Geometric shape of invariant manifolds for a class of stochastic partial differential equations *

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Abstract: Invariant manifolds play an important role in the study of the qualitative dynamical behaviors for nonlinear stochastic partial differential equations. However, the geometric shape of these manifolds is largely unclear. The purpose of the present paper is to try to describe the geometric shape of invariant manifolds for a class of stochastic partial differential equations with multiplicative white noises. The local geometric shape of invariant manifolds is approximated, which holds with significant likelihood. Furthermore, the result is compared with that for the corresponding deterministic partial differential equations.

Key words: Stochastic partial differential equation; invariant manifolds; geometric shape; analytical approximations; random dynamical systems

AMS subject classifications: 60H15, 37H05, 37L55, 37L25, 37D10.

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1 Introduction

Stochastic partial differential equations arise as models for various complex systems under random influences. There have been recent rapid progresses in this area (see [2, 6, 9, 14, 17]). For stochastic partial differential equations, random invariant manifolds play an important role in the study of dynamics because they provide a geometric structure to understand or reduce stochastic dynamics. Although the existence for such random invariant manifolds is established for certain stochastic partial differential equations (e.g., [4, 5, 7, 8, 11, 12]), the geometric shape of these manifolds is largely unclear. The purpose of the present paper is to try to describe the geometric shape of invariant manifolds for a class of stochastic partial differential equations.

We consider a class of stochastic partial differential equations in the following form

$$\frac{du}{dt} + Lu - u^p = \sigma u \circ \dot{W}, \tag{1.1}$$

subject to the homogeneous Dirichlet boundary conditions on a bounded domain with scalar white noise $\dot{W}$ of Stratonovich type. The linear operator $-L$ generates a C_0-semigroup, which is given in detail in the next section. The nonlinear power exponent p belongs to $(1, +\infty)$, and σ is a real parameter in $(0, +\infty)$.

It is well known that the theory of invariant manifolds has been developed well for deterministic dynamical systems. However, for the stochastic dynamical systems generated by stochastic partial differential equations, due to their nonclassical fluctuation of driving noise and infinite dimensionality, the theory of invariant manifolds, together with their approximation and computation, is still in its infancy.

For stochastic partial differential equation, a random invariant manifold has various samples in an infinite dimensional space. Therefore it is difficult in general to describe or "visualize" random invariant manifolds, let alone the reduction of dynamics on them. Blomker and Wang [3], and Sun et al [15] have done some work on describing such invariant manifolds. In this paper, we will consider an approximate local geometric shape of invariant manifolds for Equation (1.1).

More precisely, for Equation (1.1), we first construct a local invariant manifold. Then by approximating the invariant manifold step by step, we establish an approximate local geometric shape of the invariant manifold, which holds with probabilistic significance. Next, we study the corresponding deterministic system of Equation (1.1) (i.e., Equation (1.1) with $\sigma = 0$). Using the same method, we drive the invariant manifold and its approximating local geometric shape, which always holds.

This paper is organized as follows. In the next section, we present the assumptions of the linear operator L, introduce the basic concepts on random dynamical systems and the random evolutionary equation induced by Equation (1.1). In the third section, we show Theorem 3.1 on the existence of the local random invariant manifold for Equation (1.1). In the fourth section, we prove Theorem 4.1 on the local geometric shape of the random invariant manifold. Furthermore, we give an example to explain the local geometric shape in Remark 4.1. In the fifth section, we discuss the local geometric shape of the invariant manifold for the corresponding deterministic system of Equation (1.1). We comment on the results in the final section. We consider only unstable invariant manifolds, as stable invariant manifolds may be discussed similarly.

2 Preliminaries

2.1. Assumption of the linear operator L

Let E be a separable Hilbert space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$, and L be a closed self-adjoint linear operator with dense domain $D(L)$ in E. Let id be the identity operator on E.

Hypothesis *There exists a constant $a \geq 0$ such that $(L+a\cdot id)$ is positive and $(L+a\cdot id)^{-1}$ is compact.*

This assumption implies that the spectrum of L consists of only eigenvalues with finite multiplicities,

$$-a < \lambda_1 \leq \lambda_2 \leq \cdots, \qquad \lim_{n\to+\infty} \lambda_n = +\infty, \tag{2.1}$$

and the associated eigenfunctions $\{e_n\}_{n\in\mathbb{N}}$, $e_i \in D(L) \subset E$ form an othonormal basis of E. An example of the linear operator L is $L = -\partial_{xx} - 3\cdot id$ on $H_0^1([0,\pi])$, whose eigenvalues are $\lambda_k = k^2 - 3$ with the corresponding eigenfunctions $e_k = \sin kx$, $k = 1, 2, 3, \cdots$.

Furthermore, the positivity of $(L+a\cdot\mathrm{id})$ allows one to define the fractional power of $(L+a\cdot\mathrm{id})$, which we denote by $(L + a \cdot \mathrm{id})^\alpha$ for $\alpha \in [0, 1)$, see Henry [10] or Temam [16]. The domain of $(L + a \cdot \mathrm{id})^\alpha$, which we denote by E^α, is a Hilbert space with the scalar product $\langle u, \tilde{u}\rangle_\alpha = \langle (L + a \cdot \mathrm{id})^\alpha u, (L + a \cdot \mathrm{id})^\alpha \tilde{u}\rangle$ and corresponding norm $|\cdot|_\alpha$.

From (2.1), there exists $\lambda_N < 0$ such that $-a < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N < 0$ and $\lambda_{N+1} \ge 0$. Denote $\lambda_u := \lambda_N$ (< 0) and $\lambda_s := \lambda_{N+1}$ (≥ 0). Put $E_u := span\{e_1, \cdots, e_N\}$. Let P_u be the orthogonal projection from E to E_u and $P_s = I - P_u$. Put $L_u = P_u L$ and $L_s = P_s L$. In the following, we use the subscript "u" always for projection onto E_u and the subscript "s" for projection onto E_s. Then $E = E_u \oplus E_s$ and $E^\alpha = E_u \oplus E_s^\alpha$, where $E_s^\alpha = E_s \bigcap E^\alpha$ and $E_u \subset E^\alpha$ with $\alpha \in [0, 1)$.

From Henry [10], there exists $M > 0$ such that

$$\begin{array}{l} \|e^{-L_s t} P_s\|_{\mathcal{L}(E^\alpha, E^\alpha)} \le M e^{-\lambda_s t}, \quad t \ge 0; \\ \|e^{-L_s t} P_s\|_{\mathcal{L}(E^\alpha, E)} \le \frac{M}{t^\alpha} e^{-\lambda_s t}, \quad t \ge 0; \\ \|e^{-L_u t} P_u\|_{\mathcal{L}(E^\alpha, E^\alpha)} \le M e^{-\lambda_u t}, \quad t \le 0; \\ \|e^{-L_u t} P_u\|_{\mathcal{L}(E^\alpha, E)} \le M e^{-\lambda_u t}, \quad t \le 0, \end{array} \tag{2.2}$$

where $\mathcal{L}(X, Y)$ is the usual space of bounded linear operator from Banach space X to Banach space Y.

