1) has a unique strong solution (also a mild solution). Moreover, the following energy identities hold ([8] or [2]):

$$\begin{aligned}\|u(t)\|^2 &= \|u_0\|^2 + 2\int_0^t \langle \Delta u(s), u(s)\rangle ds + 2\int_0^t \big(f(u(s), v(s), \xi), u(s)\big)_H ds \\ &\quad +2\int_0^t \big(\sigma_1(u(s), u(s)\big)_H dW_s^1 + \int_0^t \|\sigma_1(u(s))\|^2 ds, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\|v(t)\|^2 &= \|v_0\|^2 + \frac{2}{\epsilon}\int_0^t \langle \Delta v(s), v(s)\rangle ds + \frac{2}{\epsilon}\int_0^t \big(g(u(s), v(s), \xi), v(s)\big)_H ds \\ &\quad +\frac{2}{\sqrt{\epsilon}}\int_0^t \big(\sigma_2(u(s), v(s)), v(s)\big)_H dW_s^2 + \frac{1}{\epsilon}\int_0^t \|\sigma_2(u(s), v(s))\|^2 ds. \end{aligned} \tag{3.17}$$

Similar to the case in section 2, for fixed $u_0 \in H$, $\xi \in \mathbb{R}$ we introduce a fast motion with frozen slow component

$$\begin{cases} dv(t) = [v_{xx}(t) + g(u_0, v(t), \xi)]dt + \sigma_2(u_0, v(t))dW_t^2, \\ v(x,0) = v_0(x), \quad x \in [0,1], \\ v(0,t) = v(1,t) = 0, \quad t \in [0,T]. \end{cases} \tag{3.18}$$

Under the assumptions, for any fixed $u_0 \in H$ and any $v_0 \in H$, the Eq. (3.18) has a unique strong solution (also a mild solution), which will be denoted by $v^{u_0,v_0}(t)$. By energy equality similar to (3.17), one can get

$$\mathbb{E}\|v^{u_0,v_0}(t)\|^2 \le \|v_0\|^2 - 2(\lambda_1 + \alpha)\int_0^t \mathbb{E}\|v^{u_0,v_0}(s)\|^2 ds + Ct.$$

By the Gronwall's inequality again we have

$$\mathbb{E}\|v^{u_0,v_0}(t)\|^2 \le C\Big(1 + \|v_0\|^2 e^{-2(\lambda_1+\alpha)t}\Big).$$

Let $v^{u_0,v_0'}(t)$ be the solution of Eq. (3.18) with initial value $v(0) = v_0'$. With the aid of energy equality similar to (3.17), we get that

$$\begin{aligned}\mathbb{E}\|v^{u_0,v_0}(t) - v^{u_0,v_0'}(t)\|^2 &= \|v_0 - v_0'\|^2 + 2\mathbb{E}\int_0^t \langle A(v^{u_0,v_0}(s) - v^{u_0,v_0'}(s)), v^{u_0,v_0}(s) - v^{u_0,v_0'}(s)\rangle\, ds \\ &\quad +2\mathbb{E}\int_0^t \big(g(u_0, v^{u_0,v_0}(s), \xi) - g(u_0, v^{u_0,v_0'}(s), \xi), v^{u_0,v_0}(s) - v^{u_0,v_0'}(s)\big)_H ds \\ &\quad +\mathbb{E}\int_0^t \|\sigma_2(u_0, v^{u_0,v_0}(s)) - \sigma_2(u_0, v^{u_0,v_0'}(s))\|^2 ds \\ &\le \|v_0 - v_0'\|^2 - (2\alpha_1 + 2\beta - C_{\sigma_2})\int_0^t \mathbb{E}\|v^{u_0,v_0}(s) - v^{u_0,v_0'}(s)\|^2\, ds. \end{aligned}$$

Hence

$$\mathbb{E}\|v^{u_0,v_0}(t)-v^{u_0,v_0'}(t)\|^2 \le \|v_0-v_0'\|^2 e^{-\kappa t}, \tag{3.19}$$

where $\kappa = 2\alpha_1 + 2\beta - C_{\sigma_2} > 0$.

