A variational approximation scheme for radial polyconvex elasticity that preserves the positivity of determinants

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To our colleague, teacher and friend Dave Levermore on his 60th anniversary

Abstract

We consider the equations describing the dynamics of radial motions for isotropic elastic materials; these form a system of non-homogeneous conservation laws. We construct a variational approximation scheme that decreases the total mechanical energy and at the same time leads to physically realizable motions that avoid interpenetration of matter.

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1 Introduction

The equations describing radial motions of nonlinear, isotropic, elastic materials take the form

$$w_{tt} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial \Phi}{\partial v_1}\big(w_R, \frac{w}{R}, \frac{w}{R}\big)\right) - \frac{1}{R}\sum_{i=2}^{3} \frac{\partial \Phi}{\partial v_i}\big(w_R, \frac{w}{R}, \frac{w}{R}\big)\,. \tag{1}$$

Here, y stands for a radial motion $y(x,t) = w(R,t)\frac{x}{R}$, $R = |x|$, $x \in \mathbb{R}^3$, and (1) monitors the evolution of its amplitude $w(R,t)$. A necessary condition for y to represent a physically realizable motion is $\det F > 0$ with $F = \nabla y$. In the radial case, it dictates

$$w_R (w/R)^2 > 0\,, \tag{2}$$

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and is also a sufficient condition for avoiding interpenetration of matter.

The constitutive properties of hyperelastic materials are completely determined by the stored energy function $W(F) : M^{3\times 3}_{+} \to [0,\infty)$, which - due to frame indifference - has to be invariant under rotations. For isotropic elastic materials $W(F) = \Phi(v_1, v_2, v_2)$, where Φ is a symmetric function of the principal stretches v_1, v_2, v_3 of F, see [14]. Convexity of the stored energy is, in general, incompatible with certain physical requirements and is not a natural assumption. For instance, in order to avoid interpenetration of matter the stored energy should increase without bound as $\det F \to 0^+$ so that compression of a finite volume down to a point would cost infinite energy. This behavior is inconsistent with simultaneously requiring convexity and invariance of the stored energy under rotations. As an alternative, the assumption of polyconvexity [1] is often employed, which postulates that

$$W(F) = \sigma(F, \operatorname{cof} F, \det F)$$

with σ a convex function of the null-Lagrangian vector $(F, \operatorname{cof} F, \det F)$, and encompasses certain physically realistic models (*e.g.*[5, Sec 4.9, 4.10]). In this work, we employ a specific form of polyconvex stored energy,

$$\begin{aligned} &W(F) = \Phi(v_1, v_2, v_3) \\ &= \phi(v_1) + \phi(v_2) + \phi(v_3) + g(v_2 v_3) + g(v_1 v_3) + g(v_1 v_2) + h(v_1 v_2 v_3)\,, \end{aligned} \tag{3}$$

where ϕ, g and h are convex functions and $h(\delta) \to +\infty$ as $\delta \to 0+$.

Equation (1) may be recast as a system of inhomogeneous balance laws,

$$\begin{aligned} v_t &= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial \Phi}{\partial v_1}\big(u, \frac{w}{R}, \frac{w}{R}\big)\right) - \frac{1}{R}\sum_{i=2}^{3} \frac{\partial \Phi}{\partial v_i}\big(u, \frac{w}{R}, \frac{w}{R}\big)\,. \\ u_t &= v_R \\ w_t &= v \end{aligned} \tag{4}$$

where $u = w_R$, and $v = w_t$. The system admits the entropy-entropy flux pair

$$R^2 \partial_t \left(\frac{v^2}{2} + \Phi\big(u, \frac{w}{R}, \frac{w}{R}\big)\right) - \partial_R \left(R^2\, v\, \frac{\partial \Phi}{\partial v_1}\big(u, \frac{w}{R}, \frac{w}{R}\big)\right) = 0 \tag{5}$$

which expresses the conservation of mechanical energy along smooth solutions. For polyconvex stored energies, the "entropy"

$$\eta = \frac{1}{2}v^2 + \Phi\big(u, \frac{w}{R}, \frac{w}{R}\big)$$

is not convex, what causes various difficulties in applying the general theory of conservation laws. Nevertheless, for three-dimensional elastodynamics, there are available nonlinear transport identities for the null-Lagrangians [11], which allow to view the equations of elasticity as constrained evolution of an enlarged symmetrizable system [8, 6] equipped with a relative entropy identity [10]. The enlarged system suggests a variational approximation scheme for polyconvex elasticity that dissipates the mechanical energy [8], and which, in the one-dimensional case, produces entropy weak solutions [7]. Conceptually similar structures are available in models of electromagnetism leading to augmented symmetrizable hyperbolic systems [4, 12, 13].

The above results do not take into account the constraint of positive determinant, necessary to interpret y as a physically realizable motion. In this article, we consider the equations of radial elasticity (1) and proceed to devise a variational approximation scheme that on one hand preserves the positivity of determinants (2) and on the other produces a time-discretized variant of entropy dissipation. As in [8], the scheme is based on transport identities for the null-Lagrangians. Null-Lagrangians are potential energies $\Psi(v_1, v_2, v_3; R)$ for which the functional

$$I[w] = \int_0^1 \Psi\big((w_R, \frac{w}{R}, \frac{w}{R}); R\big)\, dR \tag{6}$$

has variational derivative zero. They satisfy

$$-\partial_R\left(\Psi_{,1}\right) + R^{-1}\left(\Psi_{,2} + \Psi_{,3}\right) = 0 \quad \text{for all functions } w(R)\,. \tag{7}$$

and are computed to be the functions v_1, $v_1 v_2 R$, $v_1 v_3 R$ or $v_1 v_2 v_3 R^2$. Along solutions of the dynamical problem, each null-Lagrangian satisfies the transport identity

$$\partial_t \Psi = \partial_R\left(\Psi_{,1}\, v\right)\,, \tag{8}$$

with Ψ and $\Psi_{,i}$ are evaluated at $\Gamma = (w_R, w/R, w/R, R)$. The identities (8) allow to embed the system (4) into the symmetrizable first-order evolution (40) in Section 3.2.

The enlarged system, in the form (40), cannot handle the positivity of determinants constraint. For this reason we follow an alternative strategy, combining a change of variables suggested in Ball [3] (for the equilibrium problem) with the idea of extensions based on null-Lagrangians, and carry out an alternative extended system. We set $\rho = R^3$, $\alpha = w^3$, $\beta = w_R/R^2$, $\gamma = w^2$ and let

$$\Xi = \left(\beta\rho^{2/3}, \frac{\alpha}{\rho}, \frac{\alpha}{\rho}, \frac{\gamma}{\rho^{1/3}}, \frac{3\gamma_\rho}{2}\rho^{2/3}, \frac{3\gamma_\rho}{2}\rho^{2/3}, \alpha_\rho\rho^{2/3}\right). \tag{9}$$

The second extension has four actual unknowns v, α, β and γ, and is the symmetrizable system listed in (59) of Section 3.3 endowed with the entropy pair

$$\partial_t \left(\frac{v^2}{2} + G(\Xi) \right) - \partial_\rho \left(3\rho^{2/3}\, G_{,i}(\Xi)\, \Omega^i_{,1}(\Gamma)\, v \right) = 0 \ , \tag{10}$$

where G is defined in (46) and is (assumed) convex and Γ is as in (48).

The extended system (59) is discretized in time using an implicit-explicit scheme. It is the Euler-Lagrange equations of the variational problem: given v_0 and Ξ^0 defined via α_0, β_0 and γ_0 as in (9), minimize

$$I(\alpha, \beta, \gamma, v) = \int_0^1 \frac{1}{2}(v - v_0)^2 + G(\Xi)\, d\rho \tag{11}$$

over the set of admissible functions

$$\begin{aligned} \mathcal{A}_\lambda = \Big\{ (\alpha, \beta, \gamma, v) \in X : \alpha(0) \geqslant 0,\, \alpha(1) = \lambda,\, \alpha' > 0 \text{ a.e. } \textit{and} \\ I(\alpha, \beta, \gamma, v) < \infty,\, \frac{(\beta - \beta_0)}{h} = 3v', \\ \frac{(\alpha - \alpha_0)}{h} = 3{\alpha_0}^{2/3} v,\, \frac{(\gamma - \gamma_0)}{h} = 2{\alpha_0}^{1/3} v \Big\}. \end{aligned} \tag{12}$$

The differential constraints in (12) are affine, the condition $\alpha(1) = \lambda$ corresponds to the imposed boundary condition $y(x) = \lambda x$, $x \in \partial\mathcal{B}$, while $\alpha' > 0$ secures the positivity of determinants (2). We prove the existence and uniqueness of a minimizer for the functional I over $\mathcal{A}_\lambda$ and that the minimizer is a weak solution to the corresponding Euler-Lagrange equations, that is, a solution of the time-discrete scheme. The analysis of the minimization problem (11)-(12) uses direct methods of the calculus of variations, in the spirit of [3], with the novel element of accounting for the evolutionary constraints in (12).

In continuum physics, weak solutions of a system of conservation laws are required to satisfy entropy inequalities of the form

$$\partial_t \eta + \partial_\alpha q_\alpha \leq 0 \tag{13}$$

Such inequalities reflect irreversibility and originate from the second law of thermodynamics. For instance, admissible shocks of the elasticity equations are required to dissipate the mechanical energy. Accordingly, approximating schemes are expected to respect such behaviors and produce entropy dissipating solutions in the limit. The variational scheme studied here turns out

to satisfy a discrete version of the entropy inequality

$$\frac{\left(\frac{v^2}{2}+G(\Xi)\right)-\left(\frac{{v_0}^2}{2}+G(\Xi^0)\right)}{h}-\frac{d}{d\rho}\left(3\rho^{2/3}G_{,i}(\Xi)\Omega^i_{,1}(\Gamma^0)v\right)\leqslant 0 \qquad (14)$$

(see Section 4). In addition, the approximants satisfy $\alpha_\rho > 0$ the transformed version of (2). Finally, if the constructed approximants converge pointwise as the time-step $h \to 0$, then the limit will satisfy the mechanical energy dissipation inequality

$$\partial_t\left(\frac{v^2}{2}+G(\Xi)\right)-\partial_\rho\left(3\rho^{2/3}\,G_{,i}(\Xi)\,\Omega^i_{,1}(\Gamma)\,v\right)\leqslant 0. \qquad (15)$$

The paper is organized as follows. In Section 2 we outline the derivation of the equations of radial elasticity and list various mechanical considerations relevant to this work. Section 3 contains a discussion of null-Lagrangians and the properties of the two symmetrizable extensions of (4) pursued. Section 4 introduces the time-discrete scheme and its relation to a variational problem. In Section 5 we consider the minimization problem (11) and prove Theorems 2 and 3 regarding existence and uniqueness of minimizers. The Euler-Lagrange equations associated to the minimization problem are derived in Theorem 4 of Section 6, and the regularity of minimizers is discussed in Section 7. The fact that minimizers satisfy the time-discretized version of the entropy dissipation inequality (14) is proved in Section 4.

2 Preliminaries

We consider the equations of nonlinear elasticity

$$\begin{cases} \quad y_{tt} = \mathrm{div} S(\nabla y) & \text{in } \mathcal{B}\times(0,\infty) \\ y(x,t) = \lambda x, & \text{on } \partial\mathcal{B}\times[0,\infty) \\ \quad \det\nabla y > 0, & (x,t)\in\mathcal{B}\times[0,\infty) \end{cases} \qquad (16)$$

on the unit ball $\mathcal{B}=\{x\in\mathbb{R}^n\colon |x|<1\}$, subject to uniform stretching at the boundary and initial conditions

$$y(x,0)=y_0(x)\,,\ y_t(x,0)=v_0(x) \quad x\in\mathcal{B} \qquad (17)$$

In order for the geometric mapping $y:\mathcal{B}\times[0,\infty)\to\mathbb{R}^n$ to correspond to a physically realizable ***motion*** we have to exlude interpenetration of matter. As a minimum requirement the condition $\det\nabla y>0$ is imposed.

Let $M^{n\times n}$ be the real $n\times n$ matrices, $M^{n\times n}_+ = \{F \in M^{n\times n}\colon \det F > 0\}$, and let $SO(n)$ denote the set of proper rotations. The Piola-Kirchhoff stress is a mapping $S : M^{n\times n}_+ \to M^{n\times n}$ and for ***hyperelastic*** materials it is defined by the formula

$$S(F) = \partial W(F)/\partial F. \tag{18}$$

where $W : M^{n\times n}_+ \to \mathbb{R}^n$ is the stored-energy function of the elastic body.