2.2. Random dynamical systems

Let us recall some basic concepts in random dynamical systems as in [7]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A flow θ of mappings $\{\theta_t\}_{t \in \mathbb{R}}$ is defined on the sample space Ω such that

$$\theta : \mathbb{R} \times \Omega \to \Omega, \quad \theta_0 = id, \quad \theta_{t_1}\theta_{t_2} = \theta_{t_1 + t_2}, \tag{2.3}$$

for $t_1, t_2 \in \mathbb{R}$. This flow is supposed to be $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable, where $\mathcal{B}(\mathbb{R})$ is the σ-algebra of Borel sets on the real line $\mathbb{R}$. To have this measurability, it is not allowed to replace $\mathcal{F}$ by its $\mathbb{P}$-completion $\mathcal{F}^{\mathbb{P}}$; see Arnold [1] p. 547. In addition, the measure $\mathbb{P}$ is assumed to be ergodic with respect to $\{\theta_t\}_{t \in \mathbb{R}}$. Then $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R}, \theta)$ is called a metric dynamical system.

For our applications, we will consider a special but very important metric dynamical system induced by the Brownian motion. Let $W(t)$ be a two-sided Wiener process with trajectories in the space $C_0(\mathbb{R}, \mathbb{R})$ of real continuous functions defined on $\mathbb{R}$, taking zero value at $t = 0$. This set is equipped with the compact open topology. On this set we consider the measurable flow $\theta = \{\theta_t\}_{t\in\mathbb{R}}$, defined by $\theta_t\omega = \omega(\cdot + t) - \omega(t)$. The distribution of this process generates a measure on $\mathcal{B}(C_0(\mathbb{R},\mathbb{R}))$ which is called the Wiener measure. Note that this measure is ergodic with respect to the above flow; see the Appendix in Arnold [1]. Later on we will consider, instead of the whole $C_0(\mathbb{R},\mathbb{R})$, a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset $\Omega \subset C_0(\mathbb{R},\mathbb{R}))$ of $\mathbb{P}$-measure one and the trace σ-algebra $\mathcal{F}$ of $\mathcal{B}(C_0(\mathbb{R},\mathbb{R}))$ with respect to Ω. A set Ω is called $\{\theta_t\}_{t\in\mathbb{R}}$-invariant if $\theta_t\Omega = \Omega$ for $t \in \mathbb{R}$. On $\mathcal{F}$, we consider the restriction of the Wiener measure also denoted by $\mathbb{P}$.

The dynamics of the system on the state space E over the flow θ is described by a cocycle. For our applications it is sufficient to assume that (E, dE) is a complete metric space. A cocycle ϕ is a mapping:

$$\phi : \mathbb{R}^+ \times \Omega \times E \to E,$$

which is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(E), \mathcal{F})$-measurable such that

$$\begin{aligned}
&\phi(0, \omega, x) = x \in E,\\
&\phi(t_1 + t_2, \omega, x) = \phi(t_2, \theta_{t_1}\omega, \phi(t_1, \omega, x)),
\end{aligned}$$

for $t_1, t_2 \in \mathbb{R}^+, \omega \in \Omega$ and $x \in E$. Then ϕ together with the metric dynamical system θ forms a random dynamical system.

2.3. Random evolutionary equation

We consider a linear stochastic differential equation

$$dz + zdt = \sigma dW. \tag{2.4}$$

A solution of this equation is called an Ornstein-Uhlenbeck process. We have the following results, see Duan, Lu and Schmalfuss [7, 8].

Lemma 2.1 *(i) There exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant set $\Omega \in \mathcal{B}(C_0(\mathbb{R},\mathbb{R}))$ of full measure with sublinear growth*

$$\lim_{t\to\pm\infty} \frac{|\omega(t)|}{|t|} = 0, \quad \omega \in \Omega$$

of $\mathbb{P}$-measure one.

(ii) For $\omega \in \Omega$, the random variable

$$z(\omega) = -\sigma \int_{-\infty}^{0} e^{\tau}\omega(\tau)d\tau$$

exists and generates a unique stationary solution of Equation (2.4) given by

$$z(\theta_t\omega) = -\sigma \int_{-\infty}^{0} e^{\tau}\theta_t\omega(\tau)d\tau = -\sigma \int_{-\infty}^{0} e^{\tau}\omega(\tau+t)d\tau + \sigma\omega(t).$$

The mapping $t \to z(\theta_t\omega)$ is continuous.

(iii) In particular,

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = 0, \quad for \quad \omega \in \Omega.$$

(iv) In addition,

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_\tau\omega)d\tau = 0, \quad for \quad \omega \in \Omega.$$

We now replace $\mathcal{B}(C_0(\mathbb{R},\mathbb{R}))$ by $\mathcal{F} = \{\Omega \bigcap F| \quad F \in \mathcal{B}(C_0(\mathbb{R},\mathbb{R}))\}$ for Ω given in Lemma 2.1. The probability measure is the restriction of the Wiener measure to this new σ-algebra, which is also denoted by $\mathbb{P}$. In the following we will consider the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R}, \theta)$.

Now we show that the solution of Equation (1.1) defines a random dynamical. Firstly, the equivalent Itô equation of Equation (1.1) is given by

$$du = -Ludt + u^p dt + \frac{u}{2}dt + \sigma u dW \tag{2.5}$$

with the initial data $u(0) = u_0 \in E^\alpha$ being $\mathcal{F}_0$-measurable. Equation (2.5) can be written in the following mild integral form

$$u(t) = e^{-Lt}u_0 + \int_0^t e^{-L(t-r)}(u^p(r) + \frac{u(r)}{2})dr + \sigma \int_0^t e^{-L(t-r)}u(r)dW(r)$$

almost surely for arbitrary $u_0 \in E^\alpha$, in which the stochastic integral is to interpret in the sense of Itô.

Under the transformation of Ornstein-Uhlenbeck process (2.4), Equation (2.5) becomes a random evolutionary equation (i.e., an evolutionary equation with random coefficients)

$$\frac{dv}{dt} = -Lv + zv + e^{-z}F(e^z v) \tag{2.6}$$

with $v(0) = u_0 e^{-z(0)} := x \in E^\alpha$, where $v = ue^{-z}$ and $F(v) = v^p$ with $z = z(t) := z(\theta_t\omega)$. In contrast to the original stochastic differential equation (1.1), no stochastic integral appears here.

Then the mild integral form of (2.6) is

$$v(t) = e^{-Lt+\int_0^t z(\tau)d\tau}x + \int_0^t e^{-L(t-r)+\int_r^t z(\tau)d\tau}e^{-z(r)}F(e^{z(r)}v(r))dr$$

almost surely for any $x \in E^\alpha$.

