MAXIMAL INEQUALITY OF STOCHASTIC CONVOLUTION DRIVEN BY COMPENSATED POISSON RANDOM MEASURES IN BANACH SPACES

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ABSTRACT. Assume that E is an martingale type p Banach space with q-th, $q \geq p$, power of the norm is of C^2-class. We consider the stochastic convolution

$$u(t) = \int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz),$$

where S is a C_0-semigroup of contractions on E and $\tilde{N}$ is a compensated Poisson random measure. We derive a maximal inequality for a càdlàg modification $\tilde{u}$ of u

$$\mathbb{E} \sup_{0\leq s\leq t} |\tilde{u}(s)|_E^{q'} \leq C\, \mathbb{E} \left(\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz) \right)^{\frac{q'}{p}},$$

for every $0 < q' < \infty$ and some constant $C > 0$. Stochastic convolution and martingale type p Banach space and Poisson random measure

1. Introduction

The maximal inequality for stochastic convolutions of a contraction C_0-semigroup and right continuous martingales in Hilbert spaces was studied by Ichikawa [10], see also Tubaro [15], for more details see [14]). A submartingale type inequality for the stochastic convolutions of a contraction C_0-semigroup and square integrable martingales, also in Hilbert spaces, were obtained by Kotelenez [12]. Kotelenez also proved the existence of a càdlàg version of the stochastic convolution processes for square integrable càdlàg martingales. In the paper by Brzeźniak and Peszat [4], the authors established a maximal inequality in a certain class of Banach spaces for stochastic convolution processes driven by a Wiener process. It is of interest to know whether the maximal inequality holds also for pure jump processes. Here we extend the results from [4] to the case where the stochastic convolution is driven by a compensated Poisson random measure. We work in the framework of stochastic integrals and convolutions driven by a compensated Poisson random measures recently introduced by the first two named authours in [3].

Let us now briefly present the content of the paper. In the first section, i.e. section 2 we set up notation and terminology and then summarize without proofs some of the standard facts on stochastic integrals with values in martingale type p, $p \in (1,2]$, Banach spaces, driven by compensated Poisson random measures. In the following section 3, we proceed with the study of stochastic convolution process $(u(t))_{0\leq t\leq T}$ driven by a compensated Poisson random measure $\tilde{N}$ which is defined by the following formula

$$u(t) = \int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz), \quad t \in [0,T], \tag{1.1}$$

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where $S(t)$, $t \geq 0$ is a contraction C_0-semigroup on a martingale type p, $p \in (1,2]$, Banach space E. In particular, we show that there exists a predictable version of this stochastic convolution process u. Under some suitable assumptions we show that the process u is a unique strong solution to the following stochastic evolution equation

$$\begin{aligned} du(t) &= Au(t)dt + \int_Z \xi(t,z)\tilde{N}(dt,dz), \quad t \in [0,T], \\ u(0) &= 0, \end{aligned} \tag{1.2}$$

where A is the infinitesimal generator of the contraction C_0- semigroup $S(t)$, $t \geq 0$. In the last section 4 we present our main results. In particular, the maximal inequalities are stated and proved when the q-th power, for some q, of some equivalent norm on E is of C^2 class. We first show these inequalities for the exponent $q' \geq q$. Then we adapt some ideas from the paper of Ichikawa [10], see the proof of Theorem 1, and extend the maximal inequalities to the case of any q' in $(0, \infty)$. Thus, roughly speaking, we show that the process u has an E-valued càdlàg modification $\tilde{u}$ which satisfies the following maximal inequality, see Theorems 4.4 and 4.5,

$$\mathbb{E} \sup_{0 \leq s \leq t} |\tilde{u}(s)|_E^{q'} \leq C\, \mathbb{E} \left(\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz) \right)^{\frac{q'}{p}}, \ t \in [0,T]. \tag{1.3}$$

In the last part of section 4 we formulate and prove a different version of the maximal inequality.

Remark 1.1. *It is possible to derive inequality* (1.1) *by the same method as it has been applied to get inequality (4) in* [8] *whose authours used Szekőfalvi-Nagy's Theorem on unitary dilations. The latter result has recently been generalized to Banach space of finite cotype by Fröhlich and Weis* [6]. *However, this method works only analytic semigroups of contraction type. The results from the current paper are valid for all C_0 semigroups of contraction type. To be more precise, assuming the setting before and the additional assumption that A generates an analytic semigroup, by nearly the same lines as in* [8] *it would follow*

$$\mathbb{E} \sup_{0 \leq s \leq t} |\tilde{u}(s)|_E^{q'} \leq C\, \mathbb{E} \left(\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz) \right)^{\frac{q'}{p}}.$$

2. Stochastic integral

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual hypothesis. Let $(S, \mathcal{S})$ be a measurable space. Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ and $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Let $\mathbb{M}_{\bar{\mathbb{N}}}(S)$ denote the space of all $\bar{\mathbb{N}}$-valued measures on $(S, \mathcal{S})$. Let $\mathcal{B}(\mathbb{M}_{\bar{\mathbb{N}}}(S))$ be the smallest σ-field on $\mathbb{M}_{\bar{\mathbb{N}}}(S)$ with respect to which all the mapping $i_B : \mathbb{M}_{\bar{\mathbb{N}}}(S) \ni \mu \mapsto \mu(B) \in \bar{\mathbb{N}}$, $B \in \mathcal{S}$ are measurable.

Definition 2.1. *A Poisson random measure on $(S, \mathcal{S})$ over $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a map N such that the family $\{N(B) : B \in \mathcal{S}\}$ of random variables defined by $N(B) := i_B \circ N : \Omega \to \bar{\mathbb{N}}$ satisfies the following conditions*

(1) *for any $B \in \mathcal{S}$, $N(B)$ is a random variable with Poisson distribution, i.e.*

$$\mathbb{P}(N(B) = n) = e^{-\eta(B)} \frac{\eta(B)^n}{n!}, \qquad n = 0, 1, 2, \cdots,$$

with $\eta(B) = \mathbb{E}(N(B))$.

(2) *(independently scattered property) for any pairwise disjoint sets $B_1, \cdots, B_n \in \mathcal{S}$, the random variables*

$$N(B_1),\ \cdots,\ N(B_n)$$

are independent.

Let $(Z, \mathcal{Z})$ be a measurable space. A point function on $(Z, \mathcal{Z})$ is a mapping $\alpha : \mathcal{D}(\alpha) \subset (0, \infty) \to Z$, where the domain $\mathcal{D}(\alpha)$ is a countable subset of $(0, \infty)$. Let Π_α be the set of all point functions on Z. Let $\mathcal{Q}$ be the σ-field on Π_α generated by all mappings $\alpha \mapsto \sharp\{s \in (0, t] \cap \mathcal{D}(\alpha) : \alpha(s) \in A\}$, $A \in \mathcal{Z}$, $t > 0$.

Definition 2.2. *We call a function $\pi : \Omega \to \Pi_\alpha$ a point process on Z if it is $\mathcal{F}/\mathcal{Q}$-measurable. A point process π is said to be stationary if for every $t > 0$, π and $\theta_t\pi$ have the same probability law, where $(\theta_t\pi)(s) = \pi(s+t)$, $\mathcal{D}(\theta_t\pi) = \{s \in (0, \infty) : s+t \in \mathcal{D}(\pi)\}$. For each point process π, we define a counting measure N_π by*

$$N_\pi(t, A) := \sharp\{s \in (0, t] \cap \mathcal{D}(\pi) : \pi(s) \in A\}, \quad A \in \mathcal{Z}, \quad t \geq 0.$$

A point process π is called a Poisson point process if the counting measure N_π is a Poisson random measure. Moreover, a Poisson point process is σ-finite if there exists a sequence $\{D_n\}_{n\in\mathbb{N}} \subset \mathcal{Z}$ of increasing sets such that $\cup_n D_n = Z$ and $\mathbb{E}N_\pi(t, D_n) < \infty$ for all $0 < t \leq T$ and $n \in \mathbb{N}$. A Poisson point process is stationary if and only if there exists a nonnegative σ-finite measure on $(Z, \mathcal{Z})$ such that

$$\mathbb{E}N_\pi(t, A) = t\nu(A), \quad t \geq 0, \quad A \in \mathcal{Z}.$$

From now on, we suppose that π is a σ-finite stationary Poisson point process. For simplicity of notation, we write N instead of N_π. We employ the notation $\tilde{N}(t, A) = N(t, A) - t\nu(A)$, $t \geq 0$, $A \in \mathcal{Z}$ to denote the compensated Poisson random measure associated with the Poisson point process π. Let E be a real separable Banach space of martingale type p, $1 < p \leq 2$. That is there is a constant $K_p(E) > 0$ such that for all E-valued discrete martingale $\{M_n\}_{n=0}^N$ the following inequality holds

$$\sup_n \mathbb{E}|M_n|^p \leq K_p(E) \sum_{n=0}^N \mathbb{E}|M_n - M_{n-1}|^p$$

where we set $M_{-1} = 0$ as usual. Note that all L^q spaces, $q \geq p > 1$ are of martingale type p.

Definition 2.3. *Let us fix $0 < T < \infty$. Let $\mathcal{P}$ denote the σ-field on $[0, T] \times \Omega$ generated by all left-continuous and $\mathcal{F}_t$-adapted processes.*
Let $\hat{\mathcal{P}}$ denote the σ-field on $[0, T] \times \Omega \times Z$ generated all functions $g : [0, T] \times \Omega \times Z \to E$ satisfying the following properties

(1) *for every $0 \leq t \leq T$, the mapping $(\omega, z) \mapsto g(t, \omega, z)$ is $\mathcal{Z} \otimes \mathcal{F}_t/\mathcal{B}(E)$-measurable,*
(2) *for every (ω, z), the path $t \mapsto g(t, \omega, z)$ is left-continuous.*

We say that an E-valued process $g = (g(t))_{0\leq t\leq T}$ is predictable if the mapping $[0, T] \times \Omega \ni (t, \omega) \mapsto g(t, \omega) \in E$ is $\mathcal{P}/\mathcal{B}(E)$-measurable.
We say that a function $f : [0, T] \times \Omega \times Z \to E$ is $\mathbb{F}$-predictable if the mapping is $\hat{\mathcal{P}}/\mathcal{B}(E)$-measurable.

Proposition 2.4. *$\hat{\mathcal{P}} = \mathcal{P} \otimes \mathcal{Z}$. Furthermore they are both equal to the σ-field generated by a family $\hat{\mathcal{R}}$ defined by*

$$\hat{\mathcal{R}} = \{\{0\} \times F \times B, F \in \mathcal{F}_0, B \in \mathcal{Z}\} \cup \{(s,t] \times F \times B, F \in \mathcal{F}_s, B \in \mathcal{Z}, 0 \le s < t \le T\}.$$

Moreover, the family $\hat{R}$ is a semiring.

Let $\mathcal{M}^p([0,T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$ denote the linear space of all $\mathbb{F}$-predictable functions $f : [0,T] \times \Omega \times Z \to E$ such that

$$\int_0^T \int_Z \mathbb{E}|f(t,z)|_E^p \nu(dz)dt < \infty.$$

In this section, we shall define for all functions f in the class $\mathcal{M}^p([0,T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$ the integral

$$\int_0^T \int_Z f(t,z)\tilde{N}(dt,dz),$$

which we shall call the stochastic integral with respect to a compensated Poisson random measure.

