Finite time blow up in Kaniadakis-Quarati model of Bose-Einstein particles

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Abstract. We study a Fokker-Planck equation with linear diffusion and super-linear drift introduced by Kaniadakis and Quarati [11, 12] to describe the evolution of a gas of Bose-Einstein particles. For kinetic equation of this type it is well-known that, in the physical space $\mathbb{R}^3$, the structure of the equilibrium Bose-Einstein distribution depends upon a parameter m^*, the critical mass. We are able to describe the time-evolution of the solution in two different situations, which correspond to $m \ll m^*$ and $m \gg m^*$ respectively. In the former case, it is shown that the solution remains regular, while in the latter we prove that the solution starts to blow up at some finite time t_c, for which we give an upper bound in terms of the initial mass. The results are in favour of the validation of the model, which, in the supercritical regime, could produce in finite time a transition from a normal fluid to one with a condensate component.

Keywords. Fokker-Planck equation, Bose–Einstein condensation.

1 Introduction

The application of quantum assumptions to molecular dynamics encounters leads to some divergences from the classical kinetic theory. From Chapman and Cowling [5] one can learn that the Boltzmann Bose-Einstein equation is established by imposing that, for a gas composed of Bose-Einstein identical particles, according to quantum theory, the presence of a like particle in the velocity-range dv increases the probability that a particle will enter that range; the presence of $f(v)dv$ particles per unit volume increases this probability in the ratio $1 + \delta f(v)$. The basic assumption

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which leads to the correction in the Boltzmann collision operator, has been recently used by Kaniadakis and Quarati [11, 12] to introduce a modification of the drift term of the standard Fokker-Planck equation in presence of quantum indistinguishable particles, bosons or fermions. For Bose-Einstein particles, this model equation reads

$$\frac{\partial f}{\partial t} = \nabla \cdot \left[\nabla f + v f(1 + \delta f)\right]. \tag{1}$$

By a direct inspection, one can easily verify that equation (1) admits the Bose-Einstein distribution as stationary state. Indeed, the Bose-Einstein distribution

$$f_\infty(v) = \frac{1}{\delta}\left[e^{v^2/2+\lambda} - 1\right]^{-1} \tag{2}$$

satisfies the equation

$$\nabla f_\infty(v) + v f_\infty(v)(1 + \delta f_\infty(v)) = 0$$

for any fixed positive constant λ. The constant λ is related to the mass of Bose-Einstein distribution

$$m_\lambda = \int_{\mathbb{R}^3} \frac{1}{\delta}\left[e^{v^2/2+\lambda} - 1\right]^{-1} dv,$$

and, since the mass is decreasing as soon as λ increase, the maximum value of m_λ is attained at $\lambda = 0$. The value

$$m_c = m_0 = \int_{\mathbb{R}^3} \frac{1}{\delta}\left[e^{v^2/2} - 1\right]^{-1} dv < +\infty \tag{3}$$

defines the *critical mass.*

One of the fundamental problems related to kinetic equations that relax towards a stationary state characterized by the existence of a critical mass, is to show how, starting from an initial distribution with a supercritical mass $m > m_c$, the solution develops a singular part (the condensate). We remark that in general this phenomenon is heavily dependent of the dimension of the physical space. In dimension $d \leq 2$, in fact, the maximal mass m_0 of the Bose-Einstein distribution (2) is unbounded, and the eventual formation of a condensate is lost. The kinetics of Bose–Einstein condensation, namely the way in which the Bose fluid undergoes a transition from a normal fluid to one with a condensate component has been object of various investigations [10, 14, 20, 21, 22]. These results are mainly based on study of the Boltzmann–Nordheim kinetic equation, which describes the dynamics of weakly interacting quantum fluids. At the level of the Boltzmann–Nordheim kinetic equation, the most general and exhaustive results have been obtained by Spohn [22], who describes the precise mechanism of how the condensate is generated and annihilated.

Also, the mathematical analysis of the quantum Boltzmann equation in the space homogeneous isotropic case has shown some progresses [7, 17, 8, 9]. In dimension three of the velocity space already the issue of giving mathematical sense to the collision operator is highly non-trivial (particularly if positive measure solutions are allowed, as required by a careful analysis of the equilibrium states). All the mathematical results, however, require very strong cut-off assumptions on the cross-section [17, 9].

Accurate numerical discretizations of the quantum Boltzmann equation, which maintain the basic analytical and physical features of the continuous problem, namely, mass and energy conservation, entropy growth and equilibrium distributions have been introduced recently in [1, 18]. Related works [15, 19] in which fast methods for Boltzmann equations were derived using different techniques like multipole methods, multigrid methods and spectral methods, are relevant to quote.