For any $u \in H$ denote by P_t^u the Markov semigroup associated to Eq. (3.1) defined by

$$P_t^u f(z) = \mathbb{E}f(V_t^{u,z}), \quad t \ge 0, \quad z \in H,$$

for any $f \in \mathcal{B}_b(H)$ the space of bounded functions on H. We also recall a probability μ^u on H is called that a invariant measure for $(P_t^u)_{t\ge0}$ if

$$\int_H P_t^u f d\mu^u = \int_H f d\mu^u, \quad t \ge 0,$$

for any bounded function $f \in \mathcal{B}_b(H)$. As in [4], it is possible to show there exists an unique invariant measure μ^u for the semigroup P_t^u which satisfies

$$\int_H \|z\|\mu^u(dz) \le C(1+\|u\|). \tag{3.20}$$

Furthermore, according to Lipschitz assumption on f and (3.19) we have

$$\begin{aligned}
&\left\|\mathbb{E}f(u,V_t^{u,v},\xi) - \int_H f(u,z,\xi)\mu^u(dz)\right\| \\
= &\left\| \int_H [\mathbb{E}f(u,V_t^{u,v},\xi) - \mathbb{E}f(u,V_t^{u,z},\xi)]\mu^u(dz)\right\| \\
\le & C\int_H \mathbb{E}\|V_t^{u,v} - V_t^{u,z}\|\mu^u(dz) \\
\le & Ce^{-\frac{\kappa}{2}t}\int_H \|v-z\|\mu^u(dz) \\
\le & Ce^{-\frac{\kappa}{2}t}\Big[\|v\| + \int_H \|z\|\mu^u(dz)\Big] \\
\le & Ce^{-\frac{\kappa}{2}t}\Big[1+\|u\|+\|v\|\Big].
\end{aligned} \tag{3.21}$$

The following arguments follow [7]. First we have some mean square uniform estimates on u, v, and ξ.

Lemma 3.1. *There exists a constant $C_T > 0$ such that*

$$\sup_{0\le t\le T} \mathbb{E}|\xi|^2 \le C_T. \tag{3.22}$$

Proof. For the slow equation ξ of the microscopic system, multiplying each side with 2ξ, we get

$$\frac{d}{dt}|\xi|^2 = 2\xi\cdot b(\xi,\eta) + 2\xi\cdot\sigma_3(\xi)\dot{W}_t^3.$$

After integrating and taking expectation on both sides, we get

$$\begin{aligned}
\mathbb{E}|\xi|^2 &= x^2 + 2\mathbb{E}\int_0^t \xi\cdot b(\xi,\eta)\,ds + 2\mathbb{E}\int_0^t \xi\cdot\sigma_3(\xi)dW_s^3 \\
&\le x^2 + 2\beta t + 2\beta\int_0^t \mathbb{E}|\xi|^2\,ds.
\end{aligned}$$

Thanks to the Gronwall's inequality, we finally have

$$\begin{aligned}\mathbb{E}|\xi|^2 &\leq x^2 + 2\beta t + 2\beta \int_0^t (x^2 + 2\beta s) e^{2\beta(t-s)}\, ds \\ &= (1 + x^2) e^{2\beta t} - 1 \\ &\leq C_T.\end{aligned}$$

□

Lemma 3.2. *There exists a constant $C > 0$ such that*

$$\sup_{0 \leq t \leq T} \mathbb{E}\|v\|^2 \leq C. \tag{3.23}$$

Proof. Due to energy identity (3.17), coercivity (3.10), the assumption H2 and the Gronwall's inequality, we obtain the desired result.