Definition 2.5. *A function $f : [0,T] \times \Omega \times Z \to E$ is a step function if there is a finite sequence of numbers $0 = t_0 < t_1 < \cdots < t_n = T$ and disjoint sets A_{j-1}^k, $j = 1, \cdots, n$, $k = 1, \cdots, m$, in $\mathcal{Z}$ with $\nu(A_{j-1}^k) < \infty$ such that*

$$f(t,\omega,z) = \sum_{j=1}^n \sum_{k=1}^m \xi_{j-1}^k(\omega) 1_{(t_{j-1},t_j]}(t) 1_{A_{j-1}^k}(z), \tag{2.1}$$

where ξ_{j-1}^k is an E-valued p-integrable $\mathcal{F}_{t_{j-1}}$-measurable random variable, $j = 1, \cdots, n$ and $k = 1, \cdots, m$. The class of all such step functions belonging to $\mathcal{M}^p([0,T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$ will be denoted by $\mathcal{M}^p_{step}([0,T] \times \Omega \times \mathcal{Z}; E)$.

Notice that a function of the form $1_{\{0\}}(t)\xi(t,\omega)$ with $\xi \in \mathcal{M}^p([0,T] \times \Omega \times Z, \hat{\mathcal{P}} \otimes \mathcal{Z}, \lambda \times \mathbb{P} \times \nu; E)$ is equivalent to the identically zero process with respect to the measure $\lambda \times \mathbb{P} \times \nu$, so it has zero stochastic integral. Therefore, the inclusion or exclusion of the origin in the definition of step function is irrelevant.

Definition 2.6. *The stochastic integral of a step function f in $\mathcal{M}^p_{step}([0,T] \times \Omega \times \mathcal{Z}; E)$ of the form (2.1) with respect to $\tilde{N}$ is defined by, for $0 < t \le T$,*

$$I_t(f) := \sum_{j=1}^n \sum_{k=1}^m \xi_{j-1}^k(\omega) \tilde{N}((t_{j-1} \wedge t, t_j \wedge t] \times A_{j-1}^k).$$

Note that, for every $f \in \mathcal{M}^p_{step}([0,T] \times \Omega \times \mathcal{Z}; E)$, $I_t(f)$ does not depend on the representation (2.1) of the step function f and the process $I_t(f)$, $0 \le t \le T$ is a càdlàg martingale with mean 0. Moreover, $I_t(f)$ is linear with respect to f and satisfies the following inequality

$$\mathbb{E}\,|I_t(f)|_E^p \le C\mathbb{E} \int_0^t \int_Z |f(s,z)|_E^p \nu(dz)\,ds, \tag{2.2}$$

where C, which is independent of the function f, is the same constant as the one in the martingale type p property of the space E. Let us now extend the definition of stochastic integral to all functions f in $\mathcal{M}^p([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\times\mathbb{P}\times\nu;E)$. Take $f\in\mathcal{M}^p([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\times\mathbb{P}\times\nu;E)$. Then we can show that there exists a sequence $f^n\in\mathcal{M}^p_{step}([0,T]\times\Omega\times\mathcal{Z};E)$ such that

$$\mathbb{E}\int_0^T\int_S\|f(t,\omega,z)-f^n(t,\omega,z)\|_E^p\nu(dz)dt\to 0,\quad\text{as } n\to\infty.$$

It follows from (2.2) that

$$\mathbb{E}\,|I_T(f^n)-I_T(f^m)|_E^p\le C\mathbb{E}\int_0^t\int_Z|f^n(s,z)-f^m(s,z)|_E^p\nu(dz)\,ds\to 0,$$

as $n,m\to\infty$. In other words, $\{I_T(f^n)\}_{n=1}^\infty$ is a Cauchy sequence in $L^p(\Omega,E,\mathcal{F}_T)$. Thus the sequence $\{I_T(f^n)\}_{n=1}^\infty$ of random variables will converge in $L^p(\Omega,E,\mathcal{F}_T)$ to some particular random variable which we shall denote by $I_T(f)$. Moreover, such random variable is uniquely determined up to a set of measure zero in the variable ω. That is, it does not depend on the choice of the approximating step functions. We usually call $I_T(f)$ the stochastic integral of f with respect to a compensated Poisson random measure $\tilde{N}$. For $0\le a\le b\le T$, $B\in\mathcal{Z}$ and $f\in\mathcal{M}^p([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\times\mathbb{P}\times\nu;E)$, since $1_{(a,b]}1_Bf$ is also in $\mathcal{M}^p([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\times\mathbb{P}\times\nu;E)$, so we can define the stochastic integral from a to b of the function $f\in\mathcal{M}^p([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\times\mathbb{P}\times\nu;E)$ by

$$I_{a,b}^B(f)=\int_a^b\int_B f(t,z)\tilde{N}(dt,dz)=I_T(1_{(a,b]}1_Bf).$$

For simplicity, we denote

$$I_t(f)=\int_0^t\int_Z f(t,z)\tilde{N}(dt,dz)=I_T(1_{(0,t]}f).$$

The following result was first proven in the case $p=2$ in an important work [16] by Rüdiger.

Theorem 2.7. *([3]) Let $f\in\mathcal{M}^p([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\times\mathbb{P}\times\nu;E)$. Then $I_t(f)$, $0\le t\le T$ is a càdlàg p-integrable martingale with mean 0. More precisely, $I_t(f)$ has a modification which has $\mathbb{P}$-a.s. càdlàg trajectories. Moreover, it satisfies the following*

$$\mathbb{E}|I_t(f)|_E^p=\mathbb{E}\left|\int_0^t\int_Z f(s,z)\tilde{N}(ds,dz)\right|_E^p\le C\mathbb{E}\int_0^t\int_Z|f(t,z)|_E^p\nu(dz)\,ds. \tag{2.3}$$

From now on, while considering the stochastic process $\int_0^t\int_Z f(s,z)\tilde{N}(ds,dz)$, $0\le t\le T$, $f\in\mathcal{M}^p([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\times\mathbb{P}\times\nu;E)$, it will be assumed that the process $\int_0^t\int_Z f(s,z)\tilde{N}(ds,dz)$, $0\le t\le T$, has $\mathbb{P}$-a.s. càdlàg trajectories.

3. Stochastic convolution

Let $(S(t))_{t\ge0}$ be a contraction C_0-semigroup on E. Suppose that A is the infinitesimal generator of the C_0-semigroup $(S(t))_{t\ge0}$. If $\{A_\lambda:\lambda>0\}$ is the Yosida approximation of A, then for each λ, A_λ is a bounded operator in E and $|A_\lambda x-Ax|_E$ converges to 0 as $\lambda\to\infty$ for all $x\in E$, and uniformly convergence on bounded intervals. Let $R(\lambda,A)=(\lambda I-A)^{-1}$. By the use of Hille-Yosida Theorem (see [13]), it is easy to establish that $\lim_{\lambda\to\infty}\lambda R(\lambda,A)x=x$ and $\lambda R(\lambda,A)x\in\mathcal{D}(A)$, for all $x\in X$.

Let $\xi \in \mathcal{M}^p([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\times\mathbb{P}\times\nu;E)$. We are going to consider the following stochastic convolution process

$$u(t)=\int_0^t\int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz),\ 0\le t\le T, \tag{3.1}$$

where $\tilde{N}$ is a compensated Poisson random measure of the point process $\pi=(\pi(t))_{t\ge0}$.

We will first investigate the measurability of the process u.

Lemma 3.1. *The process $u(t)$, $0\le t\le T$ given by* (3.1) *has a predictable version.*

Proof. Let $t\in[0,T]$ be fixed. We first show that a process X defined by $X(s)=1_{(0,t]}(s)S(t-s)\xi(s,z)$, $0\le s\le T$ is predictable. Define a function $F:[0,t]\times E\ni(s,x)\mapsto S(t-s)x\in E$. Since $S(t)$, $t\ge0$ is a C_0-semigroup, so for every $x\in E$, $F(\cdot,x)$ is continuous on $[0,t]$. Also, for every $s\ge0$, $F(s,\cdot)$ is continuous. Indeed, let us fix $x_0\in E$. Then for every $x\in E$, and $0\le t\le T$,

$$|F(t,x)-F(t,x_0)|_E=|S(t-s)(x-x_0)|_E\le|x-x_0|_E,$$

as $\|S(t)\|_{\mathcal{L}(E)}\le1$. This part shows that the function F is separably continuous. Since by assumption the process ξ is $\mathbb{F}$-predictable, one can see that the mapping

$$(s,\omega,z)\mapsto(s,\xi(s,\omega,z))$$

of $[0,T]\times\Omega\times Z$ into $[0,T]\times E$ is $\mathbb{F}$-predictable. Moreover, since the process $1_{(0,t]}$ is $\mathbb{F}$-predictable and we showed that the function F is separably continuous, so the composition mapping

$$(s,\omega,z)\mapsto(s,\xi(s,\omega,z))\mapsto F(s,\xi(s,\omega,z))\mapsto 1_{(0,t]}(s)F(s,\xi(s,\omega,z))$$

is $\mathbb{F}$-predictable as well. Therefore, process $X(s)=1_{(0,t]}(s)F(s,\xi(s,z))$, $s\in[0,T]$ is $\mathbb{F}$-predictable. On the other hand, since $S(t)$, $t\ge0$ is a C_0-semigroup of contractions and ξ is in $\mathcal{M}^p([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\times\mathbb{P}\times\nu;E)$, we have

$$\mathbb{E}\int_0^T|1_{(0,t]}S(t-s)\xi(s,z)|_E^p\nu(dz)\,ds\le\mathbb{E}\int_0^T|\xi(s,z)|_E^p\nu(dz)\,ds<\infty.$$

Therefore, the process $1_{(0,t]}S(t-s)\xi(s,z)$ is of class $\mathcal{M}^p([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\times\mathbb{P}\times\nu;E)$. Hence, when the number t is fixed, the integrals

$$\int_0^r\int_Z 1_{(0,t]}S(t-s)\xi(s,z)\tilde{N}(ds,dz),\quad r\in[0,T]$$

are well defined and by Theorem 2.7, this process is a martingale. In particular, for each $r\in[0,T]$, the integral $\int_0^r\int_Z 1_{(0,t]}S(t-s)\xi(s,z)\tilde{N}(ds,dz)$ is $\mathcal{F}_r$-measurable. Take $r=t$. This gives that $\int_0^t\int_Z 1_{(0,t]}S(t-s)\xi(s,z)\tilde{N}(ds,dz)$ is $\mathcal{F}_t$-measurable.

