INVISCID LARGE DEVIATION PRINCIPLE AND THE 2D NAVIER STOKES EQUATIONS WITH A FREE BOUNDARY CONDITION

HAKIMA BESSAIH AND ANNIE MILLET

Abstract. Using a weak convergence approach, we prove a LPD for the solution of 2D stochastic Navier Stokes equations when the viscosity converges to 0 and the noise intensity is multiplied by the square root of the viscosity. Unlike previous results on LDP for hydrodynamical models, the weak convergence is proven by tightness properties of the distribution of the solution in appropriate functional spaces.

Keywords: Models of turbulence, viscosity coefficient and Navier-Stokes equations, Euler equation, stochastic PDEs, Radonifying operators, large deviations.

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1. Introduction

We are dealing with flows described by stochastic Navier Stokes equations in dimension 2 of the following form

$$\frac{\partial u_\nu(t)}{\partial t} - \nu\Delta u_\nu(t) + (u_\nu(t)\cdot\nabla)u_\nu(t) = -\nabla p + \sigma(t,u_\nu(t))\frac{\partial W(t,.)}{\partial t}, \tag{1.1}$$

in an open bounded domain D of $\mathbb{R}^2$ with a smooth boundary ∂D which satisfies the locally Lipschitz condition see [1]. Here, u_ν is the velocity of the fluid, $\nu > 0$ is its viscosity, p denotes the pressure, W is a Gaussian random field white in time, subject to the restrictions imposed below on the space correlation and σ is an operator acting on the solution. The velocity field u_ν is subject to the incompressibility condition

$$\nabla\cdot u_\nu(t,x) = 0, \quad t\in[0,T], \quad x\in D, \tag{1.2}$$

and to the boundary condition for every $t\in[0,T]$

$$u_\nu(t,.)\cdot n = 0 \;\text{ and }\; \text{curl } u_\nu(t,.) = 0 \quad \text{on } \partial D, \tag{1.3}$$

n being the unit outward normal to ∂D. The initial condition is the function ζ defined on D:

$$u_\nu(0,x) = \zeta(x) \quad x\in D. \tag{1.4}$$

In the deterministic case, for fixed $\nu > 0$ and an initial condition $\zeta\in H$, well posedness of the above system is well-known in $C([0,T];H)\cap L^2(0,T;V)$, where H and V denote respectively subspaces of $L^2(D)$ and $W^{1,2}(D)$, see [37]. The well posedness can also be found in [2] and [4] as an intermediate result for the well posedness of the deterministic and stochastic Euler equation with additive noise. Martingale solutions of the above system (with a multiplicative noise) have been studied in [5] and [9].

The aim of the present paper is to prove a Large Deviation Principle (LDP) for the stochastic 2D Navier Stokes equations (1.1) when the viscosity coefficient $\nu\to 0$ and the noise W is multiplied by the square root of the viscosity, in order to be in the Freidlin-Wentzell setting. Several recent papers have studied a LDP for the distribution of the solution to a hydro-dynamical stochastic evolution equation. We refer to [35] for the 2D Navier-Stokes equations, [21] for the Boussinesq model, [16] for more general hydro-dynamic models, [34] for tamed 3D Navier Stokes equations. All the above papers consider an equation with a (fixed) positive viscosity coefficient and study the exponential concentration to a deterministic model when the noise intensity is multiplied by a coefficient $\sqrt{\epsilon}$ which converges to 0. They deal with a multiplicative noise and use the weak convergence approach of LDP, based on the Laplace principle, developed by P. Dupuis and R. Ellis in [22].

In a recent paper [6], we consider a simpler equation driven by a multiplicative noise and a vanishing viscosity coefficient, that is a shell model of turbulence. Under certain conditions on the initial condition and the operator acting on the noise, this equation is well posed in $C([0,T];V)$ where V is a Hilbert space similar to $H^{1,2}$. However, we prove a LDP for a weaker topology, that of $L^2(0,T;\mathcal{H})$, where $\mathcal{H}$ is a subspace of V similar to $H^{\frac{1}{2},2}$, with the same scaling between the "viscosity" and the square of the noise intensity. The technique used is again the weak convergence approach, To our knowledge, this is the first paper that proves a LDP when the coefficient in front of the noise term depends on the viscosity. Let us point out that the study of the inviscid limit is an important step towards understanding turbulent fluid flows in general. Let us also refer to the paper of M. Mariani [29], where a "nonviscous" scalar equation is considered in the context of conservation laws. However the techniques used in that paper are completely different from the ones used here and in [6].

In this paper, we will generalize our result to the Navier Stokes equations (1.1) in a bounded domain of $\mathbb{R}^2$; this is technically more involved. Note that the rate function in this framework is described by the solution to a deterministic "controlled" Euler equation

$$\frac{\partial u(t)}{\partial t} + (u(t)\cdot\nabla)u(t) = -\nabla p + \sigma(t,u(t))h(t), \tag{1.5}$$

with the same incompressibility and boundary conditions, where h denotes an element of the RKHS of the noise. This equation is a deterministic counterpart of the stochastic Euler equation studied by [4] in the case of additive noise, [10] and [5] when the noise is multiplicative. There is an extensive literature for the deterministic Euler equation in dimension 2. We refer to [2], [24], [36] and the references therein and [3] for a survey paper.

The technique we use is again the weak convergence approach and thus will require to prove well posedness and apriori bounds in the space $C([0,T];L^2)\cap L^\infty(0,T;H^{1,q})$ for some $q>2$ of the solution to (1.5) for a more regular initial condition. Thus, we are able to prove the LDP in a "non-optimal" space for the Navier Stokes equations with positive viscosity, namely $L^2(0,T;\mathcal{H})$, where $\mathcal{H}$ is a Hilbert interpolation space between H and V similar to that in [6]. This is due to the fact that the Euler equation is known to be quite irregular and to require strong conditions in order to have uniqueness of the solution; this forces us to work with non Hilbert Sobolev spaces $H^{1,q}$ for $q\in(2,\infty)$ and to require that the diffusion coefficient σ is both trace class and Radonifying. Indeed, some apriori estimates have to be obtained in general Sobolev spaces uniformly in the "small" viscosity $\nu>0$ for the stochastic Navier Stokes equations (1.1) when the noise W is multiplied by $\sqrt{\nu}$ and shifted by a random element of its RKHS.

Let us finally point out that, even if the problem solved here is similar to that in [6], the final step is quite different. Indeed, unlike all the references on LDP for hydrodynamical models, the weak convergence is proven using a tightness argument and not by means of the convergence in L^2 of a properly localized sequence. Unlike in [6], no time increment has to be studied and no Hölder regularity of the map $\sigma(.,u)$ has to be imposed. Let us also point out that we replace the classical homogenous Dirichlet boundary conditions by the free boundary one. Working with the classical homogeneous Dirichlet boundary condition would lead to some boundary layers problems that are beyond the scope of this paper. For more details and explanations about the free boundary condition (1.3) we refer to [37]. Let us also mention that all our results can proved for the Stochastic Navier-Stokes equations with periodic conditions.

The paper is organized as follows: In section 2 we describe the model and establish apriori estimates in the Hilbert spaces L^2 and $H^{1,2}$ similar to known ones, except for two things: the boundary conditions are slightly different, and we have to prove estimates uniform in a "small" viscosity ν. Section 3 deals with the inviscid problem in $C([0,T];L^2)\cap L^\infty(0,T;H^{1,q})$. Section 4 proves apriori bouunds of the NS equations in $H^{1,q}$ and section 5 establishes the large deviations results. Finally, some technical results on Radonifying and Nemytski's operators are gathered in an Appendix.

2. Description of the model

For every $\nu>0$, we consider the equations of Navier-Stokes type

$$\begin{cases} \frac{\partial u}{\partial t}+(u\cdot\nabla)u+\nabla p=\nu\Delta u+\sigma(t,u)\frac{\partial W}{\partial t}, & \text{in } [0,T]\times D,\\ \nabla\cdot u=0, & \text{in } [0,T]\times D,\\ \text{curl } u=0 \text{ and } u\cdot n=0 & \text{on } [0,T]\times\partial D,\\ u|_{t=0}=\zeta, & \text{in } D,\end{cases} \tag{2.1}$$

where $\operatorname{curl} u=D_1u_2-D_2u_1$.

2.1. **Notations and hypothesis.**

Let $\mathcal{V}$ be the space of infinitely differentiable vector fields u on D with compact support strictly contained in D, satisfying $\nabla\cdot u=0$ in D and $u.n=0$ on ∂D. Let us denote by H the closure of $\mathcal{V}$ in $L^2(D;\mathbb{R}^2)$, that is

$$H=\Big\{u\in\big[L^2(D)\big]^2;\ \nabla\cdot u=0 \text{ in } D,\ u\cdot n=0 \text{ on } \partial D\Big\}.$$

The space H is a separable Hilbert space with the inner product inherited from $\big[L^2(D)\big]^2$, denoted in the sequel by $(.,.)$ and $|.|_H$ denotes the corresponding norm. For every integer $k\geq 0$ and any $q\in[1,\infty)$, let $W^{k,q}$ denote the completion of the set of $\mathcal{C}_0^\infty(\bar D,\mathbb{R})$ or of $\mathcal{C}_0^\infty(\bar D,\mathbb{R}^2)$ with respect to the norm

$$\|u\|_{W^{k,q}}=\Big(\sum_{|\alpha|\leq k}\int_D|\partial^\alpha u(x)|^q\,dx\Big)^{\frac1q}.$$

To ease notations, let $\|.\|_q:=\|.\|_{W^{0,q}}$. For $k<0$ and $q^*=q/(q-1)$, let $W^{-k,q^*}=(W^{k,q})^*$. Here, for a multi-index $\alpha=(\alpha_1,\alpha_2)$ we set $\partial^\alpha u(x)=\dfrac{\partial^{|\alpha|}u(x)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}$. For a non-negative real number $s=k+r$, where k is an integer and $0<r<1$, and for any $q\in[1,\infty)$, let $W^{s,q}$ denote the completion of the set of $\mathcal{C}_0^\infty(\bar D,\mathbb{R})$ or of $\mathcal{C}_0^\infty(\bar D,\mathbb{R}^2)$ with respect to the norm defined by:

$$\|u\|^q_{W^{s,q}}=\|u\|^q_{W^{k,q}}+\sum_{|\alpha|=k}\int_D\int_D\frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|^q}{|x-y|^{2+2r}}dxdy.$$

Given $0<\alpha<1$, let $W^{\alpha,p}(0,T;H)$ be the Sobolev space of all $u\in L^p(0,T;H)$ such that

$$\int_0^T\int_0^T\frac{|u(t)-u(s)|^p}{|t-s|^{1+\alpha p}}dtds<\infty.$$

Let us set $H^{k,q}=W^{k,q}\cap H$ for any $k\in[0,+\infty)$ and $q\in[2,\infty)$; the set $H^{k,q}$ is endowed with the norm inherited from that of $W^{k,q}$ and denoted by $\|.\|_{H^{k,p}}$. Let $V=H^{1,2}$, that is the subspace of H defined as follows:

$$V=\big\{u\in W^{1,2}(D;\mathbb{R}^2):\ \nabla\cdot u=0 \text{ in } D,\ u\cdot n=0 \text{ on } \partial D\big\}.$$

The space V is a separable Hilbert space with the inner product $((.,))$ inherited from that of $W^{1,2}(D;\mathbb{R}^2)$ and $\|.\| := \|.\|_V$ denotes the corresponding norm, defined for $u,v\in V$ by:

$$\|u\|^2 = ((u,u))\,,\ \text{and}\ ((u,v)) = \int_D \big[u(x).v(x) + \nabla u(x).\nabla v(x)\big]\,dx.$$

Identifying H with its dual space H', and H' with the corresponding natural subspace of the dual space V', we have the Gelfand triple $V\subset H\subset V'$ with continuous dense injections. We denote the dual pairing between $u\in V$ and $v\in V'$ by $\langle u,v\rangle$. When $v\in H$, we have $(u,v)=\langle u,v\rangle$. Let $b(\cdot,\cdot,\cdot): V\times V\times V\longrightarrow \mathbb{R}$ be the continuous trilinear form defined as

$$b(u,v,z) = \int_D (u(x)\cdot\nabla v(x))\cdot z(x)\,dx.$$

It is well known that there exists a continuous bilinear operator $B(\cdot,\cdot): V\times V\longrightarrow V'$ such that $\langle B(u,v),z\rangle = b(u,v,z)$, for all $z\in V$. By the incompressibility condition, for $u,v,z\in V$ we have (see e.g. [27] or [2])

$$\langle B(u,v),z\rangle = -\langle B(u,z),v\rangle \quad \text{and} \quad \langle B(u,v),v\rangle = 0. \tag{2.2}$$

Furthermore, there exits a constant C such that for any $u\in V$,

$$\|B(u,u)\|_{V'} \le C|u|_H\,\|u\|. \tag{2.3}$$

Let $a(\cdot,\cdot): V\times V\longrightarrow\mathbb{R}$ be the bilinear continuous form defined in [2] as

$$a(u,v) = \int_D \nabla u\cdot\nabla v - \int_{\partial D} k(r)u(r)\cdot v(r)dr,$$

where $k(r)$ is the curvature of the boundary ∂D at the point r, and we have the following estimates (see [26] for details):

$$\int_{\partial D} k(r)u(r)\cdot v(r)dr \le C\|u\|\|v\|, \tag{2.4}$$

and for any $\epsilon>0$ there exists a positive constant $C(\epsilon)$ such that:

$$\int_{\partial D} k(r)|u(r)|_H^2 dr \le \epsilon\|u\|^2 + C(\epsilon)|u|_H^2. \tag{2.5}$$

Moreover, we set $D(A) = \big\{u\in H^{2,2}\ :\ \text{curl } u = 0 \text{ on } \partial D\big\}$, and define the linear operator $A: D(A)\longrightarrow H$ as

$$Au = -\Delta u,\ \text{i.e.,}\ a(u,v) := (Au,v).$$

On the other hand, for all $u\in D(A)$ we have

$$(B(u,u),Au) = 0. \tag{2.6}$$

For $\beta>0$ we will denote the β-power of the operator A by A^β and its domain by $D(A^\beta)$. Here $D(A^{-\beta})$ denotes the dual of $D(A^\beta)$. Note that for $k<3/4$, we have $H^{k,2} = D(A^{k/2})$; the proof can be found in [10] Theorem 3.1. Set $\mathcal{H} = H^{1/2,2}$ and note that $\mathcal{H} = D(A^{1/4})$ and $V = D(A^{1/2})$. The continuous embedding $V\subset\mathcal{H}\subset H$ holds. Moreover, $\mathcal{H}$ is an interpolation space, that is there exists a constant $a_0>0$ such that

$$\|u\|_{\mathcal{H}}^2 \le a_0|u|_H\|u\|,\ \text{for all } u\in V. \tag{2.7}$$

Since $\mathcal{H}\subset L^4(D)$ and $\langle B(u,v),w\rangle = -\langle B(u,w),v\rangle$, we deduce

$$|\langle B(u,v),w\rangle| \le C\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}\|w\|, \tag{2.8}$$

and B can be extended as a bilinear operator from $\mathcal{H}\times\mathcal{H}\longrightarrow V'$.
In place of equations (2.1) we will consider the abstract stochastic evolution equation:

$$du(t) + \nu Au(t)dt + B(u(t),u(t))dt = \sigma(t,u(t))dW(t) \tag{2.9}$$

on the time interval $[0,T]$ with the initial condition $u(0)=\zeta$, where B satisfies conditions (2.2), (2.3), (2.6) and (2.8).