The Fokker-Planck equation (1) of Kaniadakis and Quarati has been studied only recently in [3], in dimension one of the velocity variable. In this case, indeed, the equilibrium Bose-Einstein density is a smooth function, which makes it possible to prove exponential convergence to equilibrium resorting to standard entropy methods. Other Fokker-Planck equations like the Kompaneets equation [13] have been exhaustively studied in [6].

More in details, we will describe the time-evolution of the solution of (1) in two different situations, which correspond to initial densities with a small mass $m \ll m^*$ and a big mass $m \gg m^*$ respectively. In the former case, it is shown that the L^2-norm of the solution remains uniformly bounded, excluding the formation of a condensate, while in the latter we prove that the solution starts to nucleate the condensate at some finite time t_c, for which we give an upper bound in terms of the initial mass. The results are based on various Nash-type inequalities which allow to control the evolution of the L^2-norm.

The results are in favor of the validation of the model, which, in the supercritical regime, is able to produce in finite time a transition from a normal fluid to one with a condensate component.

2 Global regularity estimates

In the rest of the paper, without loss of generality, we fix $\delta = 1$ in equation (1). Therefore we will consider the equation

$$\frac{\partial f}{\partial t} = \nabla \cdot \left[\nabla f + v f(1+f) \right]. \tag{4}$$

Indeed, if $f(v,t)$ solves equation (1), $g(v,t) = \delta f(v,t)$ solves equation (4), so that any result valid for equation (4) translates into an equivalent result for (1). Local existence and regularity of solutions for equation (4) have been proven in [4]. Let us briefly recall this result. We define $\Gamma = L^\infty(\mathbb{R}^3) \bigcap L^1_1(\mathbb{R}^3) \bigcap L^p_m(\mathbb{R}^3)$ and, for $T > 0$, $\Gamma_T := C\left([0, T); \Gamma\right)$ with norms

$$\|f(t)\|_\Gamma = \max\left\{ \|f(t)\|_{L^\infty}, \|f(t)\|_{L^1_1}, \|f(t)\|_{L^p_m} \right\}; \qquad \|f(t)\|_{\Gamma_T} = \max_{0 \le t \le T} \|f(t)\|_\Gamma,$$

where

$$\|f(t)\|_{L^p_m} = \|(1 + |v|^m) f(t)\|_{L^p}, \qquad \|f)\|_{L^p} = \left(\int_{\mathbb{R}^3} |f|^p \right)^{1/p}.$$

Then it holds [4]:

Theorem 1. *Let the initial density $f_0(v) \ge 0$ belong to Γ, with $m \ge 1$ and $p > 3$. Then there exists $T > 0$ depending only on the norm of the initial condition, such that equation (4) has a unique nonnegative solution $f(v, t)$ in $\Gamma_T := C\left([0, T); \Gamma\right)$ with $f(v, t = 0) = f_0(v)$. Moreover $\nabla f(v, t) \in BC\left((0, T), (L^p_m \bigcap L^1)\mathbb{R}^3\right)$, and the L^1-norm of $f(v, t)$ is conserved.*

By virtue of the local existence and regularity of solutions briefly resumed in Theorem 1, we can easily recover the evolution in time of the L^2-norm of the solution, provided this norm is bounded initially. Integration by parts gives

$$\frac{d}{dt} \int_{\mathbb{R}^3} f^2(v, t)\, dv = -2 \int_{\mathbb{R}^3} |\nabla f(v, t)|^2\, dv + 3 \int_{\mathbb{R}^3} f^2(v, t)\, dv + 2 \int_{\mathbb{R}^3} f^3(v, t)\, dv. \quad (5)$$

Therefore, the control of the boundedness in time of the L^2-norm of requires the control of L^3-norm of the solution in terms of the square of the L^2-norm of the gradient. In the linear Fokker-Planck equation, where the last term is absent, the standard Nash inequality

$$\left[\int_{\mathbb{R}^d} |f(v)|^2\, dv \right]^{1+2/d} \le C \|f\|_{L^1}^{4/d} \|\nabla f\|_{L^2}^2, \quad (6)$$

due to the mass conservation, guarantees that the L^2-norm of the solution is bounded by a constant independent of time. In fact, in the linear case, inequality (6) implies

$$\frac{d}{dt} \int_{\mathbb{R}^3} f^2(v, t)\, dv \le -\frac{2}{C \|f\|_{L^1}^{4/3}} \left(\int_{\mathbb{R}^3} f(v, t)^2\, dv \right)^{5/3} + 3 \int_{\mathbb{R}^3} f^2(v, t)\, dv, \quad (7)$$

and the L^2-norm of the solution can not cross the value

$$\max\left\{ \|f_0\|_{L^2}, \left(\frac{3}{2} C\right)^{3/2} \|f\|_{L^1}^2 \right\}.$$