Since our purpose is to consider the dynamical behavior of solution of Equation (2.6) in a neighborhood of the fixed point $v = 0$ in this paper, now we introduce a truncated equation of Equation (2.6) such that its nonlinear term has a small Lipschitz constant.

Let $\chi : E^\alpha \to \mathbb{R}$ be a C_0^∞ function, a cut-off function, such that

$$\chi(v) = \begin{cases} 1, & \text{if} \quad |v|_\alpha \le 1, \\ 0, & \text{if} \quad |v|_\alpha \ge 2. \end{cases}$$

For any positive parameter R, we define $\chi_R(v) = \chi(\frac{v}{R})$ for all $v \in E^\alpha$. Let $F^{(R)}(v) = \chi_R(v)F(v)$. For every $l_F > 0$ and every $\omega \in \Omega$, there must exist a positive random variable R such that

$$\|F^{(R)}(v) - F^{(R)}(\tilde{v})\| \le l_F|v - \tilde{v}|_\alpha. \tag{2.7}$$

Then the truncated equation of Equation (2.6) is as follows

$$\frac{dv}{dt} = -Lv + zv + e^{-z}F^{(R)}(e^z v). \tag{2.8}$$

By the classical evolutionary equation theory, Equation (2.8) has a unique solution for every $\omega \in \Omega$. No exceptional sets with respect to the initial conditions appear. Hence the solution mapping

$$(t, \omega, x) \mapsto \phi(t, \omega)x := v(t, \omega; x)$$

generates a continuous random dynamical system. Indeed, the mapping ϕ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(E^\alpha), \mathcal{F})$-measurable.

Introduce the transform

$$T(\omega, x) = xe^{-z(\omega)}$$

and its inverse transform

$$T^{-1}(\omega, x) = xe^{z(\omega)}$$

for $x \in E^\alpha$ and $\omega \in \Omega$. Then for the random dynamical system $v(t, \omega; x)$ generated by Equation (2.6),

$$(t, \omega, x) \mapsto T^{-1}(\theta_t \omega, v(t, \omega; T(w, x))) := u(t, \omega; x)$$

is the random dynamical system generated by Equation (1.1). For more about the relation between (1.1) and (2.6), we refer to Duan, Lu and Schmalfuss [7].

3 Existence of local invariant manifolds

In this section, we shall use the method of Duan, Lu and Schmalfuss [8] to establish the local invariant manifold of Equation (1.1).

Define a Banach space for each $\beta \in (\lambda_u, \lambda_s)$ as follows

$$C_\beta^- = \{f(\cdot) \in C((-\infty, 0]; E^\alpha)| \quad \sup_{t \le 0} e^{\beta t - \int_0^t z(\tau)d\tau} |f|_\alpha < \infty\}$$

with the norm

$$\|f\|_{C_\beta^-} = \sup_{t \le 0} e^{\beta t - \int_0^t z(\tau)d\tau} |f|_\alpha.$$

Since that

$$\begin{aligned} |e^{-L_s(t-r)+\int_r^t z(\tau)d\tau} P_s u(r)|_\alpha &\le M e^{-\beta r + \int_0^r z(\tau)d\tau} e^{-\lambda_s(t-r)+\int_r^t z(\tau)d\tau} |u(r)|_{C_\beta^-} \\ &\le M e^{(\lambda_s - \beta)r} e^{-\lambda_s t + \int_0^t z(\tau)d\tau} |u(r)|_{C_\beta^-} \\ &\longrightarrow 0, \quad \text{as} \quad r \to -\infty, \end{aligned}$$

we have that

$$P_s v(t) = \int_{-\infty}^t e^{-L_s(t-r)+\int_r^t z(\tau)d\tau} e^{-z(r)} F_s^{(R)}(e^{z(r)} v(r)) dr.$$

Then we have the following result. For the detailed proof, please see Duan, Lu and Schmalfuss [8].

Lemma 3.1 *Suppose that $v(\cdot)$ is in C_β^-. Then $v(t)$ is the solution of Equation (2.8) with the initial datum $v(0) = x$ if and only if $v(t)$ satisfies*

$$\begin{aligned} v(t) \quad = \quad & e^{-L_u t + \int_0^t z(\tau)d\tau} \xi + \int_0^t e^{-L_u(t-r)+\int_r^t z(\tau)d\tau} e^{-z(r)} F_u^{(R)}(e^{z(r)} v(r)) dr \\ & + \int_{-\infty}^t e^{-L_s(t-r)+\int_r^t z(\tau)d\tau} e^{-z(r)} F_s^{(R)}(e^{z(r)} v(r)) dr, \end{aligned} \tag{3.1}$$

where $\xi = P_u x \in E_u$.

Define

$$\begin{aligned} J(v,\xi) &= e^{-L_u t+\int_0^t z(\tau)d\tau}\xi + \int_0^t e^{-L_u(t-r)+\int_r^t z(\tau)d\tau}e^{-z(r)}F_u^{(R)}(e^{z(r)}v(r))dr \\ &\quad + \int_{-\infty}^t e^{-L_s(t-r)+\int_r^t z(\tau)d\tau}e^{-z(r)}F_s^{(R)}(e^{z(r)}v(r))dr, \end{aligned} \tag{3.2}$$

and also denote

$$SC := Ml_F[\frac{1}{\beta-\lambda_u} + \frac{\Gamma(1-\alpha)}{(\lambda_s-\beta)^{1-\alpha}}], \tag{3.3}$$

where M is the positive constant in (2.2), l_F is the Lipschitz constant in (2.7), $\Gamma(\cdot)$ is the Gamma function, $\alpha \in [0,1)$ and $\beta \in (\lambda_u, \lambda_s)$.

Then

$$\begin{aligned} \|J(v,\xi)-J(\tilde{v},\xi)\|_{C_\beta^-} &\leq \|\int_0^t e^{-L_u(t-r)+\int_r^t z(\tau)d\tau}e^{-z(r)}[F_u^{(R)}(e^{z(r)}v(r)) - F_u^{(R)}(e^{z(r)}\tilde{v}(r))]dr \\ &\quad + \int_{-\infty}^t e^{-L_s(t-r)+\int_r^t z(\tau)d\tau}e^{-z(r)}[F_s^{(R)}(e^{z(r)}v(r)) - F_s^{(R)}(e^{z(r)}\tilde{v}(r))]dr\|_{C_\beta^-} \\ &\leq Ml_F \cdot \sup_{t\leq 0}[\int_0^t e^{(\beta-\lambda_u)(t-r)}dr + \int_{-\infty}^t \frac{1}{(t-r)^\alpha}e^{(\beta-\lambda_s)(t-r)}dr]\|v-\tilde{v}\|_{C_\beta^-} \\ &\leq SC\|v-\tilde{v}\|_{C_\beta^-}. \end{aligned} \tag{3.4}$$