We assume that the stored energy function W satisfies the physical requirement of **frame-indifference** and that the elastic material is **isotropic**. Then,

$$W(QF) = W(F) = W(FQ) \quad \forall F \in M^{n\times n}_+,\ Q \in SO(n) \tag{19}$$

and (see Truesdell and Noll [14, pp 28, 317]) there exists a **symmetric** function

$$\Phi : \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i > 0\ \forall i\} \to \mathbb{R}$$

such that

$$W(F) = \Phi(v_1, \ldots, v_n) \qquad \forall F \in M^{n\times n}_+, \tag{20}$$

where $v_1, \ldots, v_n$ are the singular values of F, i.e. the eigenvalues of $(F^{\mathrm{T}}F)^{1/2}$. We note that the symmetry of Φ implies

$$\frac{\partial \Phi}{\partial v_i}(a, b, \ldots, b) = \frac{\partial \Phi}{\partial v_j}(a, b, \ldots, b), \quad i, j \geqslant 2,\ a, b \in \mathbb{R}_+. \tag{21}$$

It is easy to check that for hyperelastic, isotropic materials, frame-indifference implies

$$S(QFQ^{\mathrm{T}}) = QS(F)Q^{\mathrm{T}} \quad \text{for all} \quad Q \in SO(n). \tag{22}$$

2.1 Radial Elasticity

A function $f : \mathcal{B}\backslash\{0\} \to \mathbb{R}^n$ is called ***radial*** if

$$f(x) = w(R)\frac{x}{R}, \quad R = |x|, \tag{23}$$

where $w : [0, \infty) \to [0, \infty)$. The space of ***deformations*** of $\mathcal{B}$ is denoted by

$$\mathrm{Def}^{\,p}(\mathcal{B}) = \left\{ f \in W^p_1(\mathcal{B}, \mathbb{R}^n) \,:\, \det \nabla f > 0 \text{ a.e. } \right\}.$$

Lemma 1 (J. Ball [3]). *Let f be a radial function defined by (23). Then $f \in Def^p(\mathcal{B})$ if and only if w is absolutely continuous on $(0,1)$ and satisfies $w_R(w/R)^{n-1} > 0$ almost everywhere, and*

$$\int_0^1 \left(|w'|^p + |w/R|^p\right) R^{n-1} dR \, < \, \infty.$$

In this case the weak derivatives of f are given by

$$\nabla f = \frac{w}{R}\mathbf{I} + \left(w' - \frac{w}{R}\right)\frac{x \otimes x}{R^2} \quad a.e. \quad x \in \mathcal{B}. \tag{24}$$

Our next goal is to consider the problem (16) and to recast it for radial motions

$$y(x,t) = w(R,t)\frac{x}{R} \quad \text{for} \quad x \neq 0,$$

where $w : [0,1) \times [0,\infty) \to \mathbb{R}$ satisfies $w(R,t) \geqslant 0$. Lemma 1 implies

$$\nabla y = \frac{w}{R}\mathbf{I} + \left(w_R - \frac{w}{R}\right)\frac{x \otimes x}{R^2} \quad \text{a.e.} \quad x \in \mathcal{B}$$

and hence the eigenvalues of ∇y are expressed by

$$v_1 = w_R, \; v_2 = ... = v_n = w/R$$

The requirement

$$\det \nabla y = w_R(w/R)^{n-1} > 0 \quad \text{and} \tag{25}$$

dictates $w_R, \frac{w}{R} > 0$. We recall that $v_1, \ldots, v_n$ are actually the singular values of ∇y. Then (see e.g. J.Ball [3]) the property (22) implies that the Piola-Kirchhoff stress can be expressed as

$$S(\nabla y) = \Phi_{,2}\left(w_R, w/R, \ldots, w/R\right)\mathbf{I} + \\ \left[\Phi_{,1}\left(w_R, w/R, \ldots, w/R\right) - \Phi_{,2}\left(w_R, w/R, \ldots, w/R\right)\right]\frac{x \otimes x}{R^2}.$$

The system (16) for radial motions takes the form

$$R^{n-1}\partial_{tt} w = \frac{\partial}{\partial R}\left(R^{n-1}\Phi_{,1}(w_R, \ldots, w/R)\right) - R^{n-2}\sum_{i=2}^{n}\Phi_{,i}(w_R, \ldots, w/R) \\ w(1) = \lambda, \; w \geqslant 0, \; w_R\,(w/R)^{n-1} > 0, \;\; (R,t) \in (0,1) \times [0,\infty). \tag{26}$$

of a second order equation describing the evolution of $w(R,t)$ subject to the constraint $(26)_2$, that expresses the requirement that matter cannot interpenetrate unto itself.

2.2 Polyconvex Stored Energy for $n = 3$

From now on we fix the number of dimensions to $n = 3$ and assume that the stored energy $W : M^{3\times 3}_+ \to \mathbb{R}^3$ is ***polyconvex***, that is

$$W(F) = \bar{\bar{G}}\,(F, \operatorname{cof} F, \det F)$$

for some *convex* function $\bar{\bar{G}} : M^{3\times 3}_+ \times M^{3\times 3}_+ \times \mathbb{R}_+ \to \mathbb{R}$.

By the polar decomposition theorem any matrix $F \in M^{3\times 3}_+$ is expressed in the form $F = RU$ with $R \in SO(3)$ and $U = +\sqrt{F^T F}$. Further, $U = Q^T diag(v_1, v_2, v_3)Q$ where Q is the orthogonal matrix of eigenvectors and v_1, v_2, v_3 are the eigenvalues of U. The properties (19) of isotropy and frame-indifference imply

$$\begin{aligned} W(F) &= \bar{\bar{G}}\left(\begin{bmatrix} v_1 & & \\ & v_2 & \\ & & v_3 \end{bmatrix}, \begin{bmatrix} v_2v_3 & & \\ & v_1v_3 & \\ & & v_1v_2 \end{bmatrix}, v_1v_2v_3\right) \\ &=: \bar{G}\,(v_1, v_2, v_3, v_2v_3, v_1v_3, v_1v_2, v_1v_2v_3) \end{aligned}$$

where $\bar{G}(\Xi)$ is a convex function of $\Xi = (\xi)_{i=1\ldots 7} \in \mathbb{R}^7$.

For reasons related to the null-Lagrangian structure of an associated variational problem (outlined in the following section) the stored energy will be expressed in the form

$$\begin{aligned} W(\nabla y) &= \Phi\left(w_R, \frac{w}{R}, \frac{w}{R}\right) \\ &= \bar{G}\left(w_R, \frac{w}{R}, \frac{w}{R}, \left(\frac{w}{R}\right)^2, w_R\left(\frac{w}{R}\right), w_R\left(\frac{w}{R}\right), w_R\left(\frac{w}{R}\right)^2\right) \\ &= G\left(\Omega\left(\left(w_R, \frac{w}{R}, \frac{w}{R}\right); R\right); R\right) \end{aligned} \tag{27}$$

where Ω and G are inhomogeneous functions defined by

$$\Omega(V; R) := \left(v_1, v_2, v_3, v_2v_3R, v_1v_3R, v_1v_2R, v_1v_2v_3R^2\right). \tag{28}$$

$$G(\Xi; R) := \bar{G}\left(\xi_1, \xi_2, \xi_3, \xi_4/R, \xi_5/R, \xi_6/R, \xi_7/R^2\right). \tag{29}$$

with $V = (v_i)_{i=1\ldots 3} \in \mathbb{R}^3$ and $\Xi = (\xi)_{i=1\ldots 7} \in \mathbb{R}^7$. The convexity hypothesis on $\bar{\bar{G}}$ implies that $G(\Xi; R)$ is convex as a function of $\Xi \in \mathbb{R}^7$. In summary, we write

$$W(\nabla y) = \Phi\left(w_R, \frac{w}{R}, \frac{w}{R}\right) = G(\Omega(\Gamma; R); R), \tag{30}$$

$$\text{where} \quad \Gamma = \left(w_R, \frac{w}{R}, \frac{w}{R}\right). \tag{31}$$

For simplicity of notation, we henceforth suppress the dependence on R and write $\Omega(V) = \Omega(V;R)$ and $G(\Xi) = G(\Xi;R)$.

For three dimensional motions, (26) is rewritten as a first order system

$$\begin{aligned} R^2\partial_t v &= \frac{\partial}{\partial R}\Big(R^2\Phi_{,1}\big(w_R, \frac{w}{R}, \frac{w}{R}\big)\Big) - R\left(\Phi_{,2}+\Phi_{,3}\right)\big(w_R, \frac{w}{R}, \frac{w}{R}\big) \\ \partial_t w &= v \end{aligned} \tag{32}$$

which formally satisfies the conservation of mechanical energy identity

$$\partial_t\Big(R^2\big(\frac{v^2}{2} + \Phi\left(w_R, w/R, w/R\right)\big)\Big) = \partial_R\left(R^2 v\,\Phi_{,1}\left(w_R, w/R, w/R\right)\right). \tag{33}$$

The mechanical energy provides an entropy-entropy flux pair for (32) but the entropy is not in general convex. Using (30)-(31), the derivatives Φ_j, $j = 1,2,3$, are expressed as

$$\Phi_{,j}(v_1,v_2,v_3) = \frac{\partial}{\partial v_j}G(\Omega(V)) = \frac{\partial G}{\partial \xi_i}(\Omega(V))\frac{\partial \Omega^i}{\partial v_j}(V),$$

and $(32)_1$ is written as

$$\begin{aligned} R^2\,\partial_t v = \;& \partial_R\left(R^2\frac{\partial G}{\partial \xi_i}(\Omega(\Gamma))\frac{\partial \Omega^i}{\partial v_1}(\Gamma)\right) \\ & - R\,\frac{\partial G}{\partial \xi_i}(\Omega(\Gamma))\left(\frac{\partial \Omega^i}{\partial v_2}(\Gamma) + \frac{\partial \Omega^i}{\partial v_3}(\Gamma)\right) \end{aligned} \tag{34}$$

3 Null-Lagrangians and extensions of polyconvex radial elasticity

3.1 Null-Lagrangians

An alternative approach to derive (33) proceeds by considering the extrema of the action functional

$$J[y] = \int_0^T\int_0^1 \left(\frac{1}{2}w_t^2 - \Phi\big(w_R, \frac{w}{R}, \frac{w}{R}\big)\right) R^2\,dRdt$$

and deriving (26) (for $n = 3$) as the associated Euler-Lagrange equations. This provides a connection with the calculus of variations.

Consider the functional associated to the equilibrium problem

$$I[w] = \int_0^1 \Psi\left(w_R, w/R, w/R\,; R\right) dR$$

We ask for which functions $\Psi(v_1, v_2, v_3; R) : \mathbb{R}^4 \to \mathbb{R}$ the functional I admits variational derivative $\frac{\delta I}{\delta w} = 0$; such integrands are called *null Lagrangians* and they satisfy the Euler-Lagrange equation

$$-\partial_R(\Psi_{,1}) + R^{-1}(\Psi_{,2} + \Psi_{,3}) = 0 \quad \text{for all functions } w(R). \tag{35}$$

If $w = w(R,t)$ also depends on time, the evolution of a null Lagrangian Ψ is described by

$$\partial_t \Psi = \partial_R(\Psi_{,1}\, \partial_t w). \tag{36}$$

where Ψ and $\Psi_{,i}$ are evaluated at $(w_R, w/R, w/R, R)$.

It is easily verified that $\Psi(v_1, v_2, v_3; R)$ selected by

$$v_1, \quad v_1 v_2 R, \quad v_1 v_3 R, \quad \text{or} \quad v_1 v_2 v_3 R^2$$

are null-Lagrangians. Applying (35) to Ω^i, $i = 1, 5, 6, 7$, defined by (28) we get

$$-\partial_R\left(\Omega^i_{,1}(\Gamma)\right) + R^{-1}\left(\Omega^i_{,2}(\Gamma) + \Omega^i_{,3}(\Gamma)\right) = 0, \quad i = 1, 5, 6, 7, \tag{37}$$

with $\Gamma = (w_R, w/R, w/R)$ defined by (31).

3.2 A symmetrizable extension

The null-Lagrangian structure is used in [8] to embed the equations of 3-d elastodynamics to a hyperbolic system endowed with a convex entropy, and to construct a variational approximation scheme for the problem. We follow this procedure in order to achieve an augmented system for radial elastodynamics. The evolution in time of

$$\Omega(\Gamma) = \left(w_R, w/R, w/R, w^2/R, w_R w, w_R w, w_R w^2\right), \tag{38}$$

gives

$$\begin{aligned}
&\partial_t \Omega^1(\Gamma) = \partial_t(w_R) = \partial_R v = \partial_R\left(\Omega^1_{,1}(\Gamma) v\right)\\
&\partial_t \Omega^i(\Gamma) = \partial_t(w/R) = v/R = R^{-1}\left(\Omega^i_{,2}(\Gamma) + \Omega^i_{,3}(\Gamma)\right) v \quad \text{for } i = 2, 3\\
&\partial_t \Omega^4(\Gamma) = \partial_t\left(w^2/R\right) = 2(w/R)v = R^{-1}\left(\Omega^4_{,2}(\Gamma) + \Omega^4_{,3}(\Gamma)\right) v\\
&\partial_t \Omega^i(\Gamma) = \partial_t(w_R w) = \partial_R(wv) = \partial_R\left(\Omega^i_{,1}(\Gamma) v\right) \qquad \text{for } i = 5, 6\\
&\partial_t \Omega^7(\Gamma) = \partial_t\left(w_R w^2\right) = \partial_R(w^2 v) = \partial_R\left(\Omega^7_{,1}(\Gamma) v\right).
\end{aligned} \tag{39}$$

Note that $(39)_{1,5,6,7}$ are precisely the equations (36) describing the evolution of null Lagrangians. By contrast, $(39)_{2,3,4}$ describe the evolution of lower-order terms and do not have the structure of (36).

Equations (39) and (34) motivate an extension of radial elasticity :

$$\begin{cases} R^2\partial_t v = \partial_R\left(R^2\dfrac{\partial G}{\partial\xi_i}(\Xi)\dfrac{\partial\Omega^i}{\partial v_1}(\xi)\right) - R\dfrac{\partial G}{\partial\xi_i}(\Xi)\left(\dfrac{\partial\Omega^i}{\partial v_2}(\xi)+\dfrac{\partial\Omega^i}{\partial v_3}(\xi)\right) \\ \quad \partial_t\xi_i = \partial_R\left(\Omega^i_{,1}(\xi)v\right) \quad i=1,5,6,7 \\ \quad \partial_t\xi_i = R^{-1}\left(\Omega^i_{,2}(\xi)+\Omega^i_{,3}(\xi)\right)v \quad i=2,3,4 \\ \quad\ \xi_1 = \partial_R(R\xi_2)\,, \quad \xi_2=\xi_3 \end{cases} \tag{40}$$

$$\xi_2(1)=\xi_3(1)=\lambda,\ \xi_2,\xi_3\geqslant 0,\ \xi_7>0,\ \ (R,t)\in(0,1)\times[0,\infty), \tag{41}$$

System (40) describes the evolution of the vector (v,Ξ) where $\Xi\in\mathbb{R}^7$ and $\xi=(\xi_1,\xi_2,\xi_3)$ are the first three components of Ξ.