□

Lemma 3.3. *There exists a constant $C_T > 0$ such that*

$$\sup_{0 \leq t \leq T} \mathbb{E}\|u(t)\|^2 \leq C_T. \tag{3.24}$$

Proof. Applying energy identity (3.16), with the aid of (3.10) and the above two lemmas , we get

$$\begin{aligned}\mathbb{E}\|u(t)\|^2 &= \|u_0\|^2 + \mathbb{E}\int_0^t \langle \Delta u(s), u(s)\rangle ds + \mathbb{E}\int_0^t \Big(f(u(s), v(s), \xi), u(s)\Big)_H ds + \mathbb{E}\int_0^t \|\sigma_1(u(s))\|^2 ds \\ &\leq \|u_0\|^2 + C\int_0^t \mathbb{E}\|u(s)\|^2 ds + C\int_0^t \mathbb{E}(1 + \|u(s)\|^2 + \|v(s)\|^2 + \|\xi\|^2) ds \\ &\leq \|u_0\|^2 + C_T \int_0^t \mathbb{E}\|u(s)\|^2 ds + Ct.\end{aligned}$$

The Gronwall's inequality yields the desired estimation.

□

Lemma 3.4. *For any $h \in (0, 1)$ and $\gamma \in (0, \frac{1}{2})$, there exists a constant $C_\gamma > 0$ such that*

$$\mathbb{E}\|u(t+h) - u(t)\|^2 \leq C_\gamma h^\gamma. \tag{3.25}$$

Proof. In the mild sense

$$\begin{aligned}u(t+h) - u(t) &= [G_{t+h}u_0 - G_t u_0] + \int_t^{t+h} G_{t+h-s} f(u(s), v(s), \xi)\, ds \\ &\quad + \int_t^{t+h} G_{t+h-s}\sigma_1(X_s^\epsilon) dW_s^1 \\ &\quad + \int_0^t [G_{t+h-s} f(u(s), v(s), \xi) - G_{t-s} f(u(s), v(s), \xi)]\, ds \\ &\quad + \int_0^t [G_{t+h-s}\sigma_1(u(s)) - G_{t-s}\sigma(u(s))] dW_s^1 \\ &=: \sum_{i=1}^5 I_i.\end{aligned} \tag{3.26}$$

By the property of semigroup G_t (see [9]), we have the estimate of I_1,

$$\|I_1\|^2 \leq h^2\|\Delta u_0\|^2. \tag{3.27}$$

By the Hölder inequality and the bounded property of f, we deduce that

$$\begin{aligned}\mathbb{E}\|I_2\|^2 &\leq h\mathbb{E}\int_t^{t+h}\|G_{t+h-s}f(u(s),v(s),\xi)\|^2 ds \\ &\leq Ch\int_t^{t+h}\mathbb{E}\|f(u(s),v(s),\xi)\|^2\, ds \\ &\leq Ch^2. \end{aligned}\tag{3.28}$$

Using the Itô isometry and Hölder inequality, it yields

$$\begin{aligned}\mathbb{E}\|I_3\|^2 &= \mathbb{E}\int_t^{t+h}\|G_{t+h-s}\sigma_1(u(s))\|^2 ds \\ &\leq C\int_t^{t+h}\mathbb{E}[1+\|u(s)\|^2]\, ds \\ &\leq C_T h. \end{aligned}\tag{3.29}$$

Moreover

$$\mathbb{E}\|I_4\|^2 \leq C_\gamma h^\gamma,$$

and

$$\mathbb{E}\|I_5\|^2 \leq C_{T,\gamma} h^\gamma,$$

are obtained in the same way as those in [7], where $f(u,v,\xi)$ and $f(u,v)$ are both bounded.
As a result of (3.26)—(3.30), we obtain inequality (3.25). □