2.2. **Stochastic driving force.** Let Q be a linear positive operator in the Hilbert space H which is trace class, and hence compact. Let $H_0 = Q^{\frac12} H$; then H_0 is a Hilbert space with the scalar product

$$(\phi,\psi)_0 = (Q^{-\frac12}\phi, Q^{-\frac12}\psi),\ \forall \phi,\psi\in H_0,$$

together with the induced norm $|\cdot|_0=\sqrt{(\cdot,\cdot)_0}$. The embedding $i: H_0\to H$ is Hilbert-Schmidt and hence compact, and moreover, $i\ i^* = Q$. Let $L_Q\equiv L_Q(H_0,H)$ be the space of linear operators $S: H_0\mapsto H$ such that $SQ^{\frac12}$ is a Hilbert-Schmidt operator from H to H. The norm in the space L_Q is defined by $|S|_{L_Q}^2 = tr(SQS^*)$, where S^* is the adjoint operator of S. The L_Q-norm can also be written in the form

$$|S|_{L_Q}^2 = tr([SQ^{1/2}][SQ^{1/2}]^*) = \sum_{k\geq 1} |SQ^{1/2}\psi_k|^2 = \sum_{k\geq 1} |[SQ^{1/2}]^*\psi_k|^2 \tag{2.10}$$

for any orthonormal basis (ψ_k) in H.

Let $(W(t), t\geq 0)$ be a Wiener process defined on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P})$, taking values in H and with covariance operator Q. This means that W is Gaussian, has independent time increments and that for $s,t\geq 0,\ f,g\in H$,

$$\mathbb{E}(W(s),f)=0 \quad\text{and}\quad \mathbb{E}\big[(W(s),f)(W(t),g)\big] = \big(s\wedge t\big)\,(Qf,g).$$

Let (β_j) be standard (scalar) mutually independent Wiener processes, (e_j) be an orthonormal basis in H consisting of eigen-elements of Q, with $Qe_j = q_j e_j$. Then W has the following representation

$$W(t) = \lim_{n\to\infty} W_n(t)\ \text{ in }\ L^2(\Omega;H)\ \text{ with } W_n(t) = \sum_{1\leq j\leq n} q_j^{1/2}\beta_j(t) e_j, \tag{2.11}$$

and $Trace(Q)=\sum_{j\geq 1} q_j$. For details concerning this Wiener process see e.g. [17].

Let $k\geq 0$, $q\in(2,\infty)$ and let $R(H_0,W^{k,q})$ denote the space of all γ-radonifying mappings from H_0 into $W^{k,q}$, which are analogues of Hilbert-Schmidt operators when the Hilbert Sobolev spaces $W^{k,2}$ are replaced by the more general Banach spaces $W^{k,q}$. The definitions and some basic properties of stochastic calculus in the framework of special Banach spaces, including the case of non-Hilbert Sobolev spaces, can be found in [10]; see also [7], [19], [32] and [33]. For the sake of self-completeness, they are described in sub-section 6.2 of the Appendix. The radonifying norm $\|S\|_{R(H_0,W^{k,q})}$ of an element S of $R(H_0,W^{k,q})$ is defined in (6.9) ; it is the extension of the L_Q norm of $S\in L_Q$ which is the particular case $k=0$ and $q=2$.

2.3. **Assumptions.** Given a viscosity coefficient $\nu>0$, consider the following stochastic Navier-Stokes equations

$$du^\nu(t) + \big[\nu A u^\nu(t) + B(u^\nu(t),u^\nu(t))\big]\,dt = \sqrt{\nu}\,\sigma_\nu(t,u^\nu(t))\,dW(t), \tag{2.12}$$

where the noise intensity $\sigma_\nu : [0,T]\times V\to L_Q(H_0,H)$ of the stochastic perturbation is properly normalized by the square root of the viscosity coefficient ν. We assume that σ_ν satisfies the following growth and Lipschitz conditions:

Condition (C1): *For every $\nu>0$, $\sigma_\nu\in\mathcal{C}\big([0,T]\times V; L_Q(H_0,H)\big)$, and there exist constants $a>0$ and $K_i, L_i\geq 0$ such that for every $t\in[0,T]$, $\nu>0$ and $u,v\in V$:*

(i) $|\sigma_\nu(t,u)|_{L_Q}^2 \leq K_0 + K_1|u|_H^2 + K_2\nu^a\|u\|_V^2$,

(ii) $|\sigma_\nu(t,u)-\sigma_\nu(t,v)|_{L_Q}^2 \leq L_1|u-v|_H^2 + L_2\|u-v\|_V^2$.

For technical reasons, in order to prove a large deviation principle for the distribution of the solution to (2.12) as the viscosity coefficient ν converges to 0, we will need some precise estimates on the solution of the equation deduced from (2.12) by shifting the Brownian W by some random element of its RKHS. This cannot be deduced from similar ones on u by means of a Girsanov transformation; indeed, the Girsanov density is not uniformly bounded in $L^2(P)$ when the intensity of the noise tends to zero (see e.g. [21] or [16]).

To describe a set of admissible random shifts, we introduce the class $\mathcal{A}$ as the set of H_0-valued $(\mathcal{F}_t)-$predictable stochastic processes h such that $\int_0^T |h(s)|_0^2 ds < \infty$, a.s. For fixed $M>0$, let

$$S_M = \Big\{ h\in L^2(0,T;H_0) : \int_0^T |h(s)|_0^2 ds \leq M \Big\}.$$

The set S_M, endowed with the following weak topology, is a Polish (complete separable metric) space (see e.g. [12]): $d_1(h,k) = \sum_{k\geq 1} \frac{1}{2^k} \big| \int_0^T \big(h(s)-k(s), \tilde{e}_k(s)\big)_0 ds \big|$, where $(\tilde{e}_k(s), k\geq 1)$ is an orthonormal basis for $L^2(0,T;H_0)$. For $M>0$ set

$$\mathcal{A}_M = \{ h\in \mathcal{A} : h(\omega)\in S_M,\ a.s.\}. \tag{2.13}$$

In order to define the stochastic controlled equation, we introduce for $\nu\geq 0$ a family of intensity coefficients $\tilde{\sigma}_\nu$ which act on a random element $h\in\mathcal{A}_M$ for some $M>0$. The case $\nu=0$ will be that of an inviscid limit "deterministic" equation with no stochastic integral, and which can be dealt with for fixed ω. We assume that for any $\nu\geq 0$ the coefficient $\tilde{\sigma}_\nu$ satisfies the following condition:

Condition (C2): *For any $\nu\geq 0$, $\tilde{\sigma}_\nu \in \mathcal{C}\big([0,T]\times V; L(H_0,H)\big)$ and there exist non negative constants $\tilde{K}_{\mathcal{H}}$, $\tilde{K}_i$ and $\tilde{L}_i$ such that for every $t\in[0,T]$, $\nu\geq 0$ and $u,v\in V$:*

$$|\tilde{\sigma}_\nu(t,u)|_{L(H_0,H)} \leq \tilde{K}_0 + \tilde{K}_1 |u|_H + \sqrt{\nu}\tilde{K}_{\mathcal{H}} \|u\|_{\mathcal{H}}, \tag{2.14}$$

$$|\tilde{\sigma}_\nu(t,u) - \tilde{\sigma}_\nu(t,v)|_{L(H_0,H)} \leq \tilde{L}_1 |u-v|_H + \sqrt{\nu}\tilde{L}_2 \|u-v\|_V. \tag{2.15}$$

Examples of coefficients σ_ν and $\tilde{\sigma}_\nu$ which satisfy conditions (C1) and (C2), of Nemytski form, are provided in subsection 6.3 of the Appendix.

Let $\nu>0$, $M>0$, $h\in\mathcal{A}_M$, ζ be an H-valued random variable independent of W. Under Conditions (**C1**) and (**C2**), we consider the nonlinear SPDE

$$du_h^\nu(t) + \big[\nu\, A u_h^\nu(t) + B\big(u_h^\nu(t), (u_h^\nu(t))\big)\big]\, dt = \sqrt{\nu}\, \sigma_\nu(t, u_h^\nu(t))\, dW(t) + \tilde{\sigma}_\nu(t, u_h^\nu(t)) h(t)\, dt,$$
$$u_h^\nu(0) = \zeta, \tag{2.16}$$

with the conditions $\nabla . u_h^\nu = 0$ on $[0,T]\times D$, curl $u_h^\nu = 0$ and $u.n=0$ on $[0,T]\times\partial D$. Well posedness of the above equation as well as apriori bounds of the solution to this equation in $\mathcal{C}([0,T];H)\cap L^2(0,T;V)$ are known for fixed $\nu>0$ when $u=0$ on δD (see e.g. [35] and [16]. We will prove them uniformly in $\nu\in(0,\nu_0]$ for some small ν_0 under different boundary conditions.

Let us define the following conditions that we will use later in the paper. The following conditions **(C3)** and **(C4)** will allow to improve apriori estimates on the p-th moment of the solution to the stochastic controlled equation (2.16) in V, uniformly in time and on a "small" viscosity coefficient ν. They will also yield the existence of a solution to the inviscid deterministic equation, that is of (2.16) when $\nu=0$.

Condition (C3): *For every $\nu>0$, $\sigma_\nu\in\mathcal{C}\big([0,T]\times D(A); L_Q(H_0,V)\big)$ and there exist constants $a>0$ and $K_i, L_i\geq 0$ such that for every $t\in[0,T]$, $\nu>0$ and $u,v\in D(A)$:*

(i) $|\text{curl}\ \sigma_\nu(u)|_{L_Q}^2 \leq K_0 + K_1\|u\|_V^2 + \nu^a\, K_2\, |\nabla \text{curl}\ u|_H^2$,

(ii) $|A^{1/2}\sigma_\nu(t,u) - A^{1/2}\sigma_\nu(t,v)|_{L_Q}^2 \leq L_1 \|u-v\|_V^2 + \nu L_2 |Au-Av|_H^2$.

Condition (C4): *For every* $\nu \geq 0$, $\tilde{\sigma}_\nu \in \mathcal{C}\big([0,T] \times D(A); L(H_0, V)\big)$, *there exist non negative constants* $\tilde{K}_i$, $\tilde{L}_i$, *such that for every* $t \in [0,T]$, $\nu \geq 0$ *and* $u, v \in D(A)$*:*
(i) $|\text{curl } \tilde{\sigma}_\nu(t,u)|_{L(H_0,H)} \leq \tilde{K}_0 + \tilde{K}_1 \|u\|_V + \sqrt{\nu}\tilde{K}_2 |\nabla \text{curl } u|_H$,
(ii) $|A^{1/2}\tilde{\sigma}_\nu(t,u) - A^{1/2}\tilde{\sigma}_\nu(t,v)|_{L(H_0,H)} \leq \tilde{L}_1 \|u - v\|_V + \sqrt{\nu}\tilde{L}_2 |Au - Av|_H$.

The next condition **(C6q)** for $\nu = 0$ will enable us to prove the uniqueness of the solution to the "deterministic" inviscid equation in $H^{1,q}$ when $2 \leq q < \infty$. The following general assumptions **(C5q)** and **(C6q)** on σ_ν and $\tilde{\sigma}_\nu$ will also yield some apriori estimates for the q-th moment of the $H^{1,q}$-norm of the solution to the stochastic controlled equation.
Condition (C5q): *Let* $q \in [2,\infty)$*;* $\sigma_\nu \in \mathcal{C}\big([0,T] \times H^{2,q}; R(H_0, H^{1,q})\big)$ *for* $\nu > 0$*, and there exist non negative constants* K_i *for every* $u \in H \bigcap H^{2,q}$*, if* $\xi = \text{curl}$ *,*

$$\|\text{curl } \sigma_\nu(t,u)\|^2_{R(H_0,L^q)} \leq K_3 + K_4 \|u\|_q^2 + K_5 \|\xi\|_q^2 + K_6 \Big(\int_D |\xi(x)|^{q-2} |\nabla \xi(x)|^2 dx \Big)^{\frac{2}{q}}.$$

Condition (C6q): *Let* $q \in [2,\infty)$*;* $\tilde{\sigma}_\nu \in \mathcal{C}\big([0,T] \times H^{1,q}; L(H_0, H^{1,q})\big)$ *for* $\nu \geq 0$*, and there exist non negative constants* $\tilde{K}_i$ *such that for every* $u \in H^{1,q}$ *(resp.* $u \in H^{2,q}$ *for* $\nu = 0$*) if* $\xi = \text{curl } u$*,*

$$\|\text{curl } \tilde{\sigma}_\nu(t,u)\|_{L(H_0,L^q)} \leq \tilde{K}_3 + \tilde{K}_4 \|u\|_q + \tilde{K}_5 \|\xi\|_q + \sqrt{\nu}\tilde{K}_6 \Big(\int_D |\xi(x)|^{q-2} |\nabla \xi(x)|^2 dx \Big)^{\frac{1}{q}}.$$

Again, sub-section 6.3 of the Appendix provides examples of Nemytski operators which satisfy all the conditions above.

2.4. **Well Posedness and a priori estimates.** Let us mention in this section that the results used to obtain the well posedness of solutions are similar to known ones with different boundary conditions. However the apriori estimates are more involved since we are seeking estimates uniform in the parameter $\nu > 0$ which will be used later in Section 5 to let $\nu \to 0$.