The extension has the following properties:

(i) The constraint $(40)_4$ enforces that ξ is of the form $\xi=(w_R, w/R, w/R)$ for some function $w(R,t)$ (similarly to Γ in (31)). Moreover, $(40)_4$ is an involution: if it is satisfied for the initial data, the constraint is propagated and is satisfied for all times.

(ii) If $\Xi(\cdot,0)=\Omega(\Gamma^0)$ where $\Gamma^0=(f', f/R, f/R)$ for some $f=f(R)$, then $\Xi(R,t)$ retains the same format for all times, *i.e.* there exists w such that $\Xi=\Omega(\Gamma)$ where $\Gamma=(w_R, w/R, w/R)$. In other words, radial elasticity (32) can be viewed as a constrained evolution of (40).

(iii) The enlarged system admits an entropy pair

$$\partial_t\left(R^2\left(\frac{v^2}{2}+G(\Xi)\right)\right) - \partial_R\left(R^2\,\frac{\partial G}{\partial\xi_i}(\Xi)\,\frac{\partial\Omega^i}{\partial v_1}(Z)\,v\right)=0 \tag{42}$$

with strictly convex entropy

$$\eta(v,\Xi)=\frac{v^2}{2}+G(\Xi). \tag{43}$$

Let us remark that η is not an entropy in the usual sense of the theory of conservation laws: the identity (42) is based on the constraint $(40)_4$ together with the property (37) of null Lagrangians.

3.3 An alternative extension with a convex entropy

System (40) provides an extension of radial elasticity that is endowed with a convex entropy. Concerning the objective of achieving a variational approximation, it has the drawback that the constraint (41) of positivity for the

variables ξ_2, ξ_3 and ξ_7 is not preserved at the level of time-step approximations. Although one can control the positivity ξ_7 (the augmented variable standing for the determinant), it is not possible to control the positivity of ξ_2,ξ_3. There are also difficulties in proving that minimizers satisfy the corresponding Euler-Lagrange equations, the time-discretized system associated to (40).

For this reason, we develop an alternative extension by combining the evolution of null-Lagrangians with a change of variables used in Ball [3] for the equilibrium problem. This extension induces a variational approximation scheme that preserves the positivity of determinants.

The stored energy Φ is expressed in the form

$$\begin{aligned} \Phi\left(v_1, v_2, v_3\right) &= \bar{G}\left(v_1, v_2, v_3, v_2v_3, v_1v_3, v_1v_2, v_1v_2v_3\right) \\ &= G(\Omega(V;\rho)\,;\,\rho) \end{aligned} \tag{44}$$

where Ω and G are nonhomogeneous functions of ρ that are redefined so that

$$\Omega(V;\rho) := \left(v_1, v_2^3, v_3^3, v_2v_3\rho^{1/3}, v_1v_3\rho^{1/3}, v_1v_2\rho^{1/3}, v_1v_2v_3\rho^{2/3}\right) \tag{45}$$

$$G(\Xi;\rho) := \bar{G}\left(\xi_1, \xi_2^{1/3}, \xi_3^{1/3}, \xi_4/\rho^{1/3}, \xi_5/\rho^{1/3}, \xi_6/\rho^{1/3}, \xi_7/\rho^{2/3}\right). \tag{46}$$

It is now assumed that $G(\Xi;\rho)$ is a convex function of Ξ; this is a somewhat stronger hypothesis than polyconvexity (which is convexity of $\bar{G}$) because of the defenition of $\Omega^i(V;\rho)$, $i = 2, 3$, in (45). In the sequel any explicit ρ-dependence will be suppressed.

3.3.1 A change of variables

Following [3] we perform the change of variables

$$\rho = R^3 \quad \text{and} \quad \alpha = w^3. \tag{47}$$

Then $\Gamma = (w_R, w/R, w/R)$ is expressed as

$$\Gamma = (\alpha_\rho(\rho/\alpha)^{2/3}, (\alpha/\rho)^{1/3}, (\alpha/\rho)^{1/3}) \tag{48}$$

and the stored energy reads

$$\begin{aligned} W(\nabla y) &= \Phi\left(\alpha_\rho(\rho/\alpha)^{2/3}, (\alpha/\rho)^{1/3}, (\alpha/\rho)^{1/3}\right) \\ &= G(\Omega(\Gamma;\rho)\,;\,\rho) \end{aligned} \tag{49}$$

where Ω and G are defined in (45), (46), and $G(\cdot;\rho)$ is convex.

The system (32) takes the form

$$\begin{cases} \partial_t v = \partial_\rho \left(3\rho^{2/3} \dfrac{\partial G}{\partial \xi_i}(\Omega(\Gamma)) \dfrac{\partial \Omega^i}{\partial v_1}(\Gamma) \right) \\ \qquad\qquad - \rho^{-1/3} \dfrac{\partial G}{\partial \xi_i}(\Omega(\Gamma)) \left(\dfrac{\partial \Omega^i}{\partial v_2}(\Gamma) + \dfrac{\partial \Omega^i}{\partial v_3}(\Gamma) \right) \\ \partial_t(\alpha^{1/3}) = v \\ \quad \alpha(1) = \lambda,\ \alpha \geqslant 0,\ \alpha_\rho > 0, \quad (R,t) \in (0,1) \times [0,\infty). \end{cases} \tag{50}$$

with the last inequalities encoding the constraints for solutions to represent elastic motions. In the new variables, by (45),

$$\Omega(\Gamma) = \left(\frac{\alpha_\rho}{\alpha^{2/3}} \rho^{2/3}, \frac{\alpha}{\rho}, \frac{\alpha}{\rho}, \frac{\alpha^{2/3}}{\rho^{1/3}}, \frac{\alpha_\rho}{\alpha^{1/3}} \rho^{2/3}, \frac{\alpha_\rho}{\alpha^{1/3}} \rho^{2/3}, \alpha_\rho \rho^{2/3} \right) \tag{51}$$

and, using $(50)_2$ we compute

$$\begin{aligned} &\partial_t \, \Omega^1(\Gamma) = \partial_t \left(3\rho^{2/3} \partial_\rho(\alpha^{1/3}) \right) = 3\rho^{2/3} \partial_\rho v \\ &\partial_t \, \Omega^i(\Gamma) = \partial_t \left(\alpha/\rho \right) = 3\alpha^{2/3} v/\rho \qquad i = 2,3 \\ &\partial_t \, \Omega^4(\Gamma) = \partial_t \left(\alpha^{2/3}/\rho^{1/3} \right) = 2\alpha^{1/3} v/\rho^{1/3} \\ &\partial_t \, \Omega^i(\Gamma) = \partial_t \left((3/2)\rho^{2/3} \partial_\rho(\alpha^{2/3}) \right) = 3\rho^{2/3} \partial_\rho \left(\alpha^{1/3} v \right) \quad i = 5,6 \\ &\partial_t \, \Omega^7(\Gamma) = \partial_t \left(\alpha_\rho \rho^{2/3} \right) = 3\rho^{2/3} \partial_\rho(\alpha^{2/3} v) \end{aligned} \tag{52}$$

These equations are summarized in two groups

$$\begin{aligned} &\partial_t \Omega^i(\Gamma) = 3\rho^{2/3} \partial_\rho(\Omega^i_{,1}(\Gamma) v), \quad i = 1,5,6,7 \\ &\partial_t \Omega^i(\Gamma) = \rho^{-1/3} (\Omega^i_{,2}(\Gamma) + \Omega^i_{,3}(\Gamma)) v \,, \quad i = 2,3,4 \,, \end{aligned} \tag{53}$$

the former representing the evolution of null-Lagrangians and the latter the evolution of lower order terms. The identities (37) satisfied by null-Lagrangians become

$$-3\rho^{2/3} \partial_\rho \left(\Omega^i_{,1}(\Gamma) \right) + \rho^{-1/3} \left(\Omega^i_{,2}(\Gamma) + \Omega^i_{,3}(\Gamma) \right) = 0, \quad i = 1,5,6,7. \tag{54}$$

3.3.2 The augmented system

Next, consider the augmented system

$$\begin{cases} \partial_t v = \partial_\rho \left(3\rho^{2/3} G_{,i}(\Xi) \Omega^i_{,1}(\Gamma) \right) - \rho^{-1/3} G_{,i}(\Xi) \left(\Omega^i_{,2}(\Gamma) + \Omega^i_{,3}(\Gamma) \right) \\ \partial_t \alpha^{1/3} = v \\ \partial_t \xi_i = 3\rho^{2/3} \partial_\rho \left(\Omega^i_{,1}(\Gamma) v \right), \quad i = 1, 5, 6, 7 \\ \partial_t \xi_i = \rho^{-1/3} \left(\Omega^i_{,2}(\Gamma) + \Omega^i_{,3}(\Gamma) \right) v, \quad i = 2, 3, 4 \\ \xi_1 = 3\rho^{2/3} \partial_\rho \alpha^{1/3} \end{cases} \tag{55}$$

This is viewed as a first-order system $(55)_1$-$(55)_4$ describing the evolution of the vector (v, α, Ξ) subject to the constraint $(55)_5$. It has the following properties:

(a) The constraint $(55)_5$ is propagated by the evolution from the initial data, since $\partial_t(\xi_1 - 3\rho^{2/3}\partial_\rho \alpha^{1/3}) = 0$. We may thus write $\Omega(\Gamma)$, with Γ as in (48), and still think of (55) as a first order system.

(b) If $\Xi(\cdot, 0) = \Omega(\Gamma^0)$ with $\Gamma^0 = \left(f'(\rho/f)^{2/3}, (f/\rho)^{1/3}, (f/\rho)^{1/3} \right)$ for $f = f(\rho)$, it remains in this form $\forall t$, *i.e.* there exists $\alpha(\rho, t)$ such that Γ defined by (48) satisfies $\Gamma(., 0) = \Gamma^0$ and $\Xi = \Omega(\Gamma)\ \forall t$. In other words, radial elasticity (26) can be viewed as a constrained evolution of (55).

(c) The enlarged system admits an entropy pair

$$\partial_t \left(\frac{v^2}{2} + G(\Xi) \right) - \partial_\rho \left(3\rho^{2/3}\, G_{,i}(\Xi)\, \Omega^i_{,1}(\Gamma)\, v \right) = 0 \tag{56}$$

with (for convex G) strictly convex entropy $\eta(v, \Xi) = \frac{v^2}{2} + G(\Xi)$.

At this point we set

$$\beta = \alpha_\rho / \alpha^{2/3} \quad \text{and} \quad \gamma = \alpha^{2/3}$$
$$\Xi = \left(\beta\rho^{2/3}, \frac{\alpha}{\rho}, \frac{\alpha}{\rho}, \frac{\gamma}{\rho^{1/3}}, \frac{3\gamma_\rho}{2}\rho^{2/3}, \frac{3\gamma_\rho}{2}\rho^{2/3}, \alpha_\rho \rho^{2/3} \right), \tag{57}$$

and proceed to simplify the extended system working with α, β, γ, v as the independent variables.

Taking a closer look at the extended system we see that $\xi_2 = \xi_3$ by construction and hence equations $(55)_2$, $i = 2, 3$ are identical. Moreover,

$$\begin{aligned} \partial_t \xi_2 = 3\alpha^{2/3} v/\rho \quad &\Rightarrow \quad \partial_t \xi_7 = \rho^{2/3} \partial_\rho (\rho\, \partial_t \xi_2), \\ \partial_t \xi_4 = 2\alpha^{1/3} v/\rho^{1/3} \quad &\Rightarrow \quad \partial_t \xi_5 = \partial_t \xi_6 = \frac{3}{2} \rho^{2/3} \partial_\rho \left(\rho^{1/3} \partial_t \xi_4 \right). \end{aligned} \tag{58}$$

Hence (55) is overdetermined and extra equations $(55)_2$, $i = 5, 6, 7$ and $(55)_3$, $i = 3$ can be excluded. In explicit form the extension is written as

$$\begin{cases} \partial_t v = \partial_\rho \left(3\rho^{2/3} G_{,i}(\Xi)\, \Omega^i_{,1}(\Gamma) \right) - \rho^{-1/3} G_{,i}(\Xi) \, \left(\Omega^i_{,2}(\Gamma) + \Omega^i_{,3}(\Gamma) \right) \\ \partial_t \beta = \partial_\rho (3v) \\ \partial_t \alpha = 3\alpha^{2/3} v \\ \partial_t \gamma = 2\alpha^{1/3} v \\ \alpha(1) = \lambda, \ \alpha \geqslant 0, \ \ \alpha_\rho > 0, \ \ (\rho, t) \in (0,1) \times [0, \infty), \end{cases} \tag{59}$$

where from $(59)_3$ and $(59)_4$ we can derive the excluded equations

$$\begin{aligned} \partial_t \alpha_\rho &= \partial_\rho (3\alpha^{2/3} v) \\ \partial_t \gamma_\rho &= \partial_\rho (2\alpha^{1/3} v). \end{aligned} \tag{60}$$

4 Variational Approximation Scheme

In this section we introduce a variational approximation scheme for the radial equation of elastodynamics. The general approach is to discretize the extended system by use of implicit-explicit scheme.