In order to use a similar strategy in presence of the nonlinearity (the last integral in equation (5)), in what follows we derive Nash–type inequalities which can be applied to our problem. The proof that follows is based on Beckner inequality [2], which says that, for all $1 \le p \le 2$

$$\left\| \widehat{f} \right\|_{p'} \le (A_p)^d \left\| f \right\|_p , \qquad A_p = [p^{1/p}/p'^{1/p'}]^{1/2}. \tag{8}$$

In (8), $\widehat{f}$ is the Fourier transform of f,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(v) e^{2\pi i \xi v} \, dv$$

Taking f in $L^2(\mathbb{R}^d)$ we can write inequality (8) for f,

$$\left\| f \right\|_{p'} \le (A_p)^d \left\| \check{f} \right\|_p , \tag{9}$$

where now $\check{f}$ denotes the inverse Fourier transform

$$\check{f}(\xi) = \int_{\mathbb{R}^d} f(v) e^{-2\pi i \xi v} \, dv.$$

Let us set $p < 2$. Then, for any constant $R > 0$,

$$\int_{\mathbb{R}^d} |\check{f}(\xi)|^p \, d\xi = \int_{|\xi| \le R} |\check{f}(\xi)|^p \, d\xi + \int_{|\xi| > R} |\check{f}(\xi)|^p \, d\xi. \tag{10}$$

Since f belongs to L^1, so that $|\check{f}(\xi)| \le \|f\|_{L^1}$,

$$\int_{|\xi| \le R} |\check{f}(\xi)|^p \, d\xi \le \|f\|_{L^1}^p \int_{|\xi| \le R} d\xi \le \|f\|_{L^1} \frac{B_d}{d} R^d. \tag{11}$$

In (11) B_d denotes the measure of the surface of the unit ball in $\mathbb{R}^d$. Moreover, by Hölder inequality, provided

$$p > \frac{2d}{d+2}, \tag{12}$$

it holds

$$\int_{|\xi| > R} |\check{f}(\xi)|^p \, d\xi = \int_{|\xi| > R} \frac{1}{|\xi|^p} |\xi|^p |\check{f}(\xi)|^p \, d\xi \le$$

$$\left[\int_{|\xi| > R} |\xi|^2 |\check{f}(\xi)|^2 \right]^{p/2} \left[\int_{|\xi| > R} \frac{1}{|\xi|^{2p/(2-p)}} \right]^{1-p/2} \le c_{p,d} \left(\frac{1}{R} \right)^{p-(2-p)d/2} \|\nabla f\|_{L^2}^p. \tag{13}$$

In (13) the constant $c_{p,d}$ is

$$c_{p,d} = \left[B_d \left(\frac{2p}{2-p} - d \right) \right]^{(2-p)/2}.$$

Note that $c_{p,d}$ converges to $B_d^{(2-p)/2}$ as $p \to 2$. Substituting in (10) we obtain the inequality

$$\int_{\mathbb{R}^d} |\check{f}(\xi)|^p \, d\xi \le \frac{B_d}{d} R^d \|f\|_{L^1}^p + c_{p,d} \left(\frac{1}{R}\right)^{p-(2-p)d/2} \|\nabla f\|_{L^2}^p. \tag{14}$$

Optimizing over R we get the estimate

$$\int_{\mathbb{R}^d} |\check{f}(\xi)|^p \, d\xi \le C_{d,p} \|f\|_{L^1}^{[p(2+d)-2d]/(d+2)} \|\nabla f\|_{L^2}^{2d/(d+2)}, \tag{15}$$

where the explicitly computable constant $C_{d,p}$ depends only on the dimension d and the number $1 \le p \le 2$ which satisfies condition (12). Using now Beckner inequality (8), and using the fact that $p/p' = p-1$ we finally obtain the inequality

$$\left[\int_{\mathbb{R}^d} |f(v)|^{p'} \, dv\right]^{(p-1)(d+2)/(2d)} \le \Gamma_B \|f\|_{L^1}^{p(d+2)/(2d)-1} \|\nabla f\|_{L^2}.$$

The constant Γ_B depends only on the dimension d and the number $1 \le p \le 2$. If $p = 2$, condition (12) is satisfied for all $d \ge 1$, and inequality (16) is nothing but Nash inequality (6).