Proposition 2.1. *Let* $T > 0$, $(\sigma_\nu, \nu > 0)$ *and* $(\tilde{\sigma}_\nu, \nu > 0)$ *satisfy conditions (***C1***) and (***C2***) and let the initial condition* ζ *be such that* $\mathbb{E}|\zeta|_H^{2p} < \infty$ *for some* $p \geq 2$*. Then there exists a positive constant* $\kappa_2(p)$ *such that for* $K_2 \in [0, \kappa_2(p)]$ *and for any* $M > 0$ *there exists a constant* $\nu_0(M, \kappa_2(p), \kappa_2(p) \vee \tilde{K}^2_{\mathcal{H}}) := \nu_0 > 0$ *and positive constants* $C_1(p,M)$ *and* $\tilde{C}_1(M)$ *(depending also on* T*,* ν_0*,* $K_i, i = 0,1,2$*,* $\tilde{K}_i, i = 0,1$ *and* $\tilde{K}_{\mathcal{H}}$*), such that for any* $\nu \in (0,\nu_0]$ *and any* $h \in \mathcal{A}_M$*,* (2.16) *has a unique solution in* $\mathcal{C}([0,T]; H) \cap L^2(0,T;V)$ *which satisfies the following apriori estimates:*

$$\sup_{0<\nu\leq\nu_0} \sup_{h\in\mathcal{A}_M} \mathbb{E}\Big(\sup_{0\leq s\leq T} |u_h^\nu(s)|_H^{2p} \Big) \leq C_1(p,M) \big[1 + \mathbb{E}|\zeta|_H^{2p} \big], \tag{2.17}$$

and

$$\sup_{0<\nu\leq\nu_0} \sup_{h\in\mathcal{A}_M} \nu \int_0^T \mathbb{E}\big(\|u_h^\nu(s)\|^2 + \|u_h^\nu(s)\|^4_{\mathcal{H}} \big) ds \leq \tilde{C}_1(M) \big[1 + \mathbb{E}|\zeta|_H^4 \big]. \tag{2.18}$$

Remark 2.2. *Note that condition* (2.14) *on* $\tilde{\sigma}_\nu$ *is needed with an upper estimate written in terms of* $\|u\|_{\mathcal{H}}$ *in order to prove the existence of a solution to* (2.16) *and more precisely that the weak limit of the Galerkin approximation is a solution to* (2.16)*. However, if one knows that* (2.16) *has a unique solution, the estimates* (2.17) *and* (2.18) *can be obtained under a weaker assumption on* $\tilde{\sigma}_\nu$*. For example, if* σ_ν *satisfies condition* **(C1)** *and* $\tilde{\sigma}_\nu = C\sigma_\nu$ *for* $\nu > 0$*, then using Proposition 2.1 with* $\tilde{\sigma}_\nu = 0$ *and then using the Girsanov theorem with* $\tilde{W}_\nu(s) = W(s) - \frac{C}{\sqrt{\nu}} h_\nu(s)$ *we deduce the well-posedeness of* (2.16)*. Nevertheless, once well-posedeness is proved, the estimates* (2.17) *and* (2.18) *could not be deduced from the Girsanov theorem, since the moments of the Girsanov density are not bounded for* $\nu \in (0,\nu_0]$*. The main aim of the proof below is to establish upper estimates uniform in "small"* ν *in a general framework. Finally note that if* $\tilde{\sigma}_\nu = C\sigma_\nu$ *and* σ_ν *satisfies* **(C1)**

with $a=1$ and an arbitrary constant $K_2(1)$, then rewrittting **(C1)** *with $a=1/2$, one sees that the corresponding constant $K_2=\sqrt{\nu}K_2(1)$ can be made arbitrary small for small ν.*

Proof. The proof, which is quite classical, requires some Galerkin approximation of u_h^ν, say $u_h^{\nu,n}$, for which apriori estimates are proved uniformly in n. Using a subsequence of $(u_h^{\nu,n}, n\geq 1)$ which converges in the weak or the weak-star topologies of appropriate spaces, one can then prove that there exists a solution to (2.16) (see e.g [16] or [35]). The proof of the uniqueness is standard. To ease notation, we will replace the Galerkin approximation by the limit process u_h^ν to obtain the required apriori estimates uniformly in $n\geq 1$ and in $\nu\in(0,\nu_0]$ for some $\nu_0>0$ under slightly more general boundary conditions; the proof can then be completed as in the appendix of [16]. If the well-posedeness is already known, we use the solution u_h^ν instead of the Galerkin approximation. Finally, to prove these uniform apriori estimates, we will suppose that $\tilde{\sigma}_\nu$ satisfies **(C2)** where (2.14) is replaced by the following weaker condition for every $t\in[0,T]$, $\nu>0$ and $u\in V$:

$$\tilde{\sigma}_\nu(t,u)\leq \tilde{K}_0+\tilde{K}_1|u|_H+\sqrt{\nu}\tilde{K}_2\|u\|. \tag{2.19}$$

Let $\nu>0$, $h\in\mathcal{A}_M$; for every $N>0$, let $\tau_N=\inf\{t\geq 0,\ \ |u_h^\nu(t)|_H\geq N\}\wedge T$.

Applying Itô's formula first to $|.|_H^2$ and the process $u_h^\nu(.\wedge\tau_N)$, then to the map $x\mapsto x^p$ for $p\geq 2$ and the process $|u_h^\nu(.\wedge\tau_N)|_H^2$, we deduce:

$$|u_h^\nu(t\wedge\tau_N)|_H^{2p}+\nu 2p\int_0^{t\wedge\tau_N}|u_h^\nu(s)|_H^{2p-2}\|u_h^\nu(s)\|^2\,ds\leq |u_h^\nu(0)|_H^{2p}+J(t)+\sum_{i=1}^5 T_i(t), \tag{2.20}$$

where

$$\begin{aligned}
J(t) &= 2p\sqrt{\nu}\int_0^{t\wedge\tau_N}|u_h^\nu(s)|_H^{2p-2}\left(\sigma_\nu(s,u_h^\nu(s))dW(s),u_h^\nu(s)\right),\\
T_1(t) &= 2p\nu\int_0^{t\wedge\tau_N}|u_h^\nu(s)|_H^{2p-2}\int_{\partial D}k(r)|u_h^\nu(r)|_H^2 dr ds,\\
T_2(t) &= 2p\int_0^{t\wedge\tau_N}|u_h^\nu(s)|_H^{2p-2}\langle B(u_h^\nu(s),u_h^\nu(s)),u_h^\nu(s)\rangle ds,\\
T_3(t) &= 2p\int_0^{t\wedge\tau_N}|u_h^\nu(s)|_H^{2p-2}\big(\tilde{\sigma}_\nu(s,u_h^\nu(s))h(s),u_h^\nu(s)\big)ds,\\
T_4(t) &= \nu p\int_0^{t\wedge\tau_N}|u_h^\nu(s)|_H^{2p-2}|\sigma_\nu(s,u_h^\nu(s))|_{L_Q}^2 ds,\\
T_5(t) &= 2\nu p(p-1)\int_0^{t\wedge\tau_N}|\sigma_\nu^*(s,u_h^\nu(s))u_h^\nu(s)|_0^2\,|u_h^\nu(s)|_H^{2(p-2)}ds.
\end{aligned}$$

The incompressibility condition (2.2) implies that $T_2(t)=0$ for any $t\in[0,T]$. Using (2.5), we deduce that for any $\epsilon>0$ there exists a constant $C(\epsilon)$ such that

$$T_1(t)\leq 2\nu p\epsilon\int_0^{t\wedge\tau_N}|u_h^\nu(s)|_H^{2p-2}\|u_h^\nu(s)\|^2 ds+2\nu p\,C(\epsilon)\int_0^{t\wedge\tau_N}|u_h^\nu(s)|_H^{2p}ds.$$

Since $h\in\mathcal{A}_M$, the growth condition (2.19), the Cauchy-Schwarz and Hölder inequalities imply that for any $\tilde{\epsilon}>0$,

$$\begin{aligned}
T_3(t)&\leq 2p\int_0^{t\wedge\tau_N}|u_h^\nu(s)|_H^{2p-1}\left(\tilde{K}_0+\tilde{K}_1|u_h^\nu(s)|_H+\sqrt{\nu}\tilde{K}_2\|u_h^\nu(s)\|\right)|h(s)|_0 ds\\
&\leq 2p\int_0^{t\wedge\tau_N}\left[\tilde{K}_0+\left(\tilde{K}_0+\tilde{K}_1\right)|u_h^\nu(s)|_H^{2p}+\sqrt{\nu}\tilde{K}_2|u_h^\nu(s)|_H^{2p-1}\,\|u_h^\nu(s)\|\right]|h(s)|_0 ds
\end{aligned}$$

$$\leq 2p\tilde K_0\sqrt{MT} + 2p\Big(\tilde K_0+\tilde K_1\Big)\int_0^{t\wedge\tau_N} |u_h^\nu(s)|_H^{2p}|h(s)|_0 ds$$
$$+ 2p\Big(\int_0^{t\wedge\tau_N} \tilde\epsilon\nu |u_h^\nu(s)|_H^{2p-2}\|u_h^\nu(s)\|^2 ds\Big)^{\frac12}\Big(\int_0^{t\wedge\tau_N}\frac{\tilde K_2^2}{\tilde\epsilon}|u_h^\nu(s)|_H^{2p}|h(s)|_0^2 ds\Big)^{\frac12}.$$

Finally, the Young inequality implies that for any $\tilde\epsilon>0$, we have

$$\begin{aligned} T_3(t) \;&\leq\; 2p\tilde K_0\sqrt{MT} + p\tilde\epsilon\nu\int_0^{t\wedge\tau_N}|u_h^\nu(s)|_H^{2p-2}\|u_h^\nu(s)\|^2 ds \\ &\quad + p\int_0^{t\wedge\tau_N}|u_h^\nu(s)|_H^{2p}\Big[2\Big(\tilde K_0+\tilde K_1\Big)|h(s)|_0 + \frac{\tilde K_2^2}{\tilde\epsilon}|h(s)|_0^2\Big] ds. \end{aligned}$$

Using the growth condition **(C1)**, we deduce for $\nu\in(0,1]$:

$$\begin{aligned} T_4(t) \;&\leq\; \nu p\int_0^{t\wedge\tau_N}|u_h^\nu(s)|_H^{2p-2}\big(K_0+K_1|u_h^\nu(s)|_H^2+K_2\|u_h^\nu(s)\|^2\big)\,ds, \\ T_5(t) \;&\leq\; 2p(p-1)\nu\int_0^{t\wedge\tau_N}|u_h^\nu(s)|_H^{2p-2}\big(K_0+K_1|u_h^\nu(s)|_H^2+K_2\|u_h^\nu(s)\|^2\big)\,ds. \end{aligned}$$

Thus, the Itô formula (2.20) and the previous upper estimates of $T_i(t)$, $i=1,\cdots,5$, imply that for any $t\in[0,T]$, $\epsilon,\tilde\epsilon>0$,

$$\begin{aligned} &|u_h^\nu(t\wedge\tau_N)|_H^2 + \nu p\big(2-2\epsilon-\tilde\epsilon-K_2(2p-1)\big)\int_0^{t\wedge\tau_N}|u_h^\nu(s)|_H^{2p-2}\,\|u_h^\nu(s)\|^2\,ds \\ &\qquad \leq \tilde Z + \int_0^t\tilde\varphi(s)|u(s\wedge\tau_N)|_H^{2p}ds + J(t), \end{aligned} \tag{2.21}$$

where

$$\begin{aligned} \tilde Z &= |\zeta|_H^{2p} + 2\nu p\tilde K_0\sqrt{MT} + p(2p-1)\nu K_0 T, \\ \tilde\varphi(s) &= p\Big[2\nu C(\epsilon) + (2p-1)\nu(K_0+K_1) + 2\Big(\tilde K_0+\tilde K_1\Big)|h(s)|_0 + \frac{\tilde K_2^2}{\tilde\epsilon}|h(s)|_0^2\Big]. \end{aligned}$$

Let $\kappa_2(p)\in\big(0,\frac{2}{2p-1}\big)$; for $0\leq K_2\leq\kappa_2(p)$, let $\epsilon(K_2)=\tilde\epsilon(K_2)=\frac{1}{10}[2-K_2(2p-1)]$. For $t\in[0,T]$, set

$$X(t) := \sup_{0\leq s\leq t}|u_h^\nu(s\wedge\tau_N)|_H^{2p},\ Y(t) := \int_0^{t\wedge\tau_N}|u_h^\nu(s)|_H^{2p-2}\|u_h^\nu(s)\|^2 ds,\ \tilde I(t) := \sup_{0\leq s\leq t}J(s).$$

Let $\lambda(K_2)=\frac{1}{10}[2-K_2(2p-1)]\geq\lambda(\kappa_2(p)) := \lambda\in(0,1)$ and

$$\tilde\alpha(K_2) = \nu p\Big[2(1-\lambda(K_2)) - 2\epsilon(K_2) - \tilde\epsilon(K_2) - (2p-1)K_2\Big] \geq \tilde\alpha(\kappa_2(p)) := \tilde\alpha > 0.$$

With these notations, the inequality (2.21) yields for $\nu\in(0,1]$ and $K_2\in[0,\kappa_2(p)]$:

$$\lambda X(t) + (1-\lambda)|u_h^\nu(t\wedge\tau_N)|_H^{2p} + \tilde\alpha Y(t) \leq Z + \int_0^t\tilde\varphi(s)X(s)\,ds + \tilde I(t). \tag{2.22}$$

Furthermore, using the Burkholder-Davis-Gundy inequality, condition **(C1)**, then Cauchy-Schwarz's and Young's inequalities, we deduce that for any $\beta>0$,

$$\begin{aligned} \mathbb{E}\tilde I(t) &\leq 6\sqrt{\nu}p\mathbb{E}\left(\int_0^{t\wedge\tau_N}|u_h^\nu(s)|_H^{4p-2}|\sigma(s,u_h^\nu(s))|_{L_Q}^2 ds\right)^{1/2} \\ &\leq 6\sqrt{\nu}p\mathbb{E}\Big(\int_0^{t\wedge\tau_N}|u_h^\nu(s)|_H^{4p-2}\big[K_0+K_1|u_h^\nu(s)|_H^2+K_2\nu^a\|u_h^\nu(s)\|^2\big]ds\Big)^{1/2} \end{aligned}$$

$$\leq 6\sqrt{\nu}p\mathbb{E}\left(X(t)\int_0^{t\wedge\tau_N}\big[K_0+(K_0+K_1)|u_h^\nu(s)|_H^{2p}+K_2\nu^a|u_h^\nu(s)|_H^{2p-2}\|u_h^\nu(s)\|^2\big]ds\right)^{1/2}$$
$$\leq \tilde\beta\mathbb{E}X(t)+\tilde\gamma\mathbb{E}\int_0^t X(s)ds+\tilde\delta\mathbb{E}Y(t)+\bar C,$$

where

$$\tilde\gamma=\frac{9\nu p^2}{\tilde\beta}(K_0+K_1),\quad \tilde\delta=\frac{9\nu^{1+a}p^2}{\tilde\beta}K_2,\quad \text{and }\ \bar C=\frac{9\nu p^2}{\tilde\beta}K_0T.$$

Suppose that $0\leq K_2\leq\kappa_2(p)$ and set $\varphi=\frac{\tilde\varphi}{\lambda}$, $\alpha=\frac{\tilde\alpha}{\lambda}$, $\beta=\frac{\tilde\beta}{\lambda}$, $\gamma=\frac{\tilde\gamma}{\lambda}$, $\delta=\frac{\tilde\delta}{\lambda}$ and $I(t)=\frac{\tilde I(t)}{\lambda}$. Then

$$X(t)+\alpha Y(t)\leq\frac{Z}{\lambda}+\int_0^t\varphi(s)X(s)\,ds+I(t),$$

with

$$\mathbb{E}I(t)\leq\beta\mathbb{E}X(t)+\gamma\mathbb{E}\int_0^tX(s)ds+\delta\mathbb{E}Y(t)+\frac{\bar C}{\lambda}.$$

