Transport coefficients in the 2-dimensional Boltzmann equation

A. V. Bobylev[1] and R. Esposito[2]

Abstract

We show that a rarefied system of hard disks in a plane, described in the Boltzmann-Grad limit by the 2-dimensional Boltzmann equation, has bounded transport coefficients. This is proved by showing opportune compactness properties of the gain part of the linearized Boltzmann operator

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1 Introduction

The interest in low dimensional systems has recently increased because of their relevance in the study of nano-structures. One of the questions arising in this context is whether of not the transport coefficients are well defined. It is a common point of view, supported by experiments and numerical results, (see for example [4, 7, 8]) that heat conductivity may become infinite. Theoretical arguments, based on the Green-Kubo formula and the slow decay of time self-correlation of momentum and energy fluxes, seem also to support such a conclusion. Examples where the (un)boundedness is proved are provided by stochastic lattice particle systems (see [1] and references quoted therein). The deterministic continuous systems are out of reach of the present mathematical techniques, but for the case of rarefied gases. In this case, the Boltzmann equation has been proved to be a good approximation of the time behavior of the system in the Boltzmann-Grad limit at least for short times [5]. It is not obvious that the such a limiting procedure does not destroy the long time tails in the correlations. In this short note we will discuss this question and show that the transport coefficients are indeed bounded in the Boltzmann-Grad limit.

In low dimension the validity of the Boltzmann equation for hard spheres in a thin layer has been proved in [3], by considering a three dimensional system with one side much smaller than the others. It has been proved that, as long as the smaller side is still large compared to the interaction length, the limit equation is the Boltzmann equation with two dimensional positions and three dimensional velocities. In this case the transport coefficients, already computed by Maxwell [6], are bounded. However, the argument in [3] does not apply when the small side is of size comparable

[1]Department of Mathematics, Karlstad University, SE-651 88 Karlstad, Sweden. e-mail: alexander.bobylev@kau.se
[2]Dipartimento di Matematica, Università di L'Aquila, Coppito, 67100 AQ, Italy. e-mail: esposito@univaq.it

with the interaction length. In this case, it is shown in [3] that, in the linear case (Lorentz gas), the limiting behavior is not described by the linear Boltzmann equation. One can then consider a strictly two dimensional system, with both two dimensional positions and velocities, namely a system of hard disks moving on a plane. The Lanford proof [5] works also in this case and the limiting equation is the Boltzman equations with positions and velocities in $\mathbb{R}^2$. Therefore one can try to compute the transport coefficients for this system in the Boltzmann-Grad limit by means of the Boltzmann equation.

The coefficient of transport are obtained by solving an integral equation of the form

$$Lf = g \tag{1.1}$$

where L is the Boltzmann collision operator linearized around a Maxwellian and g are suitably chosen functions of the velocities. The linearized Boltzmann operator has the form

$$L = K - \nu,$$

where ν is a multiplication operator and K is an integral operator. Since there is no small parameter in the equation, its solution is based on the Fredholm theory [9]. This is well known when the velocities are in $\mathbb{R}^3$ (see for example [2]), while in dimension two, this requires some analysis. In next section we present the explicit expression of the kernel of the operator K for hard disks, while in Section 3 we show its compactnes in a suitable space. Then from the Fredhol alternative, we conclude that the equation (1.1) can be solved in the suitable space and as a consequence the transport coefficients are bounded.

2 Estimates on the kernels

The Boltzman equation for the probability density $f(x, v, t)$ on the phase space $\mathbb{R}^d \times \mathbb{R}^d$ is written as

$$\partial_t f + v \cdot \nabla f = Q(f, f), \tag{2.1}$$

In the following, the dimension of the position and velocity space d, will be eventually fixed to 2. The Boltzmann collision operator Q is defined as:

$$Q(f, g)(v) = \int_{\mathbb{R}^d \times S^{d-1}} dw d\omega |u \cdot \omega| \{f(v')g(w') - f(v)g(w)\}, \quad d \geq 2, \tag{2.2}$$

with

$$u = v - w, \quad v' = v - \omega(\omega \cdot u), \quad w' = w + \omega(\omega \cdot u).$$

Moreover, $\omega \in S^{d-1}$, the surface of the unit sphere in $\mathbb{R}^d$: $S^{d-1} = \{\omega \in \mathbb{R}^d \,|\, |\omega| = 1\}$. We will use the following equivalent expression for Q:

$$Q(f, g)(v) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} dw dk |k|^{3-d} \delta(k \cdot u + \frac{|k|^2}{2}) \left\{ f(v + \frac{k}{2}) g(w - \frac{k}{2}) - f(v)g(w) \right\}, \tag{2.3}$$