Successive iterates are constructed by discretizing (55) as follows: given the $(j-1)^{th}$ iterate $(\alpha_0, \beta_0, \gamma_0, v_0)$ with $\alpha_0(\rho) \geqslant 0$ and $\alpha_0'(\rho) > 0$, $\rho \in (0,1)$ we define $\Xi^0 = (\xi_i^0)_{i=1}^7$ by

$$\Xi^0(\rho) = \left(\beta_0 \rho^{2/3}, \frac{\alpha_0}{\rho}, \frac{\alpha_0}{\rho}, \frac{\gamma_0}{\rho^{1/3}}, \frac{3\gamma_0'}{2}\rho^{2/3}, \frac{3\gamma_0'}{2}\rho^{2/3}, \alpha_0' \rho^{2/3} \right) \tag{61}$$

and construct the j^{th} iterate $(\alpha, \beta, \gamma, v)$ with corresponding $\Xi = (\xi_i)_{i=1}^7$ defined by

$$\Xi(\rho) = \left(\beta \rho^{2/3}, \frac{\alpha}{\rho}, \frac{\alpha}{\rho}, \frac{\gamma}{\rho^{1/3}}, \frac{3\gamma'}{2}\rho^{2/3}, \frac{3\gamma'}{2}\rho^{2/3}, \alpha' \rho^{2/3} \right) \tag{62}$$

by solving

$$\begin{cases} (v - v_0)/h = \dfrac{d}{d\rho} \left(3\rho^{2/3} G_{,i}(\Xi) \Omega^i_{,1}(\Gamma^0) \right) \\ \qquad\qquad - \rho^{-1/3} G_{,i}(\Xi) \left(\Omega^i_{,2}(\Gamma^0) + \Omega^i_{,3}(\Gamma^0) \right) \\ (\xi_i - \xi_i^0)/h = 3\rho^{2/3} \dfrac{d}{d\rho} \left(\Omega^i_{,1}(\Gamma^0) v \right), \ \ i = 1, 5, 6, 7 \\ (\xi_i - \xi_i^0)/h = \rho^{-1/3} \left(\Omega^i_{,2}(\Gamma^0) + \Omega^i_{,3}(\Gamma^0) \right) v, \ \ i = 2, 3, 4 \\ \qquad \xi_2(1) = \xi_3(1) = \lambda, \ \xi_2, \xi_3 \geqslant 0, \ \xi_7 > 0, \ \ \rho \in (0,1), \end{cases} \tag{63}$$

where,

$$\Gamma = (\alpha'(\rho/\alpha)^{2/3}, (\alpha/\rho)^{1/3}, (\alpha/\rho)^{1/3}) \tag{64}$$

and

$$\Gamma^0 = (\alpha_0'(\rho/\alpha_0)^{2/3}, (\alpha_0/\rho)^{1/3}, (\alpha_0/\rho)^{1/3}). \tag{65}$$

As in the continuous case the discrete system (63) is over determined with extra equations

$$\begin{aligned} \left(\alpha_\rho - \alpha_{0\rho}\right)/h &= \frac{d}{d\rho}\left(3{\alpha_0}^{2/3} v\right) \\ \left(\gamma_\rho - \gamma_{0\rho}\right)/h &= \frac{d}{d\rho}\left(2{\alpha_0}^{1/3} v\right) \end{aligned} \tag{66}$$

corresponding to $(63)_2$, $i = 5, 6, 7$. Excluding them from the system above we get

$$\begin{cases} (v - v_0)/h = \dfrac{d}{d\rho}\left(3\rho^{2/3}\, G_{,i}(\Xi)\, \Omega^i_{,1}(\Gamma^0)\right) \\ \qquad\qquad - \rho^{-1/3}\, G_{,i}(\Xi)\left(\Omega^i_{,2}(\Gamma^0) + \Omega^i_{,3}(\Gamma^0)\right) \\ (\beta - \beta_0)/h = \dfrac{d}{d\rho}(3v) \\ (\alpha - \alpha_0)/h = 3{\alpha_0}^{2/3} v \\ (\gamma - \gamma_0)/h = 2{\alpha_0}^{1/3} v \\ \qquad \alpha(1) = \lambda,\ \alpha \geqslant 0, \quad \alpha' > 0, \quad \rho \in (0,1), \end{cases} \tag{67}$$

where we note that equations (66) could be obtained from $(67)_{3,4}$.

Time-step approximations capture a subtle form of dissipation associated with the undelying variational structure and the convexity of the entropy, [7, 8]. Indeed, solutions of (67) satisfy a discrete entropy inequality: To see that, consider a smooth solution (Ξ, v) of (63) associated to smooth data (Ξ^0, v_0) given by (61). Multiplying $(63)_1$ by v we get

$$\begin{aligned} \frac{v(v - v_0)}{h} + G_{,i}(\Xi)\left(3\rho^{2/3}\Omega^i_{,1}(\Gamma^0)\frac{dv}{d\rho} + \rho^{-1/3}\left(\Omega^i_{,2}(\Gamma^0) + \Omega^i_{,3}(\Gamma^0)\right) v\right) \\ = \frac{d}{d\rho}\left(3\rho^{2/3} G_{,i}(\Xi)\Omega^i_{,1}(\Gamma^0) v\right). \end{aligned} \tag{68}$$

Then denoting

$$A_i = 3\rho^{2/3}\Omega^i_{,1}(\Gamma^0)\frac{dv}{d\rho} + \rho^{-1/3}\left(\Omega^i_{,2}(\Gamma^0) + \Omega^i_{,3}(\Gamma^0)\right) v, \quad i = 1, \ldots, 7 \tag{69}$$

we claim

$$A_i = \frac{\xi_i - \xi_i^0}{h}. \tag{70}$$

Indeed, for $i = 2, 3, 4$ we have $\Omega^i_{,1} = 0$ and hence $(63)_3$ and (69) imply (70). For $i = 1, 5, 6, 7$ by the properties (54) of null Lagrangians and $(63)_2$ we get

$$\begin{aligned} A_i = v\left(-3\rho^{2/3}\frac{d}{d\rho}\left(\Omega^i_{,1}(\Gamma^0)\right) + \rho^{-1/3}\left(\Omega^i_{,2}(\Gamma^0) + \Omega^i_{,3}(\Gamma^0)\right)\right) \\ + 3\rho^{2/3}\frac{d}{d\rho}\left(\Omega^i_{,1}(\Gamma^0)v\right) = \left(\xi_i - \xi_i^0\right)/h, \quad i = 1, 5, 6, 7. \end{aligned} \tag{71}$$

Thus (68) and (70) imply

$$\frac{1}{h}\left(v(v - v_0) + G_{,i}(\Xi)\left(\xi_i - \xi_i^0\right)\right) = \frac{d}{d\rho}\left(3\rho^{2/3}G_{,i}(\Xi)\Omega^i_{,1}(\Gamma^0)v\right).$$

Now, we denote $\Theta = (v, \Xi)$ and $\Theta^0 = (v_0, \Xi^0)$. Then $\eta = 1/2v^2 + G(\Xi)$ satisfies

$$\frac{1}{h}D\eta \cdot \left(\Theta - \Theta^0\right) - \frac{d}{d\rho}\left(3\rho^{2/3}G_{,i}(\Xi)\Omega^i_{,1}(\Gamma^0)v\right) = 0.$$

For G convex the following identity holds

$$\frac{\eta(\Theta) - \eta(\Theta^0)}{h} - \frac{d}{d\rho}\left(3\rho^{2/3}G_{,i}(\Xi)\Omega^i_{,1}(\Gamma^0)v\right) \leqslant 0. \tag{72}$$

Remark 1. The second extension (59) has the property that the associated discretized system (67) propagates the relation

$$\xi_7 = \partial_\rho\left(\xi_2\rho\right)\rho^{2/3} \tag{73}$$

provided that $\xi_7^0 = \partial_\rho\left(\xi_2^0\rho\right)\rho^{2/3}$. This property is essential in showing that a minimizer associated with the functional

$$I(\alpha, \beta, \gamma, v) = \int_0^1 \frac{1}{2}(v - v_0)^2 + G(\Xi)\, d\rho$$

solves the discrete system (63).

Remark 2. We have not studied in this article the convergence as the time-step $h \to 0$. For the three-dimensional elasticity equations this process produces measure-valued solutions [8] while for one-dimensional elasticity it gives entropy weak solutions [7]. In the present case we would expect to obtain weak solutions, but the compactness properties of (4) are not at present sufficiently understood to have such a result. Nevertheless, if the iterates u^h, v^h converge strongly, the discrete entropy inequality (72) gives a weak solution dissipating the mechanical energy.

5 Existence of minimizers

Henceforth, we consider stored-energy functions (44) of the form

$$\begin{aligned}\Phi(v_1,v_2,v_3) &= \bar{G}(v_1,v_2,v_{3,2}\,v_3,v_1v_3,v_1v_2,v_1v_2v_3)\\ &= \varphi(v_1)+\varphi(v_2)+\varphi(v_3)+g(v_2v_3)+g(v_1v_3)+g(v_1v_2)+h(v_1v_2v_3).\end{aligned} \tag{74}$$

Then, the function G defined in (46) reads

$$\begin{aligned}G(\Xi;\rho) = \varphi(\xi_1)+\varphi\left(\xi_2^{1/3}\right)+\varphi\left(\xi_3^{1/3}\right)\\ +g\left(\xi_4\rho^{1/3}\right)+g\left(\xi_5\rho^{1/3}\right)+g\left(\xi_6\rho^{1/3}\right)+h(\xi_7\rho^{2/3}).\end{aligned} \tag{75}$$

Now, define $\psi(x)=\varphi(x^{1/3})$. Then, with Ξ defined in (62), the above is expressed by

$$G(\Xi)=\varphi(\beta\rho^{2/3})+2\psi\left(\alpha/\rho\right)+g\left(\gamma/\rho^{2/3}\right)+2g\left(3\gamma'\rho^{1/3}/2\right)+h(\alpha'). \tag{76}$$

We place the following assumptions on the functions φ, ψ, g, h appearing above:

(A1) $\lim_{\delta\to 0+}h(\delta)=\lim_{\delta\to+\infty}h(\delta)/\delta=+\infty$;

(A2) $\varphi,\psi,g\in C^2(\mathbb{R})$ and $h\in C^2(\mathbb{R}_+)$ satisfy

$$\varphi,\psi,g,h,\varphi'',\psi'',g''\geqslant 0 \quad\text{and}\quad h''>0; \tag{77}$$

(A3) For $1<p,q<\infty$ and some constants $c_1,c_2>0$

$$\lim_{x\to\infty}\frac{\varphi(x)}{|x|^{3p}}=\lim_{x\to\infty}\frac{\psi(x)}{|x|^p}=c_1\,,\ \lim_{x\to\infty}\frac{g(x)}{|x|^q}=c_2; \tag{78}$$

(A4) For $1<p,q<\infty$ as in (A3) and $C_1,C_2,C_3>0$

$$\limsup_{x\to\infty}\frac{|\varphi'(x)|}{|x|^{3p-1}}\leq C_1\ \ \limsup_{x\to\infty}\frac{|\psi'(x)|}{|x|^{p-1}}\leq C_2\,,\ \limsup_{x\to\infty}\frac{|g'(x)|}{|x|^q}\leq C_3; \tag{79}$$

In particular, G is *convex*.

We define spaces of functions on the interval $\rho \in (0,1)$

$$
\begin{aligned}
X_1 &= \left\{ f(\rho) \in W^{1,1}(0,1) : \quad f/\rho \in L^p(0,1) \right\}, \\
X_2 &= \left\{ f(\rho) \in L^1_{loc}(0,1) : \quad f\rho^{2/3} \in L^{3p}(0,1) \right\}, \\
X_3 &= \left\{ f(\rho) \in W^{1,1}_{loc}(0,1) : \quad f/\rho^{2/3} \in L^q,\ f'\rho^{1/3} \in L^q(0,1) \right\}, \\
Y &= \left\{ f(\rho) \in W^{1,1}_{loc}(0,1) : \quad f \in L^2,\ f'\rho^{2/3} \in L^{3p}(0,1) \right\},
\end{aligned}
$$

and

$$
X = X_1 \otimes X_2 \otimes X_3 \otimes Y.
$$

We fix a parameter $\lambda > 0$ and for the initial data $(\alpha_0, \beta_0, \gamma_0, v_0) \in X$ we require

$$
\begin{cases}
\alpha_0(1) = \lambda \quad \text{and} \quad \alpha_0 \geqslant 0,\ \alpha_0' > 0 \text{ a.e. } \rho \in (0,1) \\
\text{and} \quad \displaystyle\int_0^1 \frac{1}{2} {v_0}^2 + G(\Xi^0)\, d\rho \ < \ \infty
\end{cases}
\tag{80}
$$

Consider the problem of minimizing the functional

$$
\begin{aligned}
I(\alpha,\beta,\gamma,v) &= \int_0^1 \frac{1}{2}(v - v_0)^2 + G(\Xi)\, d\rho \\
&= \int_0^1 \frac{1}{2}(v - v_0)^2 + \varphi(\beta\rho^{2/3}) + 2\psi\left(\alpha/\rho\right) \\
&\qquad\qquad + g\left(\gamma/\rho^{2/3}\right) + 2g\left(3\gamma'\rho^{1/3}/2\right) + h(\alpha')\, d\rho
\end{aligned}
\tag{81}
$$

over the admissible set

$$
\begin{aligned}
\mathcal{A}_\lambda = \{ (\alpha,\beta,\gamma,v) \in X : &\alpha(0) \geqslant 0,\ \alpha(1) = \lambda,\ \alpha' > 0 \text{ a.e. } \textit{and} \\
&I(\alpha,\beta,\gamma,v) < \infty,\ \frac{(\beta - \beta_0)}{h} = 3v', \\
&\frac{(\alpha - \alpha_0)}{h} = 3{\alpha_0}^{2/3} v,\ \frac{(\gamma - \gamma_0)}{h} = 2{\alpha_0}^{1/3} v \}.
\end{aligned}
\tag{82}
$$

We note that I is well-defined for $(\alpha,\beta,\gamma,v) \in X$ with $\alpha' > 0$ a.e. $\rho \in (0,1)$ though might be equal to ∞.