We proved

Lemma 2. *Let $1 \le p \le 2$ satisfy condition (12). Then, there exists a constant Γ_B depending only on the dimension d and the number p such that, for all $\nabla f \in L^2(\mathbb{R}^d)$, with $f \in L^1(\mathbb{R}^d)$,*

$$\left[\int_{\mathbb{R}^d} |f(v)|^{p'} \, dv\right]^{(p-1)(d+2)/(2d)} \le \Gamma_B \|f\|_{L^1}^{p(d+2)/(2d)-1} \|\nabla f\|_{L^2}. \tag{16}$$

The result of lemma 2 can be rephrased in the following way. For any given $p' > 2$,

$$\int_{\mathbb{R}^d} |f(v)|^{p'} \, dv \le \Gamma_B \|f\|_{L^1}^{[p(2+d)-2d]/[(p-1)(d+2)]} \|\nabla f\|_{L^2}^{2d/[(p-1)(d+2)]} \tag{17}$$

In dimension $d = 3$, however, the exponent of the quantity

$$\int_{\mathbb{R}} |\nabla f(v)|^2 \, dv$$

is smaller than 1 for $p > 8/5$, namely for $p' < 8/3 < 3$. Hence, in dimension 3 inequality (2) is not enough to have a uniform control of the L^2-norm of the solution. On the other hand, $p' = 3$ is in the range of exponents for which we can have, for a supercritical mass, a steady state with a singular part. Since the singularity in the Bose–Einstein distribution is not present in the subcritical case, we aim to be able to obtain the control in dependence of the smallness of the initial mass. Let us take into account inequality (11). Instead of considering only a bound in terms of the L^1-norm of the solution, we introduce a bound in terms of both the L^1 and L^2 norms as follows. Given a positive constant $\alpha < p$,

$$\int_{|\xi|\le R} |\check{f}(\xi)|^p \, d\xi \le \|f\|_{L^1}^{p-\alpha} \int_{|\xi|\le R} |\check{f}(\xi)|^\alpha \, d\xi \le \|f\|_{L^1}^{p-\alpha} \|f\|_{L^2}^{\alpha} \left[\int_{|\xi|\le R} d\xi\right]^{(2-\alpha)/2}$$

$$\le \|f\|_{L^1}^{p-\alpha} \|f\|_{L^2}^{\alpha} \left(\frac{B_d}{d} R^d\right)^{(2-\alpha)/2}. \tag{18}$$

Hence, we can substitute inequality (14) with

$$\int_{\mathbb{R}^d} |\check{f}(\xi)|^p \, d\xi \le \|f\|_{L^1}^{p-\alpha} \|f\|_{L^2}^{\alpha} \left(\frac{B_d}{d} R^d\right)^{(2-\alpha)/2} + c_{p,d} \left(\frac{1}{R}\right)^{p-(2-p)d/2} \|\nabla f\|_{L^2}^p. \tag{19}$$

Optimizing over R we get now the estimate

$$\int_{\mathbb{R}^d} |\check{f}(\xi)|^p \, d\xi \le \tilde{C}_{d,p,\alpha} \left[\|f\|_{L^1}^{p-\alpha} \|f\|_{L^2}^{\alpha}\right]^{[2p-(2-p)d]/[2p+(p-\alpha)d]} \|\nabla f\|_{L^2}^{[pd(2-\alpha)]/[2p+(p-\alpha)d]}, \tag{20}$$

where, as before, the explicitly computable constant $\tilde{C}_{d,p,\alpha}$ depends only on the dimension d and the numbers α and p. Note that inequality (14) is a particular case of (20) corresponding to $\alpha = 0$. Hence we obtain

Lemma 3. *Let $1 \le p \le 2$ satisfy condition (12). Then, for all positive constants α with $\alpha < p$ there exists a constant Γ depending only on the dimension d and the numbers p and α such that, for all $\nabla f \in L^2(\mathbb{R}^d)$, with $f \in L^1(\mathbb{R}^d)$,*

$$\left[\int_{\mathbb{R}^d} |f(v)|^{p'} \, dv\right]^{(p-1)} \le \Gamma \left[\|f\|_{L^1}^{p-\alpha} \|f\|_{L^2}^{\alpha}\right]^{[2p-(2-p)d]/[2p+(p-\alpha)d]} \|\nabla f\|_{L^2}^{[pd(2-\alpha)]/[2p+(p-\alpha)d]}. \tag{21}$$