In fact, since for any $\psi(k)$

$$\int_{\mathbb{R}^d} dk|k|^{3-d}\delta(k\cdot u+\frac{|k|^2}{2})\psi(k)=\int_0^\infty drr^{d-1}r^{3-d}\int_{S^{d-1}} d\omega\delta(\frac{r}{2}[2\omega\cdot u+r])\psi(r\omega)=$$
$$\int_0^\infty drr^{d-1}r^{3-d}\frac{2}{r}\int_{S^{d-1}} d\omega\delta(2\omega\cdot u+r)\psi(r\omega)=$$
$$2\int_{S^{d-1}} d\omega\eta(-\omega\cdot u)2|\omega\cdot u|\psi(-2(\omega\cdot u)\omega)=$$
$$2\int_{S^{d-1}} d\omega|\omega\cdot u|\psi(-2(\omega\cdot u)\omega),$$

with $\eta(x)$ the Heaviside function, given by

$$\eta(x)=\begin{cases}1 & x\geq 0,\\ 0 & x<0,\end{cases} \tag{2.4}$$

the equivalence is immediately checked.

Let

$$M(v)=(2\pi)^{-d/2}\mathrm{e}^{-\frac{|v|^2}{2}} \tag{2.5}$$

be the standard Maxwellian such that $Q(M,M)=0$ and set

$$f=M(1+\varphi). \tag{2.6}$$

The linearized Boltzmann equation is obtained by plugging (2.6) into (2.1) and neglecting quadratic terms. It reads:

$$\partial_t\varphi+v\cdot\nabla\varphi=\hat{L}\varphi, \tag{2.7}$$

with $\hat{L}$ the linearized Boltzmann operator defined as

$$\hat{L}\varphi=M^{-1}[Q(Mf,M)+Q(M,Mf)]. \tag{2.8}$$

The operator $\hat{L}$ has the following structure:

$$\hat{L}\varphi(v)=\hat{L}_1\varphi(v)+\hat{L}_2\varphi(v)-\hat{L}_3\varphi(v)-\nu(|v|)\varphi(v) \tag{2.9}$$

where

$$\nu(|v|)=M*p(v), \tag{2.10}$$

$$\hat{L}_1\varphi(v)=\frac{1}{2}\int_{\mathbb{R}^d} dwM(w)\int_{\mathbb{R}^d} dk|k|^{3-d}\delta(k\cdot u+\frac{|k|^2}{2})\varphi(v+\frac{k}{2}) \tag{2.11}$$

$$\hat{L}_2\varphi(v)=\frac{1}{2}\int_{\mathbb{R}^d} dwM(w)\int_{\mathbb{R}^d} dk|k|^{3-d}\delta(k\cdot u+\frac{|k|^2}{2})\varphi(w-\frac{k}{2}) \tag{2.12}$$

$$\hat{L}_3\varphi(v)=(M\phi)*p(v), \tag{2.13}$$

the $*$-product denotes the convolution product

$$(f*g)(v)=\int_{\mathbb{R}^d} dwf(w)g(v-w) \tag{2.14}$$

and

$$p(v) = \int_{\mathbb{R}^d} dk|k|^{3-d}\delta(k\cdot u - \frac{|k|^2}{2}) = |v|A_d,$$

$$A_d = \int_{S^{d-1}} d\omega|\omega\cdot\omega_1| = \frac{2}{d-1}|S^{d-2}|, \tag{2.15}$$

ω_1 being any fixed unit vector and $|S^n|$ the surface of unit sphere in $\mathbb{R}^{n+1}$. Indeed,

$$A_d = 2\int_{\mathbb{R}^d} dx\delta(x^2-1)|x\cdot\omega_1| = 2\int_{-1}^{1} dzz\int_{\mathbb{R}^{d-1}} dy\delta(|z|^2+|y|^2-1) =$$
$$2\int_0^1 dz|z|(\sqrt{1-z^2})^{d-3}|S^{d-2}| = \frac{2}{d-1}|S^{d-2}|,$$

with $|S^0| = 2$, $|S^1| = 2\pi$, ...

Hence

$$\nu(|v|) = A_d\int_{\mathbb{R}^d} dwM(w)|v-w|, \quad d = 2,3\dots \tag{2.16}$$

Similarly, one can show that

$$\hat{L}_3\varphi(v) = A_d\int_{\mathbb{R}^d} dwM(w)\varphi(w)|v-w| := \int_{\mathbb{R}^d} dw\hat{K}_3(v,w)\varphi(w). \tag{2.17}$$

The operators $\hat{L}_1$ and $\hat{L}_2$ are integral operators of the form

$$\hat{L}_i\varphi(v) = \int_{\mathbb{R}^d} dw\hat{K}_i(v,w)\varphi(w), \quad i = 1,2. \tag{2.18}$$

The explicit expression of $\hat{K}_i$ will be given in the proof of Proposition 2.1 below.