Lemma 2. *The admissible set $\mathcal{A}_\lambda$ is nonempty.*

Proof. Take $(\alpha,\beta,\gamma,v)=(\alpha_0,\beta_0,\gamma_0,0)\in X$. Then (80) implies $\alpha(0)\geqslant 0$, $\alpha(1)=\lambda$, $\alpha'>0$ a.e. and

$$I(\alpha,\beta,\gamma,v)=\int\limits_0^1 \frac{1}{2}{v_0}^2+G(\Xi^0)\,d\rho \;<\; \infty.$$

Moreover the following holds: $(\beta-\beta_0)/h=0=3v'$, $(\alpha-\alpha_0)/h=0=3{\alpha_0}^{2/3}v$, and $(\gamma-\gamma_0)/h=0=2{\alpha_0}^{1/3}v$. Hence $(\alpha,\beta,\gamma,v)\in\mathcal{A}_\lambda$. □

Lemma 3 (I-bounded sequences). *Let* $\{(\alpha_n,\beta_n,\gamma_n,v_n)\}_{n\in\mathbb{N}}\subset\mathcal{A}_\lambda$ *and*

$$M=\sup_{n\in\mathbb{N}} I(\alpha_n,\beta_n,\gamma_n,v_n)\;<\;\infty. \tag{83}$$

Then $\exists\ (\alpha,\beta,\gamma,v)\in X$ *and a subsequence* $\{(\alpha_\mu,\beta_\mu,\gamma_\mu,v_\mu)\}$ *s.t.*

$$\begin{array}{rlrl}
\alpha_\mu\rightharpoonup\alpha & in\ \ W^{1,1}, & \alpha_\mu/\rho\rightharpoonup\alpha/\rho & in\ \ L^p,\\
\gamma_\mu/\rho^{2/3}\rightharpoonup\gamma/\rho^{2/3} & in\ \ L^q, & \gamma'_\mu\rho^{1/3}\rightharpoonup\gamma'\rho^{1/3} & in\ \ L^q,\\
v_\mu\rightharpoonup v & in\ \ L^2, & v'_\mu\rho^{2/3}\rightharpoonup v'\rho^{2/3} & in\ \ L^{3p},\\
\beta_\mu\rho^{2/3}\rightharpoonup\beta\rho^{2/3} & in\ \ L^{3p}. & &
\end{array} \tag{84}$$

Proof. First, $\alpha_n\geqslant 0$, $\alpha'_n>0$ a.e. and $\alpha_n(1)=\lambda$ imply that $|\alpha_n|\leqslant\lambda$. Second, from (83) it follows $\int_0^1 h(\alpha'_n)\,d\rho<M$, $\forall n$. In this case by de la Vallée Poussin criterion there exist $\alpha\in W^{1,1}$ and a subsequence $\{\alpha_s\}$ of $\{\alpha_n\}$ such that $\alpha_s\rightharpoonup\alpha$ weakly in $W^{1,1}$.

Further, by (A3) there exist constants C_1,C_2 s.t. $\varphi(x)\geqslant C_1|x|^{3p}-C_2$, $\psi(x)\geqslant C_1|x|^p-C_2$ and $g(x)\geqslant C_1|x|^q-C_2$ and hence

$$\begin{aligned}
M\geqslant I(\alpha_s,\beta_s,\gamma_s,v_s)\geqslant&\int\limits_0^1\frac{1}{2}(v_s-v_0)^2\,d\rho\\
&+C_1\int\limits_0^1|\beta_s\rho^{2/3}|^{3p}+2|\alpha_s/\rho|^p+|\gamma_s/\rho^{2/3}|^q+\frac{3}{2}|\gamma'_s\rho^{1/3}|^q\,d\rho-4C_2
\end{aligned} \tag{85}$$

for all indices s. Then $1<p,q<\infty$ and inequality above imply that $\alpha/\rho\in L^p$ and there exist $\beta\in X_2,\gamma\in X_3$, and $v\in L^2$ and a subsequence $\{\alpha_\mu,\beta_\mu,\gamma_\mu,v_\mu\}$ of $\{\alpha_s,\beta_s,\gamma_s,v_s\}$ such that $(84)_{2,3,4,5,6}$ hold.

Finally, as $(\alpha_\mu, \beta_\mu, \gamma_\mu, v_\mu) \in \mathrm{A}_\lambda$ we have $3v'_\mu\rho^{2/3} = (\beta_\mu - \beta_0)\rho^{2/3}/h$. Then by $(84)_3$ we get $3v'_\mu\rho^{2/3} \rightharpoonup (\beta - \beta_0)\rho^{2/3}/h$ in L^{3p}. Then by $(84)_6$ for each $f \in C_0^\infty(0,1)$

$$\begin{aligned}\int_0^1 vf'\,d\rho &= \lim_{\mu\to\infty}\int_0^1 v_\mu f'\,d\rho \\ &= -\lim_{\mu\to\infty}\int_0^1 v'_\mu f\,d\rho = -\int_0^1 \frac{1}{3h}(\beta-\beta_0)f\,d\rho\end{aligned} \tag{86}$$

and hence $v' = (\beta - \beta_0)/3h$. Therefore $v \in Y$ and $v'_\mu\rho^{2/3} \rightharpoonup v'\rho^{2/3}$. □

Theorem 1 (**Lower semi-continuity**). *Let* $\{(\alpha_n, \beta_n, \gamma_n, v_n)\}_{n\in\mathbb{N}} \subset \mathbb{A}_\lambda$, $(\alpha, \beta, \gamma, v) \in X$ *satisfy (83) and (84). Then* $(\alpha, \beta, \gamma, v) \in \mathbb{A}_\lambda$ *and*

$$I(\alpha, \beta, \gamma, v) \leqslant \liminf_{n\to\infty} I(\alpha_n, \beta_n, \gamma_n, v_n) = s < \infty. \tag{87}$$

Proof. By hypothesis $0 \leqslant I_n = I(\alpha_n, \beta_n, \gamma_n, v_n) \leqslant M$, $\forall n \in \mathbb{N}$ and thus $s < \infty$. Recall that $\alpha_n \rightharpoonup \alpha$ weakly in $W^{1,1}$ and along a subsequence uniformly on $C[0,1]$. Since $\alpha_n(1) = \lambda$ we obtain that $\alpha(1) = \lambda$. Moreover,

$$\lim_{n\to\infty}\int_0^1 \alpha'_n\chi_{\{\alpha'<0\}}\,d\rho = \int_0^1 \alpha'\chi_{\{\alpha'<0\}}\,d\rho. \tag{88}$$

Since $\alpha'_n > 0$ a.e. we obtain $\int_0^1 \alpha'\chi_{\{\alpha'<0\}}\,d\rho \geqslant 0$, and thus $m\{\alpha' < 0\} = 0$.

Now, denote $A = \{\rho \in (0,1) : \alpha' = 0\}$ and show that $m(A) = 0$. We will argue by contradiction. Assume that $m(A) = \varepsilon > 0$. Then (84) implies

$$\lim_{n\to\infty}\int_0^1 \alpha'_n\chi_A\,d\rho = \int_0^1 \alpha'\chi_A\,d\rho = 0. \tag{89}$$

Then, as $\alpha'_n > 0$ a.e., $\lim_{n\to\infty}\int_0^1 |\alpha'_n\chi_A|\,d\rho = 0$. Hence $\alpha'_n\chi_A \to 0$ in L^1. We extract a subsequence $\{\alpha'_{n_k}\}$ such that $\alpha'_{n_k}\chi_A \to 0$ a.e. $\rho \in (0,1)$. Now, by Egoroff's theorem there exists a measurable set $B \subset A$ such that $m(B) > \varepsilon/2$ and $\alpha'_{n_k} \to 0$ uniformly on B. Next, observe that

$$\int_0^1 h(\alpha'_{n_k})d\rho \geq \int_B h(\alpha'_{n_k})d\rho \geq m(B)\left(\inf_{\rho\in B} h(\alpha'_{n_k})\right) =: m(B)\,\mu_{n_k}$$

Since $\mu_{n_k} \to \infty$ this contradicts (83). We conclude that $m(A) = 0$.

Next we prove $\alpha \geqslant 0$ a.e. $\rho \in (0,1)$. Again $(84)_1$ implies

$$\lim_{n\to\infty} \int_0^1 \alpha_n \chi_{\{\alpha<0\}}\, d\rho = \int_0^1 \alpha \chi_{\{\alpha<0\}}\, d\rho \;\geq\; 0\,, \tag{90}$$

and thus $m\{\alpha < 0\} = 0$. This concludes that α satisfies all restrictions of membership in $\mathcal{A}_\lambda$.

Next, by (A2) we get

$$\begin{aligned}
\varphi(\beta_n \rho^{2/3}) &\geqslant \varphi(\beta\rho^{2/3}) + \varphi'(\beta\rho^{2/3})(\beta_n - \beta)\rho^{2/3},\\
\psi\left(\alpha_n/\rho\right) &\geqslant \psi\left(\alpha/\rho\right) + \psi'\left(\alpha/\rho\right)(\alpha_n - \alpha)/\rho,\\
g\left(\gamma_n/\rho^{2/3}\right) &\geqslant g\left(\gamma/\rho^{2/3}\right) + g'\left(\gamma/\rho^{2/3}\right)(\gamma_n - \gamma)/\rho^{2/3},\\
g\left(3\gamma_n'\rho^{1/3}/2\right) &\geqslant g\left(3\gamma'\rho^{1/3}/2\right) + g'\left(3\gamma'\rho^{1/3}/2\right)(\gamma_n' - \gamma)3\rho^{1/3}/2
\end{aligned} \tag{91}$$

a.e. $\rho \in (0,1)$. As $(\alpha,\beta,\gamma,v), (\alpha_n,\beta_n,\gamma_n,v_n) \in X$, from (A3) it follows that the right hand side of each of the inequalities in (91) are integrable and

$$\begin{aligned}
&\varphi'(\beta\rho^{2/3}) \in L^{\frac{3p}{3p-1}}, \quad \psi'\left(\alpha/\rho\right) \in L^{\frac{p}{p-1}},\\
&\text{and} \quad g'\left(\gamma/\rho^{2/3}\right), g'\left(3\gamma_n'\rho^{1/3}/2\right) \in L^{\frac{q}{q-1}}.
\end{aligned} \tag{92}$$

Take an arbitrary $0 < \delta < 1$ and set $A_\delta = \{\rho \in (0,1) : \delta \leqslant \alpha' \leqslant 1/\delta\}$. Then by (A2)

$$h(\alpha_n') \geqslant h(\alpha')\chi_{A_\delta} + h'(\alpha')(\alpha_n' - \alpha')\chi_{A_\delta} \;\text{ a.e.}\quad \rho \in (0,1). \tag{93}$$

Moreover, (A1) and (A2) together imply

$$\begin{aligned}
0 \;\leqslant\; & h(\alpha')\chi_{A_\delta} + |h'(\alpha')|\chi_{A_\delta}\\
&\leqslant\; 2\max(h(\delta), h(1/\delta), |h'(\delta)|, |h'(1/\delta)|).
\end{aligned}$$

Hence

$$h(\alpha')\chi_{A_\delta},\; h'(\alpha')\chi_{A_\delta} \in L^\infty. \tag{94}$$

and we conclude that the right hand side of (93) is integrable.

Finally,

$$(v_n - v_0)^2 \geqslant (v - v_0)^2 + 2(v - v_0)(v_n - v) \;\text{ a.e. } \rho \in (0,1), \tag{95}$$

where right hand side is integrable as $v, v_n, v_0 \in L^2$.

Following the discussion above, (91)-(95) imply

$$
\begin{aligned}
I_n \geqslant & \int_0^1 \frac{1}{2}(v-v_0)^2 + \varphi(\beta\rho^{2/3}) + 2\psi\left(\alpha/\rho\right) \\
& \qquad + g\left(\gamma/\rho^{2/3}\right) + 2g\left(3\gamma'\rho^{1/3}/2\right)\, d\rho + \int_0^1 h(\alpha')\chi_{A_\delta}\, d\rho \\
& + \int_0^1 (v-v_0)(v_n-v) + \varphi'(\beta\rho^{2/3})(\beta_n-\beta)\rho^{2/3} \\
& \qquad + 2\psi'\left(\alpha/\rho\right)(\alpha_n-\alpha)/\rho + g'\left(\gamma/\rho^{2/3}\right)(\gamma_n-\gamma)/\rho^{2/3} \\
& \qquad\qquad + g'\left(3\gamma'\rho^{1/3}/2\right)(\gamma_n'-\gamma)3\rho^{1/3} + h'(\alpha')\chi_{A_\delta}(\alpha_n'-\alpha')\, d\rho \\
& = J + J_\delta + J_n.
\end{aligned}
$$

Then, letting $n \to \infty$ in the inequality above, we obtain

$$
\infty \,>\, s \,=\, \liminf_{n\to\infty} I_n \,\geqslant\, J + J_\delta + \liminf_{n\to\infty} J_n.
$$

Now from (84), (92), (94), and $v - v_0 \in L^2$ it follows that $\lim_{n\to\infty} J_n = 0$ and hence

$$
\infty \,>\, s \,=\, \liminf_{n\to\infty} I_n \,\geqslant\, J + \int_0^1 h(\alpha')\chi_{A_\delta}\, d\rho. \tag{96}
$$

Now, as $\alpha' > 0$ a.e. $\rho \in (0,1)$ and $\alpha' \in L^1$, the set $\{\alpha' = 0\} \bigcup \{\alpha' = \infty\}$ is of measure zero and hence

$$
\lim_{\delta\to 0+} h(\alpha')\chi_{A_\delta} = h(\alpha')\chi_{\{0<\alpha'<\infty\}} = h(\alpha') \ \text{ a.e. } \rho \in (0,1). \tag{97}
$$

Finally, let $\delta \to 0+$. Then from (96), (97) and Monotone Convergence Theorem it follows

$$
\infty \,>\, s \,=\, \liminf_{n\to\infty} I_n \,\geqslant\, J + \int_0^1 h(\alpha')\, d\rho \,=\, I(\alpha,\beta,\gamma,v)
$$

and hence (87) holds. Since $(\alpha_n, \beta_n, \gamma_n, v_n) \in \mathcal{A}_\lambda$, and the other constraints are linear, one easily checks that the limiting $(\alpha,\beta,\gamma,v) \in \mathcal{A}_\lambda$. □

Theorem 2 (Existence). *There exists* $(\alpha,\beta,\gamma,v) \in \mathbb{A}_\lambda$ *satisfying*

$$
I(\alpha,\beta,\gamma,v) = \inf_{\mathbb{A}_\lambda} I(\bar\alpha,\bar\beta,\bar\gamma,\bar v). \tag{98}
$$

Proof. As $\mathbb{A}_\lambda$ is nonempty, we can set $s = \inf_{\mathbb{A}_\lambda} I(\bar\alpha, \bar\beta, \bar\gamma, \bar v)$. Then by definition of $\mathbb{A}_\lambda$ we have $I(\bar\alpha, \bar\beta, \bar\gamma, \bar v) < \infty$ for each $(\bar\alpha, \bar\beta, \bar\gamma, \bar v) \in \mathbb{A}_\lambda$. This implies that s is finite.