Furthermore, $C(K_2,\tilde K_2):=\int_0^T\varphi(s)ds=\Phi_1(\kappa_2(p),\tilde K_2)+\nu\Phi_2(K_2)$, where for $K_2\in[0,\kappa_2(p)]$,

$$\Phi_1(\kappa_2(p),\tilde K_2)=\frac{10p}{2-\kappa_2(p)(2p-1)}\Big[2\sqrt{MT}\Big(\tilde K_0+\tilde K_1\Big)+M\tilde K_2^2\Big],$$
$$\Phi_2(K_2)=\frac{10p}{2-\kappa_2(p)(2p-1)}\big[2C(\epsilon(K_2))+(2p-1)(K_0+K_1)\big].$$

The functions $\Phi_1(\kappa_2(p),.)$ and $\Phi_2(.)$ are clearly increasing. Let $\tilde\beta=\frac{\lambda}{4}e^{-\Phi_1(\kappa_2(p),\kappa_2(p)\vee\tilde K_2^2)}$ and choose $\nu_1\in(0,1]$ small enough to ensure $e^{\nu_1\Phi_2(\kappa_2(p))}\leq 2$. Then for any $K_2\in[0,\kappa_2(p)]$ and $\nu\in(0,\nu_1]$, we have $2\beta e^{C(K_2,\tilde K_2)}\leq 1$. Finally, let $\nu_2\in(0,\nu_1]$ be small enough to ensure that $2^53^2\nu_2^a p\kappa_2(p)e^{2\Phi_1(\kappa_2(p),\kappa_2(p)\vee\tilde K_2)}\leq 1-\big(p-\frac12\big)\kappa_2(p)$. Then for $K_2\in[0,\kappa_2(p)]$, $M>0$, $\nu\in(0,\nu_0]$ with $\nu_0:=\nu_0(M,\kappa_2(p),\kappa_2(p)\vee\tilde K_2^2)=\nu_2$, we have $2\delta e^{C(K_2,\tilde K_2)}\leq\alpha$. Thus, since $X(.)$ is bounded by N, Lemma A.1 in [16] (see also Lemma 3.9 in [21]) implies that for $\nu\in(0,\nu_0]$ and $t\in[0,T]$, we have:

$$\mathbb{E}\big[X(t)+\alpha Y(t)\big]\leq C(\mathbb{E}|\zeta|_H^{2p},\kappa_2(p),\tilde K_2,\nu_0,M,T),$$

for some constant $C(\mathbb{E}|\zeta|_H^{2p},\kappa_2(p),\tilde K_2,\nu_0,M,T)$ which does not depend on N, ν, $h\in\mathcal{A}_M$ and on the step n of the Galerkin approximation. Since the right handside in the above equation does not depend on N, letting $N\to\infty$ we obtain that $\tau_N\to T$ a.s. Hence there exists $\nu_0>0$ which does not depend on n such that the Galerkin approximation $u_h^{n,\nu}$ of u_h^ν satisfies:

$$\sup_{n\geq1}\mathbb{E}\Big(\sup_{0\leq t\leq T}|u_h^{n,\nu}(t)|_H^{2p}+\nu\int_0^T\big[\|u_h^{n,\nu}(t)\|_{\mathcal{H}}^4+\|u_h^{n,\nu}(s)\|^2\big]ds\Big)<\infty$$

for any $\nu\in(0,\nu_0]$ and any $h\in\mathcal{A}_M$. The proof is completed using a classical argument (see e.g. the Appendix of [16] for details.) □

Proposition 2.3. *Let the assumptions of Proposition 2.1 be satisfied for $p=1$ or some $p\in[2,\infty)$. Moreover, assume that the initial condition is such that $E\|\zeta\|^{2p}<\infty$ and that $(\sigma_\nu,\nu>0)$ and $(\tilde\sigma_\nu,\nu>0)$ satisfy conditions* **(C3)** *and* **(C4)**. *Then there exists a constant $\bar\kappa_2(p)>0$ and given any $M>0$, there exists $\bar\nu_0:=\bar\nu_0(M,\bar\kappa_2(p)\vee\tilde K_{\mathcal{H}}\vee\tilde K_2)\in(0,\nu_0]$ and* *a positive constant $C_2(p,M)$ such that for $K_2\in[0,\bar\kappa_2(p)]$, $0<\nu\le\bar\nu_0$ and $h\in\mathcal{A}_M$, the solution to* (2.16) *satisfies:*

$$\mathbb{E}\Big(\sup_{0\le t\le T}\|u_h^\nu(t)\|^{2p}+\nu\int_0^T|Au_h^\nu(s)|_H^2\,ds\Big)\le C_2(p,M)\big(1+\mathbb{E}\|\zeta\|^{2p}\big). \tag{2.23}$$

Proof. Let $\xi_h^\nu=\text{curl }u_h^\nu$, then it is a classical result that u_h^ν is solution of the following elliptic problem (see e.g. [5] and the references therein),

$$\begin{cases}-\Delta u_h^\nu=\nabla^\perp\xi_h^\nu & \text{in } D,\\ u_h^\nu\cdot n=\xi_h^\nu=0 & \text{on }\partial D,\end{cases} \tag{2.24}$$

where $\nabla^\perp=(D_2,-D_1)$. Using the equation (2.24), we get that

$$-(\Delta u_h^\nu,\Delta u_h^\nu)=(\nabla^\perp\xi_h^\nu,\Delta u_h^\nu)=-(\nabla^\perp\xi_h^\nu,\nabla^\perp\xi_h^\nu).$$

Hence

$$|\Delta u_h^\nu|_H^2=|\nabla^\perp\xi_h^\nu|_H^2=\|D_2\xi_h^\nu\|_{L^2(D)}^2+\|D_1\xi_h^\nu\|_{L^2(D)}^2=|\nabla\xi_h^\nu|_H^2.$$

Hence, using (6.3) we see that it is enough to prove that for $\nu_0>0$ small enough, there exists a constant $C(M,T,K_i,\tilde K_i):=C_3$ such that for any $\nu\in(0,\nu_0]$ and $h\in\mathcal{A}_M$,

$$\mathbb{E}\Big(\sup_{0\le t\le T}|\xi_h^\nu(t)|_H^{2p}+\nu\int_0^T|\nabla\xi_h^\nu(s)|_H^2ds\Big)\le C_3(1+\mathbb{E}|\text{ curl }\zeta|_H^{2p}). \tag{2.25}$$

We at first prove this inequality for the Galerkin approximation of the solution ; a standard argument extends it to u_h^ν and hence ξ_h^ν. Fix $N>0$ and set $\bar\tau_N=\inf\{t\ge0:|\xi_h^\nu(t)|_H\ge N\}\wedge T$. Applying the curl to the evolution equation (2.16) yields $\xi_h^\nu(0)=\text{curl }\zeta$ and

$$\begin{aligned}d\xi_h^\nu(t)+\nu A\xi_h^\nu(t)dt+&\text{curl }B(u_h^\nu(t),u_h^\nu(t))dt=\\ &\sqrt{\nu}\,\text{curl }\sigma_\nu(s,u_h^\nu(t))\,dW(t)+\text{curl }\tilde\sigma_\nu(s,u_h^\nu(t))h(t)\,dt.\end{aligned} \tag{2.26}$$

Recall that equation (6.7) with $q=2$, implies $(\text{ curl }B(u_h^\nu,u_h^\nu),\xi_h^\nu)=0$ for $u\in D(A)$. Using Itô's formula for the square of the H norm, and then for the map $x\to|x|_H^p$ for $p\in[2,\infty)$, we obtain for $t\in[0,T]$:

$$|\xi_h^\nu(s\wedge\bar\tau_N)|_H^{2p}+2p\nu\int_0^{t\wedge\bar\tau_N}|\nabla\xi_h^\nu(s)|_H^2\,|\xi_h^\nu(s)|_H^{2p-2}ds=|\text{curl }\zeta|_H^{2p}+\bar J(t)+\sum_{i=1}^3\bar T_i(t), \tag{2.27}$$

where

$$\begin{aligned}\bar J(t)&=2p\sqrt{\nu}\int_0^{t\wedge\bar\tau_N}|\xi_h^\nu(s)|_H^{2p-2}\,\big(\text{curl }\sigma_\nu(s,u_h^\nu(s))dW(s)\,,\,\xi_h^\nu(s)\big),\\ \bar T_1(t)&=2p\int_0^{t\wedge\bar\tau_N}|\xi_h^\nu(s)|_H^{2p-2}\,\big(\text{curl }\tilde\sigma_\nu(s,u_h^\nu(s))h(s)\,,\,\xi_h^\nu(s)\big)\,ds,\\ \bar T_2(t)&=\nu p\int_0^{t\wedge\bar\tau_N}|\xi_h^\nu(s)|_H^{2p-2}\,|\text{curl }\sigma_\nu(s,u_h^\nu(s))|_{L_Q}^2\,ds,\\ \bar T_3(t)&=2\nu p(p-1)\int_0^{t\wedge\bar\tau_N}|\xi_h^\nu(s)|_H^{2p-4}\,\big|\big(\text{curl }\sigma_\nu(s,u_h^\nu(s))\big)^*\xi_h^\nu(s)\big|_{H_0}^2\,ds.\end{aligned}$$

Using the Cauchy-Schwarz inequality, (**C4**) and (6.3) with $q=2$, we get that

$$\begin{aligned}\bar T_1(t)&\le 2p\int_0^{t\wedge\bar\tau_N}|\xi_h^\nu(s)|_H^{2p-1}\,|\text{curl }\tilde\sigma_\nu(s,u)|_{L(H_0,H)}\,|h(s)|_0\,ds\\ &\le 2p\int_0^{t\wedge\bar\tau_N}\Big[\tilde K_0+\Big(\tilde K_0+2\,C\tilde K_1\Big)|\xi_h^\nu(s)|_H^{2p}+\tilde K_1|u_h^\nu(s)|_H|\xi_h^\nu|_H^{2p-1}\end{aligned}$$

$$+ \sqrt{\nu}\tilde{K}_2|\nabla\xi_h^\nu(s)|_H|\xi_h^\nu|_H^{2p-1}\Big]\,|h(s)|_0\,ds$$

Using Cauchy-Schwarz's, Hölder's and Young's inequalities, we deduce that for any $\bar\epsilon > 0$, we have:

$$\begin{aligned}
\bar{T}_1(t) &\leq 2p\tilde{K}_0\sqrt{MT} + 2p\Big(\tilde{K}_0 + 2\,C\tilde{K}_1\Big)\int_0^{t\wedge\bar\tau_N} |\xi_h^\nu(s)|_H^{2p}\,|h(s)|_0\,ds\\
&\quad + 2p\tilde{K}_1\Big(\int_0^{t\wedge\bar\tau_N} |u_h(s)|_H^{2p}|h(s)|_0 ds\Big)^{\frac{1}{2p}}\Big(\int_0^{t\wedge\bar\tau_N} |\xi_h^\nu(s)|_H^{2p}\,|h(s)|_0\,ds\Big)^{\frac{2p-1}{2p}}\\
&\quad + 2\Big(\int_0^{t\wedge\bar\tau_N} \nu p\bar\epsilon|\xi_h^\nu(s)|_H^{2p-2}\,|\nabla\xi_h^\nu(s)|_H^2\,ds\Big)^{\frac12}\Big(\int_0^{t\wedge\bar\tau_N} \frac{p\tilde{K}_2^2}{\bar\epsilon}|\xi_h^\nu(s)|_H^{2p}\,|h(s)|_0^2\,ds\Big)^{\frac12}\\
&\leq 2p\tilde{K}_0\sqrt{MT} + \tilde{K}_1^{2p}\sqrt{MT}\sup_{0\leq s\leq T}|u_h^\nu(s)|_H^{2p} + \int_0^{t\wedge\bar\tau_N}\psi_1(s)|\xi_h^\nu(s)|_H^{2p}ds\\
&\quad + p\nu\bar\epsilon\int_0^{t\wedge\bar\tau_N} |\xi_h^\nu(s)|_H^{2p-2}\,|\nabla\xi_h^\nu(s)|_H^2\,ds,
\end{aligned}$$

where

$$\psi_1(s) := 2p\Big(\tilde{K}_0 + 2\,C\tilde{K}_1\Big) + (2p-1)|h(s)|_0 + \frac{p\tilde{K}_2^2}{\bar\epsilon}\,|h(s)|_0^2.$$

Furthermore, $\bar{T}_3(t)$ can be upper estimated in terms of $\bar{T}_2(t)$ as follows:

$$\bar{T}_3(t) \leq 2\nu p(p-1)\int_0^{t\wedge\bar\tau_N} |\text{curl }\sigma_\nu(u_h^\nu(s))|_{L_Q}^2|\xi_h^\nu(s)|_H^{2p-2}\,ds = 2(p-1)\bar{T}_2(t).$$

Finally, condition **(C3)**, (6.3) with $q=2$, Hölder's and Young's inequalities, we obtain for $\nu\in(0,1]$:

$$\begin{aligned}
\bar{T}_2(t) + \bar{T}_3(t) &\leq \nu p(2p-1)\int_0^{t\wedge\bar\tau_N}|\xi_h^\nu(s)|_H^{2p-2}\Big[K_0 + K_1\big(|u_h^\nu(s)|_H^2 + 4C^2|\xi_h^\nu(s)|_H^2\big)\\
&\qquad + K_2|\nabla\xi_h^\nu(s)|_H^2\Big]\,ds\\
&\leq \nu p(2p-1)\Big[K_0T + \int_0^{t\wedge\bar\tau_N}[K_0 + 4K_1C^2]|\xi_h^\nu(s)|_H^{2p}ds + K_2\int_0^{t\wedge\bar\tau_N}|\xi_h^\nu(s)|_H^{2p-2}|\nabla\xi_h^\nu(s)|_H^2ds\\
&\quad + K_1\Big(\int_0^{t\wedge\bar\tau_N}|\xi_h^\nu(s)|_H^{2p}ds\Big)^{\frac{p-1}{p}}\Big(\int_0^{t\wedge\bar\tau_N}|u_h^\nu(s)|_H^{2p}ds\Big)^{\frac1p}\Big]\\
&\leq \nu(2p-1)T\Big[pK_0 + K_1\sup_{0\leq s\leq T}|u_h^\nu(s)|_H^{2p}\Big] + \int_0^{t\wedge\bar\tau_N}\nu\,\psi_2\,|\xi_h^\nu(s)|_H^{2p}\,ds\\
&\quad + \nu p(2p-1)K_2\int_0^{t\wedge\bar\tau_N}|\xi_h^\nu(s)|_H^{2p-2}|\nabla\xi_h^\nu(s)|_H^2ds,
\end{aligned}$$

where

$$\psi_2 = (2p-1)\big[p(K_0 + 4K_1C^2) + (p-1)K_1\big].$$

Let $\bar\kappa_2(p)\in\big[0,\kappa_2(p)\big]$ where $\kappa_2(p) < \frac{2}{2p-1}$ is defined in Proposition 2.1 and for $0\leq K_2\leq\bar\kappa_2(p)$, set

$$\bar\epsilon(K_2) = \bar\lambda(K_2) = \frac14\big(2 - K_2(2p-1)\big) \geq \bar\lambda(\bar\kappa_2(p)) := \bar\epsilon = \bar\lambda.$$