Let $SC < 1$. Then by the uniform contraction mapping principle, for each $\xi \in E_u$, $J(v,\xi)$ has a unique fixed point $v^*(t,\omega;\xi) \in C_\beta^-$. Put $h(\omega,\xi) = P_s v^*(0,\omega;\xi)$. Thus

$$h(\omega,\xi) = \int_{-\infty}^0 e^{L_s r+\int_r^0 z(\tau)d\tau}e^{-z(r)}F_s^{(R)}(e^{z(r)}v(r))dr. \tag{3.5}$$

Lemma 3.2 *Let R be a positive random variable such that l_F satisfies $SC < 1$. For the unique fixed point $v^* = v^*(t,\omega;\xi) = J(v^*) \in C_\beta^-$ of the operator J, there exists a positive constant C such that*

$$\|v^*(t,\omega;\xi_1) - v^*(t,\omega;\xi_2)\|_{C_\beta^-} \leq C|\xi_1-\xi_2|_\alpha.$$

Moreover,

$$\|h(\omega,\xi_1) - h(\omega,\xi_2)\|_{C_\beta^-} \leq C|\xi_1-\xi_2|_\alpha.$$

Lemma 3.3 *Let R be a positive random variable such that l_F satisfies $SC < 1$. Then there exists a positive constant C such that*

$$\begin{aligned} &\|v^*(t,\omega;\xi)\|_{C_\beta^-} \leq C|\xi|_\alpha, \\ &\|v_s^*(t,\omega;\xi)\|_{C_\beta^-} \leq C|\xi|_\alpha, \\ &\|v_u^*(t,\omega;\xi)\|_{C_\beta^-} \leq C|\xi|_\alpha, \end{aligned} \tag{3.6}$$

where $v_s^* = P_s v^*$ *and* $v_u^* = P_u v^*$.

Proof. Firstly, for $t \le 0$, since $\beta \in (\lambda_u, \lambda_s)$, we have

$$\begin{aligned} \|J(0,\xi)\|_{C_\beta^-} &= \|e^{-L_u t+\int_0^t z(\tau)d\tau}\xi\|_{C_\beta^-} \le \sup_{t\in(-\infty,0]} e^{\beta t-\int_0^t z(\tau)d\tau}e^{-L_u t+\int_0^t z(\tau)d\tau}|\xi|_\alpha \\ &\le \sup_{t\in(-\infty,0]} Me^{(\beta-\lambda_u)t}|\xi|_\alpha \le C|\xi|_\alpha. \end{aligned} \tag{3.7}$$

It follows from Lemma 3.2, (3.4) and (3.7) that

$$\|v^*(t,\omega;\xi)\|_{C_\beta^-} \le \|J(v^*,\xi)-J(0,\xi)\|_{C_\beta^-} + \|J(0,\xi)\|_{C_\beta^-} \le SC\|v^*(t,\xi)\|_{C_\beta^-} + C|\xi|_\alpha,$$

which implies that $\|v^*(t,\omega;\xi)\|_{C_\beta^-} \le \frac{C}{1-SC}|\xi|_\alpha$.

Meanwhile,

$$\begin{aligned} \|v_s^*(t,\omega;\xi)\|_{C_\beta^-} &= \|P_s v^*(t,\omega;\xi)\|_{C_\beta^-} \\ &= \| \int_{-\infty}^t e^{-L_s(t-r)+\int_r^t z(\tau)d\tau}e^{-z(r)}F_s^{(R)}(e^{z(r)}v(r))dr\|_{C_\beta^-} \\ &\le \int_{-\infty}^t e^{\beta t-\int_0^t z(\tau)d\tau}\frac{M}{(t-r)^\alpha}e^{-\lambda_s(t-r)}e^{\int_r^t z(\tau)d\tau}e^{-\beta r+\int_0^r z(\tau)d\tau}l_F\|v(r)\|_{C_\beta^-}dr \\ &\le Ml_F\|v(r)\|_{C_\beta^-}\int_{-\infty}^t e^{(\beta-\lambda_s)(t-r)}\frac{1}{(t-r)^\alpha}dr \\ &\le C\|v(r)\|_{C_\beta^-} \\ &\le C|\xi|_\alpha. \end{aligned}$$

Therefore

$$\|v_u^*(t,\omega;\xi)\|_{C_\beta^-} = \|v^*(t,\omega;\xi)-v_s^*(t,\omega;\xi)\|_{C_\beta^-} \le \|v^*(t,\omega;\xi)\|_{C_\beta^-} + \|v_s^*(t,\omega;\xi)\|_{C_\beta^-} \le C|\xi|_\alpha.$$

The proof is complete. ■

Lemma 3.4 *Let R be a positive random variable such that l_F satisfies $SC < 1$. Then*

$$\mathcal{M}(\omega) = \{\xi + h(\omega,\xi) | \quad \xi \in E_u\} \tag{3.8}$$

is a local invariant manifold for Equation (2.6).

Theorem 3.1 (Existence of local random invariant manifold)

Let R be a positive random variable such that l_F satisfies $SC < 1$ as in the inequality (3.3). Then

$$\widetilde{\mathcal{M}}(\omega) = T^{-1}\mathcal{M}(\omega) = \{\xi + e^{z(\omega)}h(\omega, e^{-z(\omega)}\xi) | \quad \xi \in E_u\} \tag{3.9}$$

is a local invariant manifold for Equation (1.1). Namely, the graph of $e^{z(\omega)}h(\omega,e^{-z(\omega)}\xi)$ is the local random invariant manifold $\widetilde{\mathcal{M}}(\omega)$ for Equation (1.1).

Lemma 3.2, Lemma 3.4 and Theorem 3.1 can be proved as in Duan, Lu and Schmalfuss [8].

4 Local geometric shape of invariant manifolds

In this section, we approximate the random invariant manifold $M(\omega)$ step by step to derive the local geometric shape of the invariant manifold, as inspired by Blomker and Wang [3].

Define

$$\hbar_1(t) = \int_{-\infty}^{t} e^{-L_s(t-r)+\int_r^t z(\tau)d\tau} e^{-z(r)} F_s^{(R)}(e^{z(r)} v_u(r))dr. \tag{4.1}$$

Lemma 4.1 *There exists a positive constant C such that*

$$\|v_s^*(t) - \hbar_1(t)\|_{C_\beta^-} \leq C|\xi|_\alpha, \quad \textit{for arbitray} \quad t \leq 0. \tag{4.2}$$