Now we show that the process u is continuous in p-mean. Observe that from the inequality $|a+b|^p\le2^p|a|^p+2^p|b|^p$, inequality (2.3) and the contraction property of the semigroup $S(t)$, $t\ge0$, we have, for $0 \le r < t \le T$,

$$\begin{aligned}\mathbb{E}|u(t)-u(r)|_E^p &= \mathbb{E}\left|\int_0^t\int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz) - \int_0^r\int_Z S(r-s)\xi(s,z)\tilde{N}(ds,dz)\right|_E^p \\ &\le 2^p\mathbb{E}\left|\int_r^t\int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz)\right|_E^p \\ &\qquad + 2^p\mathbb{E}\left|\int_0^r\int_Z \Big(S(t-s)-S(r-s)\Big)\xi(s,z)\tilde{N}(ds,dz)\right|_E^p\end{aligned}$$

$$\begin{aligned}&\le 2^pC_p\mathbb{E}\int_r^t\int_Z |S(t-s)\xi(s,z)|_E^p\nu(dz)\,ds \\ &\qquad + 2^pC_p\mathbb{E}\int_0^r\int_Z |\Big(S(t-s)-S(r-s)\Big)\xi(s,z)|_E^p\nu(dz)\,ds \\ &\le 2^pC_p\mathbb{E}\int_r^t\int_Z |\xi(s,z)|_E^p\nu(dz)\,ds \\ &\qquad + 2^pC_p\mathbb{E}\int_0^r\int_Z |\Big(S(t-s)-S(r-s)\Big)\xi(s,z)|_E^p\nu(dz)\,ds \\ &= 2^pC_p\mathbb{E}\int_0^T\int_Z 1_{(r,t]}(s)|\xi(s,z)|_E^p\nu(dz)\,ds \\ &\qquad + 2^pC_p\mathbb{E}\int_0^T\int_Z |1_{(0,r]}\Big(S(t-s)-S(r-s)\Big)\xi(s,z)|_E^p\nu(dz)\,ds.\end{aligned}$$

Here we note that $1_{(r,t]}(s)|\xi(s,z)|_E^p$ converges to 0 for all $(s,\omega,z) \in [0,T]\times\Omega\times Z$, as $t\downarrow r$ or $r\uparrow t$. So by the Lebesgue Dominated converges theorem, the first term on the right hand side of above inequality converges to 0 as $t\downarrow r$ or $r\uparrow t$. For the second term, by the continuity of C_0-semigroup $S(t)$, $t\ge 0$, the integrand $1_{(0,r]}\Big(S(t-s)-S(r-s)\Big)\xi(s,z)$ converges to 0 pointwise on $[0,T]\times\Omega\times Z$. Moreover we find that

$$|1_{(0,r]}S(t-s)-S(r-s)\Big)\xi(s,z)|_E \le |\xi(s,z)|_E$$

So, again by the Lebesgue Dominated Convergence Theorem, the second term also converges to 0 as $t\downarrow r$ or $r\uparrow t$. Therefore, the process u is continuous in the p-mean. Since by Proposition 3.6 in [5], every adapted and stochastically continuous process on an interval $[0,T]$ has a predictable version on $[0,T]$, we conclude that the process $u(t)$, $0\le t\le T$ has a predictable version.

□

Assume that A is the infinitesimal generator of a C_0-semigroup $S(t)$, $t\ge 0$ of contractions on the Banach space E and that ξ is a function belonging to $\mathcal{M}^p([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\times\mathbb{P}\times\nu;E)$.

Consider the problem (1.2) which for the convenience of the reader we write aagin below.

$$\begin{aligned}du(t) &= Au(t)dt + \int_Z \xi(t,z)\tilde{N}(dt,dz) \\ u(0) &= 0,\end{aligned} \tag{3.2}$$

Definition 3.2. *Suppose that* $\mathbb{E}\int_0^T\int_Z|\xi(s,z)|_E^p\nu(dz)dt<\infty$. *A strong solution to Problem* (1.2) *is a* $\mathcal{D}(A)$*-valued predictable stochastic process* $(u(t))_{0\le t\le T}$ *such that*

(1) $u(0)=0$ *a.s.*

(2) *For any* $t \in [0,T]$ *the equality*

$$u(t) = \int_0^t Au(s)\,ds + \int_0^t \int_Z \xi(s,z)\tilde{N}(ds,dz) \tag{3.3}$$

holds $\mathbb{P}$*-a.s..*

Lemma 3.3. *Let* $\xi \in \mathcal{M}^p([0,T]\times\Omega\times Z, \hat{\mathcal{P}}, \lambda\times\mathbb{P}\times\nu; \mathcal{D}(A))$*. Then the process* u *defined by*

$$u(t) = \int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz),\ \ t\in[0,T], \tag{3.4}$$

is a unique strong solution of equation (1.2).

Proof. Let us us fix $t \in [0,T]$. First we need to show that $u(t) \in \mathcal{D}(A)$. For this, Let $R(\lambda,A) = (\lambda I - A)^{-1}$, $\lambda > 0$, be the resolvent of A. Since $AR(\lambda,A) = \lambda R(\lambda,A) - I_E$, $AR(\lambda,A)$ is bounded. Hence, since $\xi \in \mathcal{M}^p([0,T]\times\Omega\times Z, \hat{\mathcal{P}}, \lambda\times\mathbb{P}\times\nu; \mathcal{D}(A))$, we obtain

$$\begin{aligned} R(\lambda,A)\int_0^t\int_Z AS(t-s)\xi(s,z)\tilde{N}(ds,dz) &= \int_0^t\int_Z R(\lambda,A)AS(t-s)\xi(s,z)\tilde{N}(ds,dz) \\ &= \lambda R(\lambda,A)\int_0^t\int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz) \\ &\quad - \int_0^t\int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz). \end{aligned}$$

Thus, it follows that

$$\begin{aligned} &\int_0^t\int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz) \\ &\qquad = R(\lambda,A)\left[\lambda\int_0^t\int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz) - \int_0^t\int_Z AS(t-s)\xi(s,z)\tilde{N}(ds,dz)\right]. \end{aligned}$$

Since $Rng(R(\lambda,A)) = \mathcal{D}(A)$, we infer that $\int_0^t\int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz) \in \mathcal{D}(A)$. On the other hand, let us take $h \in (0,t)$ and observe that since $\frac{S(h)-I}{h}$ is bounded, we get the following equality

$$\begin{aligned} &\frac{S(h)-I}{h}\int_0^t\int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz) \\ &\qquad = \int_0^t\int_Z \frac{S(h)-I}{h} S(t-s)\xi(s,z)\tilde{N}(ds,dz). \end{aligned}$$

So by applying the triangle inequality and inequality (2.3), we find that

$$\begin{aligned} &\mathbb{E}\left|A\int_0^t\int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz) - \int_0^t\int_Z AS(t-s)\xi(s,z)\tilde{N}(ds,dz)\right|^p \\ &\quad \le 2^p\mathbb{E}\left|A\int_0^t\int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz) - \frac{S(h)-I}{h}\int_0^t\int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz)\right|^p \\ &\qquad + 2^p\mathbb{E}\left|\int_0^t\int_Z AS(t-s)\xi(s,z)\tilde{N}(ds,dz) - \int_0^t\int_Z \frac{S(h)-I}{h}S(t-s)\xi(s,z)\tilde{N}(ds,dz)\right|^p \\ &\quad \le 2^p\mathbb{E}\left|\Big(A - \frac{S(h)-I}{h}\Big)\int_0^t\int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz)\right|^p \\ &\qquad + C_p\mathbb{E}\int_0^t\int_Z \left|AS(t-s)\xi(s,z) - \frac{1}{h}\Big(S(h)-I\Big)S(t-s)\xi(s,z)\right|_E^p \nu(dz)\,ds \\ &\quad := \mathrm{I}(h) + \mathrm{II}(h). \end{aligned} \tag{3.5}$$

For the integrand of I(h), since $\xi(s,z)\in\mathcal{D}(A)$, we observe that $\frac{S(h)-I}{h}S(t-s)\xi(s,z)=\frac{1}{h}\int_0^h S(r)AS(t-s)\xi(s,z)dr$, so we have $\left|\frac{S(h)-I}{h}S(t-s)\xi(s,z)\right|_E^p\leq|A\xi(s,z)|_E^p$. Hence we infer that the integrand

$$\left|AS(t-s)\xi(s,z)-\frac{1}{h}\Big(S(h)-I\Big)S(t-s)\xi(s,z)\right|_E^p$$

of I(h) is bounded by a function $C_1|A\xi(s,z)|_E^p$ which is in $\mathcal{M}^p([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\times\mathbb{P}\times\nu;E)$ by assumption. Since A is the infinitesimal generator of the C_0-semigroup $S(t)$, $t\geq 0$, the integrand

$$\left|AS(t-s)\xi(s,z)-\frac{1}{h}\Big(S(h)-I\Big)S(t-s)\xi(s,z)\right|_E^p$$

converges to 0 pointwisely on $[0,t]\times\Omega\times Z$. Therefore, by the Lebesgue Dominated convergence theorem, the term II(h) of above inequality (3.5) converges to 0 as $h\downarrow 0$.
Since we have already shown that $\int_0^t\int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz)\in\mathcal{D}(A)$, it is easy to see that the term I(h) of (3.5) converges to 0 as $h\downarrow 0$ as well. Hence by inequality (3.5) we conclude that

$$A\int_0^t\int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz)=\int_0^t\int_Z AS(t-s)\xi(s,z)\tilde{N}(ds,dz),\quad \mathbb{P}\text{-a.s.} \tag{3.6}$$

In order to verify equality (3.3), by the Fubini's theorem and equality (3.6) we find that

$$\begin{aligned}\int_0^t Au(s)\,ds &= \int_0^t\int_0^s\int_Z AS(s-r)\xi(r,z)\tilde{N}(dr,dz)\,ds\\ &= \int_0^t\int_Z\int_r^t AS(s-r)\xi(r,z)\,ds\tilde{N}(dr,dz)\\ &= \int_0^t\int_Z\int_r^t \frac{dS(s-r)\xi(r,z)}{ds}ds\tilde{N}(dr,dz)\\ &= \int_0^t\int_Z\left(S(t-r)\xi(r,z)-\xi(r,z)\right)\tilde{N}(dr,dz)\\ &= \int_0^t\int_Z S(t-r)\xi(r,z)\tilde{N}(dr,dz)-\int_0^t\int_Z\xi(r,z)\tilde{N}(dr,dz)\\ &= u(t)-\int_0^t\int_Z\xi(r,z)\tilde{N}(dr,dz),\quad\mathbb{P}\text{-a.s.}\end{aligned}$$

which shows equality (3.3).