The exponent of the term $\|\nabla f\|_{L^2}$ in (21) is decreasing with respect to α for $p \ge 2/3$. Therefore, the minimum value of the exponent is reached for $\alpha = p$. In this case, if $p = 3/2$, in dimension $d = 3$ the inequality reduces to

$$\int_{\mathbb{R}^3} |f(v)|^3 \, dv \le \Gamma^2 \|f\|_{L^2} \|\nabla f\|_{L^2}^{3/2}. \tag{22}$$

Going back to (5), or $\int_{\mathbb{R}^3} |f(v)|^3 \, dv \le \|\nabla f\|_{L^2}^2$, so that

$$\frac{d}{dt} \int_{\mathbb{R}^3} f^2(v,t) \, dv \le 3 \int_{\mathbb{R}^3} f^2(v,t) \, dv \tag{23}$$

or $\int_{\mathbb{R}^3} |f(v)|^3\, dv > \|\nabla f\|^2_{L^2}$, and by means of (22)

$$\int_{\mathbb{R}^3} |f(v)|^3\, dv \le \Gamma^2 \|f\|_{L^2} \|\nabla f\|^{3/2}_{L^2} \le \Gamma^2 \|f\|_{L^2} \left(\int_{\mathbb{R}^3} |f(v)|^3\, dv \right)^{3/4},$$

that implies

$$\int_{\mathbb{R}^3} |f(v)|^3\, dv \le \Gamma^8 \|f\|^4_{L^2}. \tag{24}$$

In this second case, we obtain from (5)

$$\frac{d}{dt} \int_{\mathbb{R}^3} f^2(v,t)\, dv = -2 \int_{\mathbb{R}^3} |\nabla f(v,t)|^2\, dv + 3 \int_{\mathbb{R}^3} f^2(v,t)\, dv + 2\Gamma^8 \left(\int_{\mathbb{R}^3} f^2(v,t)\, dv \right)^2. \tag{25}$$

Finally, Nash inequality (6) shows that the L^2-norm of the solution to equation (4) satisfies

$$\frac{d}{dt} \|f\|^2_{L^2} \le -\frac{2}{C\|f\|_1^{4/3}} \|f\|^{10/3}_{L^2} + 3\|f\|^2_{L^2} + 2\Gamma^8 \|f\|^4_{L^2}. \tag{26}$$

By the L^1 contraction property of equation (1) [3, 4], inequality (26) is enough to guarantee that, for a given initial datum $f_0(v) \in L^1(\mathbb{R}^3) \bigcap L^2(\mathbb{R}^3)$, there exists a time T, which depends both on the L^1 and L^2 norms of the initial datum, such that in the time interval $(0,T)$ equation (4) has a unique solution $f(v,t)$ which conserves the mass and belongs to $L^2(\mathbb{R}^3)$. Thus, we can weaken the local existence theorem 1. It holds

Theorem 4. *Let the initial density $f_0(v) \ge 0$ belong to $L^1(\mathbb{R}^3) \bigcap L^2(\mathbb{R}^3)$. Then there exists $T > 0$ depending only on the norm of the initial condition, such that equation (4) has a unique nonnegative solution $f(v,t)$ in $L^1(\mathbb{R}^3) \bigcap L^2(\mathbb{R}^3)$ with $f(v,t=0) = f_0(v)$. Moreover the L^1-norm of $f(v,t)$ is conserved.*

A direct inspection of inequality (26) shows that the interval of existence is inversely proportional to the L^1-norm of the initial datum. This remark indicates that, provided the initial L^1-norm is sufficiently small, the local existence theorem 4 gives a global existence result. If $p = 3/2$ and $d = 3$, the choice $\alpha = 1$ leads to

$$\int_{\mathbb{R}^3} |f(v)|^3\, dv \le \Gamma^2 \|f\|^{1/3}_{L^1} \|f\|^{2/3}_{L^2} \|\nabla f\|^2_{L^2}. \tag{27}$$

Inequality (27) implies that the L^3-norm of the solution can be bounded in terms of the square of the L^2-norm of the gradient.