The transport coefficients are computed in the Chapmann-Enskog expansion by solving the integral equation

$$\hat{L}\varphi = g, \tag{2.19}$$

and one has to choose, with $\alpha,\beta = 1,\dots,d$

$$g = v_\alpha v_\beta - \frac{1}{d}|v|^2\delta_{\alpha,\beta} \tag{2.20}$$

to compute the viscosity coefficient and

$$g = \frac{1}{2}v_\alpha(|v|^2-(d+2)) \tag{2.21}$$

to compute the heat conductivity.

It is convenient to symmetrize the operator $\hat{L}$ by setting $\tilde{\psi}(v) = \sqrt{M}\varphi(v)$, $\tilde{h}(v) = \sqrt{M}g(v)$, so that (2.19) becomes

$$L\tilde{\psi} = \tilde{h}, \tag{2.22}$$

with

$$L\tilde{\psi}(v) = \sqrt{M}\hat{L}\left[\frac{1}{\sqrt{M}}\tilde{\psi}\right](v) = -\nu(|v|)\tilde{\psi}(v) + \int_{\mathbb{R}^d} dw\Big[\tilde{K}_1(v,w)+\tilde{K}_2(v,w)-\tilde{K}_3(v,w)\Big]\tilde{\psi}(w), \tag{2.23}$$

$$\tilde{K}_i(v,w) = \hat{K}_i(v,w)\exp\left[\frac{w|^2 - |v|^2}{2}\right]. \tag{2.24}$$

The operator so defined is symmetric in $L_2(\mathbb{R}^d)$.

Since the operator L has a non trivial null space, spanned by the function $\sqrt{M}$, $v_1\sqrt{M}$, $v_2\sqrt{M}$, $\dots$, $v_d\sqrt{M}$ and $|v|^2\sqrt{M}$, in order that the equation (2.22) has solutions, the right hand side h has to be orthogonal to the null space, a condition which is fulfilled by the functions $\tilde{h} = \sqrt{M}g$ with g given by (2.20) and (2.21).

Therefore it only remains to establish the sufficient conditions to apply the Fredhom alternative theorems [9].

We set $\psi = \sqrt{\nu}\tilde{\psi}$, $\tilde{h} = \sqrt{\nu}h$ and (2.22) becomes

$$h = -\psi + (L_1 + L_2 - L_3)\psi \tag{2.25}$$

with

$$L_i\psi(v) = \int_{\mathbb{R}^d} K_i(v,w)\psi(w), \quad i = 1,2,3, \tag{2.26}$$

and

$$K_i(v,w) = \tilde{K}_i(v,w)(\sqrt{\nu(v)\nu(w)})^{-1}. \tag{2.27}$$

We are interested in the case $d = 2$. The explicit expressions of the kernels $\tilde{K}_i(v,w)$ and $K_i(v,w)$ in $d = 2$ are given in the following:

Proposition 2.1. *For $d = 2$ the kernels $\tilde{K}_i(v,w)$ read*

$$\tilde{K}_1(v,w) = a\exp\left[-\frac{A(v,w)}{8}\right], \tag{2.28}$$

$$\tilde{K}_2(v,w) = ab\frac{\exp\left[-\frac{A(v,w)}{8}\right]}{|v-w|}B\left(\frac{|v||w|}{|v-w|}\sqrt{1-\left(\frac{v\cdot v}{|v||w|}\right)^2}\right), \tag{2.29}$$

$$\tilde{K}_3(v,w) = 4\sqrt{M(v)}\sqrt{M(w)}|v-w|, \tag{2.30}$$

with

$$M(v) = (2\pi)^{-1}\exp\left[-\frac{|v|^2}{2}\right], \quad A(v,w) = |u|^2 + \frac{(|v|^2-|w|^2)^2}{|u|^2}, \tag{2.31}$$

$$u = v - w, \quad B(x) = \mathrm{e}^{-\dfrac{x^2}{2}} + x\int_0^x dy\mathrm{e}^{-\dfrac{y^2}{2}}, \tag{2.32}$$

and $a = b = \sqrt{\dfrac{2}{\pi}}$. Moreover

$$K_1(v,w) = a\frac{\exp\left[-\frac{A(v,w)}{8}\right]}{\sqrt{\nu(|v|)\nu(|w|)}}, \tag{2.33}$$

$$K_2(v,w) = ab\frac{\exp\left[-\frac{A(v,w)}{8}\right]}{|v-w|\sqrt{\nu(|v|)\nu(|w|)}}B\left(\frac{|v||w|}{|v-w|}\sqrt{1-\left(\frac{v\cdot v}{|v||w|}\right)^2}\right) \tag{2.34}$$

$$K_3(v,w) = 4\frac{\sqrt{M(v)}\sqrt{M(w)}}{\sqrt{\nu(|v|)\nu(|w|)}}|v-w|. \tag{2.35}$$

Proof. The proof of Proposition 2.1 is given in Appendix □

The following estimates are based on the above explicit formulas.