Next, we introduce an auxiliary process $(\hat{u}(t),\hat{v}(t)) \in H\times H$. Fix a positive number δ and do a partition of time interval $[0,T]$ of size δ. We construct a process $\hat{v}(t)$ by means of the equations

$$\begin{aligned}\hat{v}(t) = v(k\delta) \quad &+ \quad \frac{1}{\varepsilon}\int_{k\delta}^t \Delta\hat{v}(s) + \frac{1}{\varepsilon}\int_{k\delta}^t g(u(k\delta),\hat{v}(s),\xi(s))\, ds \\ &+ \quad \frac{1}{\sqrt{\varepsilon}}\int_{k\delta}^t \sigma_2(u(k\delta),\hat{v}(s))dW_s^2, \end{aligned}\tag{3.30}$$

for $t \in \big[k\delta, \min\big((k+1)\delta, T\big)\big)$.
Also define the process $\hat{u}(t)$ by linear equation with additive noise

$$\begin{aligned}\hat{u}(t) = u_0 \quad &+ \quad \int_0^t \Delta\hat{u}(s)\, ds + \int_0^t f(u([s/\delta]\delta),\hat{v}(s),\xi(s))\, ds \\ &+ \int_0^t \sigma_1(u(s))\, dW_s^1, \end{aligned}\tag{3.31}$$

for $t \in [0,T]$.

Similar to the mean square uniform estimates on v, we have

Lemma 3.5. *There exists a constant $C > 0$ such that*

$$\sup_{0 \le t \le T} \mathbb{E}\|\hat{v}\|^2 \le C. \tag{3.32}$$

We now are ready to establish mean-square convergence of the auxiliary processes $\hat{v}(t)$ and $\hat{u}(t)$ to the fast solution process $v(t)$ and slow $u(t)$, respectively.

Lemma 3.6. *For any $\gamma \in (0, \frac{1}{2})$, there exist constants $C_{T,\gamma} > 0$ such that*

$$\sup_{0 \le t \le T} \mathbb{E}\|v(t) - \hat{v}(t)\|^2 \le C_{T,\gamma} \frac{\delta^{1+\gamma}}{\varepsilon} e^{\frac{C\delta}{\varepsilon}}.$$

Proof. For $t \in [0, T]$ with $t \in [k\delta, (k+1)\delta)$, by energy identity (3.17) , (3.10) and the Lipschitz condition of $g(u, v, \xi)$ that

$$|g(u(s), v(s), \xi) - g(u(k\delta), \hat{v}(s), \xi)|^2 \le K_g(\|u(s) - u(k\delta)\|^2 + \|v(s) - \hat{v}(s)\|^2),$$

we get the desired result.

□

The next lemma is by the same argument with the help of

$$|f(u, v, \xi) - f(u([t/\delta]\delta), \hat{v}, \xi)|^2 \le K_f(\|u - u([t/\delta]\delta)\|^2 + \|v - \hat{v}\|^2).$$

Lemma 3.7. *For any $\gamma \in (0, \frac{1}{2})$, there exists constant $C_{T,\gamma} > 0$ such that*

$$\sup_{0 \le t \le T} \mathbb{E}\|u(t) - \hat{u}(t)\|^2 \le C_{T,\gamma}(\delta^\gamma + \frac{\delta^{1+\gamma}}{\varepsilon} e^{\frac{C\delta}{\varepsilon}}).$$

In the following we prove the averaging principle that the slow component process $u(t)$ converges in mean-square sense to an effective dynamics equation as follows

$$\begin{cases} d\bar{u}(t) = \Delta\bar{u}(t)dt + \bar{f}(\bar{u}(t), \bar{\xi})dt + \sigma_1(\bar{u}(t))dW_t^1, \\ \bar{u}(x, 0) = u_0(x). \end{cases} \tag{3.33}$$

The following lemma formulates mean-square convergence of the auxiliary process $\hat{u}(t)$ to the averaged solution process $\bar{u}(t)$.