Proof. Note that $v_s^* = P_s v^*$. Then it follows from Lemma 3.3 that

$$\begin{aligned}
& \|v_s^*(t) - \hbar_1(t)\|_{C_\beta^-} \\
= \;& \| \textstyle\int_{-\infty}^t e^{-L_s(t-r)+\int_r^t z(\tau)d\tau} [e^{-z(r)} F_s^{(R)}(e^{z(r)} v^*(r)) - e^{-z(r)} F_s^{(R)}(e^{z(r)} v_u^*(r))]dr \|_{C_\beta^-} \\
\leq \;& \textstyle\int_{-\infty}^t e^{\beta t - \int_0^t z(\tau)d\tau} \frac{M}{(t-r)^\alpha} e^{-\lambda_s(t-r)} e^{\int_r^t z(\tau)d\tau} e^{-\beta r + \int_0^r z(\tau)d\tau} l_F \|v^*(r) - v_u^*(r)\|_{C_\beta^-} dr \\
= \;& M l_F \|v_s^*(r)\|_{C_\beta^-} \textstyle\int_{-\infty}^t \frac{1}{(t-r)^\alpha} e^{(\beta-\lambda_s)(t-r)} dr \\
= \;& M l_F \|v_s^*(r)\|_{C_\beta^-} \frac{\Gamma(1-\alpha)}{(\lambda_s-\beta)^{1-\alpha}} \\
\leq \;& C|\xi|_\alpha.
\end{aligned}$$

The proof is complete. ■

Lemma 4.2 *There exists a positive constant C such that*

$$\|v_u^*(t) - e^{-L_u t + \int_0^t z(\tau)d\tau} \xi\|_{C_\beta^-} \leq C|\xi|_\alpha, \quad \textit{for arbitrary} \quad t \leq 0. \tag{4.3}$$

Proof. Firstly, we note that

$$v_u^* = P_u v^* = e^{-L_u t + \int_0^t z(\tau)d\tau} \xi + \int_0^t e^{-L_u(t-r)+\int_r^t z(\tau)d\tau} e^{-z(r)} F_u^{(R)}(e^{z(r)} v(r))dr. \tag{4.4}$$

Then for $t \leq 0$, from Lemma 3.3, we have

$$\begin{aligned}
\|v_u^*(t) - e^{-L_u t + \int_0^t z(\tau)d\tau} \xi\|_{C_\beta^-} &= \| \textstyle\int_0^t e^{-L_u(t-r)+\int_r^t z(\tau)d\tau} e^{-z(r)} F_u^{(R)}(e^{z(r)} v(r))dr \|_{C_\beta^-} \\
&\leq \textstyle\int_0^t e^{\beta t - \int_0^t z(\tau)d\tau} e^{-\lambda_u(t-r)} e^{\int_r^t z(\tau)d\tau} e^{-\beta r + \int_0^r z(\tau)d\tau} l_F \|v(r)\|_{C_\beta^-} dr \\
&\leq M l_F C|\xi|_\alpha \textstyle\int_0^t e^{\beta t} e^{-\lambda_u(t-r)} e^{-\beta r} dr \\
&\leq M l_F C \cdot \frac{1}{\beta-\lambda_u} \cdot |\xi|_\alpha \\
&\leq C|\xi|_\alpha.
\end{aligned}$$

This completes the proof. ■

Define

$$\hbar_2 = \int_{-\infty}^{0} e^{L_s r + \int_r^0 z(\tau)d\tau} e^{-z(r)} F_s^{(R)}(e^{z(r)} e^{-L_u r + \int_0^r z(\tau)d\tau} \xi) dr. \tag{4.5}$$

Lemma 4.3 *There exists a positive constant C such that*

$$\|\hbar_1(0) - \hbar_2\| \leq C|\xi|_\alpha. \tag{4.6}$$

Proof. It follows from (4.1), (4.5) and Lemma 4.2 that

$$\begin{aligned}
\|\hbar_1(0) - \hbar_2\| &= \| \int_{-\infty}^0 e^{L_s r + \int_r^0 z(\tau)d\tau} [e^{-z(r)} F_s^{(R)}(e^{z(r)} v_u^*(r)) - e^{-z(r)} F_s^{(R)}(e^{z(r)} e^{-L_u r + \int_0^r z(\tau)d\tau} \xi)] dr \| \\
&\leq \int_{-\infty}^0 \frac{M}{(-r)^\alpha} e^{-\lambda_s(-r)} e^{\int_r^0 z(\tau)d\tau} e^{-\beta r + \int_0^r z(\tau)d\tau} l_F \| v_u^*(r) - e^{-L_u r + \int_0^r z(\tau)d\tau} \xi \|_{C_\beta^-} dr \\
&\leq M l_F C |\xi|_\alpha \int_{-\infty}^0 \frac{1}{(-r)^\alpha} e^{(\lambda_s - \beta) r} dr \\
&\leq M l_F C \cdot \frac{\Gamma(1-\alpha)}{(\lambda_s - \beta)^{1-\alpha}} |\xi|_\alpha \\
&\leq C|\xi|_\alpha.
\end{aligned}$$

The proof is thus complete. ■

Lemma 4.4[3] *There is a random variable $K_1(\omega)$ such that $K_1(\omega) - 1$ has a standard exponential distribution and*

$$\int_0^t z(\tau)d\tau + z(t) = z(0) + \sigma\omega(t) \leq \sigma(K_1(\omega) + |t|), \quad \textit{for arbitrary} \quad t \leq 0,$$

where z satisfies Equation (2.4). Also,

$$|\omega(t)| \leq \max\{\omega(t), -\omega(t)\} \leq K^{\pm}(\omega) + |t|, \quad \textit{for arbitrary} \quad t \leq 0,$$

where $K^{\pm}(\omega) = K_1(\omega) + K_1(-\omega)$ and $K_1(-\omega)$ has the same law as $K_1(\omega)$. Furthermore, for $|z(0)|$ a similar estimate is true.

Define

$$K_2(\omega) = \sup_{\tau \leq 0} |\frac{1 - e^{-\lambda_u \tau + \sigma\omega(\tau)}}{\gamma e^{\delta|\tau|}}|,$$

where γ and δ are the positive constants.

Lemma 4.5 *Choose two positive real parameters γ and δ satisfying $\gamma \geq \max\{-\lambda_u, \sigma\}$ and $\delta > -\lambda_u + \sigma$. Then there is a constant C such that*

$$K_2(\omega) \leq C e^{\sigma K^{\pm}(\omega)}(1 + K^{\pm}(\omega)).$$

Proof. Using $|1-e^x| \le |x|e^{|x|}$ and Lemma 4.4, we get

$$\begin{aligned} K_2(\omega) &= \sup_{\tau\le 0} |\tfrac{1-e^{-\lambda_u\tau+\sigma\omega(\tau)}}{\gamma e^{\delta|\tau|}}| \\ &\le \sup_{\tau\le 0} \tfrac{|\lambda_u||\tau|+|\sigma||\omega(\tau)|}{\gamma} e^{|\lambda_u||\tau|+|\sigma||\omega(\tau)|} e^{-\delta|\tau|} \\ &\le e^{\sigma K^{\pm}(\omega)} \sup_{\tau\le 0}(|\tau|+|\omega(\tau)|)e^{(-\lambda_u+\sigma-\delta)|\tau|} \\ &\le Ce^{\sigma K^{\pm}(\omega)} \sup_{\tau\le 0}(|\tau|e^{(-\lambda_u+\sigma-\delta)|\tau|}+K^{\pm}(\omega)e^{(-\lambda_u+\sigma-\delta)|\tau|}) \\ &\le Ce^{\sigma K^{\pm}(\omega)}(1+K^{\pm}(\omega)). \end{aligned}$$