Lemma 2.2. *For B and ν defined above, we have*

1. *There is a constant $c > 0$ such that*

$$B\left(\frac{|v||w|}{|v-w|}\sqrt{1-\left(\frac{v\cdot v}{|v||w|}\right)^2}\right) \le c[1+\min\{|v|,|w|\}; \tag{2.36}$$

2. *There is a constant $\nu_0 > 0$ such that*

$$\nu(|v|) \ge \nu_0(1+|v|). \tag{2.37}$$

Proof. We have

$$B(x) = \mathrm{e}^{-\dfrac{x^2}{2}} + x\int_0^x dy \mathrm{e}^{-\dfrac{y^2}{2}} \le 1 + \sqrt{\frac{\pi}{2}}x, \tag{2.38}$$

$$\begin{aligned}
&\frac{|v|^2|w|^2}{|v-w|^2}\left\{1-\left(\frac{v\cdot v}{|v||w|}\right)^2\right\} = \frac{|v|^2|w|^2}{|v-w|^2}\left\{1-\cos^2\theta\right\}\\
&= \frac{|v|^2|w|^2}{|v|^2+|w|^2}\left\{\frac{1-\cos^2\theta}{1-2(\cos\theta)\frac{|v||w|}{|v|^2+|w|^2}}\right\} \le \frac{|v|^2|w|^2}{|v|^2+|w|^2}\left\{\frac{1-\cos^2\theta}{1-\cos\theta}\right\}\\
&= \frac{|v|^2|w|^2}{|v|^2+|w|^2}\left\{1+\cos\theta\right\} \le 2\frac{|v|^2|w|^2}{|v|^2+|w|^2}.
\end{aligned}$$

Moreover, if $|v| \ge |w|$

$$\frac{|v|^2|w|^2}{|v|^2+|w|^2} \le \frac{|w|^2}{1+\frac{|w|^2}{|v|^2}} \le |w|^2, \tag{2.39}$$

Hence the statement 1) holds with $c = \sqrt{\pi}$.

From (2.16)

$$\nu(|v|) = \frac{2}{\pi}\int dw|v-w|\mathrm{e}^{-w^2/2} = \frac{2}{\pi}\int dw|w|\mathrm{e}^{-(v-w)^2/2}, \tag{2.40}$$

$$\nabla_v\nu(|v|) = \frac{\nu'(|v|)}{|v|}v = -\frac{2}{\pi}\int dw|w|(v-w)\mathrm{e}^{-(v-w)^2/2}. \tag{2.41}$$

Taking the inner product with $v/|v|$, we get

$$\nu'(|v|) = -\frac{2}{\pi|v|}\int dw|w|(v-w)\cdot v\mathrm{e}^{-(v-w)^2/2} = -\frac{2}{\pi|v|}\int dw|v-w|(w\cdot v)\mathrm{e}^{-w^2/2} \tag{2.42}$$

Therefore

$$\frac{1}{2}\pi\nu'(|v|) = -\int_0^\infty dy y^2\mathrm{e}^{-y^2/2}g(y,|v|), \tag{2.43}$$

with

$$g(y,|v|) = 2\int_0^\pi d\theta \cos\theta \sqrt{y^2 + |v|^2 - 2y|v|\cos\theta}. \tag{2.44}$$

It results in $g(y,|v|) < 0$, because, for $\theta \in [0, \frac{\pi}{2}]$

$$\sqrt{y^2 + |v|^2 - 2y|v|\cos\theta} < \sqrt{y^2 + |v|^2 + 2y|v|\cos\theta} \tag{2.45}$$

for $|y| > \epsilon$, $|v| > \epsilon$, $|\theta - \pi/2| > \epsilon$. Hence $\nu'(|v|) > 0$. Therefore, since obviously $\nu(0) > 0$, there is $\nu_0 > 0$ such that $\nu(|v|) > \nu_0(1 + |v|)$. □

As a consequence of Proposition 2.1 and Lemma 2.2, we have the following estimate for the kernels K_1 and K_2:

Proposition 2.3. *There are constant $\mathcal{C}_1$ and $\mathcal{C}_2$ such that*

$$K_1(v,w) \le \mathcal{C}_1 \frac{\exp\left[-\dfrac{A(v,w)}{8}\right]}{\sqrt{(1+|v|)(1+|w|)}}, \tag{2.46}$$

$$K_2(v,w) \le \mathcal{C}_2 \frac{\exp\left[-\dfrac{A(v,w)}{8}\right]}{|v-w|}, \tag{2.47}$$

where

$$A(v,w) = |u|^2 + \frac{(|v|^2 - |w|^2)^2}{|u|^2}. \tag{2.48}$$

Proof. It is enough to note that

$$\frac{1 + \min\{|v|, |w|\}}{\sqrt{(1+|v|)(1+|w|)}} \le 1. \tag{2.49}$$

□

Next proposition contains the essential estimates to prove the compactness of the operators L_1 and L_2:

Proposition 2.4. *Let $g_i(|v|) = \int_{\mathbb{R}^2} dw K_i(v,w)$, $i = 1,2$. Then*

$$g_1(|v|) \le h_1(|v|) := 8\sqrt{\pi}\mathcal{C}_1 \int_{-1}^{1} \frac{dz}{\sqrt{1-z^2}} \exp\left[-\frac{|v|^2 z^2}{4}\right], \tag{2.50}$$

$$g_2(|v|) \le h_2(|v|) := \frac{8\pi}{\sqrt{1+|v|}}\mathcal{C}_2 \tag{2.51}$$

The functions h_i, $i = 1,2$, are bounded, monotone and go to 0 as $|v| \to \infty$.

Proof. By (2.47) and (2.48)

$$g_2(|v|) \le \mathcal{C}_2 \frac{1}{\sqrt{1+|v|}} \int_{\mathbb{R}^2} dw \exp\left[-\frac{(v-w)^2}{8}\right] = \frac{8\pi}{\sqrt{1+|v|}}\mathcal{C}_2. \tag{2.52}$$

By (2.46) and (2.48), using $u = v - w$,

$$\frac{1}{\mathcal{C}_1} g_1(|v|) = \int_{\mathbb{R}^2} \frac{du}{|u|} \exp\left[-\frac{A(v, v-u)}{8}\right] = \int_{\mathbb{R}^2} \frac{du}{|u|} \exp\left[-\frac{|u|^2 + \frac{(|u|^2 - 2u\cdot v)^2}{|u|^2})}{8} = \right]$$

$$2\int_0^\infty d\rho \exp\left[-\frac{\rho^2}{8}\right] \int_0^{2\pi} d\theta \exp\left[-\frac{(\rho - 2|v|\cos\theta)^2}{8}\right]$$

$$= 2\int_0^{2\pi} d\theta \int_{-|v|\cos\theta}^{\infty} d\zeta \exp\left[-\frac{\zeta^2 + |v|^2\cos^2\theta}{4}\right] \le 4\sqrt{\pi}\int_0^{2\pi} d\theta \exp\left[-\frac{|v|^2\cos^2\theta}{4}\right] = \frac{h_1(|v|)}{\mathcal{C}_1}.$$

Obviously $h_1(|v|) < h_1(0) = 4\pi\mathcal{C}_1$. $h_1'(|v|) \le 0$. Finally, $h_(|v|) \to 0$ as $|v| \to \infty$ because

$$h_1(|v|) \le 16\sqrt{\pi}\mathcal{C}_1 \left\{ \sqrt{\pi} \int_0^{\frac{1}{\sqrt{2}}} dz \exp\left[-\frac{|v|^2 z^2}{4}\right] + \frac{\pi}{2}\exp\left[-\frac{|v|^2}{8}\right] \right\}. \tag{2.53}$$

□

3 Compactness

To prove the compactness of the operators L_1 and L_2, we introduce, for any $R > 0$, $\chi_R(x) = \eta(R-x)$ and $\chi_R^c = 1 - \chi_R$. Clearly

$$1 = (\chi_R(|v|) + \chi_R^c(|v|))(\chi_R(|w|) + \chi_R^c(|w|)) = \chi_R(|v|)\chi_R(|w|) + \chi_R^c(|v|) + \chi_R(|v|)\chi_R^c(|w|). \tag{3.1}$$

We introduce the following kernels for $i = 1, 2$:

$$\mathcal{Q}_i^R(v, w) = \chi_R(|v|)\chi_R(|w|)K_i(v, w), \tag{3.2}$$

$$\mathcal{S}_i^R = \chi_R^c(|v|)K_i(v, w), \quad \mathcal{P}_i^R(v, w) = \chi_R(|v|)\chi_R^c(|w|)K_i(v, w), \tag{3.3}$$

and denote by $\mathbb{Q}_i^R$, $\mathbb{S}_i^R$ and $\mathbb{P}_i^R$ the corresponding operators on $L_2(\mathbb{R}^2)$.