Lemma 3.8. *For any $\gamma \in (0, \frac{1}{2})$, there exist constants $C_{T,\gamma} > 0$ such that*

$$\mathbb{E}\|\hat{u}(t) - \bar{u}(t)\|^2 \le C_{T,\gamma}(\delta^\gamma + \frac{\varepsilon}{\delta} + \frac{\delta^{1+\gamma}}{\varepsilon} e^{\frac{C\delta}{\varepsilon}}).$$

Proof. In the mild sense, we have

$$\begin{aligned}
\hat{u}(t)-\bar{u}(t) &= \int_0^t G_{t-s}[f(u([s/\delta]\delta),\hat{v}(s),\xi)-\bar{f}(u(s),\xi)]\,ds+\int_0^t G_{t-s}[\bar{f}(u(s),\xi)-\bar{f}(\hat{u}(s),\xi)]\,ds \\
&\quad +\int_0^t G_{t-s}[\bar{f}(\hat{u}(s),\xi)-\bar{f}(\bar{u}(s),\bar{\xi})]\,ds+\int_0^t G_{t-s}[\sigma_1(u(s))-\sigma_1(\hat{u}(s))]\,dW_s^1 \\
&\quad +\int_0^t G_{t-s}[\sigma_1(\hat{u}(s))-\sigma_1(\bar{u}(s))]\,dW_s^1 \\
&:= \sum_{i=1}^{5} J_i(t).
\end{aligned}$$

In view of the Hölder inequality, the Lipschitz condition of $\bar{f}(u,\xi)$ and contraction of the semigroup G_t, it follows from Lemma 3.7 that

$$\mathbb{E}\|J_2(t)\|^2 \;\le\; C_{T,\gamma}(\delta^\gamma+\frac{\delta^{1+\gamma}}{\varepsilon}e^{\frac{C\delta}{\varepsilon}}).$$

For J_3 , because of the Lipschitz continuity of $\bar{f}$ we have

$$\begin{aligned}
\mathbb{E}\|J_3(t)\|^2 &\le C_T\mathbb{E}\int_0^t\|\bar{f}(\hat{u}(s),\xi)-\bar{f}(\bar{u}(s),\bar{\xi})\|^2\,ds \\
&\le C_T\int_0^t \mathbb{E}(\|\hat{u}(s)-\bar{u}(s)\|^2+|\xi-\bar{\xi}|^2)\,ds. \qquad (3.34)
\end{aligned}$$

Furthermore, J_4, J_5 are estimated using the properties of G_t and Lemma (3.7),

$$\mathbb{E}\|J_4(t)\|^2 \;\le\; C_{T,\gamma}(\delta^\gamma+\frac{\delta^{1+\gamma}}{\varepsilon}e^{\frac{C\delta}{\varepsilon}}),$$

$$\mathbb{E}\|J_5(t)\|^2 \;\le\; C\int_0^t\mathbb{E}\|\hat{u}(s)-\bar{u}(s)\|^2\,ds.$$

For $\mathbb{E}\|J_1(t)\|^2$ with $t\in[k\delta,(k+1)\delta)$, we write

$$\begin{aligned}
J_1(t) &= \sum_{p=0}^{k-1}\int_{p\delta}^{(p+1)\delta} G_{t-s}[f(u(p\delta),\hat{v}(s),\xi)-\bar{f}(u(p\delta),\xi)]\,ds \\
&\quad +\sum_{p=0}^{k-1}\int_{p\delta}^{(p+1)\delta} G_{t-s}[\bar{f}(u(p\delta),\xi)-\bar{f}(u(s),\xi)]\,ds \\
&\quad +\int_{k\delta}^{t} G_{t-s}[f(u(p\delta),v(s),\xi)-\bar{f}(u(s),\xi)]\,ds \qquad (3.35) \\
&:= J_1'(t)+J_2'(t)+J_3'(t).
\end{aligned}$$