Let

$$\bar{X}(t) = \sup_{0\leq s\leq t}|\xi_h^\nu(s\wedge\bar\tau_N)|_H^{2p}\,,\quad \bar{Y}(t) = \int_0^{t\wedge\bar\tau_N}|\xi_h^\nu(s)|_H^{2p-2}\,|\nabla\xi_h^\nu(s)|_H^2\,ds.$$

Then for

$$\bar\alpha(K_2)=\nu p\big[2(1-\bar\lambda)-\bar\epsilon-K_2(2p-1)\big]=\frac{\nu p}{4}\big[2-K_2(2p-1)\big]\geq \bar\alpha(\bar\kappa_2(p)):=\bar\alpha,$$
$$\bar Z:=|\text{curl }\zeta|_H^{2p}+2p\tilde K_0\sqrt{MT}+\nu p(2p-1)TK_0 + \big(\tilde K_1^{2p}\sqrt{M}+\nu(2p-1)TK_1\big)\sup_{0\leq s\leq T}|u_h^\nu(s)|_H^{2p},$$

equation (2.27) and the upper bounds of $\bar T_i(t)$ imply that for any $t\in[0,T]$ and $\nu\in(0,\nu_0(M)]$, where $\nu_0(M)\in(0,1]$ is defined in Lemma 2.1:

$$\bar\lambda\bar X(t)+\bar\alpha\bar Y(t)+(1-\bar\lambda)|\xi_h^\nu(s\wedge\bar\tau_N)|_H^{2p}\leq \bar Z+\int_0^t\big[\psi_1(s)+\nu\psi_2\big]\bar X(s)ds+\bar I(t), \qquad (2.28)$$

where $\bar I(t)=\sup_{0\leq s\leq t}|\bar J(s)|$.
The Davies inequality, condition **(C3)**, (6.3) for $q=2$, Cauchy-Schwarz's, Hölder's and Young's inequalities imply that for any $\bar\beta>0$,

$$\begin{aligned}
\mathbb{E}\bar I(t)&\leq 6p\sqrt{\nu}\mathbb{E}\Big(\int_0^{t\wedge\bar\tau_N}|\xi_h^\nu(s)|_H^{4p-2}\,|\text{curl }\sigma_\nu(u_h^\nu(s)|_{L_Q}^2\,ds\Big)^{\frac12}\\
&\leq 6p\sqrt{\nu}\mathbb{E}\Big[\sup_{0\leq s\leq t}|\xi_h^\nu(s\wedge\bar\tau_N)|_H^p\Big(\int_0^{t\wedge\bar\tau_N}|\xi_h^\nu(s)|_H^{2p-2}\big[K_0+K_1|u_h^\nu(s)|_H^2\\
&\qquad +4C^2K_1|\xi_h^\nu(s)|_H^2+\nu^aK_2|\nabla\xi_h^\nu(s)|_H^2\big]\,ds\Big)^{\frac12}\Big]\\
&\leq \bar\beta\mathbb{E}\bar X(t)+\frac{9p^2\nu}{\bar\beta}K_0T+\frac{9p^2\nu}{\bar\beta}(K_0+4C^2K_1)\mathbb{E}\int_0^{t\wedge\bar\tau_N}|\xi_h^\nu(s)|_H^{2p}ds\\
&\qquad+\frac{9p^2\nu}{\bar\beta}K_1\Big(\mathbb{E}\int_0^{t\wedge\bar\tau_N}|\xi_h^\nu(s)|_H^{2p}ds\Big)^{\frac{p-1}{p}}\Big(T\mathbb{E}\sup_{0\leq s\leq T}|u_h^\nu(s)|_H^{2p}\Big)^{\frac1p}\\
&\qquad+\frac{9p^2\nu^{a+1}}{\bar\beta}K_2\mathbb{E}\int_0^{t\wedge\bar\tau_N}|\xi_h^\nu(s)|_H^{2p-2}\,|\nabla\xi_h^\nu(s)|_H^2\,ds\\
&\leq \bar\beta\mathbb{E}\bar X(t)+\bar\gamma\mathbb{E}\int_0^{t\wedge\bar\tau_N}\bar X(s)ds+\bar\delta\mathbb{E}\bar Y(t)+\tilde Z,
\end{aligned}\qquad (2.29)$$

where $\bar\gamma=\frac{9p^2\nu}{\bar\beta}\big[p(K_0+4C^2K_1)+(p-1)K_1\big]$, $\bar\delta=\frac{9p^2\nu^{a+1}}{\bar\beta}K_2$ and $\tilde Z:=\frac{9p\nu}{\bar\beta}\big[pK_0T+K_1\mathbb{E}\big(\sup_{0\leq s\leq t}|u_h^\nu(s)|_H^{2p}\big)\big]$. Set $\varphi(s)=\frac{\psi_1(s)+\nu\psi_2}{\bar\lambda}$, $\alpha=\bar\alpha/\bar\lambda$, $\beta=\bar\beta/\bar\lambda$, $\gamma=\bar\gamma/\bar\lambda$, $\delta=\bar\delta/\bar\lambda$ and $I(t)=\bar I(t)/\bar\lambda$. Then for $0\leq K_2\leq\bar\kappa_2(p)$ and $0<\nu\leq\nu_0(M)$ we have for $t\in[0,T]$,

$$\bar X(t)+\alpha\bar Y(t)\leq\int_0^t\varphi(s)\bar X(s)ds+I(t)+\bar Z(t)/\bar\lambda,$$

and

$$\mathbb{E}I(t)\leq\beta\mathbb{E}\bar X(t)+\gamma\mathbb{E}\int_0^t\bar X(s)ds+\delta\mathbb{E}\bar Y(t)+\tilde Z/\bar\lambda.$$

Furthermore, $C(\tilde K_2):=\int_0^T\varphi(s)ds=\bar\Phi_1(\bar\kappa_2(p),\tilde K_2)+\nu\bar\Phi_2$, where $\bar\Phi_2=\psi_2T/\bar\lambda(\bar\kappa_2(p))$ and

$$\bar\Phi_1(\bar\kappa_2(p),\tilde K_2)=2p\Big[\big(\tilde K_0+2C\tilde K_1\big)T+(2p-1)\sqrt{MT}+\frac{p\tilde K_2^2M}{\bar\epsilon(\bar\kappa_2(p))}\Big]/\bar\lambda(\bar\kappa_2(p)).$$

The map $\bar\Phi_1(\bar\kappa_2(p),.)$ is clearly increasing. Set $\bar\beta=\frac{\bar\lambda}{4}e^{-\bar\Phi_1(\bar\kappa_2(p),\tilde K_2\vee\sqrt{\bar\kappa_2(p)})}$ and let $\bar\nu_0\in(0,\nu_0]$ be such that $e^{\bar\nu_0\bar\Phi_2}\leq 2$ and $2^63^2\bar\kappa_2(p)\bar\nu_0^ae^{2\bar\Phi_1(\bar\kappa_2(p),\tilde K_2\vee\sqrt{\bar\kappa_2(p)})}\leq 2-\bar\kappa_2(p)(2p-1)$.

Then for $K_2 \in [0, \bar\kappa_2(p)]$ and $\nu \in (0, \bar\nu_0]$, we have $2\beta e^{C(\tilde K_2)} \le 1$ and $2\delta e^{C(\tilde K_2)} \le \alpha$. Hence, applying Lemma A.1 in [16] we deduce

$$\sup_N \mathbb{E}\Big(\sup_{0\le t\le T} |\xi_h^\nu(t\wedge \bar\tau_N)|_H^{2p} + \nu \int_0^T |\nabla \xi_h^\nu(s)|_H^2 ds\Big) \le C(\mathbb{E}|\text{curl } \zeta|_H^{2p}, M, T),$$

for every $h \in \mathcal{A}_M$ and $\nu \in (0, \bar\nu_0]$. Since the previous upper bound is uniform in N, we deduce that $\bar\tau_N \to T$ as $N \to \infty$. Thus, using the monotone convergence we get (2.25), which concludes the proof. □

The following well-posedeness result for problem (2.16) follows from Propositions 2.1 and 2.3; the proof is not given and we refer to [16] and [35] for details.

Theorem 2.4. *Let $(\sigma_\nu, \nu > 0)$ and $(\tilde\sigma_\nu, \nu > 0)$ satisfy conditions* **(C1)** - **(C4)** *some $p \in [2, \infty)$ such that $\mathbb{E}(\|\zeta\|^{2p}) < \infty$. There exists a constant $\kappa_2(p) > 0$ such that for $K_2 \in \big[0, \kappa_2(p)\big]$ and for every $M > 0$ and $T > 0$, there exists $\bar\nu_0 := \bar\nu_0(M, \kappa_2(p), \kappa_2(p) \vee \tilde K_2^2 \vee \tilde K_{\mathcal{H}}^2) > 0$ such that for any $\nu \in (0, \bar\nu_0]$ and $h \in \mathcal{A}_M$, there exists a pathwise unique "weak" solution u_h^ν of equation* (2.9) *with initial condition $u_h^\nu(0) = \zeta \in V$ such that* a.s. *$u_h^\nu \in \mathcal{C}\big([0,T]; V\big)$, the inequalities* (2.17), (2.18) *and* (2.23) *hold, and such that for every $v \in D(A)$ and $t \in [0,T]$, one has* a.s.

$$\begin{aligned}(u_h^\nu(t), v) - (\zeta, v) + \int_0^t \big[\nu(u_h^\nu(s), Av) + \langle B(u_h^\nu(s), v), u_h^\nu(s)\rangle\big] ds \\ = \sqrt{\nu} \int_0^t (\sigma_\nu(s, u_h^\nu(s)) dW(s), v) + \int_0^t (\tilde\sigma_\nu(s, u_h^\nu(s)) h(s), v) ds. \qquad (2.30)\end{aligned}$$

3. Well posedness of the inviscid problem

The aim of this section is to deal with the inviscid case $\nu = 0$, that is with the Euler evolution equation

$$du_h^0(t) + B(u_h^0(t), u_h^0(t))\, dt = \tilde\sigma_0(t, u_h^0(t))\, h(t)\, dt\,, \quad u_h^0(0) = \zeta \qquad (3.1)$$

in $[0,T] \times D$ with the incompressibility condition $\nabla . u_h^0 = 0$ in $[0,T] \times D$ and curl $u = 0$, $u.n = 0$ on $[0,T] \times \partial D$.

Theorem 3.1. *Let us assume that $\zeta \in V$ and that $\tilde\sigma_0$ satisfies conditions* **(C2)** *and* **(C4)***. Then for all $M > 0$, $h \in \mathcal{A}_M$ and $T > 0$, there exists* a.s. *a solution $u_h^0 \in C\big([0,T]; H\big) \bigcap L^\infty\big(0,T; V\big)$ for the equation* (3.1) *with the initial condition $u_h^0 = \zeta$, such that for all $v \in V$ and $t \in [0,T]$*

$$\big(u_h^0(t), v\big) + \int_0^t \langle B(u_h^0(s), v), u_h^0(s)\rangle ds = \int_0^t \big(\tilde\sigma_0(s, u_h^0(s))\, h(s), v\big) ds, \quad \text{a.s.} \qquad (3.2)$$

Moreover, there exists a positive constant $C_3(M)$ (which also depends on $\tilde K_0$, $\tilde K_1$ and T) such that for every $h \in \mathcal{A}_M$, one has a.s.

$$\sup_{0\le t\le T} \|u_h^0(t)\| \le C_3(M)(1 + \|\zeta\|). \qquad (3.3)$$

Proof. For $\mu > 0$, let us approximate equation (3.1) by the solution $u_h^{0\mu}$ to the following Navier Stokes evolution equation:

$$du_h^{0\mu}(t) + \big[\mu A u_h^{0\mu}(t) + B(u_h^{0\mu}(t), u_h^{0\mu}(t))\big]\, dt = \tilde\sigma_0(t, u_h^{0\mu}(t))\, h(t)\, dt\,, \quad u_h^{0\mu}(0) = \zeta, \quad (3.4)$$

with the same incompressibility and boundary conditions. If $\zeta \in H$, $\tilde\sigma_0$ satisfies the condition **(C2)** and $h \in \mathcal{A}_M$ for $M > 0$, then Lemma 2.1 shows that a.s. equation (3.4) has a unique solution $u_h^{0\mu} \in C\big([0,T]; H\big) \bigcap L^2\big(0,T; V\big)$ (see also [35] or [16]). Moreover, if

$\zeta \in V$ and $\tilde\sigma_0$ satisfies **(C4)**, then Proposition 2.3 implies that a.s. $u_h^{0\mu} \in C\big([0,T];V\big)$. In order to prove the existence of solutions for equation (3.1), we need some estimates on $u_h^{0\mu}$ uniform in $\mu>0$. Multiply the equation (3.4) by $2u_h^{0\mu}$ and integrate over $[0,t]\times D$; then (6.5), the Cauchy-Schwarz and Young inequalities and assumption **(C2)**, yield for every $\mu>0$:

$$\begin{aligned}|u_h^{0\mu}(t)|_H^2 + 2\mu\int_0^t \|u_h^{0\mu}(s)\|^2 ds &\le |\zeta|_H^2 + 2\int_0^t \big(\tilde\sigma_0(s,u_h^{0\mu}(s))h(s), u_h^{0\mu}(s)\big)ds\\ &\le |\zeta|_H^2 + 2\int_0^t |\tilde\sigma_0(s,u_h^{0\mu}(s))|_{L(H_0,H)}\,|h(s)|_0\,|u_h^{0\mu}(s)|_H\,ds\\ &\le |\zeta|_H^2 + 2\tilde K_0\sqrt{MT} + 2\Big(\tilde K_0+\tilde K_1\Big)\int_0^t |u_h^{0\mu}(s)|_H^2\,|h(s)|_0 ds.\end{aligned}$$

Hence, by Gronwall's lemma, we deduce the existence of a constant $\tilde C_1$ which depends on $M,T,\tilde K_0$ and $\tilde K_1$ such that:

$$\sup_{\mu>0}\sup_{0\le t\le T}|u_h^{0\mu}(t)|_H^2 \le \tilde C_1(1+|\zeta|_H^2). \tag{3.5}$$