Previous estimates show that the operators $\mathbb{S}_i^R$ and $\mathbb{P}_i^R$ go to 0 as $R \to \infty$ in the uniform operator norm. To prove this we need to show the estimates

$$\|\mathbb{S}_i^R f\|^2\| \le o(R)\|f\|^2, \quad \|\mathbb{P}_i^R f\|^2 \le o(R)\|f\|^2, \tag{3.4}$$

with $\|\cdot\|$ the $L_2(\mathbb{R}^2)$-norm and $o(R) \to 0$ as $R \to \infty$. Let $\phi(v, w) \ge 0$ be one of the these kernels. We have:

$$\begin{aligned}
& \int_{\mathbb{R}^2} dv \left[\int_{\mathbb{R}^2} dw \phi(v, w) f(w)\right]^2 = \\
& \int_{\mathbb{R}^2} dv \left[\int_{\mathbb{R}^2} dw \sqrt{\phi(v, w)}\sqrt{\phi(v, w)} f(w)\right]^2 \\
\le\ & \int_{\mathbb{R}^2} dv \left[\int_{\mathbb{R}^2} dw' \phi(v, w')\right] \left[\int_{\mathbb{R}^2} dw \phi(v, w) f^2(w)\right] \\
\le\ & \int_{\mathbb{R}^2} dw f^2(w) \int_{\mathbb{R}^2} dv \phi(v, w) \left[\int_{\mathbb{R}^2} dw' \phi(v, w')\right].
\end{aligned}$$

Therefore, setting

$$c = \sup_{w\in\mathbb{R}^2} \int_{\mathbb{R}^2} dv\phi(v,w)\left[\int_{\mathbb{R}^2} dw'\phi(v,w')\right], \tag{3.5}$$

for any $f \in L_2(\mathbb{R}^2)$ we have

$$\int_{\mathbb{R}^2} dv \left|\int_{\mathbb{R}^2} dw\phi(v,w)f(w)\right|^2 \le c\|f\|^2. \tag{3.6}$$

Let us now consider the case $\phi(v,w) = \mathcal{S}_i^R$. Remember the definition of g_i in Proposition 2.3

$$\int_{\mathbb{R}^2} dv\mathcal{S}_i^R(v,w)\left[\int_{\mathbb{R}^2} dw'\mathcal{S}_i^R(v,w')\right] = \int_{|v|>R} dvK_i(v,w)g_i(v) \le h_i(R)h_i(0) \to 0 \tag{3.7}$$

If $\phi(v,w) = \mathcal{P}_i^R$, then

$$\begin{aligned}&\int_{\mathbb{R}^2} dv\mathcal{P}_i^R(v,w)\left[\int_{\mathbb{R}^2} dw'\mathcal{P}_i^R(v,w')\right] = \int_{|v|\le R} dvK(v,w)\chi_R^c(w)\left[\int_{\mathbb{R}^2} dw'K_i(v,w')\chi_R^c(w')\right]\\ &\le h_i(0)h_i(R)\chi_R^c(w) \to 0.\end{aligned}$$

The compactness of the operators $\mathbb{Q}_i^R$, for fixed R is standard (see [9], pages 229-231). Indeed, from Proposition 2.3 the kernel $\mathbb{Q}_1$ is a continuous function on a bounded set, hence uniformly continuous and this is sufficient for the compactness. As for the kernel $\mathbb{Q}_2^R$ we have to take care of the singularity $v = w$. To do this we proceed in the same spirit as before. Since now R is fixed, we drop the apex R and introduce a (smooth version) of the characteristic function $\chi_\varepsilon(v) = \eta(\varepsilon - |v|)$ and $\chi_\varepsilon^c(|v|) = \eta(|v| - \varepsilon)$ and write

$$\mathbb{Q}_2(v,w) = \chi_\varepsilon^c(|v-w|)\mathbb{Q}_2(v,w) + \chi_\varepsilon(|v-w|)\mathbb{Q}_2(v,w). \tag{3.8}$$

Clearly for the part $\chi_\varepsilon^c(|v-w|)\mathbb{Q}_2(v,w)$ we can use the same argument used for $\mathbb{Q}_1$. Now we show that $\chi_\varepsilon(|v-w|)\mathbb{Q}_2(v,w)$ goes to 0 as $\varepsilon \to 0$ in the uniform operator norm.