For the uniqueness, suppose that u^1 and u^2 are two strong solutions of Problem (1.2). Let $w=u^1-u^2$. Then we infer that

$$w(t)=u^1(t)-u^2(t)=\int_0^t A(u^1(s)-u^2(s))\,ds=A\int_0^t w(s)\,ds.$$

Put $v(t)=\int_0^t w(s)\,ds$. Then $v(t)$ is continuously differentiable on $[0,T]$ and $v(t)\in\mathcal{D}(A)$. Now applying Itô's formula to the function $f(s)=S(t-s)v(s)$ yields

$$\begin{aligned}\frac{df(s)}{ds} &= -AS(t-s)v(s)+S(t-s)\frac{dv(s)}{ds}\\ &= -AS(t-s)v(s)+S(t-s)w(s)=-AS(t-s)v(s)+S(t-s)Av(s)=0.\end{aligned}$$

So we infer $v(t)=f(t)=f(0)=S(t)v(0)=0$ a.s.. Therefore, $w(s)=0$ a.s.. That is $u^1(t)=u^2(t)$ a.s. $t\in[0,T]$. □

4. Maximal inequalities for stochastic convolution

Assumption 4.1. *Suppose that E is a real separable Banach space of martingale type p, $1 < p \leq 2$. In addition we assume that the Banach space E satisfies the following condition:*
(Cond. 1) *There exists an equivalent norm $|\cdot|_E$ on E and $q \in [p, \infty)$ such that the function $\phi : E \ni x \mapsto |x|_E^q \in \mathbb{R}$, is of class C^2 and there exists constant k_1, k_2 such that for every $x \in E$, $|\phi'(x)| \leq k_1 |x|_E^{q-1}$ and $|\phi''(x)| \leq k_2 |x|_E^{q-2}$.*

Remark 4.1. *It can be proved that if E satisfies condition **(Cond. 1)** for some q and $q_2 > q$, then E satisfies condition **(Cond. 1)** for q_2.*

Remark 4.2. *Notice that the Sobolev space $H^{s,p}$ with $p \in [2, \infty)$ and $s \in \mathbb{R}$ satisfies above condition Cond. 1 and L^r-spaces with $r \geq q$ also satisfies condition Cond. 1.*

Now we proceed with the study of the stochastic convolution

$$u(t) = \int_0^t \int_Z S(t-s)\xi(s,z)\tilde{N}(ds,dz), \ t \in [0,T]. \tag{4.1}$$

Before proving the main theorem, we first need the following Lemmas.

Lemma 4.2. *For all $x \in D(A)$, $\phi'(x)(Ax) \leq 0$.*

Proof. Take $0 \leq r < t < \infty$. We have

$$\begin{aligned} |S(t)x|_E^q - |S(r)x|_E^q &= |S(t-r)S(r)x|_E^q - |S(r)x|_E^q \\ &\leq |S(t-r)|_{\mathcal{L}(E)}^q |S(r)x|_E^q - |S(r)x|_E^q \\ &\leq |S(r)x|_E^q - |S(r)x|_E^q = 0, \quad \text{for all } x \in E. \end{aligned}$$

Thus the function $t \mapsto \phi(x)(S(t)x)$ is decreasing. Also, observe that for $x \in D(A)$,

$$\left.\frac{d\phi(S(t)x)}{dt}\right|_{t=0} = \phi'(S(0)x)(Ax) = \phi'(x)(Ax).$$

Hence $\phi'(x)(Ax) = \left.\frac{d\phi(S(t)x)}{dt}\right|_{t=0} \leq 0$ which shows the Lemma. □

Lemma 4.3. *The random variable $\sup_{0\leq t\leq T} |u(t)|$ is measurable.*

Proof. Since we have shown in Lemma 3.1 the stochastically continuity of the process u, applying Theorem 5.3 in [20], we can find a version $\tilde{u}$ of u which is separable. That is there exists a countable subset T_0 which is everywhere dense in $[0,T]$ such that $\tilde{u}(t)$ belongs to the set of partial limits $\lim_{s\in T_0, s\to t} \tilde{u}(s)$ for all $t \in [0,T]\backslash T_0$. Hence

$$\sup_{t\in[0,T]} |\tilde{u}(t)| = \sup_{t\in[0,T]} \lim_{s_n\to t, s_n\in T_0} |\tilde{u}(s_n)| = \sup_{s_n\in T_0} |\tilde{u}(s_n)|,$$

where $\sup_{s_n\in T_0} |\tilde{u}(s_n)|$ is measurable. Therefore, the random variable $\sup_{t\in[0,T]} |\tilde{u}(t)|$ is also measurable. □

Henceforth, when we study the stochastic convolution process u, we refer to the version of u such that it is predictable and its supremum over $[0,T]$ is measurable.

Theorem 4.4. *Suppose E is an martingale type p Banach space satisfying Assumption 4.1. Then there exists a càdlàg modification $\tilde{u}$ of u and a constant C such that for every $0 < t \leq T$,*

$$\mathbb{E} \sup_{0\leq s\leq t} |\tilde{u}(s)|_E^{q'} \leq C\, \mathbb{E} \left(\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz) \right)^{\frac{q'}{p}}, \tag{4.2}$$

where $q' \geq q$ and q is the number from Assumption 4.1.
From now on, A denotes the infinitesimal generator of the C_0-semigroup $(S(t))_{t\geq 0}$ of contractions.

Proof. **Case I.** First suppose that $\xi \in \mathcal{M}^p([0,T]\times\Omega\times Z, \hat{\mathcal{P}}, \lambda\times\mathbb{P}\times\nu; \mathcal{D}(A))$. We will prove

$$\mathbb{E} \sup_{0\leq s\leq t} |u(s)|_E^{q} \leq C\, \mathbb{E} \left(\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz) \right)^{\frac{q}{p}}, \tag{4.3}$$

We have shown in Lemma 3.3 that the process u is a unique strong solution to the following problem

$$\begin{aligned} du(t) &= Au(t)dt + \int_Z \xi(t,z)\tilde{N}(dt,dz), \quad t\in[0,T],\\ u(0) &= 0. \end{aligned} \tag{4.4}$$

Moreover, it can be written as

$$u(t) = \int_0^t Au(s)\,ds + \int_0^t \int_Z \xi(s,z)\tilde{N}(ds,dz), \; t\in[0,T]. \tag{4.5}$$

We shall note here that in view of the right continuity of the right hand side of (4.5), the cádlàg property of the function $u(t)$, $0\leq t\leq T$ follows immediately. Notice that the function $\phi : E \ni x \mapsto |x|_E^q$ is of C^2 class by assumption. Thus, one may apply the Itô formula [9] to the process u and get for $t\in[0,T]$,

$$\begin{aligned} \phi(u(t)) &= \int_0^t \phi'(u(s))(Au(s))\,ds + \int_0^t\int_Z \phi'(u(s-))(\xi(s,z))\tilde{N}(ds,dz)\\ &\quad + \int_0^t\int_Z \Big[\phi(u(s-)+\xi(s,z)) - \phi(u(s-) - \phi'(u(s-))(\xi(s,z))\Big] N(ds,dz) \;\; \mathbb{P}\text{-a.s..} \end{aligned} \tag{4.6}$$

Since by Lemma $\phi'(x)(Ax)\leq 0$, for all $x\in D(A)$, we infer that for $t\in[0,T]$,

$$\begin{aligned} \phi(u(t)) &\leq \int_0^t\int_Z \phi'(u(s-))(\xi(s,z))\tilde{N}(ds,dz)\\ &\quad + \int_0^t\int_Z \Big[\phi(u(s-)+\xi(s,z)) - \phi(u(s-)) - \phi'(u(s-))(\xi(s,z))\Big] N(ds,dz) \;\; \mathbb{P}\text{-a.s..} \end{aligned} \tag{4.7}$$

Taking the supremum over the set $[0,t]$ and then the expectation to both sides of above inequality yields

$$\begin{aligned} \mathbb{E}\sup_{0\leq s\leq t}\phi(u(s)) &\leq \mathbb{E}\sup_{0\leq s\leq t}\int_0^s\int_Z \phi'(u(r))(\xi(r,z))\tilde{N}(dr,dz)\\ &\quad + \mathbb{E}\sup_{0\leq s\leq t}\int_0^s\int_Z \Big[\phi(u(r-)+\xi(r,z)) - \phi(u(r-)) - \phi'(u(r-))(\xi(r,z))\Big] N(dr,dz)\\ &=: I_1(t) + I_2(t). \end{aligned}$$

Applying the Davis inequality and the Jensen inequality to I_1 we obtain for some constant C that

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t} I_1(s) &\le C\mathbb{E}\left(\int_0^t\int_Z |\phi'(u(s-))(\xi(s,z))|^p N(ds,dz)\right)^{\frac{1}{p}}\\ &\le k_1 C\mathbb{E}\sup_{0\le s\le t}|u(s)|^{q-1}\left(\int_0^t\int_Z |\xi(s,z)|^p N(ds,dz)\right)^{\frac{1}{p}}.\end{aligned}$$

Firstly we are going to estimate the integral $I_2(t)$. Note that for every $s\in[0,t]$,

$$\begin{aligned}&\int_0^s\int_Z \Big|\phi(u(r)+\xi(r,z))-\phi(u(r))-\phi'(u(r-))(\xi(r,z))\Big|_E N(dr,dz)\\ &\qquad = \sum_{r\in(0,s]\cap\mathcal{D}(\pi)}\Big|\phi(u(r-)+\xi(r,\pi(r)))-\phi(u(r-))-\phi'(u(r-))(\xi(r,\pi(r)))\Big|_E, \quad \mathbb{P}\text{-a.s.}.\end{aligned}$$

Let us recall that by the assumption the function ϕ is of C^2 class. Applying the mean value Theorem, see [11], to the function ϕ, for each $r\in[0,s]$ we can find $0<\theta<1$ such that

$$\Big|\phi(u(r-)+\xi(r,\pi(r)))-\phi(u(r-))\Big|_E = |\xi(r,\pi(r))|_E\Big|\phi'(u(r-)+\theta\xi(r,\pi(r)))\Big|_{\mathcal{L}(E)}.$$

By the assumptions $|\phi'(x)|\le k_1|x|_E^{q-1}$, $x\in E$ and the fact that $|x+\theta y|_E\le\max\{|x|_E,|x+y|_E\}$ for all $x,y\in E$, we obtain

$$\begin{aligned}\Big|\phi'(u(r-)+\theta\xi(r,\pi(r)))\Big|_{\mathcal{L}(E)} &\le k_1\big|u(r-)+\theta\xi(r,\pi(s))\big|_E^{q-1}\\ &\le k\max\big\{|u(r-)|_E^{q-1},\big|u(r-)+\xi(r,\pi(r))\big|_E^{q-1}\big\}.\end{aligned}$$

Observe that for all $0\le r\le s\le t$,

$$|u(r-)|_E^{q-1}\le\sup_{0\le\rho\le s}|u(\rho-)|_E^{q-1}\le\sup_{0\le\rho\le t}|u(\rho-)|_E^{q-1}=\sup_{0\le\rho\le t}|u(\rho)|_E^{q-1}.$$

Moreover, since $u(r-)+\xi(r,\phi(r))=u(r)$, we get

$$|u(r-)+\xi(r,\pi(r))|_E^{q-1}\le\sup_{0\le r\le s}|u(r)|_E^{q-1}\le\sup_{0\le s\le t}|u(s)|_E^{q-1}.$$

Therefore, we infer that for each $r\in[0,s]$,

$$\Big|\phi(u(r-)+\xi(r,\pi(r)))-\phi(u(r-))\Big|_E\le k_1|\xi(r,\pi(r))|_E\sup_{0\le s\le t}|u(s)|_E^{q-1}.$$

It follows that

$$\begin{aligned}&\Big|\phi(u(r-)+\xi(r,\pi(r)))-\phi(u(r-))-\phi'(u(r-))(\xi(r,\pi(r)))\Big|_E\\ &\qquad\le\Big|\phi(u(r)+\xi(r,\pi(r)))-\phi(u(r))\Big|_E+\Big|\phi'(u(r-))(\xi(r,\pi(r)))\Big|_E\\ &\qquad\le 2k_1|\xi(r,\pi(r))|_E\sup_{0\le s\le t}|u(s)|_E^{q-1}.\end{aligned}$$