Use the Nash–type inequality (27) into (5). The L^2-norm of the solution satisfies

$$\frac{d}{dt} \int_{\mathbb{R}^3} f^2\, dv \le -2 \int_{\mathbb{R}^3} |\nabla f|^2\, dv + 3 \int_{\mathbb{R}^3} f^2\, dv + \frac{2}{3} \Gamma^2 m^{1/3} \|f\|^{2/3}_{L^2} \int_{\mathbb{R}^3} |\nabla f|^2\, dv =$$

$$-2\int_{\mathbb{R}^3}|\nabla f|^2\,dv\left[1-\frac{1}{3}\Gamma^2 m^{1/3}\int_{\mathbb{R}^3}f^2\,dv\right]+3\int_{\mathbb{R}^3}f^2\,dv. \tag{28}$$

Clearly, $m \ll 1$ enters to control the growth of the L^2-norm. In fact, if at time t

$$\frac{1}{3}\Gamma^2 m^{1/3}\int_{\mathbb{R}^3}f^2(v,t)\,dv \le 1, \tag{29}$$

the coefficient of the L^2-norm of the gradient in (28) is nonnegative, and Nash inequality (6) implies that

$$\frac{d}{dt}\|f\|^2_{L^2} \le -\frac{2}{Cm^{4/3}}\|f\|^{10/3}_{L^2}\left[1-\frac{1}{3}\Gamma^2 m^{1/3}\|f\|^{2/3}_{L^2}\right]+\|f\|^2_{L^2} \tag{30}$$

Look for the evolution of $y(t)$

$$\frac{d}{dt}y(t) \le y\left[-\frac{2}{Cm^{4/3}}y^{2/3}+\frac{2\Gamma^2}{3Cm}y+1\right]=y(t)z(y), \tag{31}$$

with the constraint induced by (29)

$$y(0) < y(m) = \frac{3}{\Gamma^2 m^{1/3}}$$

Let $m \ll 1$ such that $z(y_m)=0$. Note that this choice is always possible, due to the fact that in the negative term in $z(y(m))$ the exponent of the mass m is bigger than in the positive one. Since $z(y)$ is nonincreasing in the interval $0 \le y \le \bar{y} = (2/\Gamma^2)^3\, m^{-1}$, the choice $y(m) < \bar{y}$ then implies $y(t) \le y(m)$. The condition $y(m) < \bar{y}$ is satisfied provided

$$m < \left(\frac{8}{3}\right)^{3/2}\frac{1}{\Gamma^6}. \tag{32}$$

By the previous computations we get

Theorem 5. *Let the initial mass m satisfy the smallness condition (32). Then, if the initial density f_0 further satisfies*

$$\int_{\mathbb{R}^3} f_0^2\,dv < \frac{3}{\Gamma^2 m^{1/3}},$$

Kaniadakis-Quarati model has a unique global solution. This solution remain regular for all times, and

$$\int_{\mathbb{R}^3} f^2(v,t)\,dv < \frac{3}{\Gamma^2 m^{1/3}}.$$

3 Blow up in the super-linear case

In addition to the hypotheses of Theorem 4, let us assume $f_0 \in L^1_2(\mathbb{R}^3)$, so that

$$E(0) = \int_{\mathbb{R}^3} |v|^2 f_0(v)\, dv = E_0 < +\infty. \tag{33}$$

Then, in its interval of existence, the second moment of the solution remains bounded, and it satisfies

$$\frac{d}{dt}\int_{\mathbb{R}^3} |v|^2 f(v,t)\, dv \le 6\int_{\mathbb{R}^3} f(v,t)\, dv - 2\int_{\mathbb{R}^3} |v|^2 f(v,t)\, dv - 2\int_{\mathbb{R}^3} |v|^2 f^2(v,t)\, dv. \tag{34}$$

Following [4], let us introduce a sequence $(\vartheta_n)_{n\ge1}$ of smooth cut-off functions such that $0 \le \vartheta_n \le 1$, $\vartheta_n(v) = 1$ if $|v| \le n$, $\vartheta_n(v) = 0$ if $|v| \ge 2n$, while $|\nabla\vartheta_n| \le 1/n$ and $|\Delta\vartheta_n| \le 1/n^2$. By multiplying equation (4) times $|v|^2\vartheta_n(v)$ and integrating over $\mathbb{R}^3$ we get

$$\frac{d}{dt}\int_{\mathbb{R}^3} |v|^2\vartheta_n(v) f(t)\, dv = \int_{\mathbb{R}^3} |v|^2\vartheta_n(v)\nabla f(t)\, dv + \int_{\mathbb{R}^3} |v|^2\vartheta_n(v)\Delta\cdot(vf(t)(1+f(t))\ dv =$$

$$\int_{\mathbb{R}^3} \left[\nabla\vartheta_n|v|^2 + 4\Delta\vartheta_n\cdot v + 6\vartheta_n\right] f(t)\, dv + \int_{\mathbb{R}^3} \nabla\vartheta_n\cdot v|v|^2 f(t)(1+f(t))\, dv$$