The proof is complete. ■

Lemma 4.6 *Let $e^{z(0)}|\xi|_\alpha \le R$. Then there exists a positive constant C such that*

$$|\chi_R(e^{z(r)}e^{-L_u r+\int_0^r z(\tau)d\tau}\xi)-1| \le \frac{C}{R}e^{z(0)}K_2(\omega)\gamma e^{-\delta r}|\xi|_\alpha, \quad \textit{for arbitrary} \quad r\le 0. \tag{4.7}$$

Proof. Note that $e^{z(0)}|\xi|_\alpha \le R$. Then $\chi_R(e^{z(0)}\xi)=1$. Therefore, for $r\le 0$, it follows from Lemma 4.4 that

$$\begin{aligned} |\chi_R(e^{z(r)}e^{-L_u r+\int_0^r z(\tau)d\tau}\xi)-1| &\le |\chi_R(e^{z(r)}e^{-\lambda_u r+\int_0^r z(\tau)d\tau}\xi)-\chi_R(e^{z(0)}\xi)| \\ &\le \tfrac{C}{R}|e^{-\lambda_u r+z(r)+\int_0^r z(\tau)d\tau}\xi - e^{z(0)}\xi|_\alpha \\ &\le \tfrac{C}{R}|\xi|_\alpha\cdot|e^{-\lambda_u r+z(0)+\sigma\omega(r)}-e^{z(0)}| \\ &\le \tfrac{C}{R}e^{z(0)}|\xi|_\alpha\cdot|1-e^{-\lambda_u r+\sigma\omega(r)}| \\ &\le \tfrac{C}{R}e^{z(0)}K_2(\omega)\gamma e^{-\delta r}|\xi|_\alpha. \end{aligned} \tag{4.8}$$

The proof is complete. ■

We further define

$$\hbar_3 = \int_{-\infty}^0 e^{L_s r+\int_r^0 z(\tau)d\tau}e^{-z(r)}F_s(e^{z(r)}e^{-L_u r+\int_0^r z(\tau)d\tau}\xi)dr. \tag{4.9}$$

Lemma 4.7 *Let $0<\sigma<\frac{\lambda_s-(p-1)\lambda_u}{p}$ and $e^{z(0)}|\xi|_\alpha \le R$. Then there exists a positive constant C such that*

$$\|\hbar_2-\hbar_3\| \le Ce^{z(0)}K_2(\omega)e^{(p-1)\sigma K_1(\omega)}|\xi|_\alpha^2. \tag{4.10}$$

Proof. Firstly, from the condition $\sigma<\frac{\lambda_s-(p-1)\lambda_u}{p}$, we know that

$$\lambda_s - p\lambda_u - (p-1)\sigma > -\lambda_u+\sigma,$$

which implies that there must exist a constant δ satisfying

$$\lambda_s - p\lambda_u - (p-1)\sigma > \delta > -\lambda_u + \sigma. \tag{4.11}$$

Also, we note that $F(u) = u^p$, $F^{(R)}(u) = \chi_R(u)F(u)$, $e^{z(0)}|\xi|_\alpha \le R$ and (2.7). Therefore, from Lemma 4.7 and Lemma 4.4, we get

$$\begin{array}{rl} & \|\hbar_2 - \hbar_3\| \\ \le & \| \int_{-\infty}^0 e^{L_s r + \int_r^0 z(\tau)d\tau} e^{-z(r)} [F_s^{(R)}(e^{z(r)} e^{-L_u r + \int_0^r z(\tau)d\tau}\xi) - F_s(e^{z(r)} e^{-L_u r + \int_0^r z(\tau)d\tau}\xi)]dr\| \\ \le & \| \int_{-\infty}^0 e^{L_s r + \int_r^0 z(\tau)d\tau} e^{(p-1)z(r)} e^{-pL_u r + p\int_0^r z(\tau)d\tau} \xi_s^p [\chi_R(e^{z(r)} e^{-L_u r + \int_0^r z(\tau)d\tau}\xi) - 1]dr\| \\ \le & \frac{C}{R} e^{-(p-1)z(0)} K_2(\omega)\gamma|\xi|_\alpha \int_{-\infty}^0 e^{-\delta r} e^{L_s r + \int_r^0 z(\tau)d\tau} e^{(p-1)z(r)} e^{-pL_u r + p\int_0^r z(\tau)d\tau} \|(e^{z(0)}\xi_s)^p\| dr \\ \le & \frac{C}{R} e^{-(p-2)z(0)} K_2(\omega)\gamma|\xi|_\alpha^2 M^2 l_F \int_{-\infty}^0 \frac{1}{(-r)^\alpha} e^{(p-1)[z(r) + \int_0^r z(\tau)d\tau]} e^{(\lambda_s - p\lambda_u - \delta)r} dr \\ \le & \frac{C}{R} e^{-(p-2)z(0)} K_2(\omega)\gamma|\xi|_\alpha^2 M^2 l_F e^{(p-1)\sigma K_1(\omega)} \int_{-\infty}^0 \frac{1}{(-r)^\alpha} e^{(\lambda_s - p\lambda_u - \delta - (p-1)\sigma)r} dr, \end{array}$$

which, from (4.11), immediately implies that

$$\begin{array}{rl} \|\hbar_2 - \hbar_3\| & \le \frac{C}{R} e^{-(p-2)z(0)} K_2(\omega)\gamma|\xi|_\alpha^{p+1} M^2 l_F e^{(p-1)\sigma K_1(\omega)} \frac{\Gamma(1-\alpha)}{(\lambda_s - p\lambda_u - \delta - (p-1)\sigma)^{1-\alpha}} \\ & \le CK_2(\omega) e^{(p-1)\sigma K_1(\omega) - (p-2)z(0)} |\xi|_\alpha^2. \end{array}$$

The proof is complete. ■

Lemma 4.8 *Let*

$$K_3(\omega) = \sup_{r \le 0} \left| \frac{1 - e^{(p-1)\sigma\omega(r)}}{\gamma_1 e^{(p-1)\delta_1 |r|}} \right|. \tag{4.12}$$

If $\gamma_1 > \sigma$ and $\delta_1 > \sigma$, then

$$K_3(\omega) \le Ce^{(p-1)\sigma K^\pm(\omega)}(1 + K^\pm(\omega)).$$

Furthermore,

$$|1 - e^{(p-1)\sigma\omega(r)}| \le K_3(\omega)\gamma_1 e^{-(p-1)\delta_1 r}, \quad \textit{for arbitrary} \quad r \le 0. \tag{4.13}$$