On the other hand side, we can also find some $0<\delta<1$ such that

$$\begin{aligned}\Big|\phi(u(r-)+\xi(r,\pi(r)))-\phi(u(r-))-\phi'(u(r-))(\xi(r,\pi(r)))\Big|_E &= \frac{1}{2}|\xi(r,\pi(r))|_E^2|\phi''(u(r-)+\theta\xi(r,\pi(r)))|\\ &\le\frac{k_2}{2}|\xi(r,\pi(r))|_E^2|u(r-)+\theta\xi(r,\pi(r))|_E^{q-2}.\end{aligned}$$

A similar argument as above, we obtain

$$\Big|\phi(u(r-)+\xi(r,\pi(r)))-\phi(u(r-))-\phi'(u(r-))(\xi(r,\pi(r)))\Big|_E\le\frac{k_2}{2}|\xi(r,\pi(r))|_E^2\sup_{0\le s\le t}|u(s)|_E^{q-2}.$$

Summing up, we have

$$\begin{aligned}
\Big|\phi(u(r-)+\xi(r,\pi(r)))&-\phi(u(r-))-\phi'(u(r-))(\xi(r,\pi(r)))\Big|_E \\
&= \Big|\phi(u(r-)+\xi(r,\pi(r)))-\phi(u(r-))-\phi'(u(r-))(\xi(r,\pi(r)))\Big|_E^{(2-p)+(p-1)} \\
&\le \Big(2k_1|\xi(r,\pi(r))|_E \sup_{0\le s\le t}|u(s)|_E^{q-1}\Big)^{2-p}\Big(\frac{k_2}{2}|\xi(r,\pi(r))|_E^2 \sup_{0\le s\le t}|u(s)|_E^{q-2}\Big)^{p-1} \\
&\le K|\xi(r,\pi(r))|_E^p \sup_{0\le s\le t}|u(s)|_E^{q-p},
\end{aligned}$$

where $K=(2k_1)^{2-p}(k_1/2)^{p-1}$.

Hence,

$$\begin{aligned}
\sum_{r\in(0,t]\cap\mathcal{D}(\pi)} \Big|\phi(u(r-)+\xi(r,\pi(r)))&-\phi(u(r-))-\phi'(u(r-))(\xi(r,\pi(r)))\Big|_E \\
&\le K\sup_{0\le s\le t}|u(s)|_E^{q-p}\sum_{r\in(0,t]\cap\mathcal{D}(\pi)}|\xi(r,\pi(r))|_E^p \\
&= K\sup_{0\le s\le t}|u(s)|_E^{q-p}\int_0^s\int_Z|\xi(r,z)|_E^p\;N(dr,dz),
\end{aligned}$$

which also shows that the integral $\int_0^t\int_Z\Big[\phi(u(s-)+\xi(s,z))-\phi(u(s-))-\phi'(u(s-))(\xi(s,z))\Big]N(ds,dz)$ is well defined since $\xi\in\mathcal{M}^p([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\times\mathbb{P}\times\nu;\mathcal{D}(A))$. Therefore, we infer

$$\begin{aligned}
\int_0^s\int_Z\Big|\phi(u(r-)+\xi(r,z))-\phi(u(r-))&-\phi'(u(r-))(\xi(r,z))\Big|_E N(dr,dz) \\
&\le K\sup_{0\le s\le t}|u(s)|_E^{q-p}\int_0^s\int_Z|\xi(r,z)|_E^p\;N(dr,dz).
\end{aligned}$$

Hence, we get the following estimate for $I_2(t)$

$$I_2(t)\le K\sup_{0\le s\le t}|u(s)|_E^{q-p}\int_0^s\int_Z|\xi(r,z)|_E^p\;N(dr,dz),\quad t\in[0,T],$$

where the constant K only depends on k_1, k_2, p and q. Now applying Hölder's and Young's inequalities to $I_1(t)$ yields

$$\begin{aligned}
I_1(t) &\le k_1C\Big[\Big(\mathbb{E}\Big(\sup_{0\le s\le t}|u(s)|_E^{q-1}\Big)^{\frac{q}{q-1}}\Big)^{\frac{q-1}{q}}\Big(\mathbb{E}\Big(\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz)\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} \\
&\le k_1C\Big[\Big(\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^q\Big)^{\frac{q-1}{q}}\Big(\mathbb{E}\Big(\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz)\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} \\
&= k_1C\Big[\Big(\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^q\;\varepsilon\Big)^{\frac{q-1}{q}}\Big(\mathbb{E}\Big(\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz)\Big)^{\frac{q}{p}}\Big(\frac{1}{\varepsilon}\Big)^{q-1}\Big)^{\frac{1}{q}} \\
&\le k_1C\left[\frac{q-1}{q}\;\varepsilon\;\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^q+\frac{1}{\varepsilon^{q-1}q}\mathbb{E}\Big(\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz)\Big)^{\frac{q}{p}}\right] \\
&= k_1C\frac{q-1}{q}\;\varepsilon\;\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^q+k_1C\frac{1}{\varepsilon^{q-1}q}\mathbb{E}\Big(\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz)\Big)^{\frac{q}{p}}.
\end{aligned}$$

In the same manner for the integral $I_2(t)$ we can see that

$$\begin{aligned}
I_2(t) &= K\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^{q-p}\int_0^t\int_Z|\xi(s,z)|_E^p N(ds,dz)\\
&\le K\left(\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^{(q-p)\frac{q}{q-p}}\right)^{\frac{q-p}{q}}\left(\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^p N(ds,dz)\right)^{\frac{q}{p}}\right)^{\frac{p}{q}}\\
&\le K\left(\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^{q}\right)^{\frac{q-p}{q}}\left(\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^p N(ds,dz)\right)^{\frac{q}{p}}\right)^{\frac{p}{q}}\\
&= K\left(\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^{q}\ \varepsilon\right)^{\frac{q-p}{q}}\left(\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E N(ds,dz)\right)^{q}\left(\frac{1}{\varepsilon}\right)^{\frac{q-p}{p}}\right)^{\frac{p}{q}}\\
&\le K\frac{q-p}{q}\varepsilon\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^q + K\frac{p}{q}\frac{1}{\varepsilon^{\frac{q-p}{q}}}\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E N(ds,dz)\right)^{q}
\end{aligned}$$

where we used Hölder's inequality in the first and fourth inequalities and Young's inequality in the third inequality.
It then follows that

$$\begin{aligned}
\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^q &\le k_1C\frac{q-1}{q}\ \varepsilon\ \mathbb{E}\sup_{0\le s\le t}|u(s)|_E^q + k_1C\frac{1}{\varepsilon^{q-1}q}\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^p N(ds,dz)\right)^{\frac{q}{p}}\\
&\quad +K\frac{q-p}{q}\varepsilon\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^q + K\frac{p}{q}\frac{1}{\varepsilon^{\frac{q-p}{q}}}\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E N(ds,dz)\right)^{q}\\
&= \left(k_1C\frac{q-1}{q}+K\frac{q-p}{q}\right)\varepsilon\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^q\\
&\quad +\left(k_1C\frac{1}{\varepsilon^{q-1}q}+K\frac{p}{q}\frac{1}{\varepsilon^{\frac{q-p}{q}}}\right)\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^p N(ds,dz)\right)^{\frac{q}{p}}.
\end{aligned}$$

Now we can choose a suitable number ε such that

$$\left(k_1C\frac{q-1}{q}+K\frac{q-p}{q}\right)\varepsilon=\frac{1}{2}.$$

Consequently, there exists C which is independent of A such that

$$\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^q \le C\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^p N(ds,dz)\right)^{\frac{q}{p}}. \tag{4.8}$$

Case II. Suppose $\xi\in\mathcal{M}^p([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\times\mathbb{P}\times\nu;E)$. Set $R(n,A)=(nI-A)^{-1}$, $n\in\mathbb{N}$. Then we put $\xi^n(t,\omega)=nR(n,A)\xi(t,\omega)$ on $[0,T]\times\Omega$. Since A is the infinitesimal generator of the C_0-semigroup $S(t)$, $t\ge0$ of contractions, by the Hille-Yosida Theorem, $\|R(n,A)\|\le\frac{1}{n}$ and $\xi^n(t,\omega)\in\mathcal{D}(A)$, for every $(t,\omega)\in[0,T]\times\Omega$. Moreover, $\xi^n(t,\omega)\to\xi(t,\omega)$ pointwise on $[0,T]\times\Omega$. Also, we observe that $|\xi^n-\xi|=|nR(n,A)\xi-\xi|\le2|\xi|$. Therefore, it follows by applying the Lebesgue Dominated Convergence Theorem that

$$\int_0^T\int_Z|\xi^n(t,z)-\xi(t,z)|^p\nu(dz)dt$$

converges to 0 as $n\to\infty$, $\mathbb{P}$-a.s.. Since the poisson random measure N is a $\mathbb{P}$-a.s. positive measure and we have

$$\mathbb{E}\int_0^T\int_Z|\xi^n(t,z)-\xi(t,z)|^pN(dt,dz)=\mathbb{E}\int_0^T\int_Z|\xi^n(t,z)-\xi(t,z)|^p\nu(dz)dt,$$

we infer

$$\int_0^T\int_Z|\xi^n(t,z)-\xi(t,z)|^pN(dt,dz)\to 0,\quad \text{as } n\to\infty \ \mathbb{P}\text{-a.s..}$$

One can also easily show that $\xi^n\in\mathcal{M}^p([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\times\mathbb{P}\times\nu;\mathcal{D}(A))$.

Define, for each $n\in\mathbb{N}$, a process u^n by

$$u^n(t)=\int_0^t S(t-s)\xi^n(s,z)\tilde{N}(ds,dz),\ t\in[0,T].$$

As we have already noted in case 1, function $u_n(t)$ can also be formulated in a way of strong solutions so that $u_n(t)$ is càdlàg for each $n\in\mathbb{N}$. By the discussion in case 1, for each $n\in\mathbb{N}$, $u^n(t)$, $0\le t\le T$ satisfies the following

$$\mathbb{E}\sup_{0\le t\le T}|u^n(t)|^q\le C\mathbb{E}\left(\int_0^t\int_Z|\xi^n(s,z)|_E^pN(ds,dz)\right)^{\frac{q}{p}}.$$

On the other hand, since by Theorem 2.7, we have

$$\begin{aligned}\mathbb{E}|u^n(t)-u(t)|_E^p&=\mathbb{E}|u^n(t)-u(t)|_E^p\\&=\mathbb{E}\left|\int_0^t\int_Z\Big(S(t-s)\xi^n(s,z)-S(t-s)\xi(s,z)\Big)\tilde{N}(ds,dz)\right|_E^p\\&\le C_p\mathbb{E}\int_0^T\int_Z|\xi^n(s,z)-\xi(s,z)|^p\nu(dz)\,ds,\end{aligned}$$

we infer that $u^n(t)$ converges to $u(t)$ in $L^p(\Omega)$ for every $t\in[0,T]$. Moreover, from case 1, we know that

$$\mathbb{E}\sup_{0\le s\le t}|u^n(s)-u^m(s)|_E^q\le C\mathbb{E}\left(\int_0^t\int_Z|\xi^n(s,z)-\xi^m(s,z)|_E^pN(ds,dz)\right)^{\frac{q}{p}}.$$