Let $\xi_h^{0\mu}(t) := \text{curl }u_h^{0\mu}(t)$; then applying the curl operator to equation (3.4) and using (6.6) we obtain the following evolution equation

$$d\xi_h^{0\mu}(t) + \mu A\xi_h^{0\mu}(t) + B(u_h^{0\mu}(t),\xi_h^{0\mu}(t))\,dt = \text{curl }\tilde\sigma_0(t,u_h^{0\mu}(t))\,h(t)\,dt\ , \tag{3.6}$$

with the initial condition $\xi_h^{0\mu}(0)=\text{curl }\zeta$ and the boundary condition $\xi_h^{0\mu}=0$ on ∂D. Multiply the equation (3.6) by $2\xi_h^{0\mu}$ and integrate over $[0,T]\times D$; since $\tilde\sigma_0$ satisfies the condition **(C4)**, using (6.7) for $q=2$, (6.3), Cauchy-Schwarz's and Young's inequalities, we deduce

$$\begin{aligned}&|\xi_h^{0\mu}(t)|_H^2 + 2\mu\int_0^t\|\xi_h^{0\mu}(s)\|^2ds\\ &\le |\text{curl }\zeta|_H^2 + 2\int_0^t|\text{curl }\tilde\sigma_0(s,u_h^{0\mu}(s))|_{L(H_0,H)}\,|h(s)|_0\,|\xi_h^{0\mu}(s)|_H\,ds\\ &\le |\text{curl }\zeta|_H^2 + 2\int_0^t\Big(\tilde K_0+\tilde K_1|u_h^{0\mu}(s)|_H + 2C\tilde K_1|\xi_h^{0\mu}(s)|_H\Big)\,|h(s)|_0|\xi_h^{0\mu}(s)|_H ds\\ &\le \|\zeta\|^2 + \Big(2\tilde K_0 + \tilde K_1\sup_{s\in[0,T]}|u_h^{0\mu}(s)|_H^2\Big)\sqrt{MT} + 2\int_0^t\Big(\tilde K_0+(2C+1)\tilde K_1\Big)\,|h(s)|_0|\xi_h^{0\mu}(s)|_H^2 ds.\end{aligned}$$

Thus, (3.5), Gronwall's lemma yields the existence of a constant $\tilde C_2$ which depends on $M,T,\tilde K_0$ and $\tilde K_1$ such that for every $h\in\mathcal{A}_M$:

$$\sup_{\mu>0}\sup_{0\le t\le T}|\xi_h^{0\mu}(t)|_H^2 \le \tilde C_2(1+\|\zeta\|^2)\quad\text{a.s.} \tag{3.7}$$

Combining the estimates (3.5), (3.7) and (6.3), we deduce the existence of a constant $\tilde C_3$ depending on $M,T,\tilde K_0$ and $\tilde K_1$ such that for any $h\in\mathcal{A}_M$ one has:

$$\sup_{\mu>0}\sup_{0\le t\le T}\|u_h^{0\mu}\| \le \tilde C_3(1+\|\zeta\|)\ \text{ a.s.} \tag{3.8}$$

Furthermore, we have $u_h^{0\mu}\in C\big([0,T];H\big)\bigcap L^\infty\big(0,T;V\big)$ a.s. for every $\mu>0$, and

$$u_h^{0\mu}(t) = \zeta - \mu\int_0^t Au_h^{0\mu}(s) - \int_0^t B(u_h^{0\mu}(s),u_h^{0\mu}(s))\,ds + \int_0^t\tilde\sigma_0(s,u_h^{0\mu}(s))\,h(s)\,ds.$$

Using the estimates (3.5), (3.8), assumptions **(C4)** on $\tilde\sigma_0$ and (6.4) for $q=2$ and $r=1$, we deduce the existence of a constant $\tilde C_4$ depending on $M, T, \tilde K_0$ and $\tilde K_1$ such that the following estimate holds for any $\mu\in(0,1]$ and $h\in\mathcal{A}_M$:

$$\|u_h^{0\mu}\|_{W^{1,2}(0,T;V')}\le \tilde C_4(1+\|\zeta\|).\ \text{ a.s.} \tag{3.9}$$

By classical compactness arguments, we can extract a subsequence (still denoted $u_h^{0\mu}$) and prove the existence of a function $v\in W^{1,2}\big(0,T;V'\big)\bigcap L^\infty\big(0,T;V\big)$ such that as $\mu\to 0$:

$$\begin{aligned}
&u_h^{0\mu}\to v \text{ weakly in } L^2\big(0,T;V\big)\,,\\
&u_h^{0\mu}\to v\ \text{ strongly in } L^2\big(0,T;H\big)\,,\\
&u_h^{0\mu}\to v\ \text{ in the weak star topology of }\ L^\infty\big(0,T;V\big)\,,\\
&u_h^{0\mu}\to v\ \text{ weakly }\ \text{in } W^{1,2}\big(0,T;V'\big)\,.
\end{aligned}$$

Letting $\mu\to 0$ in equation (3.4), we deduce that the above limit v is solution of the equation (3.1), that is $v=u_h^0$. Moreover, the estimate (3.8) being uniform in $\mu>0$, we deduce the estimate (3.3). □

The following theorem shows that if curl ζ is bounded, then the solution to (3.1) is unique.

Theorem 3.2. *Let us assume that the assumptions of Theorem 3.1 are satisfied. Moreover, let us assume that* curl $\zeta\in\big(L^\infty(D)\big)^2$ *and that condition* **(C6q)** *holds for every* $q\in(2,\infty)$. *Then, for every* $M\ge 0$ *and* $h\in\mathcal{A}_M$, *the solution of equation* (3.1) *with the initial condition* $u_h^0=\zeta$ *is* a.s. *unique in* $C\big([0,T];H\big)\bigcap L^\infty\big(0,T;H^{1,q}\big)$ *for every* $q\in(2,\infty)$ *and every* $T>0$. *Moreover, there exist positive constants* $C_4(M)$ *and* $\bar C_4(M)$ *(which also depend on* T, $\tilde K_i$ *and* $\|\zeta\|_{L^\infty(D)^2}$*), such that for every* $h\in\mathcal{A}_M$ *and* $q\in(2,\infty)$, *one has a.s.*

$$\sup_{0\le t\le T}\|\text{curl } u_h^0(t\|_q\le C_4(M)(1+\|\zeta\|+\|\text{curl } \zeta\|_q), \tag{3.10}$$

$$\sup_{0\le t\le T}\|\nabla u_h^0(t)\|_q\le \bar C_4(M)q(1+\|\zeta\|+\|\text{curl } \zeta\|_q). \tag{3.11}$$

Proof. The first step of the proof will establish the estimate (3.10). The second step will prove the uniqueness of the solution u_h^0.

Step1. (Existence) Using (6.3) one sees that the proof of (3.11) reduces to that of (3.10), that is to check $L^q(D)$ upper bounds for $\xi_h^0(t):=\text{curl } u_h^0(t)$. Replacing u_h^0 by its Galerkin approximation $u_{h,n}^0$, we may assume that $u_{h,n}^0\in H^{2,q}$ and deduce the desired inequality by proving upper bounds which do not depend on n. To ease notations in the sequel, we skip the index n.

Let us apply the curl to the equation (3.1); the identity (6.6) yields

$$d\xi_h^0(t)+(u_h^0(t)\cdot\nabla)\xi_h^0(t))\,dt=\text{curl } \tilde\sigma_0(t,u_h^0(t))\,h(t)\,dt\ ,\quad \xi_h^0(0)=\text{curl } \zeta. \tag{3.12}$$

Let us multiply the equation (3.12) by $q|\xi_h^0(t)|^{q-2}\xi_h^0(t)$ and integrate over $[0,t]\times D$; we obtain

$$\begin{aligned}
\|\xi_h^0(t)\|_q^q+q\int_0^t\!\!\int_D(u_h^0(s)\cdot\nabla)\xi_h^0(s))|\xi_h^0(s)|^{q-2}\xi_h^0(s)dxds&=\|\text{curl } \zeta\|_q^q\\
+q\int_0^t\!\!\int_D \text{curl } \tilde\sigma_0(s,u_h^0(s))h(s)|\xi_h^0(s)|^{q-2}\xi_h^0(s)dxds&.
\end{aligned}$$

Using (6.7), we deduce that for every s, $\int_D (u_h^0(s)\cdot\nabla)\xi_h^0(s))|\xi_h^0(s)|^{p-2}\xi_h^0(s)dx=0$. On the other side, using the Hölder and Young inequalities, condition **(C6q)** yields

$$\begin{aligned}\|\xi_h^0(t)\|_q^q &\leq \|\text{curl }\zeta\|_q^q + q\int_0^t \big(\tilde K_3+\tilde K_4\|u_h^0(s)\|_q+\tilde K_5\|\xi_h^0(s)\|_q\big)|h(s)|_0\|\xi_h^0(s)\|_q^{q-1}ds\\ &\leq \|\text{curl }\zeta\|_q^q + \Big(q\tilde K_3+\tilde K_4\sup_{0\leq s\leq T}\|u_h^0(s)\|_q^q\Big)\sqrt{MT}\\ &\quad + q\big(\tilde K_3+\tilde K_4+\tilde K_5)\big)\int_0^t |h(s)|_0\|\xi_h^0(s)\|_q^q ds.\end{aligned}$$

Finally, the inclusion $V=H^{1,2}\subset L^q(D)$ given by (6.1), the control of the V norm proven in (3.3) (which clearly also holds for the Galerkin approximation $u_{h,n}^0$ with an upper bound which does not depend on n) and Gronwall's lemma imply the existence of a non negative constant $\tilde C_5(M)$, depending on $T,M,\tilde K_3,\tilde K_4$ such that for any $n\geq 1$ and $h\in\mathcal{A}_M$, we have

$$\begin{aligned}\sup_{0\leq t\leq T}\|\xi_{h,n}^0(t)\|_q^q &\leq \Big(\|\text{curl }\zeta\|_q^q+\Big[q\tilde K_3+\tilde K_4\sup_{0\leq t\leq T}\|u_h^0(t)\|_q^q\Big]\Big)\,e^{q\tilde C_5(M)} \qquad (3.13)\\ &\leq \Big(\|\text{curl }\zeta\|_q^q+\big[q\tilde K_3+\tilde K_4 C(q)^q C_3(M)^q\,2^{q-1}(1+\|\zeta\|^q)\big]\Big)\,e^{q\tilde C_5(M)}.\end{aligned}$$

Since $\sup\{q^{\frac{1}{q}}\,:\,2\leq q<\infty\}<\infty$, letting $n\to\infty$, classical arguments conclude that $\sup_{0\leq t\leq T}\|\xi_{h,n}^0(t)\|_q\leq\|\text{curl }\zeta\|_q+C(T,M,q)\big(1+\|\zeta\|\big)$ for every $h\in\mathcal{A}_M$ and $n\geq 1$. Using (6.3) for some $q_0\in(2,q)$, we deduce that $\sup_{0\leq t\leq T}\|\nabla u_{h,n}^0(t)\|_{q_0}\leq\tilde C_5(M)\big(1+\|\text{curl }\zeta\|_{q_0}+\|\zeta\|\big)$ a.s. Thus, the Sobolev embedding (6.2) yields the existence of a constant $\tilde C_6(M)$ such that $\sup_{0\leq t\leq T}\|u_{h,n}^0(t)\|_{L^\infty(D)}\leq\tilde C_6(M)\big(1+\|\text{curl }\zeta\|_{q_0}+\|\zeta\|\big)$ a.s. for any $n\geq 1$ and $h\in\mathcal{A}_M$. Since D is bounded, using this inequality in (3.13), we deduce the existence of a constant $\tilde C_7(M)$ such that $\sup_{0\leq t\leq T}\|\xi_{h,n}^0(t)\|_q^q\leq\exp(q\tilde C_5(M))\big[\|\text{curl }\zeta\|_q^q+q\tilde C_7(M)\big(1+\|\zeta\|^q+\|\text{curl }\zeta\|_{q_0}^q\big)\big]$ a.s. for every integer $n\geq 1$ and any $h\in\mathcal{A}_M$. Since $q_0\leq q$ and D is bounded, we deduce $\|\text{curl }\zeta\|_{q_0}^q\leq[\lambda(D)\vee 1]^q\|\text{curl }\zeta\|_q^q$. Letting $n\to\infty$ and using classical arguments, we conclude the proof of (3.10).

Step 2. (Uniqueness) Let us mention that the proof of the uniqueness is based on [2] and [36] adapted to the nonhomogeneous random case. Using the estimate (3.11) for some $q>2$ (such as $q=4$) and (6.2), we deduce that any solution u_h^0 to (3.1) belongs to $L^\infty((0,T)\times D)$. Let u_h^0 and v_h^0 be two solutions for equation (3.1) with the same initial condition and let us denote by $z:=u_h^0-v_h^0$; then z is solution of $z(0)=0$ and

$$dz(s)+\big[B(u_h^0(s),u_h^0(s))-B(v_h^0(s),v_h^0(s))\big]\,ds=\big[\tilde\sigma_0(s,u_h^0(s))-\tilde\sigma_0(s,v_h^0(s))\big]\,h(s)ds.$$

Let us multiply the above equation by $z(t)$ and integrate on D, use assumption **(C2)** on $\tilde\sigma_0$, the Schwarz and Hölder inequalities and (3.11). This yields for any $q\in(1,\infty)$, when $q'=\frac{q}{q-1}$ denotes the conjugate exponent of q:

$$\begin{aligned}\frac12\frac{d}{dt}|z(t)|_H^2 &= -(B(z(t),u_h^0(t)),z(t))+\big(\big[\tilde\sigma_0(t,u_h^0(t))-\tilde\sigma_0(t,v_h^0(t))\big]\,h(t),z(t)\big)\\ &\leq\int_D|z(t)|^2(x)|\nabla u_h^0(t)|(x)\,dx+|\big(\tilde\sigma_0(t,u_h^0(t))-\tilde\sigma_0(t,v_h^0(t))\big)|_{L(H_0,H)}|h(t)|_0\,|z(t)|_H\\ &\leq\|\nabla u_h^0(t)\|_q\|z(t)\|_{L^\infty(D)}^{\frac{2}{q}}|z(t)|_H^{\frac{2}{q'}}+\tilde L_1|u_h^0(t)-v_h^0(t)|_H|h(t)|_0|z(t)|_H.\end{aligned}$$

Set $Z:=\sup_{0\leq t\leq T}\|z(t)\|_{L^\infty(D)}$ and $X(t):=|z(t)|_H^2$. Since D is bounded, $\|\text{curl }\zeta\|_q\leq C\|\text{curl }\zeta\|_\infty$ for some constant $C\geq 1$ and all $q\in[2,\infty)$; then $X(0)=0$ and for $t\in[0,T]$, (3.11) yields

$$X'(t) \le 2Cq\,\bar C_4(M)[1+\|\zeta\|+\|\text{curl }\zeta\|_{L^\infty(D)}]Z^{\frac{2}{q}}X(t)^{1-\frac1q}+2\tilde L_1|h(t)|_0X(t),$$

which leads to

$$\int_0^t \frac{X'(s)}{X(s)^{1-1/q}}ds \le 2Cq\,\bar C_4(M)[1+\|\zeta\|+\|\text{curl }\zeta\|_{L^\infty(D)}]Z^{\frac2q}t+\int_0^t 2\tilde L_1|h(s)|_0X(s)^{1/q}ds.$$

Hence, using Gronwall's lemma, we deduce that for $q\in[2,\infty)$ and $t\in[0,T]$,

$$\begin{aligned} X(t)^{\frac1q} &\le 2\bar C_4(M)[1+\|\zeta\|+\|\text{curl }\zeta\|_{L^\infty(D)}]Z^{\frac2q}t+\frac2q\int_0^t \tilde L_1|h(s)|_0X(s)^{\frac1q}ds\\ &\le 2\bar C_4(M)[1+\|\zeta\|+\|\text{curl }\zeta\|_{L^\infty(D)}]Z^{\frac2q}t\exp(\tilde L_1\sqrt{MT}). \end{aligned}$$

Finally, we get the following estimate for any $T^*\in[0,T]$ and $q\in(2,\infty)$:

$$\sup_{0\le t\le T^*}|z(t)|_H^2 \le \Big(2\bar C_4(M)[1+\|\zeta\|+\|\text{curl }\zeta\|_{L^\infty(D)}]T^*\exp(2\tilde L_1\sqrt{MT})\Big)^q Z^2. \tag{3.14}$$

Thus, choosing $T_1^*>0$ small enough and letting $q\longrightarrow\infty$, we deduce that $|z(t)|_H^2=0$ for every $t\in[0,T_1^*]$. Repeating this argument with $u_h^0(T_1^*)=v_h^0(T_1^*)$ instead of ζ and using (6.2), (3.11) and (3.3), we conclude that there exists $T^*>0$ such that $|z(t)|_H^2=0$ for every integer $k=0,1,\cdots$ and any $t\in[T_1^*+kT^*,T_1^*+(k+1)T^*]\cap[0,T]$. This concludes the proof of the uniqueness. □

4. Apriori bounds of the stochastic controlled equation in $H^{1,q}$

In order to prove the large deviation principle for the solution u to (2.1), we need to obtain more regularity and apriori bounds for the solution u_h^ν to the stochastic controlled equation (2.16) in non Hilbert Sobolev spaces, such as $H^{1,q}$ for $q>2$. This requires some more conditions on the diffusion coefficient σ_ν and the stochastic calculus in Banach spaces briefly described in the subsection 6.2 of the Appendix.