Next, by definition of s there exists $\{(\alpha_n, \beta_n, \gamma_n, v_n)\}_{n\in\mathbb{N}} \in \mathbb{A}_\lambda$ such that $\lim_{n\to\infty} I_n = s$ with $I_n = I(\alpha_n, \beta_n, \gamma_n, v_n)$. Then, as $\{I_n\}_{n\in\mathbb{N}}$ is bounded, lemma 3 and Theorem 1 imply that $\exists(\alpha, \beta, \gamma, v) \in \mathbb{A}_\lambda$ satisfying $I(\alpha, \beta, \gamma, v) \leqslant \liminf_{n\to\infty} I_n = s$. In this case the definition of s implies $I(\alpha, \beta, \gamma, v) = s$. □

Theorem 3 (Uniqueness). *A minimizer $(\alpha, \beta, \gamma, v) \in \mathbb{A}_\lambda$ of I over $\mathbb{A}_\lambda$ is unique.*

Proof. We will argue by contradiction. Assume $(\alpha, \beta, \gamma, v), (\bar\alpha, \bar\beta, \bar\gamma, \bar v) \in \mathbb{A}_\lambda$ are two distinct minimizers. Then we consider $(\frac{\alpha+\bar\alpha}{2}, \frac{\beta+\bar\beta}{2}, \frac{\gamma+\bar\gamma}{2}, \frac{v+\bar v}{2})$ and notice that it also belongs to $\mathbb{A}_\lambda$.

Define $A = \{\rho \in (0,1) : \alpha' \neq \bar\alpha'\}$. Then $mA > 0$. Indeed, if $\alpha' = \bar\alpha'$ a.e., then $\alpha(1) = \bar\alpha(1) = \lambda$ implies $\alpha = \bar\alpha$. In turn, this implies $v = \bar v'$, $\beta = \bar\beta$ and $\gamma = \bar\gamma$, which contradicts to the assumption that $(\alpha, \beta, \gamma, v)$ and $(\bar\alpha, \bar\beta, \bar\gamma, \bar v)$ are distinct.

Now, as $h'' > 0$, we have

$$\frac{h(\alpha') + h(\bar\alpha')}{2} > h\left(\frac{\alpha' + \bar\alpha'}{2}\right), \quad \rho \in A.$$

and hence, as mA is positive, we must have

$$\int_0^1 \frac{h(\alpha') + h(\bar\alpha')}{2}\, d\rho > \int_0^1 h\left(\frac{\alpha' + \bar\alpha'}{2}\right) d\rho.$$

Let $s = \inf_{\mathbb{A}_\lambda} I(\tilde\alpha, \tilde\beta, \tilde\gamma, \tilde v)$. Then by the inequality above and convexity of φ, ψ and g we obtain

$$s = \frac{I(\alpha, \beta, \gamma, v) + I(\bar\alpha, \bar\beta, \bar\gamma, \bar v)}{2} > I\left(\frac{\alpha+\bar\alpha}{2}, \frac{\beta+\bar\beta}{2}, \frac{\gamma+\bar\gamma}{2}, \frac{v+\bar v}{2}\right). \quad (99)$$

which, since $\left(\frac{\alpha+\bar\alpha}{2}, \frac{\beta+\bar\beta}{2}, \frac{\gamma+\bar\gamma}{2}, \frac{v+\bar v}{2}\right) \in \mathbb{A}_\lambda$, contradicts the definition of s. Hence $(\alpha, \beta, \gamma, v) = (\bar\alpha, \bar\beta, \bar\gamma, \bar v)$. □

6 Euler-Lagrange Equations

In this section we show that the minimizer of I satisfies the system (63) a.e. $\rho \in (0,1)$. For this, in addition to (80) we assume that initial iterate $(\alpha_0, \beta_0, \gamma_0, v_0)$ satisfies the following: for each $\delta \in (0,1)$

$$\alpha_0' \in L^{3p}(\delta,1) \bigcap L^q(\delta,1). \tag{100}$$

Theorem 4 (Weak Form). *Let $(\alpha, \beta, \gamma, v) \in \mathbb{A}_\lambda$ be the minimizer of I over $\mathbb{A}_\lambda$ and initial iterate $(\alpha_0, \beta_0, \gamma_0, v_0)$ satisfy (80) and (100). Let also*

$$G_1(\rho) = G_{,i}(\Xi)\Omega^i_{,1}(\Gamma^0) \tag{101}$$

and

$$G_2(\rho) = G_{,i}(\Xi)\left(\Omega^i_{,2}(\Gamma^0) + \Omega^i_{,3}(\Gamma^0)\right) \tag{102}$$

Then for each $\delta \in (0,1)$

$$\rho^{2/3}G_1(\rho) \in W^{1,1}(\delta,1) \quad \textit{and} \quad \rho^{-1/3}G_2(\rho) \in L^1(\delta,1)$$

and a.e. $\rho \in (0,1)$

$$3\rho^{2/3}G_1(\rho) = \int_1^\rho \left(s^{-1/3}G_2(s) + \frac{v(s) - v_0(s)}{h} \right) ds + \textit{const.} \tag{103}$$

Moreover, for each $\delta \in (0,1)$

$$\alpha' \in L^{3p}(\delta,1) \bigcap L^q(\delta,1). \tag{104}$$

Proof. Fix $k \in \mathbb{N}$ and define $S_k = \{\rho \in [1/k, 1) : \ 1/k < \alpha' < k\}$. Let also take $f \in L^\infty$ with $\int_{S_k} f\, d\rho = 0$. We denote $\chi_k = \chi_{S_k}$, $l_k = \alpha_0(1/k)$ and set

$$\mu(\rho) = \int_0^\rho \chi_k(s) f(s)\, ds. \tag{105}$$

Before proceeding further we make the following remark. Let $t \in \mathbb{R}$ and $F(x) = x^t, x \in \mathbb{R}_+$. Take $\delta \in (0,1)$. Then, as $\alpha_0 \in W^{1,1}$, $\alpha_0 \geqslant 0$ and $\alpha_0' > 0$ a.e. $\rho \in (0,1)$ we must have $0 < \alpha_0(\delta) \leqslant \alpha_0 \leqslant \lambda$ for all $\rho \in (\delta, 1)$. Hence $|F'(\alpha_0)| \leqslant t\left(\alpha_0(\delta) + \lambda\right)^{t-1}$ for all $\rho \in (\delta,1)$. Therefore we conclude that for each $t \in \mathbb{R}$ and $\delta \in (0,1)$

$$\alpha_0{}^t \in W^{1,1}(\delta,1) \quad \text{with} \quad \frac{d}{d\rho}\left(\alpha_0{}^t\right) = t\alpha_0{}^{t-1}\alpha_0'. \tag{106}$$

(i) *Step 1. Definition of the variation.* For $|\varepsilon| < \frac{1}{6k(\|f\|_\infty+1)}$ we define $(\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, v_\varepsilon)$ by

$$\begin{aligned} v_\varepsilon &= v + \varepsilon \frac{\mu}{h{\alpha_0}^{2/3}} \\ \alpha_\varepsilon &= \alpha_0 + h\left(3v_\varepsilon {\alpha_0}^{2/3}\right) = \alpha + 3\varepsilon\mu \\ \beta_\varepsilon &= \beta_0 + h\left(3v_\varepsilon'\right) = \beta + 3\varepsilon\left(\frac{\mu}{{\alpha_0}^{2/3}}\right)' \\ \gamma_\varepsilon &= \gamma_0 + h\left(2v_\varepsilon {\alpha_0}^{1/3}\right) = \gamma + 2\varepsilon\frac{\mu}{{\alpha_0}^{1/3}} \end{aligned} \tag{107}$$

Due to (106), $(\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, v_\varepsilon)$ is well-defined. We next prove:

Lemma 4. *The variation* $(\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, v_\varepsilon) \in \mathbb{A}_\lambda$.

Proof. First, we notice that

$$(\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, v_\varepsilon) = (\alpha, \beta, \gamma, v) \quad \text{if} \quad \rho \in (0, 1/k). \tag{108}$$

Then we check that

$$\alpha_\varepsilon(1) = \alpha(1) + 3\varepsilon \int_{S_k} f(s)\, ds = \lambda.$$

Next, we see that $\alpha_\varepsilon' = \alpha' + 3\varepsilon\chi_k f$ and therefore

$$\begin{aligned} \alpha_\varepsilon' &= \alpha', \quad \rho \notin S_k, \\ \frac{1}{2k} &\leqslant \alpha_\varepsilon' \leqslant k+1, \quad \rho \in S_k. \end{aligned} \tag{109}$$

This implies that $\alpha_\varepsilon > 0$ a.e. $\rho \in (0,1)$ and hence (108) implies $\alpha_\varepsilon \geqslant 0$.

Now we make the following estimates. First, we see that

$$|\mu'| + \left|\frac{\mu}{\rho}\right| + \left|\frac{\mu}{\rho^{2/3}{\alpha_0}^{1/3}}\right| + \left|\frac{\mu}{h{\alpha_0}^{2/3}}\right| \leqslant \|f\|_\infty \left(1 + k + \frac{k^{2/3}}{l_k^{1/3}} + \frac{1}{h l_k^{2/3}}\right).$$

and for $j = 1, 2$

$$\left|\left(\frac{\mu}{{\alpha_0}^{j/3}}\right)'\right| \leqslant \|f\|_\infty \left(l_k^{-j/3} + l_k^{-(1+j/3)}\left|\alpha_0'\right|\right).$$

Thus we conclude that there exists C such that $\forall \rho \in (1/k, 1)$

$$|\alpha_\varepsilon' - \alpha'| + |\alpha_\varepsilon/\rho - \alpha/\rho| + |\gamma_\varepsilon/\rho^{2/3} - \gamma/\rho^{2/3}| + |v - v_\varepsilon| \leqslant \varepsilon C \tag{110}$$

and

$$|\beta_\varepsilon \rho^{2/3} - \beta \rho^{2/3}| + |\gamma'_\varepsilon \rho^{1/3} - \gamma' \rho^{1/3}| \leqslant \varepsilon C \left(1 + |\alpha'_0|\right). \tag{111}$$

As $(\alpha, \beta, \gamma, v) \in X$, last two inequalities imply $(\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, v_\varepsilon) \in X$.

Further, by (A3), (110) and (111) we conclude that there exists C such that for all $\rho \in (1/k, 1)$

$$\begin{aligned}
\psi(\alpha_\varepsilon/\rho) &\leqslant C\left(|\alpha/\rho|^p + 1\right) \\
\varphi(\beta_\varepsilon \rho^{2/3}) &\leqslant C\left(|\beta \rho^{2/3}|^{3p} + |\alpha'_0|^{3p} + 1\right) \\
g(\gamma_\varepsilon/\rho^{2/3}) &\leqslant C\left(|\gamma/\rho^{2/3}|^q + 1\right) \\
g(3\gamma'_\varepsilon \rho^{1/3}/2) &\leqslant C\left(|\gamma' \rho^{1/3}|^q + |\alpha'_0|^q + 1\right).
\end{aligned}$$

By (109) we also have

$$\begin{aligned}
&h(\alpha'_\varepsilon) = h(\alpha'), \quad \rho \notin S_k, \\
&h(\alpha'_\varepsilon) \leqslant \max_{\frac{1}{2k} \leqslant \delta \leqslant k} |h(\delta)| = M_k, \quad \rho \in S_k
\end{aligned} \tag{112}$$

and hence

$$h(\alpha'_\varepsilon) \leqslant h(\alpha') + M_k, \quad \rho \in (0, 1). \tag{113}$$

Now, similarly to (62), set

$$\Xi_\varepsilon = \left(\beta_\varepsilon \rho^{2/3}, \frac{\alpha_\varepsilon}{\rho}, \frac{\alpha_\varepsilon}{\rho}, \frac{\gamma_\varepsilon}{\rho^{1/3}}, \frac{3\gamma'_\varepsilon}{2}\rho^{2/3}, \frac{3\gamma'_\varepsilon}{2}\rho^{2/3}, \alpha'_\varepsilon \rho^{2/3}\right). \tag{114}$$

Then, by the discussion above, it follows that

$$G(\Xi_\varepsilon) + \frac{(v_\varepsilon - v_0)^2}{2} = G(\Xi) + \frac{(v - v_0)^2}{2}, \quad \rho \in (0, 1/k) \tag{115}$$

and there exists C such that for all $\rho \in (1/k, 1)$

$$\begin{aligned}
G(\Xi_\varepsilon) + \frac{(v_\varepsilon - v_0)^2}{2} \leqslant C\Big(1 + |\beta \rho^{2/3}|^{3p} + |\alpha'_0|^{3p} + |\alpha/\rho|^p + |\gamma/\rho^{2/3}|^q \\
+ |\gamma' \rho^{1/3}|^q + |\alpha'_0|^q + |v|^2 + |v_0|^2 + h(\alpha')\Big).
\end{aligned} \tag{116}$$

Then, as $I(\alpha, \beta, \gamma, v) < \infty$, (115) and (116) imply $I(\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, v_\varepsilon) < \infty$ and hence by construction and the above discussion we get $(\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, v_\varepsilon) \in \mathbb{A}_\lambda$. □

Step 2. The next objective is to validate the formal identity

$$\frac{d}{d\varepsilon} I(\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, v_\varepsilon)\Big|_{\varepsilon=0} = \int_0^1 \frac{d}{d\varepsilon}\left(\frac{(v_\varepsilon - v_0)^2}{2} + G(\Xi_\varepsilon)\right)\Big|_{\varepsilon=0} d\rho = 0. \quad (117)$$

This will require several detailed estimations presented below.