From above discussion, we know that the right hand-side of above inequality converges to 0 as $n,m\to\infty$. In this case, it is possible to construct a sequence $\{n_k\}_{k=1}^\infty$ of $\{n\}_{n=1}^\infty$ for which the following is satisfied

$$\mathbb{E}\sup_{0\le s\le T}|u^{n_{k+1}}(s)-u^{n_k}(s)|^q<\frac{1}{k^{2q+2}}.$$

Hence, on the basis of Chebyshev inequality, we obtain

$$\mathbb{P}\left\{\sup_{0\le s\le T}|u^{n_{k+1}}(s)-u^{n_k}(s)|>\frac{1}{k^2}\right\}\le k^{2q}\mathbb{E}\sup_{0\le s\le T}|u^{n_{k+1}}(s)-u^{n_k}(s)|^q<\frac{1}{k^2}.$$

Then the series $\sum_{k=1}^\infty\mathbb{P}\left\{\sup_{0\le s\le T}|u^{n_{k+1}}(s)-u^{n_k}(s)|>\frac{1}{k}\right\}$ will converges. It follows from the Borel-Cantelli Lemma that with probability 1 there exists an integer beyond which the inequality

$$\sup_{0\le s\le T}|u^{n_{k+1}}(s)-u^{n_k}(s)|\le\frac{1}{k^2}$$

holds. Consequently, the series of càdlàg functions

$$\sum_{k=1}^\infty[u^{n_{k+1}}(s)-u^{n_k}(s)]$$

converges uniformly on $[0,T]$ with probability 1 to a càdlàg function which we shall define by $\tilde u = (\tilde u(t))_{t\in[0,T]}$. Moreover, we have

$$\mathbb{E}\sup_{0\le t\le T}|u^n(t)-\tilde u(t)|^q \to 0, \quad \text{as } n\to\infty.$$

Therefore, by the Minkowski Inequality we have

$$\begin{aligned}\left[\mathbb{E}\sup_{0\le s\le t}|\tilde u(s)|^q\right]^{\frac1q} &\le \left[\mathbb{E}\sup_{0\le s\le t}|\tilde u(s)-u^n(s)|^q\right]^{\frac1q} + \left[\mathbb{E}\sup_{0\le s\le t}|u^n(s)|^q\right]^{\frac1q}\\ &\le \left[\mathbb{E}\sup_{0\le s\le t}|\tilde u(s)-u^n(s)|^q\right]^{\frac1q} + \left[C\mathbb{E}\left(\int_0^t\int_Z|\xi^n(s,z)|_E^p N(ds,dz)\right)^{\frac qp}\right]^{\frac1q}.\end{aligned}$$

Note that the constant C on the right hand side of above inequality does not depend on operator A. So the constant C remains the same for every n. It follows by letting $n\to\infty$ in above inequality that

$$\mathbb{E}\sup_{0\le s\le t}|\tilde u(s)|^q \le C\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^p N(ds,dz)\right)^{\frac qp}$$

Also, we have for every $t\in[0,T]$, by Minkowski inequality that

$$\begin{aligned}(\mathbb{E}|\tilde u(t)-u(t)|_E^p)^{\frac1p} &\le (\mathbb{E}|\tilde u(t)-u_n(t)|_E^p)^{\frac1p} + (\mathbb{E}|u(t)-u_n(t)|_E^p)^{\frac1p}\\ &\le (\mathbb{E}|\tilde u(t)-u_n(t)|_E^q)^{\frac1q} + (\mathbb{E}|u(t)-u_n(t)|_E^p)^{\frac1p}\\ &\le \left(\mathbb{E}\sup_{0\le t\le T}|\tilde u(t)-u_n(t)|_E^q\right)^{\frac1q} + (\mathbb{E}|u(t)-u_n(t)|_E^p)^{\frac1p}.\end{aligned}$$

Letting $n\to\infty$, it follows that $u(t)=\tilde u(t)$ in $L^p(\Omega)$ for any $t\in[0,T]$. This shows the inequailty (4.2) for $q'=q$. The case $q'>q$ follows from the fact that if the martingale type p Banach space E satisfies Assumption 4.1 for some q, then Condition 1 is also satisfied with $q'>q$.

□

The following result could be derived immediately from the proof of above theorem.

Corollary 4.1. *Let Let E be a martingale type p Banach space, $1<p\le 2$ satisfying Assumption 4.1. Then the stochastic convolution process u has càdlàg modification.*

Corollary 4.2. *Let E be a martingale type p Banach space, $1<p\le 2$. There exists a càdlàg modification $\tilde u$ of u such that for some constant C and every stopping time $\tau>0$ and $t>0$,*

$$\mathbb{E}\sup_{0\le s\le t\wedge\tau}|\tilde u(s)|_E^q \le C\,\mathbb{E}\left(\int_0^{t\wedge\tau}\int_Z|\xi(s,z)|_E^p N(ds,dz)\right)^{\frac qp}. \tag{4.9}$$

Proof. Let us first consider the case when $\xi \in \mathcal{M}^p([0,T]\times\Omega\times Z, \hat{\mathcal{P}}, \lambda\times\mathbb{P}\times\nu; \mathcal{D}(A))$. A similar argument as in Theorem 4.4 gives the following

$$\begin{aligned}\phi(u(t)) &= \int_0^t \phi'(u(s))(Au(s))\,ds + \int_0^t\int_Z \phi'(u(s-))(\xi(s,z))\tilde{N}(ds,dz)\\ &\quad + \int_0^t\int_Z \Big[\phi(u(s-)+\xi(s,z)) - \phi(u(s-) - \phi'(u(s-))(\xi(s,z))\Big]N(ds,dz)\\ &\le \int_0^t\int_Z \phi'(u(s-))(\xi(s,z))\tilde{N}(ds,dz)\\ &\quad + \int_0^t\int_Z \Big[\phi(u(s-)+\xi(s,z)) - \phi(u(s-) - \phi'(u(s-))(\xi(s,z))\Big]N(ds,dz) \quad \mathbb{P}\text{-a.s..}\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t\wedge\tau}\phi(u(s)) &= \mathbb{E}\sup_{0\le s\le t\wedge\tau}|u(s\wedge\tau)|_E^q\\ &\le \mathbb{E}\sup_{0\le s\le t\wedge\tau}\int_0^{s\wedge\tau}\int_Z \phi'(u(r-))(\xi(r,z))\tilde{N}(dr,dz)\\ &\quad + \mathbb{E}\sup_{0\le s\le\wedge\tau}\int_0^{s\wedge\tau}\Big[\phi(u(r-)+\xi(r,z)) - \phi(u(r-) - \phi'(u(r-))(\xi(r,z))\Big]N(dr,dz)\\ &= \mathbb{E}\sup_{0\le s\le t\wedge\tau}\int_0^s\int_Z 1_{(0,\tau]}(r)\phi'(u(r-))(\xi(r,z))\tilde{N}(dr,dz)\\ &\quad + \mathbb{E}\sup_{0\le s\le t\wedge\tau}\int_0^s\int_Z 1_{(0,\tau]}(r)\Big|\phi(u(r-)+\xi(r,z)) - \phi(u(r-) - \phi'(u(r-))(\xi(r,z))\Big|_E N(ds,dz)\\ &= I_1 + I_2.\end{aligned}$$

Now we consider integral I_2. By the definition of Lebesgue-Stieltges inegral, we have

$$\begin{aligned}&\int_0^s\int_Z \Big|\phi(u(r-)+\xi(r,z)) - \phi(u(r-) - \phi'(u(r-))(\xi(r,z))\Big|_E 1_{(0,\tau]}(r)N(dr,dz)\\ &\qquad = \sum_{0<r\le s}\Big|\phi(u(r-)+\xi(r,\xi(r))) - \phi(u(r-) - \phi'(u(r-))(\xi(r,\xi(r)))\Big|_E 1_{(0,\tau]}(r),\end{aligned}$$

Notice that the function $\phi(\cdot) = |\cdot|^q$ is of class C^2. Applying Taylor formula to function ϕ we get for some $0<\theta,\delta<1$,

$$\begin{aligned}&\Big|\phi(u(r-)+\xi(r,\pi(r))) - \phi(u(r-))\Big|_E 1_{(0,\tau]}(r)\\ &\qquad\qquad \le |\xi(r,\pi(r))|_E\Big|\phi'(u(r-)+\theta\xi(r,\pi(r)))\Big|1_{(0,\tau]}(r),\\ &\Big|\phi(u(r-)+\xi(r,\pi(r))) - \phi(u(r-) - \phi'(u(r-))(\xi(r,\pi(r)))\Big|_E 1_{(0,\tau]}(r)\\ &\qquad\qquad \le \frac{1}{2}|\xi(r,\pi(r))|_E^2|\phi''(u(r-)) + \delta\xi(r,\pi(r))|1_{(0,\tau]}(r)\end{aligned}$$

Moreover we know that $|\phi'(x)|_{\mathcal{L}(E)} \le k_1|x|_E^{q-1}$, so we obtain

$$\begin{aligned}\Big|\phi'(u(r-)+\theta\xi(r,\pi(r)))\Big|_E 1_{(0,\tau]}(r) &\le k_1\big|u(r-)+\theta\xi(r,\pi(r))\big|_E^{q-1}1_{(0,\tau]}(r)\\ &\le k_1\max\big\{|u(r-)|_E^{q-1}1_{(0,\tau]}(r), \big|u(r-)+\xi(r,\pi(r))\big|_E^{q-1}1_{(0,\tau]}(r)\big\}.\end{aligned}$$

Observe that

$$|u(r-)|_E^{q-1}1_{(0,\tau]}(r) \le \sup_{0\le r\le s}|u(r-)|_E^{q-1}1_{(0,\tau]}(r) \le \sup_{0\le s\le t\wedge\tau}|u(s-)|_E^{q-1} = \sup_{0\le s\le t\wedge\tau}|u(s)|_E^{q-1},$$

and

$$|u(r-)+\xi(r,\pi(r))|_E^{q-1}1_{(0,\tau]}(r) \le \sup_{0\le r\le s}|u(r)|_E^{q-1}1_{(0,\tau]}(r) \le \sup_{0\le s\le t\wedge\tau}|u(s)|_E^{q-1},$$

where $q\ge 2$. Therefore, we infer

$$\begin{aligned}\Big|\phi(u(r-)+\xi(r,\pi(r)))-\phi(u(r-))\Big|_E 1_{(0,\tau]}(r) &\le |\xi(r,\pi(r))|_E 1_{(0,\tau]}(r)\Big|\phi'(u(r-)+\theta\xi(r,\pi(r)))\Big|_{\mathcal{L}(E)}\\ &\le k_1|\xi(r,\pi(r))|_E 1_{(0,\tau]}(r)\sup_{0\le s\le t\wedge\tau}|u(s)|_E^{q-1}.\end{aligned}$$

Similarly, from the assumption $|\phi''(x)|\le k_2|x|_E^{q-2}$ we obtain

$$|\phi''(u(r-))+\delta\xi(r,\pi(r))|_E 1_{(0,\tau]}(r)\le k_2\sup_{0\le s\le t\wedge\tau}|u(s)|_E^{q-2}1_{(0,\tau]}(r).$$