Indeed, by the same argument used to obtain (3.6), we have

$$\begin{aligned}&\int_{|v|\le R} dv \left|\int_{|w|\le R} dw\chi_\varepsilon(|v-w|)\mathbb{Q}_2(v,w)f(w)\right|^2 \le\\ &\mathcal{C}_2^2\left(\int_{\{z\in\mathbb{R}^2, |z|<\varepsilon\}} \frac{dz}{|z|}\right)^2 \|f\|^2 \le 4\pi^2\mathcal{C}_2^2\varepsilon^2\|f\|^2 \to 0,\end{aligned}$$

as $\varepsilon \to 0$.

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A Appendix

Proof of Proposition 2.1. For the moment we keep d unspecified.
By using (2.11),

$$\hat{L}_1\varphi(v) = \int_{\mathbb{R}^d} dk|k|^{3-d} I(k,v)\varphi(v+\frac{k}{2}). \tag{A.1}$$

We compute $I(k,v)$:

$$I(k,v) = \frac{1}{2}\int_{\mathbb{R}^d} dwM(w)\delta(k\cdot[v-w+\frac{k}{2}]) = J(v+\frac{k}{2},k), \tag{A.2}$$

$$J(v,k) = \frac{1}{2}\int_{\mathbb{R}^d} dwM(w)\delta(k\cdot(v-w)). \tag{A.3}$$

We set $w = \hat{k}w_\parallel + w_\perp$, $\hat{k} = k|k|^{-1}$, $w_\parallel = w\cdot\hat{k}$. Hence

$$J(v,k) = \frac{(2\pi)^{-d/2}}{2|k|}\int_{-\infty}^{\infty} dw_\parallel \delta(v_\parallel - w_\parallel)\int_{\mathbb{R}^{d-1}} dw_\perp \exp\left[-\frac{w_\parallel^2 + |w_\perp|^2}{2}\right] = \frac{1}{2\sqrt{2\pi}|k|}\exp\left[-\frac{v_\parallel^2}{2}\right]$$

Therefore

$$L_1\varphi(v) = \frac{1}{2\sqrt{2\pi}}\int_{\mathbb{R}^d} dk|k|^{2-d}\exp\left[-\frac{((v+\frac{k}{2})\cdot\hat{k})^2}{2}\right]\varphi(v+\frac{k}{2}) \tag{A.4}$$

So we get

$$\hat{K}_1(v,w) = \sqrt{\frac{2}{\pi}}|v-w|^{2-d}\exp\left[-\frac{1}{2}\left(w\cdot\frac{v-w}{|v-w|}\right)^2\right]. \tag{A.5}$$

By using (2.12) we have (with the notation $u = v-w$, $\hat{u} = u/|u|$, $w_\parallel = (w\cdot\hat{u})\hat{u}$ and $w_\perp = w - w_\parallel$)

$$\begin{aligned}
\hat{K}_2(v,w) = \frac{1}{2}\int_{\mathbb{R}^d} dkM(w+\frac{k}{2})\delta(k\cdot(v-w))|k|^{3-d} = \\
\frac{1}{2(2\pi)^{d/2}}\int_{\mathbb{R}^d} dk|k|^{3-d}\delta(k\cdot u)\exp\left[-\frac{(w+\frac{k}{2})^2}{2}\right] = \\
\frac{2}{(2\pi)^{d/2}}\frac{1}{|u|}\int_{\mathbb{R}^d} dk|k|^{3-d}\delta(k\cdot\hat{u})\exp\left[-\frac{|w|^2+|k|^2-2k\cdot w}{2}\right] \\
= \frac{1}{(2\pi)^{d/2}}\frac{2}{|u|}\int_{-\infty}^{\infty} dk_\parallel\delta(k_\parallel)\int_{\mathbb{R}^{d-1}} dk_\perp|k_\perp|^{3-d}\exp\left[-\frac{(|w_\parallel|^2+|w_\perp-k_\perp|^2)}{2}\right].
\end{aligned} \tag{A.6}$$

Hence

$$\hat{K}_2(v,w) = \sqrt{\frac{2}{\pi}}\frac{1}{|v-w|}\exp\left[-\frac{1}{2}\left(w\cdot\frac{v-w}{|v-w|}\right)^2\right]H_d(w_\perp), \tag{A.7}$$

with H_d defined on $\mathbb{R}^{d-1}$ as

$$H_d(x) = \frac{1}{(2\pi)^{\frac{d-1}{2}}}\int_{\mathbb{R}^{d-1}} dy\exp\left[-\frac{|y|^2}{2}\right]|x-y|^{3-d}. \tag{A.8}$$