Proof. Using $|1 - e^x| \le |x|e^{|x|}$ and Lemma 4.4, we get

$$\begin{array}{rl} K_3(\omega) & = \sup_{r \le 0} \left| \frac{1 - e^{(p-1)\sigma\omega(r)}}{\gamma_1 e^{(p-1)\delta_1 |r|}} \right| \\ & \le \sup_{r \le 0} \frac{(p-1)\sigma|\omega(r)|}{\gamma_1} e^{(p-1)\sigma|\omega(r)|} e^{-(p-1)\delta_1 |r|} \\ & \le (p-1)e^{(p-1)\sigma K^\pm(\omega)} \sup_{r \le 0} (|r| + K^\pm(\omega)) e^{(p-1)(\sigma - \delta_1)|r|} \\ & \le (p-1)e^{(p-1)\sigma K^\pm(\omega)} \sup_{r \le 0} (|r| e^{(p-1)(\sigma - \delta_1)|r|} + K^\pm(\omega) e^{(p-1)(\sigma - \delta_1)|r|}) \\ & \le Ce^{\sigma K^\pm(\omega)}(1 + K^\pm(\omega)). \end{array}$$

The proof is complete. ■

Lemma 4.9 *Let $0 < \sigma < -\lambda_u$ and $e^{z(0)}|\xi|_\alpha \le R$. Then there exists a positive constant C such that*

$$\|\hbar_3 - e^{(p-1)z(0)}(L_s - pL_u)^{-1}\xi_s^p\| \le Ce^{(p-1)z(0)}K_3(\omega)|\xi|_\alpha. \tag{4.14}$$

Proof. It follows from (4.9) and Lemma 4.4 that

$$\hbar_3 = e^{(p-1)z(0)}\xi_s^p \int_{-\infty}^0 e^{L_s r} e^{-pL_u r} e^{(p-1)\sigma\omega(r)} dr.$$

From the condition $\sigma < -\lambda_u$, there must exist a parameter δ_1 satisfying $\sigma < \delta_1 < -\lambda_u$, which implies that

$$\lambda_s - p\lambda_u - (p-1)\delta_1 > 0. \tag{4.15}$$

Also note that

$$e^{(p-1)z(0)}\xi_s^p \int_{-\infty}^0 e^{L_s r} e^{-pL_u r} dr = e^{(p-1)z(0)}(L_s - pL_u)^{-1}\xi_s^p.$$

Therefore, it follows from Lemma 4.8 and (4.15) that

$$\begin{array}{rl} & \|\hbar_3 - e^{(p-1)z(0)}(L_s - pL_u)^{-1}\xi_s^p\| \\ = & \|e^{-z(0)}(e^{z(0)}\xi_s)^p \int_{-\infty}^0 e^{L_s r} e^{-pL_u r}[e^{(p-1)\sigma\omega(r)} - 1]dr\| \\ \le & |\xi|_\alpha K_3(\omega)\gamma_1 M^2 l_F \int_{-\infty}^0 \frac{1}{(-r)^\alpha} e^{(\lambda_s - p\lambda_u - (p-1)\delta_1)r} dr \\ \le & |\xi|_\alpha K_3(\omega)\gamma_1 M^2 l_F \frac{\Gamma(1-\alpha)}{(\lambda_s - p\lambda_u - (p-1)\delta_1)^{1-\alpha}} \\ \le & CK_3(\omega)|\xi|_\alpha. \end{array}$$

The proof is complete. ■

Lemma 4.10 *Let $0 < \sigma < \min\{\frac{\lambda_s - (p-1)\lambda_u}{p}, -\lambda_u\}$ and $e^{z(0)}|\xi|_\alpha \le R$. Then there exists a positive constant C such that*

$$\|h(\omega,\xi) - e^{(p-1)z(0)}(L_s - pL_u)^{-1}\xi_s^p\| \le C[1 + K_3(\omega) + e^{(p-1)\sigma K_1(\omega) - (p-2)z(0)}K_2(\omega)|\xi|_\alpha]\cdot|\xi|_\alpha. \tag{4.16}$$

This lemma is directly from Lemma 4.1, Lemma 4.3, Lemma 4.7 and Lemma 4.9.

Finally we have the following main result.

Theorem 4.1 (Local geometric shape of random invariant manifold)

Let $0 < \sigma < \min\{\frac{\lambda_s-(p-1)\lambda_u}{p}, -\lambda_u\}$ *and* $|\xi|_\alpha \leq R$*. Then there exists a positive constant* C *such that*

$$\|e^{z(\omega)}h(\omega, e^{-z(\omega)}\xi) - (L_s - pL_u)^{-1}\xi_s^p\| \leq C(|\xi|_\alpha + |\xi|_\alpha^2) \tag{4.17}$$

holds with probability larger than $1 - Ce^{-\frac{1}{\sigma}}$*. Therefore in a neighborhood of zero for Equation (1.1), the graph* $(\xi, e^{z(\omega)}h(\omega, e^{-z(\omega)}\xi))$ *of the invariant manifold* $\widetilde{\mathcal{M}}(\omega)$ *is approximately given by* $(\xi, (L_s - pL_u)^{-1}\xi_s^p)$ *with probability larger than* $1 - Ce^{-\frac{1}{\sigma}}$*.*

Proof. Define $\Omega_K = \{\omega \in \Omega| \quad K^{\pm}(\omega) > \frac{1}{\sigma}\}$. By Lemma 4.4 this set has probability less than $Ce^{-\frac{1}{\sigma}}$. Therefore, on the complement Ω_K^C, there must exist a positive constant C such that

$$K_1(\omega) \leq C, K_2(\omega) \leq C, K_3(\omega) \leq C.$$

Therefore, it follows from Lemma 4.10 that Theorem 4.1 holds. The proof is complete. ■

Remark 4.1 *Here we present an example to explain Theorem 4.1. Consider Equation (1.1) with the line operator* $L = -\partial_{xx} - 3 \cdot id$ *on* $[0, \pi]$ *with the homogeneous Dirichlet boundary condition. Then the eigenvalues of* L *are*

$$\lambda_1 = -2,\ \lambda_2 = 1,\ \lambda_3 = 6,\ \cdots, \lambda_k = k^2 - 3,\ \cdots$$

with the corresponding eigenfunctions

$$e_1 = \sin x,\ e_2 = \sin 2x,\ e_3 = \sin 3x,\ \cdots, e_k = \sin kx,\ \cdots$$

Therefore, Theorem 4.1 affords a local unstable invariant manifold for Equation (1.1). In this case, $L_u = -2 \cdot id$ *and* $E_u = span\{e_1\}$*. Then we can write* $\xi = r \cdot e_1$ *with* $r \in \mathbb{R}$*. We denote* $e_1^{\perp} := (L_s + 2p \cdot id)^{-1}P_s e_1^p$*. Then*

$$(L_s - pL_u)^{-1}\xi_s^p = r^p \cdot (L_s + 2p \cdot id)^{-1}P_s e_1^p = r^p \cdot e_1^{\perp}.$$

Therefore in the state space spanned by the coordinate variables e_1 *and* $e_1^{\perp}$*, the geometric shape of* $\widetilde{\mathcal{M}}(\omega)$ *is given by* (r, r^p)*.*

5 Results for the corresponding deterministic system

In this section, we briefly comment on invariant manifolds for the corresponding deterministic system of Equation (1.1)(i.e. Equation (1.1) with $\sigma = 0$). We consider the local (unstable) invariant manifold and its local geometric shape, for this deterministic system. Since we use the same method as in Section 3 and in Section 4 above, we omit the proofs of the results but only highlight some differences.