It then follows that

$$\begin{aligned}&\sum_{0<r\le s}\Big|\phi(u(r-)+\xi(r,\xi(r)))-\phi(u(r-)-\phi'(u(r-))(\xi(r,\xi(r)))\Big|_E 1_{(0,\tau]}(r)\\ &\qquad=\sum_{0<r\le s}\Big|\phi(u(r-)+\xi(r,\xi(r)))-\phi(u(r-)-\phi'(u(r-))(\xi(r,\xi(r)))\Big|_E^{(2-p)+(p-1)}1_{(0,\tau]}(r)\\ &\qquad\le\left(2k_1|\xi(r,\pi(r))|_E\sup_{0\le s\le t\wedge\tau}|u(s)|_E^{q-1}1_{(0,\tau]}(r)\right)^{2-p}\left(k_2|\xi(r,\pi(r))|^2\sup_{0\le s\le t\wedge\tau}|u(s)|_E^{q-2}1_{(0,\tau]}(r)\right)^{p-1}\\ &\qquad=K\sup_{0\le s\le t\wedge\tau}|u(s)|_E^{q-p}\sum_{0<r\le s}|\xi(r,\pi(r))|_E^p1_{(0,\tau]}(r).\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^s\int_Z\Big|\phi(u(r-)+\xi(r,z))-\phi(u(r-))-\phi'(u(r-))(\xi(r,z))\Big|_E 1_{(0,\tau]}(r)N(dr,dz)\\ &\qquad\le K\sup_{0\le s\le t\wedge\tau}|u(s)|_E^{q-p}\int_0^s\int_Z|\xi(r,z)|_E^p1_{(0,\tau]}(r)\ N(dr,dz).\end{aligned}$$

Hence, for integral I_2, we can estimate as follows

$$I_2\le K\mathbb{E}\sup_{0\le s\le t\wedge\tau}|u(s)|_E^{q-p}\int_0^s\int_Z|\xi(r,z)|_E^p1_{(0,\tau]}(r)\ N(dr,dz).$$

For integral I_1, applying the stopped Davis' inequality yields the following

$$\begin{aligned}I_1&\le C\mathbb{E}\left(\int_0^s\int_Z|\phi'(u(r-))(\xi(r,z))|_E^p1_{(0,\tau]}(r)N(dr,dz)\right)^{\frac{1}{p}}\\ &\le k_1C\mathbb{E}\sup_{0\le s\le t\wedge\tau}|u(s)|_E^{q-1}\left(\int_0^{t\wedge\tau}\int_Z|\xi(r,z)|_E^pN(ds,dz)\right)^{\frac{1}{p}}.\end{aligned}$$

The rest argument goes without any difference with the proof of Theorem 4.4.

□

Theorem 4.5. *Let E be an martingale type p Banach space, $1<p\le 2$, satisfying Assumption 4.1. Then there exists a constant C such that for every process u there exists a càdlàg modification $\tilde u$ of u such that for all $0\le t\le T$ and $0<q'<\infty$,*

$$\mathbb{E}\sup_{0\le s\le t}|\tilde u(s)|_E^{q'}\le C\ \mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz)\right)^{\frac{q'}{p}},\tag{4.10}$$

Proof. The inequality (4.10) has already been shown for $q' \geq q$ in Theorem 4.4. Now we are in a position to show it for $0 < q' < p$. Let us fix q' such that $0 < q' < q$. Take $\lambda > 0$. Define a stopping time

$$\tau := \inf \left\{ t : \left(\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz) \right)^{\frac{1}{p}} > \lambda \right\}.$$

Since the process $\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz)$, $0 < t \leq T$ is right continuous, the random time τ is indeed a $\mathcal{F}_{t+}$-stopping time. Moreover, we find that $\int_0^t \int_Z |\xi(s,z)|_E^p N(ds,dz) \leq \lambda$, for $0 < t < \tau$, and $\int_0^\tau \int_Z |\xi(s,z)|_E^p N(ds,dz) \geq \lambda$ when $\tau < \infty$. Also, we observe that for every $0 < t \leq T$,

$$\mathbb{E} \int_0^t \int_Z f(s,z) \tilde{N}(ds,dz) = \mathbb{E} \int_0^{t-} \int_Z f(s,z) \tilde{N}(ds,dz). \tag{4.11}$$

This equality can be verified first for step functions, then for every function f in $\xi \in \mathcal{M}^p([0,T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$ we can approximate it by step functions in $\mathcal{M}^p_{step}([0,T] \times \Omega \times \mathcal{Z}; E)$, so the equality (4.11) holds for every $f \in \xi \in \mathcal{M}^p([0,T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$. Therefore, by using Chebyshev's inequaliy and Corollary 4.2 to Theorem 4.4, we obtain

$$\begin{aligned}
\mathbb{P}\left(\sup_{0 \leq s \leq t \wedge \tau} |u(s)| > \lambda \right) &\leq \frac{1}{\lambda^q} \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau} |u(s)|^q \\
&\leq \frac{C}{\lambda^q} \mathbb{E} \left(\int_0^{t \wedge \tau} \int_Z |\xi(s,z)|^p N(ds,dz) \right)^{\frac{q}{p}} \\
&= \frac{C}{\lambda^q} \mathbb{E} \left(\int_0^{(t \wedge \tau)-} \int_Z |\xi(s,z)|^p N(ds,dz) \right)^{\frac{q}{p}} \\
&\leq \frac{C}{\lambda^q} \mathbb{E} \left[\left(\int_0^t \int_Z |\xi(s,z)|^p N(ds,dz) \right)^{\frac{q}{p}} \wedge \lambda^q \right]. \qquad (4.12)
\end{aligned}$$

On the other hand, since $\{\sup_{0 \leq s \leq t} |u(s)| > \lambda, \tau \geq t\} \subset \{\sup_{0 \leq s \leq t \wedge \tau} |u(s)| > \lambda\}$, we have

$$\begin{aligned}
\mathbb{P}(\sup_{0 \leq s \leq t} |u(s)| > \lambda) &= \mathbb{P}(\sup_{0 \leq s \leq t} |u(s)| > \lambda, \tau \geq t) + \mathbb{P}(\sup_{0 \leq s \leq t} |u(s)| > \lambda, \tau < t) \\
&\leq \mathbb{P}(\sup_{0 \leq s \leq t} |u(s)| > \lambda, \tau \geq t) + \mathbb{P}(\tau < t) \\
&\leq \mathbb{P}(\sup_{0 \leq s \leq t \wedge \tau} |u(s)| > \lambda) + \mathbb{P}(\tau < t).
\end{aligned} \tag{4.13}$$

Substituting (4.12) into (4.13) results in

$$\begin{aligned}
\mathbb{P}(\sup_{0 \leq s \leq t} |u(s)| > \lambda) \leq \frac{C}{\lambda^q} \mathbb{E} &\left[\left(\int_0^t \int_Z |\xi(s,z)|^p N(ds,dz) \right)^{\frac{q}{p}} \wedge \lambda^q \right] \\
&+ \mathbb{P} \left[\left(\int_0^t \int_Z |\xi(s,z)|^p N(ds,dz) \right)^{\frac{1}{p}} > \lambda \right].
\end{aligned}$$

Integrating both sides of the last inequality with respect to measure $q'\lambda^{q'-1}d\lambda$ and applying the equality $\mathbb{E}|X|^{q'} = \int_0^\infty q'\lambda^{q-1}\mathbb{P}(|X|>\lambda)d\lambda$, see [10], we infer that

$$
\begin{aligned}
\mathbb{E}\sup_{0\le s\le t}|u(s)|^{q'} &= \int_0^\infty \mathbb{P}(\sup_{0\le s\le t}|u(s)|>\lambda)q'\lambda^{q'-1}d\lambda\\
&\le \int_0^\infty \frac{C}{\lambda^q}\mathbb{E}\left[\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{q}{p}}\wedge\lambda^q\right]q'\lambda^{q'-1}d\lambda\\
&\qquad + \int_0^\infty \mathbb{P}\left[\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{1}{p}}>\lambda\right]q'\lambda^{q'-1}d\lambda\\
&= \int_0^\infty \frac{C}{\lambda^q}\mathbb{E}\left[\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{q}{p}}\wedge\lambda^q\right]q'\lambda^{q'-1}d\lambda\\
&\qquad + \mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{q'}{p}}.
\end{aligned}\tag{4.14}
$$

Let us denote $\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{1}{p}}$ by X. The first term on the right hand side of (4.14) becomes

$$
\begin{aligned}
&\frac{C}{\lambda^q}\int_0^\infty \mathbb{E}\left[\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{q}{p}}\wedge\lambda^q\right]q'\lambda^{q'-1}d\lambda\\
&\qquad = C\int_0^\infty \mathbb{E}(X^q\wedge\lambda^q)q'\lambda^{q'-q-1}d\lambda\\
&\qquad = C\mathbb{E}\int_0^\infty (X^q\wedge\lambda^q)q'\lambda^{q'-q-1}d\lambda\\
&\qquad = C\mathbb{E}\int_0^X \lambda^q q'\lambda^{q'-q-1}d\lambda + C\mathbb{E}\int_X^\infty |X|^q q'\lambda^{q'-q-1}d\lambda\\
&\qquad = C\mathbb{E}X^{q'} + C\mathbb{E}X^q\int_X^\infty q'\lambda^{q'-q-1}d\lambda\\
&\qquad = C(1+\frac{q'}{q-q'})\mathbb{E}X^{q'}\\
&\qquad = \frac{Cq}{q-q'}\mathbb{E}X^{q'}\\
&\qquad = \frac{Cq}{q-q'}\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{q'}{p}}
\end{aligned}
$$

Therefore, we conclude that

$$
\begin{aligned}
\mathbb{E}\sup_{0\le s\le t}|u(s)|^{q'} &\le \frac{Cq}{q-q'}\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{q'}{p}} + \mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{q'}{p}}\\
&= \left(1+\frac{Cq}{q-q'}\right)\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|^pN(ds,dz)\right)^{\frac{q'}{p}},
\end{aligned}
$$

which completes the proof.