Proposition 4.1. *Suppose that $\mathbb{E}|\zeta|^{2p}<\infty$ for some $p\in[2,\infty)$ and let $q\in[2,\infty)$ be such that $E\|\zeta\|_{H^{1,q}}^q<\infty$. Let σ_ν and $\tilde\sigma_\nu$ satisfy conditions* **(C1)-(C4)**, **(C5q)** *and* **(C6q)** *with $K_2\le\bar\kappa_2(p)$, where $\bar\kappa_2(p)$ is defined in Proposition 2.3. Then for every $M>0$ there exists $K_6(M)$ and $\tilde\nu_0:=\tilde\nu_0(M,K_6(M))>0$ such that for $0\le K_6\vee\tilde K_6\le K_6(M)$, $\nu\in(0,\tilde\nu_0]$, and $h\in\mathcal{A}_M$, the solution u_h^ν to (2.16) belongs to $L^\infty(0,T;H^{1,q})$ a.s. Furthermore, there exists a constant $C_5(M,q)$ such that*

$$\sup_{0<\nu\le\tilde\nu_0}\ \sup_{h\in\mathcal{A}_M}\mathbb{E}\Big(\sup_{0\le t\le T}\|u_h^\nu(t)\|_{H^{1,q}}^q\Big)\le C_5(M,q)\big(1+\mathbb{E}\|\zeta\|_{H^{1,q}}^q\big). \tag{4.1}$$

Proof. The Sobolev embedding inequality (6.1) and Proposition 2.3 imply that for $0<\nu\le\bar\nu_0$ and $h\in\mathcal{A}_M$, $\mathbb{E}\big(\sup_{0\le t\le T}\|u_h^\nu(t)\|_q^q\big)\le C(q)^qC_2(q,M)\big(1+\mathbb{E}\|\zeta\|^q\big)$. Using the inequality (6.3), one sees that the proof of (4.1) reduces to check that if $\zeta_h^\nu=\text{curl }u_h^\nu$,

$$\sup_{0<\nu\le\tilde\nu_0(M)}\ \sup_{h\in\mathcal{A}_M}\mathbb{E}\Big(\sup_{0\le t\le T}\|\xi_h^\nu(t)\|_q^q\Big)\le\bar C_6(M,q)\big(1+\mathbb{E}\|\text{curl }\zeta\|_q^q\big). \tag{4.2}$$

We use once more the Galerkin approximation $u_{h,n}^\nu$ of u_h^ν and prove an estimate similar to (4.2) for $\xi_{h,n}^\nu=\text{curl }u_{h,n}^\nu$ with a constant $C_6(M,q)$ which does not depend on n. The process $\xi_{h,n}^\nu$ satisfies an equation similar to (2.26) and once more to ease notations, we will skip the index n. Let $\langle .,.\rangle$ denote the duality between $L^p(D)$ and $L^{p/(p-1)}(D)$ for some

$p \in (1,\infty)$. For fixed $N>0$, let $\tau_N = \inf\{t\geq 0 : \|\xi^\nu_{h,n}(t)\|_q \geq N\}\wedge T$. The Itô formula (6.11) and the upper estimate (6.12) yield

$$\|\xi^\nu_h(t\wedge\tau_N)\|_q^q \leq \|\mathrm{curl}\ \zeta\|_q^q + J(t) + \sum_{1\leq i\leq 4} T_i(t), \tag{4.3}$$

where we have:

$$\begin{aligned}
J(t) &= q\sqrt{\nu}\int_0^{t\wedge\tau_N} \langle |\xi^\nu_h(s)|^{q-2}\xi^\nu_h(s)\,,\,\mathrm{curl}\ \sigma_\nu(s,u^\nu_h(s))dW(s)\rangle,\\
T_1(t) &= -q\nu\int_0^{t\wedge\tau_N}\langle |\xi^\nu_h(s)|^{q-2}\xi^\nu_h(s)\,,\,A\xi^\nu_h(s)\rangle ds,\\
T_2(t) &= -q\int_0^{t\wedge\tau_N}\langle |\xi^\nu_h(s)|^{q-2}\xi^\nu_h(s)\,,\,\mathrm{curl}\ B(u^\nu_h(s),u^\nu_h(s)\rangle ds,\\
T_3(t) &= q\int_0^{t\wedge\tau_N}\langle |\xi^\nu_h(s)|^{q-2}\xi^\nu_h(s)\,,\,\mathrm{curl}\,\tilde{\sigma}_\nu(s,u^\nu_h(s))h(s)\rangle ds,\\
T_4(t) &= \frac{q}{2}(q-1)\nu\int_0^{t\wedge\tau_N}\|\mathrm{curl}\ \sigma_\nu(s,u^\nu_h(s))\|^2_{R(H_0,L^q)}\|\xi^\nu_h(s)\|_q^{q-2}\,ds.
\end{aligned}$$

Since $\xi^\nu_h=0$ on ∂D and $A=-\Delta$, we have:

$$\begin{aligned}
T_1(t) &= -q\nu\int_0^{t\wedge\tau_N} ds\int_D \langle\nabla\big(|\xi^\nu_h(s)|^{q-2}\xi^\nu_h(s)\big)\,,\,\nabla\xi^\nu_h(s)\rangle\,dx\\
&= -q(q-1)\nu\int_0^{t\wedge\tau_N}ds\int_D |\xi^\nu_h(s)|^{q-2}\,|\nabla\xi^\nu_h(s)|^2dx.
\end{aligned}$$

Equation (6.7) implies that $T_2(t)=0$. Hölder's and Young's inequalities and the assumption **(C6q)** yield for $q'=\frac{q}{q-1}$ and for any $\epsilon>0$,

$$\begin{aligned}
T_3(t) &\leq q\int_0^{t\wedge\tau_N}\||\xi^\nu_h(s)|^{q-1}\|_{q'}\,\|\mathrm{curl}\ \tilde{\sigma}_\nu(s,u^\nu_h(s))h(s)\|_q\,ds\\
&\leq q\int_0^{t\wedge\tau_N}\|\xi^\nu_h(s)\|_q^{q-1}\Big[\tilde{K}_3+\tilde{K}_4\|u^\nu_h(s)\|_q+\tilde{K}_5\|\xi^\nu_h(s)\|_q\\
&\qquad +\sqrt{\nu}\tilde{K}_6\Big(\int_D|\xi^\nu_h(s)|^{q-2}|\nabla\xi^\nu_h(s)|^2dx\Big)^{\frac{1}{q}}\Big]\,|h(s)|_0ds\\
&\leq q\tilde{K}_3\sqrt{MT}+\tilde{K}_4\int_0^{t\wedge\tau_N}\|u^\nu_h(s)\|_q^q\,|h(s)|_0\,ds+\nu\epsilon\int_0^{t\wedge\tau_N}ds\int_D|\xi^\nu_h(s)|^{q-2}|\nabla\xi^\nu_h(s)|^2dx\\
&\quad+\int_0^{t\wedge\tau_N}\|\xi^\nu_h(s)\|_q^q\Big(q\big[\tilde{K}_3+\tilde{K}_5+\tilde{K}_4\big]|h(s)|_0+(q-1)\frac{\tilde{K}_6^{q/(q-1)}}{\epsilon^{1/(q-1)}}\nu^{\frac{q-2}{2q-2}}|h(s)|_0^{\frac{q}{q-1}}\Big)ds.
\end{aligned}$$

Condition **(C5q)**, Hölder's and Young's inequalities imply that for any $\eta\in(0,\frac{q}{2}-1)$, $\bar{\eta}=1-\eta\frac{2}{q-2}\in(0,1)$ and $\nu\in(0,1]$,

$$\begin{aligned}
T_4(t) &\leq \frac{q(q-1)}{2}\nu\int_0^{t\wedge\tau_N}\|\xi^\nu_h(s)\|_q^{q-2}\Big[K_3+K_4\|u^\nu_h(s)\|_q^2+K_5\|\xi^\nu_h(s)\|_q^2\\
&\qquad + K_6\Big(\int_D|\xi^\nu_h(s)|^{q-2}|\nabla\xi^\nu_h(s)|^2dx\Big)^{\frac{2}{q}}\Big]ds\\
&\leq \frac{q(q-1)}{2}\nu K_3T+\nu(q-1)K_4\int_0^{t\wedge\tau_N}\|u^\nu_h(s)\|_q^qds\\
&\quad+\frac{q-1}{2}\int_0^{t\wedge\tau_N}\big[\nu q(K_3+K_5)+\nu K_4(q-2)+\nu^{\bar{\eta}}(q-2)K_6^{\frac{q}{q-2}}\big]\|\xi^\nu_h(s)\|_q^qds
\end{aligned}$$

$$+ (q-1)\nu^{1+\eta} \int_0^{t\wedge\tau_N} ds \int_D |\xi_h^\nu(s)|^{q-2} |\nabla \xi_h^\nu(s)|^2 dx.$$

For $t \in [0,T]$, let

$$X(t) = \sup_{0\le s\le t} \|\xi_h^\nu(s\wedge\tau_N)\|_q^q \quad \text{and} \quad Y(t) = \int_0^{t\wedge\tau_N} ds \int_D |\xi_h^\nu(s)|^{q-2} |\nabla \xi_h^\nu(s)|^2 dx.$$

Then for any $\lambda \in (0,1)$, the inequality (4.3) and the above estimates of $T_i(t)$ imply that

$$\lambda X(t) + (1-\lambda)\|\xi_h^\nu(t\wedge\tau_N)\|_q^q + \nu\big[q(q-1)(1-\lambda) - \epsilon - (q-1)\nu^\eta\big] Y(t) \le Z + \int_0^t \varphi(s) X(s) ds + I(t),$$

where

$$I(t) = \sup_{0\le s\le t} J(s),$$

$$Z = \|\text{curl } \zeta\|_q^q + q\tilde{K}_3\sqrt{MT} + \frac{q(q-1)}{2}\nu K_3 T + \int_0^{t\wedge\tau_N} \big[\nu K_4(q-1) + \tilde{K}_4|h(s)|_0\big] \|u_h^\nu(s)\|_q^q ds,$$

$$\varphi(s) = [q(\tilde{K}_3 + \tilde{K}_4 + \tilde{K}_5)|h(s)|_0 + (q-1)\frac{\tilde{K}_6^{q/(q-1)}}{\epsilon^{1/(q-1)}} \nu^{\frac{q-2}{2q-2}} |h(s)|_0^{\frac{q}{q-1}}$$
$$+ \frac{q-1}{2}\big[\nu q(K_3 + K_4 + K_5) + \nu^{\bar\eta}(q-2) K_6^{\frac{q}{q-2}}\big].$$

Set $\lambda = \frac12$, $\epsilon = \frac{q(q-1)}{8}$, let $\nu_0 \in (0, \bar\nu_0(M)]$ be such that $\nu_0^\eta \le \frac{q}{8}$, and let $\tilde\eta = \frac{q-2}{2q-2} \wedge \bar\eta$. Thus $C(\nu, K_6, \tilde{K}_6) = \int_0^T \varphi(s) ds = \Phi_1(M, \tilde{K}_6) + \nu^{\tilde\eta}\Phi_2(M, K_6, \tilde{K}_6)$. For $\nu \in (0,\nu_0]$, we have:

$$\frac12 X(t) + \frac{\nu}{4} q(q-1) Y(t) \le Z + \int_0^t \varphi(s) X(s) ds + I(t).$$

Furthermore, using the Burkholder-Davies-Gundy inequality (6.10), condition **(C5q)**, Hölder's and Young's inequalities, we deduce that for any $\beta > 0$,

$$\mathbb{E} I(t) \le \sqrt{\nu} C_1 q \mathbf{E}\Big(\int_0^{t\wedge\tau_N} \|\text{curl } \sigma_\nu(s, u_h^\nu(s))\|_{R(H_0,L_q)}^2 \|\xi_h^\nu(s)\|_q^{2(q-1)} ds\Big)^{\frac12}$$
$$\le \sqrt{\nu} C_1 q \mathbb{E}\Big(\sup_{0\le s\le t} \|\xi_h^\nu(s\wedge\tau_N)\|_q^{\frac{q}{2}} \Big[\int_0^{t\wedge\tau_N} \|\xi_h^\nu(s)\|_q^{q-2} \Big\{ K_3 + K_4 \|u_h^\nu(s)\|_q^2$$
$$+ K_5 \|\xi_h^\nu(s)\|_q^2 + K_6 \Big(\int_D |\xi_h^\nu(s)|^{q-2} |\nabla \xi_h^\nu(s)|^2 dx\Big)^{\frac{2}{q}} \Big\} ds\Big]^{\frac12}\Big)$$
$$\le \beta \mathbb{E} X(t) + \gamma \mathbf{E} \int_0^t X(s) ds + \delta \mathbf{E} \int_0^t Y(s) ds + \bar{Z},$$

where

$$\gamma = \frac{1}{4\beta}\nu C_1^2\big[q^2(K_3 + K_5) + qK_4(q-2) + K_6 q(q-2)\big], \quad \delta = \frac{1}{2\beta}\nu q C_1^2 K_6,$$
$$\bar{Z} = \frac{K_4}{2\beta}\nu q C_1^2 \mathbb{E} \int_0^{t\wedge\tau_N} \|u_h^\nu(s)\|_q^q ds.$$

Set $\alpha = \frac{\nu}{4} q(q-1)$, and choose $\beta > 0$ such that $2\beta e^{\Phi_1(M,1)} = 1/2$. Choosing $K_6(M) \le 1$ small enough, we may have for $0 \le K_6 \vee \tilde{K}_6 \le K_6(M)$: $16 K_6 e^{2\Phi_1(M,\tilde{K}_6)} \frac{q}{q-1} C_1^2 \le \frac12$. Then choosing $\tilde\nu_0(M) \in (0,\nu_0]$ small enough, we may have $e^{2\tilde\nu_0(M)^{\tilde\eta}\Phi_2(M, K_6(M), K_6(M))} \le 2$. This yields $2\delta e^{C(\nu, K_6, \tilde{K}_6)} \le \alpha$ for $\nu \in (0, \tilde\nu_0(M)]$ and $K_6 \vee \tilde{K}_6 \le K_6(M)$. Thus, using Lemma A1 in [16], we conclude that (4.2) holds for the Galerkin approximation $\xi_{h,n}^\nu$ of ξ_h^ν with a constant $C_6(M,q)$ which does not depend on n. A classical weak convergence argument concludes the proof. $\square$

5. Large deviations

We will prove a large deviation principle using a weak convergence approach [11, 12], based on variational representations of infinite dimensional Wiener processes.