$$-2\int_{\mathbb{R}^3} |v|^2\vartheta_n(v) f(t)(1+f(t))\, dv$$

$$\le 6\int_{\mathbb{R}^3} f(v,t)\, dv - 2\int_{\mathbb{R}^3} |v|^2 f(v,t)\,(1+f(t))\ dv$$

$$+5\int_{n<|v|<2n} f(t)\, dv + \int_{n<|v|<2n} |v|^2 f(t)\, dv. \tag{35}$$

Let $n\to\infty$. Since the sequences $(f\chi_{n<|v|<2n})_{n\ge1}$ and $(|v|^2 f\chi_{n<|v|<2n})_{n\ge1}$ converge pointwise to zero and are bounded by f and $|v|^2 f$ respectively, with $f \in \Gamma_T$, we conclude via the Lebesgue dominated convergence theorem that the last two integrals in (35) converge to zero, and the differential inequality (34) holds true.

Let us examine in more details the last integral on the right-hand side of (34). Let us set

$$h_\epsilon(v) = \left(\frac{1}{2\epsilon}\right)^d \prod_{i=1}^d \chi(-\epsilon \le v_i \le \epsilon),$$

where $\chi(E)$ denotes the characteristic function of the set E. Then

$$\int_{\mathbb{R}^d} h_\epsilon(v)\, dv = 1,$$

and the function $h_\epsilon(v)$ collapses into a mass concentrated in $v = 0$ as $\epsilon \to 0$. Consequently,

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^d} v^2 h_\epsilon(v)\, dv = 0.$$

On the other hand , since

$$\int_{\mathbb{R}^d} v^2 h_\epsilon^{p+1}(v)\, dv = \epsilon^{2+d-d(p+1)} \int_{\mathbb{R}^d} v^2 h_1(v)\, dv$$

the behavior of the integral as $\epsilon \to 0$ depends upon the sign of the exponent of ϵ. In case $p > 2/d$,

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^d} v^2 h_\epsilon^{p+1}(v)\, dv = +\infty. \tag{36}$$

The previous example indicates that in (34), which corresponds to $p = 1$ and $d = 3$, so that $p > 2/d$, the last integral dominates in presence of a mass concentrating in $v = 0$. This suggests to look for a lower bound on the last integral in (34) in terms of the second moment. We prove

Lemma 6. *Let $f(v)$ be a nonnegative function in $L_1(\mathbb{R}^d)$, $d \ge 1$, of finite second moment. Then, if $p > 2/d$, the following inequality holds*

$$\int_{\mathbb{R}^d} v^2 f^{p+1}(v)\, dv \ge B_{p,d} \frac{\left(\int_{\mathbb{R}^d} f(v)\, dv\right)^{[p(d+2)]/2}}{\left(\int_{\mathbb{R}^d} v^2 f(v)\, dv\right)^{(pd-2)/2}}. \tag{37}$$

`Proof`

For a given positive constant R, one has

$$\int_{\mathbb{R}^d} f(v)\, dv \le \int_{|v|\le R} f(v)\, dv + \frac{1}{R^2} \int_{\mathbb{R}^d} v^2 f(v)\, dv. \tag{38}$$

On the other hand Hölder inequality implies

$$\int_{|v|\le R} f(v)\, dv = \int_{|v|\le R} |v|^{-2/(p+1)} \left(|v|^{2/(p+1)} f(v)\right)\, dv \le$$

$$\left(\int_{|v|\le R} v^2 f^{p+1}(v)\, dv\right)^{1/(p+1)} \left(\int_{|v|\le R} |v|^{-2/p}\, dv\right)^{p/(p+1)}. \tag{39}$$

Since $p > 2/d$, denoting by S_d the measure of the unit ball in $\mathbb{R}^d$, we obtain

$$\int_{|v|\le R} |v|^{-2/p}\, dv = S_d \int_{\rho\le R} \rho^{-2/p+d-1}\, d\rho = \frac{pS_d}{pd-2} R^{d-2/p}.$$

Substituting into (38) gives

$$\int_{\mathbb{R}^d} f(v)\,dv \le \left(\int_{\mathbb{R}^d} v^2 f^{p+1}(v)\,dv\right)^{1/(p+1)} \left(\frac{pS_d}{pd-2}\right)^{p/(p+1)} R^{\frac{pd-2}{p+1}} + \frac{1}{R^2}\int_{\mathbb{R}^d} v^2 f(v)\,dv. \tag{40}$$

Optimizing over R inequality (40) we finally get

$$\int_{\mathbb{R}^d} f(v)\,dv \le c_{p,d} \left(\int_{\mathbb{R}^d} v^2 f^{p+1}(v)\,dv\right)^{2/[p(d+2)]} \left(\int_{\mathbb{R}^d} v^2 f(v)\,dv\right)^{(pd-2)/[p(d+2)]}. \tag{41}$$

The explicitly computable constant $c_{p,d}$ reads

$$c_{p,d} = \left[\left(\frac{2}{\alpha}\right)^{\alpha/(2+\alpha)} + \left(\frac{\alpha}{2}\right)^{2/(2+\alpha)}\right] \left(\frac{pS_d}{pd-2}\right)^{2p/(pd-2)},$$

where $\alpha = (pd-2)/(p+1)$.