□

Corollary 4.3. *Let E be an martingale type p Banach space, $1<p\le 2$ satisfying Assumption 4.1. Then there exists an E-valued càdlàg modification $\tilde{u}$ of u such that for some constant $C>0$,* *independent of* u, *and all* $t\in[0,T]$ *and* $0<q'\le p$,

$$\mathbb{E}\sup_{0\le s\le t}|\tilde u(s)|_E^{q'}\le C\;\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^p\nu(dz)\,ds\right)^{\frac{q'}{p}} \tag{4.15}$$

Proof of Corollary 4.3. First, we consider the case $q'=p$. Since $\xi\in\mathcal{M}^p([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\times\mathbb{P}\times\nu;E)$, so both integrals $\int_0^t\int_Z|\xi(s,z)|_E^p\nu(dz)ds$ and $\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz)$ are well defined as Lebesgue-Stieltjes integrals. We can obtain from Theorem 4.5 with $q'=p$ that

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}|u(s)|_E^p&\le C\;\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^pN(ds,dz)\right)\\&=C\;\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^p\nu(dz)ds\right).\end{aligned}$$

This shows (4.15) for $q'=p$. Now we are in a position to show Inequality (4.15) for $0<q'<p$. Let q' be fixed. Take $\lambda>0$. Define stopping time

$$\tau=\inf\{t\in[0,T]:\left(\int_0^t\int_Z|\xi(s,z)|^p\nu(dz)\,ds\right)^{\frac1p}>\lambda\}.$$

The random variable τ is a stopping time. Indeed the process $\int_0^t\int_Z|\xi(s,z)|^p\nu(dz)\,ds$, $0\le t\le T$ is a continuous process and so the claim follows immediately. It follows from Chebyshev's inequaliy and Corollary 4.2 that

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le s\le t\wedge\tau}|u(s)|>\lambda\right)&=\mathbb{E}1_{\{\sup_{0\le s<t\wedge\tau}|u(s)|>\lambda\}}\\&\le\frac{1}{\lambda^q}\mathbb{E}\sup_{0\le s<t\wedge\tau}|u(s)|^q\\&\le\frac{C}{\lambda^q}\mathbb{E}\left(\int_0^{t\wedge\tau}\int_Z|\xi(s,z)|^p\nu(dz)\,ds\right)^{\frac qp}\\&\le\frac{C}{\lambda^q}\mathbb{E}\left[\left(\int_0^t\int_Z|\xi(s,z)|^p\nu(dz)\,ds\right)^{\frac qp}\wedge\lambda^q\right],\end{aligned} \tag{4.16}$$

where we used the definition of stopping time τ and the increasing property of process $\int_0^t\int_Z|\xi(s,z)|^p\nu(dz)\,ds$, $0\le t\le T$. The rest of the proof can be done exactly in the same manner as in the proof of Theorem 4.5. □

Corollary 4.4. *Let* E *be an martingale type* p *Banach space,* $1<p\le2$ *satisfying Assumption 4.1. Then for any* $n\in\mathbb{N}$ *there exists a constant* $C=C(n)$ *such that for every every* $\xi\in\bigcap_{k=1}^n\mathcal{M}^{p^k}([0,T]\times\Omega\times Z,\hat{\mathcal{P}},\lambda\times\mathbb{P}\times\nu;E)$ *and* $t\in[0,T]$ *we have*

$$\mathbb{E}\sup_{0\le s\le t}|\tilde u(s)|_E^{p^n}\le C\;\sum_{k=1}^n\mathbb{E}\left(\int_0^t\int_Z|\xi(s,z)|_E^{p^k}\nu(dz)ds\right)^{p^{n-k}}. \tag{4.17}$$

where $\tilde u$ *is the càdlàg modification of* u *as before.*

The proof of Corollary 4.4 is similar to the proof Lemma 5.2 in Bass and Cranston [2] or of Lemma 4.1 in Protter and Talay [19]. Essential ingredients of that proof are the following two results. The first of them being about integration of real valued processes.

Lemma 4.6. *Let E be an martingale type p Banach space, $1 < p \le 2$, satisfying Assumption 4.1. For any $0 < q' < \infty$, there exists a constant C such that for all $\xi \in \mathcal{M}^p([0,T]\times\Omega\times Z, \hat{\mathcal{P}}, \lambda\times\mathbb{P}\times\nu; E)$ we have*

$$\mathbb{E}\sup_{0\le s\le t}\left(\int_0^s \xi(r,z)\tilde{N}(dr,dz)\right)_E^{q'} \le C\,\mathbb{E}\left(\int_0^t |\xi(s,z)|_E^p N(ds,dz)\right)^{\frac{q'}{p}}, \quad t\in[0,T]. \tag{4.18}$$

Proof of Lemma 4.6. This result is a special case of Theorem 4.5 when $S(t) = I$, $0 \le t \le T$. □

Lemma 4.7. *For any $n \in \mathbb{N}$ there exists a constant $D_n > 0$ such for any process*

$$f \in \bigcap_{k=1}^n \mathcal{M}^{p^k}([0,T]\times\Omega\times Z, \hat{\mathcal{P}}, \lambda\times\mathbb{P}\times\nu; \mathbb{R})$$

and $t \in [0,T]$, the following inequality

$$\mathbb{E}\sup_{0\le s\le t}\left(\int_0^s\int_Z f(r,z)\tilde{N}(dr,dz)\right)^{p^n} \le D_n\sum_{k=1}^n \mathbb{E}\left(\int_0^t\int_Z |f(s,z)|^{p^k}\nu(dz)ds\right)^{p^{n-k}} \tag{4.19}$$

holds.

Proof of Lemma 4.7. We shall show this Lemma by induction. The case $n = 1$. This follows from [3]. Now we assume that the assertion in the Claim is true for $n-1$, where $n \in \mathbb{N}$ and $n \ge 2$. We will show that it is true for n. Since by assumption $f \in \mathcal{M}^p([0,T]\times\Omega\times Z, \hat{\mathcal{P}}, \lambda\times\mathbb{P}\times\nu; \mathbb{R})$, so both integrals $\int_0^t\int_Z |f(s,z)|^p N(ds,dz)$ and $\int_0^t\int_Z |f(s,z)|^p \nu(dz)ds$ are well defined as Lebesgue-Stieltjes integrals. Moreover, we have

$$\int_0^t\int_Z |f(s,z)|^p\tilde{N}(ds,dz) = \int_0^t\int_Z |f(s,z)|^p N(ds,dz) - \int_0^t\int_Z |f(s,z)|^p\nu(dz)ds. \tag{4.20}$$

Hence by applying first inequality (4.18) and next the equality (4.20) we infer that

$$\begin{aligned}
\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s\int_Z f(r,z)\tilde{N}(dr,dz)\right|^{p^n} &\le C\mathbb{E}\left|\int_0^t\int_Z |f(s,z)|^p N(ds,dz)\right|^{p^{n-1}}\\
&\le 2^{p^{n-1}}C\left\{\mathbb{E}\left(\int_0^t\int_Z |f(s,z)|^p\,\tilde{N}(ds,dz)\right)^{p^{n-1}} + \mathbb{E}\left(\int_0^t\int_Z |f(s,z)|^p\,\nu(dz)\,ds\right)^{p^{n-1}}\right\}.
\end{aligned}$$

Next, by the inductive assumption applied to the real valued process $|f|^p \in \bigcap_{k=1}^{n-1}\mathcal{M}^{p^k}([0,T]\times\Omega\times Z, \hat{\mathcal{P}}, \lambda\times\mathbb{P}\times\nu; \mathbb{R})$, we have

$$\begin{aligned}
&\mathbb{E}\left|\int_0^t\int_Z f(s,z)\tilde{N}(ds,dz)\right|^{p^n}\\
&\le 2^{p^{n-1}}C\left(D_{n-1}\sum_{i=1}^{n-1}\mathbb{E}\left(\int_0^t\int_Z |f(s,z)|^{p^{i+1}}\,\nu(dz)\,ds\right)^{p^{n-1-i}} + \mathbb{E}\left(\int_0^t\int_Z |f(s,z)|^p\,\nu(dz)\,ds\right)^{p^{n-1}}\right)\\
&\le D_n\sum_{k=1}^n\mathbb{E}\Big(\int_0^t\int_Z |f(s,z)|^{p^k}\,\nu(dz)\,ds\Big)^{p^{n-k}}.
\end{aligned}$$

This proves the validity of the assertion in the Lemma for n what completes the whole proof. □

Proof of Corollary 4.4. Let us take $n \in \mathbb{N}$. By applying first Theorem 4.5 and next the equality (4.20) when $\xi \in \mathcal{M}^p([0,T]\times\Omega\times Z, \hat{\mathcal{P}}, \lambda\times\mathbb{P}\times\nu; E)$, we infer that for all $t \in [0,T]$,

$$\begin{aligned}
\mathbb{E}\sup_{0\le s\le t} |\tilde{u}(s)|_E^{p^n} &\le C\,\mathbb{E}\left(\int_0^t\int_Z |\xi(s,z)|_E^p\, N(ds,dz)\right)^{p^{n-1}} \\
&\le 2^{p^{n-1}} C\,\mathbb{E}\left(\int_0^t\int_Z |\xi(s,z)|_E^p\, \tilde{N}(ds,dz)\right)^{p^{n-1}} \\
&\quad +2^{p^{n-1}} C\,\mathbb{E}\left(\int_0^t\int_Z |\xi(s,z)|_E^p\, \nu(dz)\,ds\right)^{p^{n-1}} \\
&\le 2^{p^{n-1}}\, C\,D_{n-1}\sum_{k=1}^{n-1}\mathbb{E}\left(\int_0^t\int_Z |\xi(s,z)|_E^{p^{k+1}}\nu(dz)ds\right)^{p^{n-1-k}} \\
&\quad +2^{p^{n-1}}\, C\,\mathbb{E}\,\left(\int_0^t\int_Z |\xi(s,z)|_E^p\, \nu(dz)\,ds\right)^{p^{n-1}} \\
&\le C(n)\sum_{k=1}^{n}\mathbb{E}\left(\int_0^t\int_Z |\xi(s,z)|_E^{p^k}\nu(dz)ds\right)^{p^{n-k}},
\end{aligned}$$

where we used in the third inequality Lemma 4.6 with f replaced by real-valued process $|\xi|_E^p \in \bigcap_{k=1}^{n-1}\mathcal{M}^{p^k}([0,T]\times\Omega\times Z, \hat{\mathcal{P}}, \lambda\times\mathbb{P}\times\nu; \mathbb{R})$. This completes the proof of Corollary 4.4. □

5. Final comments

It is possible to derive inequality (1.1) by the method used by the second named authour and Seidler in [8], see as inequality (4) therein. These authours used the Szekőfalvi-Nagy's Theorem on unitary dilations in Hilbert spaces. The latter result has recently been extended by Fröhlich and Weis [6] to Banach spaces of finite cotype. However, this method works only analytic semigroups of contraction type while the results from the current paper are valid for all C_0 semigroups of contraction type. Let us now formulate the following result whose proof is a clear combination of the proofs from [8] and [6]. For the explanation of the terms used we refer the reader to the latter work. Similar observation for processes driven by a Wiener process was made independently by Seidler [18].

Theorem 5.1. *Let E be an martingale type p Banach space, $1 < p \le 2$. Let $-A$ be a generator of a bounded analytic semigroup in E such that for some $\theta < \frac{1}{2}\pi$, the operator A has a bounded $H^\infty(S_\theta)$ calculus. Then, for any $0 < q' < \infty$, there exists a constant C such that for all $\xi \in \mathcal{M}^p([0,T]\times\Omega\times Z, \hat{\mathcal{P}}, \lambda\times\mathbb{P}\times\nu; E)$ we have*

$$\mathbb{E}\sup_{0\le s\le t}\left(\int_0^s \xi(r,z)\tilde{N}(dr,dz)\right)_E^{q'} \le C\,\mathbb{E}\left(\int_0^t |\xi(s,z)|_E^p N(ds,dz)\right)^{\frac{q'}{p}}, \quad t\in[0,T]. \tag{5.1}$$

The following result could be derived immediately from the proof of above theorem.

Corollary 5.1. *Let E be a martingale type p Banach space, $1 < p \le 2$. Let $-A$ be a generator of a bounded analytic semigroup in E such that for some $\theta < \frac{1}{2}\pi$ the operator A has a bounded $H^\infty(S_\theta)$ calculus. Then, the stochastic convolution process u defined by (1.1) has càdlàg modification.*

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