At this point, let us make estimates of the following difference quotients on the interval $\rho \in (1/k, 1)$. First, by (110) we get

$$\begin{aligned} \frac{1}{\varepsilon}|(v_\varepsilon - v_0)^2 - (v - v_0)^2| =& \frac{1}{\varepsilon}|v_\varepsilon - v||v_\varepsilon + v - 2v_0| \\ &\leqslant C\left(|v| + |v_0| + 1\right). \end{aligned} \quad (118)$$

Further, by the Mean Value Theorem

$$\frac{1}{\varepsilon}|\varphi(\beta_\varepsilon \rho^{2/3}) - \varphi(\beta\rho^{2/3})| = \frac{1}{\varepsilon}|\varphi'(\tau_\varepsilon)||\beta_\varepsilon\rho^{2/3} - \beta\rho^{2/3}|,$$

where $\min(\beta, \beta_\varepsilon)\rho^{2/3} \leqslant \tau_\varepsilon \leqslant \max(\beta, \beta_\varepsilon)\rho^{2/3}$. Hence from (111) it follows $|\tau_\varepsilon| \leqslant |\beta\rho^{2/3}| + \varepsilon C(|\alpha_0'| + 1)$ and therefore (A4) implies

$$|\varphi'(\tau_\varepsilon)| \leqslant C\left(|\beta\rho^{2/3}|^{3p-1} + |\alpha_0'|^{3p-1} + 1\right).$$

Thus

$$\begin{aligned} &\frac{1}{\varepsilon}|\varphi(\beta_\varepsilon\rho^{2/3}) - \varphi(\beta\rho^{2/3})| \\ &\qquad \leqslant C\left(|\beta\rho^{2/3}|^{3p-1} + |\alpha_0'|^{3p-1} + 1\right)\left(|\alpha_0'| + 1\right). \end{aligned} \quad (119)$$

Similarly,

$$\frac{1}{\varepsilon}|\psi(\alpha_\varepsilon/\rho) - \psi(\alpha/\rho)| = \frac{1}{\varepsilon}|\psi'(\tau_\varepsilon)||\alpha_\varepsilon/\rho - \alpha/\rho|,$$

where $\min(\alpha_\varepsilon, \alpha)/\rho \leqslant \tau_\varepsilon \leqslant \max(\alpha_\varepsilon, \alpha)/\rho$. Hence $|\tau_\varepsilon| \leqslant |\alpha/\rho| + \varepsilon C$ and (A4) implies

$$|\psi'(\tau_\varepsilon)| \leqslant C\left((|\alpha/\rho| + 1)^{p-1} + 1\right)$$

and hence

$$\frac{1}{\varepsilon}|\psi(\alpha_\varepsilon/\rho) - \psi(\alpha/\rho)| \leqslant C\left((|\alpha/\rho| + 1)^{p-1} + 1\right). \quad (120)$$

Next,

$$\frac{1}{\varepsilon}|g(\gamma_\varepsilon/\rho^{2/3}) - g(\gamma/\rho^{2/3})| = \frac{1}{\varepsilon}|g'(\tau_\varepsilon)||\gamma_\varepsilon/\rho^{2/3} - \gamma/\rho^{2/3}|,$$

where $\min(\gamma_\varepsilon,\gamma)/\rho^{2/3} \leqslant |\tau_\varepsilon| \leqslant \max(\gamma_\varepsilon,\gamma)/\rho^{2/3}$ and hence $|\tau_\varepsilon| \leqslant |\gamma/\rho^{2/3}| + \varepsilon C$. Then by (A4)

$$|g'(\tau_\varepsilon)| \leqslant C\left((|\gamma/\rho^{2/3}|+1)^{q-1}+1\right)$$

and hence

$$\frac{1}{\varepsilon}|g(\gamma_\varepsilon/\rho^{2/3}) - g(\gamma/\rho^{2/3})| \leqslant C\left((|\gamma/\rho^{2/3}|+1)^{q-1}+1\right). \tag{121}$$

Further,

$$\frac{1}{\varepsilon}|g(3\gamma'_\varepsilon\rho^{1/3}/2) - g(3\gamma'\rho^{1/3}/2)| = \frac{3}{2\varepsilon}|g'(\tau_\varepsilon)||\gamma'_\varepsilon\rho^{1/3} - \gamma'\rho^{1/3}|,$$

where $\frac{3}{2}\min(\gamma'_\varepsilon,\gamma')\rho^{1/3} \leqslant |\tau_\varepsilon| \leqslant \frac{3}{2}\max(\gamma'_\varepsilon,\gamma')\rho^{1/3}$. Hence we must have $|\tau_\varepsilon| \leqslant \frac{3}{2}\left(|\gamma'\rho^{1/3}| + \varepsilon C(|\alpha'_0|+1)\right)$. Then (A4) implies

$$|g'(\tau_\varepsilon)| \leqslant C\left((|\gamma'\rho^{1/3}| + |\alpha'_0| + 1)^{q-1}+1\right)$$

and hence

$$\begin{aligned}\frac{1}{\varepsilon}|g(3\gamma'_\varepsilon\rho^{1/3}/2) - g(3\gamma'\rho^{1/3}/2)| \\ \leqslant C\left((|\gamma'\rho^{1/3}| + |\alpha'_0| + 1)^{q-1}+1\right)\left(|\alpha'_0|+1\right).\end{aligned} \tag{122}$$

Finally, if $\rho \notin S_k$, then $\frac{1}{\varepsilon}|h(\alpha'_\varepsilon) - h(\alpha')| = 0$ and if $\rho \in S_k$, we get

$$\frac{1}{\varepsilon}|h(\alpha'_\varepsilon) - h(\alpha')| = \frac{1}{\varepsilon}|h'(\tau_\varepsilon)||\alpha'_\varepsilon - \alpha'|,$$

where $\min(\alpha'_\varepsilon,\alpha') \leqslant \tau_\varepsilon \leqslant \max(\alpha'_\varepsilon,\alpha)$. Then by $(109)_2$ we get $\frac{1}{2k} \leqslant \tau_\varepsilon \leqslant k+1$ and hence

$$|h'(\tau_\varepsilon)| \leqslant \max_{\frac{1}{2k}\leqslant\delta\leqslant k+1}|h'(\delta)|.$$

Thus by (110) we conclude that for $\rho \in (1/k, 1)$

$$\frac{1}{\varepsilon}|h(\alpha'_\varepsilon) - h(\alpha')| \leqslant C. \tag{123}$$

Thus (115),(118)-(123) and the assumptions on the initial iterate (80) and (100) imply that

$$\frac{1}{\varepsilon}\left|G(\Xi_\varepsilon) + \frac{(v_\varepsilon - v_0)^2}{2} - G(\Xi) - \frac{(v-v_0)^2}{2}\right|$$

is bounded on $(0,1)$ by an integrable function. Letting $\varepsilon \to 0$, and using the Dominated Convergence theorem, (A2) and the fact that $(\alpha, \beta, \gamma, v)$ is the minimizer, we obtain the identity (117).

Step 3. Conclusion of the computation. The last step is to compute the right hand side of (117). Note first that

$$\begin{aligned}
\frac{d\Xi^1_\varepsilon}{d\varepsilon} &= \frac{d}{d\varepsilon}\beta_\varepsilon \rho^{2/3} = 3\left(\frac{\mu}{{\alpha_0}^{2/3}}\right)' \rho^{2/3} \\
\frac{d\Xi^2_\varepsilon}{d\varepsilon} &= \frac{d\Xi^3_\varepsilon}{d\varepsilon} = \frac{d}{d\varepsilon}\left(\frac{\alpha_\varepsilon}{\rho}\right) = \frac{3\mu}{\rho} \\
\frac{d\Xi^4_\varepsilon}{d\varepsilon} &= \frac{d}{d\varepsilon}\left(\frac{\gamma_\varepsilon}{\rho^{1/3}}\right) = \frac{2\mu}{{\alpha_0}^{1/3}\rho^{1/3}} \\
\frac{d\Xi^5_\varepsilon}{d\varepsilon} &= \frac{d\Xi^6_\varepsilon}{d\varepsilon} = \frac{d}{d\varepsilon}\left(\frac{3}{2}\gamma'_\varepsilon \rho^{2/3}\right) = 3\left(\frac{\mu}{{\alpha_0}^{1/3}}\right)' \rho^{2/3} \\
\frac{d\Xi^7_\varepsilon}{d\varepsilon} &= \frac{d}{d\varepsilon}\left(\alpha'_\varepsilon \rho^{2/3}\right) = 3\mu' \rho^{2/3}
\end{aligned}$$

and

$$\frac{dv_\varepsilon}{d\varepsilon} = \frac{\mu}{h{\alpha_0}^{2/3}}.$$

Then the integrand in (117) is expressed by

$$(v - v_0)\frac{dv_\varepsilon}{d\varepsilon}\bigg|_{\varepsilon=0} + G_{,i}(\Xi)\ \frac{d\Xi^i_\varepsilon}{d\varepsilon}\bigg|_{\varepsilon=0} = a\mu + b\mu',$$

where

$$\begin{aligned}
a(\rho) = &-G_{,1}(\Xi)\frac{2\alpha'_0}{{\alpha_0}^{5/3}}\rho^{2/3} + G_{,2}(\Xi)\frac{3}{\rho} + G_{,3}(\Xi)\frac{3}{\rho} \\
&+ G_{,4}(\Xi)\frac{2}{{\alpha_0}^{1/3}\rho^{1/3}} - \left(G_{,5}(\Xi) + G_{,6}(\Xi)\right)\frac{\alpha'_0}{{\alpha_0}^{4/3}}\rho^{2/3} + \frac{(v - v_0)}{h{\alpha_0}^{2/3}}
\end{aligned} \tag{124}$$

and

$$b(\rho) = \frac{3\rho^{2/3}}{{\alpha_0}^{2/3}}\left(G_{,1}(\Xi) + G_{,5}(\Xi){\alpha_0}^{1/3} + G_{,6}(\Xi){\alpha_0}^{1/3} + G_{,7}(\Xi){\alpha_0}^{2/3}\right). \tag{125}$$

Thus by (117) we have $(a\mu + b\mu') \in L^1$ and

$$\int_{1/k}^{1} (a\mu + b\mu')\, d\rho = 0. \tag{126}$$

Now, we claim $a \in L^1(1/k, 1)$. By (A3) and definition (75) of G it follows that for $\rho \in (1/k, 1)$

$$\left| G_{,1}(\Xi) \frac{\alpha_0'}{{\alpha_0}^{5/3}} \rho^{2/3} \right| \leqslant \left| \frac{\varphi'(\beta\rho^{2/3}){\alpha_0}'}{l_k^{5/3}} \right| \leqslant C \left(|\beta\rho^{2/3}|^{3p-1} + 1 \right) |\alpha_0'|,$$

$$\frac{1}{\rho} \left| G_{,2}(\Xi) + G_{,3}(\Xi) \right| \leqslant 2k \left| \psi'(\alpha/\rho) \right| \leqslant C \left(|\alpha/\rho|^{p-1} + 1 \right),$$

and

$$\left| G_{,4}(\Xi) \frac{1}{{\alpha_0}^{1/3}\rho^{1/3}} \right| \leqslant \frac{k^{1/3}}{l_k^{1/3}} \left| g'(\gamma/\rho^{2/3}) \right| \leqslant C \left(|\gamma/\rho^{2/3}|^{q-1} + 1 \right).$$

As the right hand sides of the inequalities above are integrable on $(1/k, 1)$ we have $a \in L^1(1/k, 1)$ and this, in turn, implies $b\,\mu' \in L^1(1/k, 1)$. Now, we set $z(\rho) = \int_1^\rho a(s)\,ds$ for $\rho \in (1/k, 1)$. Then z is absolutely continuous and so is μz. As $(\mu z)|_{\rho=1/k} = (\mu z)|_{\rho=1} = 0$ we get

$$0 = \int_{1/k}^1 (\mu z)'\,d\rho = \int_{1/k}^1 \left(\mu' \int_1^\rho a(s)\,ds + \mu a \right) d\rho.$$

Then (126) becomes

$$\int_{S_k} \left(-\int_1^\rho a(s)\,ds + b \right) f\,d\rho = 0. \tag{127}$$

By the properties of f we obtain that for some constant c_k

$$b - \int_1^\rho a(s)\,ds = c_k \;\; a.e.\ \rho \in S_k.$$

As k was arbitrary, the above equality is valid for all $k \in \mathbb{N}$. In this case $S_k \subset S_{k+1}$ implies that $c_k = c_{k+1}$. As $\bigcup_k S_k = \{\rho \in (0,1) : 0 < \alpha' < \infty\}$ and $m\left((0,1) \backslash \bigcup_k S_k\right) = 0$, we conclude

$$b - \int_1^\rho a(s)\,ds = \text{const.} \;\; \text{a.e. } \rho \in (0,1). \tag{128}$$