Setting $d = 3$ and $p = 1$ into (37) gives

$$\int_{\mathbb{R}^3} v^2 f^2(v)\,dv \ge \frac{m^{5/2}}{b\left(\int_{\mathbb{R}^3} v^2 f(v)\,dv\right)^{1/2}}, \tag{42}$$

where the constant b can be explicitly computed to give $b = 2\pi(4^{2/5}+1)$.

From now on, let us suppose that the solution $f(v,t)$ to (4) belongs to $L_1(\mathbb{R}^3)$ in some time interval $[0,T)$. If this is the case, inserting the lower bound (42) into (34) gives

$$\frac{d}{dt}E(t) \le 6m - 2E(t) - \frac{m^{5/2}}{\pi(4^{2/5}+1)E(t)^{1/2}} = \Phi(E), \tag{43}$$

where we denoted by m the initial (preserved in time) mass and by $E(t)$ the second moment at time t. The function $\Phi(E)$ attains the maximum value in

$$\bar{E} = \left[\frac{m^{5/2}}{\pi(4^{2/5}+1)}\right]^{2/3},$$

and in this point

$$\Phi(\bar{E}) = 6m - \left(\frac{2}{(4\pi)^{2/3}} + 1\right)\frac{m^{5/3}}{\pi(4^{2/5}+1)^{2/3}}. \tag{44}$$

Since the exponent of the mass m in the negative term in (44) has the exponent strictly bigger than 1, choosing m sufficiently large we obtain

$$\Phi(\bar{E}) = -\rho < 0,$$

that implies, at time $t > 0$

$$E(t) \le E_0 - \rho t.$$

Therefore, if the initial mass m is sufficiently large, and the initial second moment is bounded, the second moment of the solution to (4) decays to zero in finite time. The critical value of m as given in consequence of inequality (42) can be obtained from (44)

$$\bar{m} = \frac{24\pi^{5/2}\sqrt{6}(4^{2/5}+1)}{((4\pi)^{2/3}+2)^{3/2}}. \tag{45}$$

The hypothesis which leads to the finite in time decay to zero of the second moment, namely the complete condensation of the solution, is a consequence of the regularity assumption. In other words, or the solution starts to blow up after a finite time t_1, or the solution $f(v,t)$ belongs to $L_1(\mathbb{R}^3)$, for $T > t_1$ and a complete condensation occurs at some subsequent finite time t_2.

It is interesting to remark that, independently of the size of the initial mass, blow up in finite time also occurs when initially the initial second moment is suitably small, compared to the mass

$$E_0^{1/2} < \frac{m^{3/2}}{6\pi(4^{2/5}+1)}. \tag{46}$$

In this case, in fact, since inequality (43) implies the (weaker) inequality

$$\frac{d}{dt}E(t) < 6m - \frac{m^{5/2}}{\pi(4^{2/5}+1)E(t)^{1/2}}, \tag{47}$$

if E_0 satisfies (46), the right-hand side of (47) is strictly negative at time $t = 0$, said $-\rho$, and the second moment start to decrease, decaying to zero in finite time. Consequently, if the second moment is initially sufficiently small there is formation of a condensate in finite time. The two situations describe nicely the physical picture, in which a Bose fluid develops a condensate part not only when the initial density is super-critical, but also in the case in which the initial temperature is sufficiently low. We collect these results into the following

Theorem 7. *Let $f_0(v)$ be the initial value of equation (4). If the initial mass is sufficiently large, that is $m_0 > \bar{m}$, where*

$$\bar{m} = \frac{24\pi^{5/2}\sqrt{6}(4^{2/5}+1)}{((4\pi)^{2/3}+2)^{3/2}},$$

or the initial energy E_0 sufficiently small, that is $E_0 < \bar{E}$, where

$$\bar{E} < \left[\frac{m^{3/2}}{6\pi(4^{2/5}+1)}\right]^2,$$

the solution blows up in finite time.

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