Consider the deterministic system

$$\frac{du}{dt} + Lu - u^p = 0. \tag{5.1}$$

with the initial data $u(x, 0) = u_0 = x \in E^\alpha$.

Define a Banach space for each $\beta \in (\lambda_u, \lambda_s)$ as follows

$$\mathfrak{C}_\beta^- = \{f(\cdot) \in C((-\infty, 0]; E^\alpha) | \quad \sup_{t\leq 0} e^{\beta t} |f|_\alpha < \infty\}$$

with the norm

$$\|f\|_{\mathfrak{C}_\beta^-} = \sup_{t\leq 0} e^{\beta t} |f|_\alpha.$$

Lemma 5.1 *Assume that $u(\cdot)$ is in $\mathfrak{C}_\beta^-$. Then $u(t)$ is the local solution of Equation (5.1) with the initial datum $u(0) = x$ if and only if $u(t)$ satisfies*

$$u(t) = e^{-L_u t}\xi + \int_0^t e^{-L_u(t-r)} F_u^{(R)}(u(r))dr + \int_{-\infty}^t e^{-L_s(t-r)} F_s^{(R)}(u(r))dr, \tag{5.2}$$

where $\xi = P_u x \in E_u$.

Let $\mathfrak{J}(u, \xi)$ denote the right hand side of (5.2), and $h(\xi) = \int_{-\infty}^0 e^{-L_s(t-r)} F_s^{(R)}(u(r))dr$. Then we have following results.

Lemma 5.2 *Let R be a positive real number such that l_F satisfies $SC < 1$. Then $\mathfrak{J}(u, \xi)$ has a unique fixed point $u^* = u^*(t; \xi) = \mathfrak{J}(u^*) \in \mathfrak{C}_\beta^-$. Furthermore, there exist a positive constant C such that*

$$\|u^*(t; \xi_1) - u^*(t; \xi_2)\|_{\mathfrak{C}_\beta^-} \leq C|\xi_1 - \xi_2|_\alpha,$$

$$\|u^*(t; \xi)\|_{\mathfrak{C}_\beta^-} \leq C|\xi|_\alpha, \quad \|u_s^*(t; \xi)\|_{\mathfrak{C}_\beta^-} \leq C|\xi|_\alpha, \quad \textit{and} \quad \|u_u^*(t; \xi)\|_{\mathfrak{C}_\beta^-} \leq C|\xi|_\alpha,$$

where $u_s^* = P_s u^*$ *and* $u_u^* = P_u u^*$. *Moreover,*

$$\|h(\xi_1) - h(\xi_2)\|_{\mathfrak{C}_\beta^-} \le C|\xi_1 - \xi_2|_\alpha.$$

.

Theorem 5.1 (Existence of local invariant manifold for the corresponding deterministic system)

Let R *be a positive real number such that* l_F *satisfies* $SC < 1$ *as in (3.3). Then*

$$\widetilde{\mathcal{M}} = \{\xi + h(\xi)| \quad \xi \in E_u\} \tag{5.3}$$

is a local invariant manifold for the deterministic Equation (5.1). Namely, the graph of $h(\xi)$ *is the local deterministic invariant manifold* $\widetilde{\mathcal{M}}$ *for Equation (5.1).*

In the following, we approximate the local invariant manifold $\widetilde{\mathcal{M}}$ for Equation (5.1). Define

$$\hbar_1(t) = \int_{-\infty}^{t} e^{-L_s(t-r)} F_s^{(R)}(u_u(r))dr, \tag{5.4}$$

$$\hbar_2 = \int_{-\infty}^{0} e^{L_s r} F_s^{(R)}(e^{-L_u r}\xi)dr, \tag{5.5}$$

and

$$\hbar_3 = \int_{-\infty}^{0} e^{L_s r} F_s(e^{-L_u r}\xi)dr. \tag{5.6}$$

Lemma 5.3 *Let* $|\xi|_\alpha \le R$. *There exists a positive constant* C *such that*

$$\|u_s^*(t) - \hbar_1(t)\|_{\mathfrak{C}_\beta^-} \le C|\xi|_\alpha, \quad \textit{for arbitray} \quad t \le 0,$$

$$\|u_u^*(t) - e^{-L_u t + \int_0^t z(\tau)d\tau}\xi\|_{\mathfrak{C}_\beta^-} \le C|\xi|_\alpha, \quad \textit{for arbitrary} \quad t \le 0,$$

$$|\chi_R(e^{-L_u t}\xi) - 1| \le \frac{C}{R}(1 - e^{-\lambda_u t})|\xi|_\alpha, \quad \textit{for arbitrary} \quad t \le 0,$$

and

$$\|\hbar_1(0) - \hbar_2\| \le C|\xi|_\alpha, \quad \|\hbar_2 - \hbar_3\| \le C|\xi|_\alpha^2.$$

Furthermore, noting that $\hbar_3 = (L_s - pL_u)^{-1}\xi_s^p$ and Lemma 5.3, we have the following result about local geometric shape of invariant manifold for Equation (5.1).

Theorem 5.2 (Local geometric shape of invariant manifold for the corresponding deterministic system)

Let $|\xi|_\alpha \le R$. Then there exists a positive constant C such that

$$\|h(\xi) - (L_s - pL_u)^{-1}\xi_s^p\| \le C(|\xi|_\alpha + |\xi|_\alpha^2). \tag{5.7}$$

Therefore in a neighborhood of zero for Equation (5.1), the graph $(\xi, h(\xi))$ of the invariant manifold $\widetilde{\mathcal{M}}$ is approximately given by $(\xi, (L_s - pL_u)^{-1}\xi_s^p)$.

6 Conclusions

For a class of stochastic partial differential equations, after establishing the existence of the local unstable random invariant manifold (see Theorem 3.1), we derive an approximation for the local geometric shape of this random invariant manifold (see Theorem 4.1). The local geometric shape approximation holds with significant probability. Furthermore, with the noise intensity σ decreasing, this significant probability is increasing. In fact, as noise intensity $\sigma \searrow 0$, the probability $1 - Ce^{-\frac{1}{\sigma}} \nearrow 1$. On the other hand, when $\sigma = 0$, Equation (1.1) is a deterministic system, the local geometric shape approximation of the corresponding deterministic invariant manifold is the same but holds surely (see Theorem 5.2).

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