Now, let us fix $\delta \in (0,1)$. By the above argument $a \in L^1(\delta, 1)$ and (128) implies $b \in W^{1,1}(\delta, 1)$ with the weak derivative $b' = a$. Moreover, by (106)

we have ${\alpha_0}^{2/3} \in W^{1,1}(\delta,1)$ and hence $b{\alpha_0}^{2/3} \in W^{1,1}(\delta,1)$. At this point, we compute

$$D\Omega(\Gamma^0) = \begin{bmatrix} 1 & 0 & 0 & 0 & {\alpha_0}^{1/3} & {\alpha_0}^{1/3} & {\alpha_0}^{2/3} \\ 0 & 3\left(\frac{\alpha_0}{\rho}\right)^{2/3} & 0 & {\alpha_0}^{1/3} & 0 & \frac{\alpha_0'\rho}{{\alpha_0}^{2/3}} & \frac{\alpha_0'\rho}{{\alpha_0}^{1/3}} \\ 0 & 0 & 3\left(\frac{\alpha_0}{\rho}\right)^{2/3} & {\alpha_0}^{1/3} & \frac{\alpha_0'\rho}{{\alpha_0}^{2/3}} & 0 & \frac{\alpha_0'\rho}{{\alpha_0}^{1/3}} \end{bmatrix}$$

and notice that, definitions (124) and (125) of a and b respectively imply that

$$b{\alpha_0}^{2/3} = 3\rho^{2/3} G_{,i}(\Xi)\Omega^i_{,1}(\Gamma^0) = 3\rho^{2/3} G_1(\rho)$$

and its weak derivative is expressed as

$$\begin{aligned} \frac{d}{d\rho} b{\alpha_0}^{2/3} &= a{\alpha_0}^{2/3} + b\frac{2\alpha_0'}{3{\alpha_0}^{1/3}} = \\ &= \rho^{-1/3} G_{,i}(\Xi)\left(\Omega^i_{,2}(\Gamma^0) + \Omega^i_{,3}(\Gamma^0)\right) + \frac{v - v_0}{h} \\ &= \rho^{-1/3} G_2(\rho) + \frac{v - v_0}{h}. \end{aligned}$$

We conclude that for $\delta \in (0,1)$

$$\rho^{2/3} G_1(\rho) \in W^{1,1}(\delta,1) \quad \text{and} \quad \rho^{-1/3} G_2(\rho) \in L^1(\delta,1) \tag{129}$$

and for almost every $\rho \in (0,1)$

$$3\rho^{2/3} G_1(\rho) = \int_1^\rho \left(s^{-1/3} G_2(s) + \frac{v(s) - v_0(s)}{h} \right) ds + \text{const.} \tag{130}$$

Finally, to prove (104), we compute

$$\begin{aligned} (\alpha - \alpha_0)' = h\left(3{\alpha_0}^{2/3} v\right)' &= h\left(\frac{2\alpha_0'}{{\alpha_0}^{1/3}} v + 3{\alpha_0}^{2/3} v'\right) \\ &= (\alpha - \alpha_0)\frac{2\alpha_0'}{3\alpha_0} + (\beta - \beta_0){\alpha_0}^{2/3} \end{aligned}$$

and hence

$$\alpha' = \frac{\alpha_0'}{3}\left(1 + \frac{2\alpha}{\alpha_0}\right) + (\beta - \beta_0){\alpha_0}^{2/3}. \tag{131}$$

Similarly,

$$\begin{aligned} (\gamma - \gamma_0)' = h\left(2{\alpha_0}^{1/3} v\right)' &= \frac{2}{3}\left(\frac{\alpha - \alpha_0}{{\alpha_0}^{1/3}}\right)' \\ &= \frac{2}{3{\alpha_0}^{1/3}}\left(\alpha' - \frac{\alpha_0'}{3}\left(2 + \frac{\alpha}{\alpha_0}\right)\right) \end{aligned}$$

and hence

$$\alpha' = \frac{\alpha_0'}{3}\left(2 + \frac{\alpha}{\alpha_0}\right) + \frac{3}{2}(\gamma' - \gamma_0{}')\alpha_0{}^{1/3}. \tag{132}$$

Now, take $\delta \in (0,1)$. Then from (131) and (132) it follows that for all $\rho \in (\delta, 1)$

$$|\alpha'| \leqslant \frac{|\alpha_0'|}{3}\left(1 + \frac{2\lambda}{\alpha_0(\delta)}\right) + |\beta - \beta_0|\lambda^{2/3} \tag{133}$$

and

$$|\alpha'| \leqslant \frac{|\alpha_0'|}{3}\left(2 + \frac{\lambda}{\alpha_0(\delta)}\right) + \frac{3}{2}|\gamma' - \gamma_0{}'|\lambda^{1/3}. \tag{134}$$

Since δ is arbitrary and $\beta - \beta_0 \in L^{3p}(\delta, 1)$, $\gamma' - \gamma_0' \in L^q(\delta, 1)$, the assumption (100) and last two inequalities imply that for each $\delta \in (0,1)$

$$\alpha' \in L^{3p}(\delta, 1) \bigcap L^q(\delta, 1). \tag{135}$$

This completes the proof. □

7 Regularity

First, we claim that for each representative of the minimizer $(\alpha, \beta, \gamma, v) \in \mathbb{A}_\lambda$ in the theorem (4) we can alter α' on a set of measure zero such that functions G_1 and G_2 defined in (101) and (102) satisfy

$$3\rho^{2/3}G_1(\rho) = \int_1^\rho s^{-1/3}G_2(s) + \frac{v(s) - v_0(s)}{h}\, ds + C_0, \quad \text{for all } \rho \in (0,1].$$

Indeed, let us fix representatives $(\alpha, \beta, \gamma, v)$ and $(\alpha_0, \beta_0, \gamma_0, v_0)$. Define

$$z(\rho) = \frac{1}{3\rho^{2/3}}\int_1^\rho s^{-1/3}G_2(s) + \frac{v(s) - v_0(s)}{h}\, ds + C_0 \tag{136}$$

and let $A = \{\rho \in (0,1] : G_1(\rho) \neq z(\rho)\}$. Take any $\rho_0 \in A$ and define

$$y_0 = \left(z(\rho) - \varphi'(\beta\rho^{2/3}) - 2g'(3\gamma'\rho^{1/3})(\alpha_0/\rho)^{1/3}\right)\Big|_{\rho=\rho_0}.$$

Then by (A1) and (A2) it follows that there exists a unique x_0 such that $h'(x_0) = y_0\,(\rho_0/\alpha_0(\rho_0))^{2/3}$. Now, by definition of G_1 we have for all $\rho \in (0,1]$

$$G_1(\rho) = \varphi'(\beta\rho^{2/3}) + 2g'(3\gamma'\rho^{1/3}/2)(\alpha_0/\rho)^{1/3} + h'(\alpha')(\alpha_0/\rho)^{2/3}. \tag{137}$$

Thus assigning $\alpha'(\rho_0) = x_0$ we get $G_1(\rho_0) = z(\rho_0)$. In the end, after altering this way α' on the set A, we get that $G_1(\rho) = z(\rho)$ for all $\rho \in (0, 1]$. Moreover by (103) we have $mA = 0$ and this finishes the proof.

Second, let us require smoother initial iterate than before. In particular

Lemma 5 (Regularity). *Let $(\alpha, \beta, \gamma, v) \in \mathbb{A}_\lambda$ be the minimizer of I over $\mathbb{A}_\lambda$. Assume that initial iterate $(\alpha_0, \beta_0, \gamma_0, v_0)$ satisfies (80) and*

$$\alpha_0, \gamma_0 \in C^1(0, 1] \quad \textit{and} \quad \beta_0 \in C(0, 1]. \tag{138}$$

Then

$$\alpha, \gamma, v \in C^1(0, 1] \quad \textit{and} \quad \beta \in C(0, 1]. \tag{139}$$

Proof. Clearly, we can pick a representative $(\alpha, \beta, \gamma, v)$ such that $\alpha, \gamma, v \in C(0, 1]$. Then, proceeding as in (131) and (132), constrains $\frac{\alpha-\alpha_0}{h} = 3{\alpha_0}^{2/3}v$, $\frac{\gamma-\gamma_0}{h} = 2{\alpha_0}^{1/3}v$ and $\frac{\beta-\beta_0}{h} = 3v'$ imply a.e. $\rho \in (0, 1)$

$$\beta\rho^{2/3} = \alpha'(\rho/\alpha_0)^{2/3} + f_1(\rho) \tag{140}$$

and

$$\frac{3}{2}\gamma'\rho^{1/3} = \alpha'(\rho/\alpha_0)^{1/3} + f_2(\rho), \tag{141}$$

where

$$f_1(\rho) = \beta_0\rho^{2/3} - \frac{\alpha_0'\rho^{2/3}}{3{\alpha_0}^{2/3}}\left(1 + \frac{2\alpha}{\alpha_0}\right)$$
$$f_2(\rho) = \frac{3}{2}\gamma_0'\rho^{1/3} - \frac{\rho^{1/3}}{\alpha_0^{1/3}}\left(2 + \frac{\alpha}{\alpha_0}\right).$$

We note that (138) implies that f_1 and f_2 are continuous on $(0, 1]$ functions.

First, we alter β and γ' so that equality in (140) and (141) holds for all $\rho \in (0, 1)$. Hence by (137) we have for all $\rho \in (0, 1]$

$$\begin{aligned} G_1(\rho) = {} & \varphi'\left(\alpha'(\rho/\alpha_0)^{2/3} + f_2(\rho)\right) \\ & + 2g'\left(\alpha'(\rho/\alpha_0)^{1/3} + f_1(\rho)\right)(\alpha_0/\rho)^{1/3} \\ & + h'(\alpha')(\alpha_0/\rho)^{2/3}. \end{aligned} \tag{142}$$

and this suggests to define $f : \mathbb{R}_+ \times (0, 1] \to \mathbb{R}$ by

$$\begin{aligned} f(x, \rho) = {} & \varphi'\left(x(\rho/\alpha_0)^{2/3} + f_2(\rho)\right) \\ & + 2g'\left(x(\rho/\alpha_0)^{1/3} + f_1(\rho)\right)(\alpha_0/\rho)^{1/3} \\ & + h'(x)(\alpha_0/\rho)^{2/3}. \end{aligned} \tag{143}$$

Now, define $A = \{\rho \in (0,1] : G_1(\rho) \neq z(\rho)\}$. Clearly, $mA = 0$ and note that from (142) it follows

$$G_1(\rho) = f(\alpha', \rho) = z(\rho), \quad \rho \notin A. \tag{144}$$

Take $\rho_0 \in A$. Then, as $\rho_0 > 0$ and $\alpha_0(\rho_0) > 0$, properties (A1)-(A3) imply that $f_x(x, \rho_0) > 0$ for all $x \in \mathbb{R}_+$; moreover, $\lim_{x\to 0+} f(x, \rho_0) = -\infty$ and $\lim_{x\to+\infty} f(x, \rho_0) = +\infty$. Hence there exists unique $x_0 \in \mathbb{R}_+$ such that $f(x_0, \rho_0) = z(\rho_0)$.

At this point we are ready to assign new values for α', β and γ'. Define

$$\alpha'(\rho_0) = x_0, \quad \beta(\rho_0) = \frac{x_0}{\alpha_0(\rho_0)^{2/3}} + \frac{f_1(\rho_0)}{\rho_0^{2/3}}$$

and

$$\gamma'(\rho_0) = \frac{2}{3}\left(\frac{x_0}{\alpha_0(\rho_0)^{1/3}} + \frac{f_2(\rho_0)}{\rho_0^{1/3}}\right).$$

This implies that (140) and (141) hold at $\rho = \rho_0$ and hence by (137)

$$G_1(\rho_0) = f(x_0, \rho_0) = f(\alpha'(\rho_0), \rho_0) = z(\rho_0). \tag{145}$$

As $\rho_0 \in A$ was arbitrary (144) and (145) imply

$$G_1(\rho) = f(\alpha', \rho) = z(\rho), \quad \rho \in (0,1]. \tag{146}$$

Hence G_1 is continuous on $(0,1]$ and therefore $\alpha' > 0$ for all $\rho \in (0,1]$.

Now, let us assume $\rho_k \to \rho_0$ and $\alpha'(\rho_k) \to l \in [0,\infty]$ with $\rho_k, \rho_0 \in (0,1]$, $k \in \mathbb{N}$. First, we claim that $l \in (0,\infty)$. Indeed, assume that $l = 0$ or $l = +\infty$. Then by continuity of α_0 we have $\alpha_0(\rho_k) \to \alpha_0(\rho_0) > 0$ and hence properties (A1)-(A3), together with continuity of f_1 and f_2, imply $\lim_{k\to\infty} f(\alpha'(\rho_k), \rho_k) = \mp\infty$ respectively. Thus by continuity of G_1 and (146) we have

$$G_1(\rho_0) = \lim_{k\to\infty} G_1(\rho_k) = \lim_{k\to\infty} f(\alpha'(\rho_k), \rho_k) = \mp\infty \tag{147}$$

which is a contradiction. Therefore we assume $l \in (0,\infty)$. Then, as functions f_1 and f_2 are continuous on $(0,1]$, we must have $\lim_{k\to\infty} f(\alpha'(\rho_k), \rho_k) = f(l, \rho_0)$ and therefore by (146) we get

$$\begin{aligned} f(\alpha'(\rho_0), \rho_0) = G_1(\rho_0) &= \lim_{k\to\infty} G_1(\rho_k) \\ &= \lim_{k\to\infty} f(\alpha'(\rho_k), \rho_k) = f(l, \rho_0). \end{aligned} \tag{148}$$

Then by strict monotonicity of $f(\cdot, \rho_0)$ we get $\alpha_0(\rho_0) = l$. Thus we conclude that α' must be continuous on $(0, 1]$.

Finally, from the discussion above it follows that equalities (140) and (141) hold for all $\rho \in (0, 1]$. In this case by the continuity of f_1, f_2 and α' we get $\beta, \gamma' \in C(0, 1]$. Moreover, as $\frac{\alpha - \alpha_0}{h} = 3{\alpha_0}^{2/3} v$ for all $\rho \in (0, 1]$, we obtain $v \in C^1(0, 1]$. This finishes the proof. □

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