Clearly H_d depends only on $|x|$. The calculation of H_d gives the following result: For $d\geq 3$,

$$H_d(x) = \frac{|S^{d-3}|}{(2\pi)^{(d-1)/2}}\int_0^{\infty} drr^{d-2}\exp\left[-\frac{r^2}{2}\right]\int_{-1}^{1} dz(1-z^2)^{(d-4)/2}(|x|^2+r^2-2r|x|z)^{-(d-3)/2}; \tag{A.9}$$

For $d = 2$

$$H_2(x) = \sqrt{\frac{2}{\pi}} \left[\exp\left[-\frac{x^2}{2}\right] + x \int_0^x dy \exp\left[-\frac{y^2}{2}\right]\right]. \tag{A.10}$$

Then we need to compute $|w_\perp|$. We set $\cos\theta = \dfrac{v \cdot w}{|v||w|}$. Then

$$|w_\perp|^2 = \frac{|v|^2|w|^2}{|v-w|^2}(1 - \cos^2\theta). \tag{A.11}$$

In fact

$$\begin{aligned}
|w_\perp|^2 = |w|^2 + |\hat{u}\cdot w|^2 - 2(\hat{u}\cdot w)^2 = |w|^2 - |\hat{u}\cdot w|^2 = \\
\frac{1}{|u|^2}\left[|w|^2|v-w|^2 - (v\cdot w)^2 - |w|^4 + 2|w|^2(v\cdot w)\right] = \\
\frac{1}{|u|^2}\left[|w|^2(|v|^2+|w|^2) - 2|w|^2(v\cdot w) - (v\cdot w)^2 - |w|^4 + 2|w|^2(v\cdot w)\right] = \\
\frac{1}{|v-w|^2}[|w|^2|v|^2 - (v\cdot w)^2 = |w|^2|v|^2(1-\cos^2\theta)].
\end{aligned} \tag{A.12}$$

Therefore

$$|w_\perp| = \frac{|v||w|}{|v-w|}\sqrt{1 - \left(\frac{v\cdot w}{|v||w|}\right)}. \tag{A.13}$$

We conclude that

$$\hat{K}_2(v,w) = G_d(v,w) H_d\left(\frac{|v||w|}{|v-w|}\sqrt{1-\left(\frac{v\cdot w}{|v||w|}\right)}\right), \tag{A.14}$$

and

$$G_d(v,w) = \sqrt{\frac{2}{\pi}}\frac{1}{|v-w|}\exp\left[-\frac{1}{2}\left(w\cdot\frac{v-w}{|v-w|}\right)^2\right]. \tag{A.15}$$

To compute K_i we have to multiply $\hat{K}_i$ by $\sqrt{M(v)}/\sqrt{M(w)}$.

We have the identity

$$2\left(w\cdot\frac{v-w}{|v-w|}\right)^2 - |w|^2 + |v|^2 = \frac{1}{2}|u|^2 + \frac{1}{2}\frac{(|v|^2-|w|^2)^2}{|u|^2}. \tag{A.16}$$

Proof. Proof of the identity

$$\begin{aligned}
&2\left(w\cdot\frac{v-w}{|v-w|}\right)^2 - |w|^2 + |v|^2 = \frac{1}{|v-w|^2}\left\{(|v|^2+|w|^2-2v\cdot w)(|v|^2-|w|^2) + 2(|w|^2 - v\cdot w)^2\right\} \\
&= \frac{1}{|v-w|^2}\left\{|v|^4 - |w|^4 - 2(v\cdot w)(|v|^2-|w|^2) + 2|w|^4 + 2(v\cdot w)^2 - 4(v\cdot w)|w|^2\right\} \\
&= \frac{1}{|v-w|^2}\left\{-2(|v|^2+|w|^2)(v\cdot w) + |w|^4 + |v|^4 + 2(v\cdot w)^2\right\} \\
&= \frac{1}{|v-w|^2}\left\{(v\cdot(v-w))^2 + (w\cdot(v-w))^2\right\} = (v\cdot\hat{u})^2 + (w\cdot\hat{u})^2.
\end{aligned}$$

Moreover

$$|u|^2[2(v\cdot\hat u)^2+2(w\cdot\hat u)^2-|u|^2]=2(|v|^2-v\cdot w)^2+2(|w|^2-v\cdot w)^2-(|v|^2+|w|^2-2v\cdot w)^2=$$
$$-4(v\cdot w)(|v|^2+|w|^2)+4(v\cdot w)(|v|^2+|w|^2)+4(v\cdot w)^2-4(v\cdot w)^2$$
$$+2(|v|^4+|w|^4)-(|v|^2+|w|^2)^2=|v|^4+|w|^4-2|v|^2|w|^2=(|v|^2-|w|^2)^2.$$

□

Therefore, the exponent in the exponentials is exactly $A(v,w)$ and this concludes the proof of Proposition 2.1. □

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