Due to (3.25), we conclude

$$\mathbb{E}\|J_2'(t)\|^2 \;\le\; C_{T,\gamma}\delta^\gamma,$$

with $\gamma \in (0, \frac{1}{2})$.
According to the mean square uniform estimates on u, $\hat{v}$, ξ and the linear growth conditions of f and $\bar{f}$, we get

$$\begin{aligned}\mathbb{E}\|J_3'(t)\|^2 &= \mathbb{E}\|\int_{k\delta}^{t} G_{t-s}[f(u(k\delta), \hat{v}(s), \xi) - \bar{f}(u(s), \xi)]\, ds\|^2 \\ &\leq \delta \mathbb{E}\int_{k\delta}^{t} \|f(u(k\delta), \hat{v}(s), \xi) - \bar{f}(u(s), \xi)\|^2\, ds \\ &\leq C\delta \int_{k\delta}^{t} \mathbb{E}[1 + \|u(k\delta)\|^2 + \|\hat{v}(s)\|^2 + \|u(s)\|^2 + |\xi|^2]\, ds \\ &\leq C_T\delta^2. \qquad (3.36)\end{aligned}$$

The argument of the estimate of

$$\mathbb{E}\|J_1'\|^2 \leq C_T\frac{\varepsilon}{\delta}, \qquad (3.37)$$

is the same as that in [7], except that the coefficient f has an extra parameter ξ, which can be handled using (3.21) and the boundedness conditions for f.

Combing (3.36), (3.36) and (3.37) it yields

$$\mathbb{E}\|J_1(t)\|^2 \leq C_{T,\gamma}\delta^{\gamma} + C_T\frac{\varepsilon}{\delta}. \qquad (3.38)$$

Therefore, combining together (3.34)—(3.35) and (3.38) we obtain

$$\mathbb{E}\|\hat{u}(t) - \bar{u}(t)\|^2 \leq C_{T,\gamma}(\delta^{\gamma} + \frac{\varepsilon}{\delta} + \frac{\delta^{1+\gamma}}{\varepsilon}e^{\frac{C\delta}{\varepsilon}} + \sup_{0\leq t\leq T}\mathbb{E}|\xi - \bar{\xi}|^2) + C_T\int_0^t \mathbb{E}\|\hat{u}(s) - \bar{u}(s)\|^2\, ds \qquad (3.39)$$

and thus

$$\mathbb{E}\|\hat{u}(t) - \bar{u}(t)\|^2 \leq C_{T,\gamma}(\delta^{\gamma} + \frac{\varepsilon}{\delta} + \frac{\delta^{1+\gamma}}{\varepsilon}e^{\frac{C\delta}{\varepsilon}} + \sup_{0\leq t\leq T}\mathbb{E}|\xi - \bar{\xi}|^2).$$

This proves the lemma. □

Finally we have the following theorem.

Theorem 3.1. *(Effective dynamical reduction II)*
Under the Hypotheses (H1)—(H5), the system (3.13) *is an effective description of the original system* (3.1). *That is, for any* $T > 0$,

$$\lim_{\epsilon\to 0}\sup_{0\leq t\leq T}\mathbb{E}\|u(t) - \bar{u}(t)\|^2 \to 0. \qquad (3.40)$$

This says that u converges to $\bar{u}$ in mean-square, uniformly on finite time intervals.

Remark 3.1. *According to* (3.40) *and by the Chebyshev inequality, there is a direct consequence that u converges to $\bar{u}$ in probability.*

Proof. By Lemma (3.7) and Lemma (3.8) and take $\delta = \epsilon[-\ln \epsilon]^{\frac{1}{2}}$, we have

$$\begin{aligned}\sup_{0\le t\le T} \mathbb{E}\|u(t)-\bar{u}(t)\|^2 &\le \sup_{0\le t\le T} \mathbb{E}\|u(t)-\bar{u}(t)\|^2 + \sup_{0\le t\le T} \mathbb{E}\|u(t)-\bar{u}(t)\|^2 \\ &\le C_{T,\gamma}(\delta^\gamma + \frac{\varepsilon}{\delta} + \frac{\delta^{1+\gamma}}{\varepsilon} e^{\frac{C\delta}{\varepsilon}} + \sup_{0\le t\le T} \mathbb{E}|\xi - \bar{\xi}|^2) \\ &\to 0,\end{aligned}$$

as $\varepsilon \to 0$. □

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