Let $\sigma : [0,T]\times V \to L_Q$, and for every $\nu>0$ let $\bar\sigma_\nu : [0,T]\times Dom(A)\to L_Q$ satisfy the following:

Condition (C7)

(i) For every $q\in[2,\infty)$ there exist non negative constants, $\bar K_i$, $i=0,...,4$ and $\bar L_1$ such that for all $u,v\in H^{1,q}$ and $t\in[0,T]$:

$$|\sigma(t,u)|^2_{L_Q}\le \bar K_0+\bar K_1\,|u|^2_H,\quad \big|\text{curl }\sigma(t,u)\big|^2_{L_Q}\le \bar K_0+\bar K_1\,\|u\|^2_V,$$
$$\|\text{curl }\sigma(t,u)\|^2_{R(H_0,L^q)}\le \bar K_2+\bar K_3\|u\|^2_q+\bar K_4\|\text{curl }u\|^2_q,$$
$$|\sigma(t,u)-\sigma(t,v)|^2_{L_Q}\le \bar L_1\,|u-v|^2_H,\quad \big|A^{\frac12}\sigma(t,u)-A^{\frac12}\sigma(t,v)\big|^2_{L_Q}\le \bar L_1\,\|u-v\|^2_V,$$

(ii) For every $q\in[2,\infty)$ there exist non negative constants $\bar K_0$, $\bar K_5$, $\bar K_6$ and $\bar L_2$ such that for $\nu>0$, $s,t\in[0,T]$, and $u,v\in Dom(A)\bigcap H^{2,q}$, if $|.|$ denotes the absolute value,

$$|\bar\sigma_\nu(t,u)|^2_{L_Q}\le \big(\bar K_0+\bar K_5\,\|u\|^2_V\big),\quad \big|\text{curl }\bar\sigma_\nu(t,u)\big|^2_{L_Q}\le\big(\bar K_0+\bar K_5\,|\nabla\text{curl }u|^2_H\big),$$
$$\|\text{curl }\bar\sigma_\nu(t,u)\|^2_{R(H_0,L^q)}\le \bar K_0+\bar K_6\Big(\int_D|\text{curl }u(x)|^{q-2}|\nabla\text{curl }u(x)|^2dx\Big)^{\frac2q},$$
$$|\bar\sigma_\nu(t,u)-\bar\sigma_\nu(t,v)|^2_{L_Q}\le \bar L_2\,\|u-v\|^2_V,\ \big|A^{\frac12}\bar\sigma_\nu(t,u)-A^{\frac12}\bar\sigma_\nu(t,v)\big|^2_{L_Q}\le \bar L_2\,|Au-Av|^2_H.$$

Note that as in the case of Hilbert-Schmidt operators with Hilbert spaces, we have $\|T\|_{L(H_0,L^q)}\le C\|T\|_{R(H_0,L^q)}$. Set

$$\sigma_\nu=\tilde\sigma_\nu=\sigma+\sqrt{\nu}\bar\sigma_\nu\ \text{ for }\quad \nu>0,\quad\text{and}\quad \tilde\sigma_0=\sigma. \tag{5.1}$$

Then for $\nu\ge0$, the coefficients σ_ν and $\tilde\sigma_\nu$ satisfy the conditions **(C1)**-**(C6q)** with appropriate coefficients. Indeed, note that conditions **(C1)** and **(C3)** hold with $a=1$ and that $K_2\le 2\bar K_5$, $\tilde K_2=\bar K_5$, $L_2\le 2\bar L_2$, $\tilde L_2=\bar L_2$, $K_6\le 2\nu\bar K_6$ and $\tilde K_6=C\bar K_6$. Using Remark 2.2 we deduce that for any coefficients $\bar K_2$ and $\bar K_6$, the conditions of Propositions 2.1, 2.3 and 4.1 are satisfied for small enough $\bar\nu_0$ and $\nu\in(0,\bar\nu_0]$.

Let $\mathcal B$ denote the Borel $\sigma-$field of the Polish space

$$\mathcal X=C\big([0,T];H\big)\bigcap L^\infty\big(0,T;H^{1,q}\cap V\big)\bigcap L^2\big(0,T;\mathcal H\big) \tag{5.2}$$

endowed with the norm

$$\|u\|_{\mathcal X}:=\Big(\int_0^T\|u(t)\|^2_{\mathcal H}dt\Big)^{1/2}$$

and

$$\mathcal Y=\{\zeta\in V,\ \text{such that curl }\zeta\in L^\infty(D)\} \tag{5.3}$$

endowed with the norm $\|.\|_{\mathcal Y}$ defined by:

$$\|\zeta\|^2_{\mathcal Y}:=\|\zeta\|^2+\|\text{curl}\zeta\|^2_{L^\infty}.$$

Note that using (6.3) and (6.1) we deduce that $\mathcal Y\subset H^{1,q}$ for any $q\in[2,\infty)$. We will establish a LDP in the set $\mathcal X$ for the family of distributions of the solutions $u^\nu=\mathcal G^\nu_\zeta(\sqrt{\nu}W)$ to the evolution equation (2.12) with initial condition $u^\nu(0)=\zeta\in\mathcal Y$.

Definition 5.1. *The random family (u^ν) is said to satisfy a large deviation principle on $\mathcal{X}$ with the good rate function I if the following conditions hold:*

I is a good rate function. *The function $I : \mathcal{X} \to [0,\infty]$ is such that for each $M \in [0,\infty[$ the level set $\{\phi \in \mathcal{X} : I(\phi) \le M\}$ is a compact subset of $\mathcal{X}$.*
For $A \in \mathcal{B}$, set $I(A) = \inf_{u\in A} I(u)$.

Large deviation upper bound. *For each closed subset F of $\mathcal{X}$:*

$$\limsup_{\nu\to 0} \nu \log \mathbb{P}(u^\nu \in F) \le -I(F).$$

Large deviation lower bound. *For each open subset G of $\mathcal{X}$:*

$$\liminf_{\nu\to 0} \nu \log \mathbb{P}(u^\nu \in G) \ge -I(G).$$

Let $\mathcal{C}_0 = \{\int_0^\cdot h(s)ds \; : \; h \in L^2(0,T;H_0)\} \subset \mathcal{C}([0,T];H_0)$. Given $\zeta \in \mathcal{Y}$ define $\mathcal{G}^0_\zeta : \mathcal{C}([0,T];H_0) \to \mathcal{X}$ by $\mathcal{G}^0_\zeta(g) = u^0_h$ where $g = \int_0^\cdot h(s)ds \in \mathcal{C}_0$ and u^0_h is the solution to the (inviscid) control equation (3.1) with initial condition ζ, and $\mathcal{G}^0_\zeta(g) = 0$ otherwise. The following theorem is the main result of this section.

Theorem 5.2. *Let $\zeta \in \mathcal{Y}$, and let $\sigma_\nu = \sigma + \sqrt{\nu}\bar{\sigma}_\nu$ where the coefficients σ and $\bar{\sigma}_\nu$ satisfy condition* (**C7**). *Then the solution $(u^\nu, \nu > 0)$ to* (2.12) *with initial condition ζ satisfies a large deviation principle in $\mathcal{X}$ with the good rate function*

$$I(u) = \inf_{\{h\in L^2(0,T;H_0):\, u=\mathcal{G}^0_\zeta(\int_0^\cdot h(s)ds)\}} \Big\{ \frac{1}{2} \int_0^T |h(s)|_0^2\, ds \Big\}. \tag{5.4}$$

In order to prove this theorem, fix $q, p \in [4,\infty)$, $M > 0$ and let $\nu_0(M) \in (0, \nu_0 \wedge \bar{\nu}_0 \wedge \tilde{\nu}_0]$ be small enough to ensure that for $\nu \in (0, \nu_0(M)]$, $2\nu \bar{K}_5 \le \kappa_2(p) \wedge \bar{\kappa}_2(p)$, $2\nu \bar{K}_6 \le K_6(M)$, where $\kappa_2(p)$, $\bar{\kappa}_2(p)$, $K_6(M)$, ν_0, $\bar{\nu}_0$ and $\tilde{\nu}_0$ are defined in Propositions 2.1, 2.3 and 4.1 applied with q.

Let $(h_\nu, 0 < \nu \le \tilde{\nu}_0(M))$ be a family of random elements taking values in the set $\mathcal{A}_M$ defined by (2.13). Let $u^\nu_{h_\nu}$ be the solution of the following corresponding stochastic controlled equation

$$du^\nu_{h_\nu}(t) + \big[\nu A u^\nu_{h_\nu}(t) + B(u^\nu_{h_\nu}(t), u^\nu_{h_\nu}(t))\big]dt = \sqrt{\nu}\, \sigma_\nu(t, u^\nu_{h_\nu}(t))\, dW(t) + \sigma_\nu(t, u^\nu_{h_\nu}(t)) h_\nu(t)dt, \tag{5.5}$$

with initial condition $u^\nu_{h_\nu}(0) = \xi \in \mathcal{Y}$. Note that $u^\nu_{h_\nu} = \mathcal{G}^\nu_\xi\Big(\sqrt{\nu}\big(W_\cdot + \frac{1}{\sqrt{\nu}} \int_0^\cdot h_\nu(s)ds\big)\Big)$ due to the uniqueness of the solution. The following proposition establishes the weak convergence of the family $(u^\nu_{h_\nu})$ as $\nu \to 0$.

Proposition 5.3. *Let us assume that $\sigma_\nu = \sigma + \sqrt{\nu}\bar{\sigma}_\nu$ where the coefficients σ and $\bar{\sigma}_\nu$ satisfy condition* (**C7**). *Let ζ be $\mathcal{F}_0$-measurable such that $\mathbb{E}\big(|\zeta|_H^4 + \|\zeta\|^2_{\mathcal{Y}}\big) < +\infty$, and let h_ν converge to h in distribution as random elements taking values in $\mathcal{A}_M$, where this set is defined by* (2.13) *and endowed with the weak topology of the space $L_2(0,T;H_0)$. Then, as $\nu \to 0$, the solution $u^\nu_{h_\nu}$ of* (5.5) *converges in distribution in $\mathcal{X}$ to the solution u^0_h of* (3.1). *That is, as $\nu \to 0$, the process $\mathcal{G}^\nu_\zeta\Big(\sqrt{\nu}\big(W_\cdot + \frac{1}{\sqrt{\nu}} \int_0^\cdot h_\nu(s)ds\big)\Big)$ converges in distribution to $\mathcal{G}^0_\xi\big(\int_0^\cdot h(s)ds\big)$ in $\mathcal{X}$.*

Proof. **Step 1:** Let us decompose $u^\nu_{h_\nu} = \zeta + \sum_{i=1}^4 J_i$, where

$$J_1 = -\nu \int_0^t A u^\nu_{h_\nu}(s) ds, \qquad J_2 = -\int_0^t B(u^\nu_{h_\nu}(s), u^\nu_{h_\nu}(s)) ds,$$

$$J_3 = \sqrt{\nu} \int_0^t \sigma_\nu(s, u^\nu_{h_\nu}(s)) dW(s), \qquad J_4 = \int_0^t \sigma_\nu(s, u^\nu_{h_\nu}(s)) h_\nu(s) ds.$$

For $\nu\in(0,\nu_0(M)]$ we have using Minkowski's and Cauchy-Schwarz's inequalities

$$\begin{aligned}\|J_1\|^2_{W^{1,2}(0,T;H)} &= \nu\int_0^T\Big|\int_0^t Au^\nu_{h_\nu}(s)ds\Big|^2_H dt+\nu\int_0^T|Au^\nu_{h_\nu}(t)|^2_H dt\\ &\le C(T)\nu\int_0^T|Au^\nu_{h_\nu}(s)|^2_H ds.\end{aligned}$$

Hence, using the estimate (2.23), we get that

$$\mathbb{E}\|J_1\|^2_{W^{1,2}(0,T;H)}\le\tilde C_1(M,T)[1+\mathbb{E}\|\zeta\|^4]. \tag{5.6}$$

Similarly, the upper estimate (2.23) implies that for all $p\in[2,\infty)$,

$$\begin{aligned}\mathbb{E}\|J_1\|^p_{W^{1,p}(0,T;V')} &\le \nu C(T)\mathbb{E}\int_0^T\|Au^\nu_{h_\nu}(s)\|^p_{V'}ds\\ &\le \nu C(T)\mathbb{E}\int_0^T\|u^\nu_{h_\nu}(s)\|^p ds\le C(T)[1+\mathbb{E}\|\zeta\|^p].\end{aligned} \tag{5.7}$$

Using again Minkowski's and Hölder's inequalities and the estimate (6.8), we deduce that for $4\le p<q<\infty$

$$\|J_2\|^p_{W^{1,p}(0,T;H)}\le C(T)\int_0^T\|u^\nu_{h_\nu}(t)\|^p_{H^{1,q}}\|u^\nu_{h_\nu}(t)\|^p dt.$$

Thus Hölder's inequality with the conjugate exponents q/p and $q/(q-p)$ and the upper estimates (2.23) and (4.1) yield

$$\mathbb{E}\|J_2\|^p_{W^{1,p}(0,T;H)}\le C(T,M,p,q)[1+\mathbb{E}\|\zeta\|^{pq/(q-p)}]^{1-p/q}\,[1+\mathbb{E}\|\zeta\|^q_{H^{1,q}}]^{p/q}. \tag{5.8}$$

The Minkowski and Cauchy Schwarz inequalities and condition **(C7)** imply

$$\begin{aligned}\|J_4\|^2_{W^{1,2}(0,T;H)} &\le C(T)\int_0^T\|\sigma_\nu(s,u^\nu_{h_\nu}(s))\|^2_{L_Q}|h_\nu(s)|^2_0 ds\\ &\le C(T)M\Big[1+\sup_{0\le t\le T}|u^\nu_{h_\nu}(t)|^2_H+\nu\sup_{0\le t\le T}\|u^\nu_{h_\nu}(t)\|^2\Big].\end{aligned}$$

Thus the upper estimates (2.17) and (2.23) yield that for $\nu\in(0,\bar\nu_0]$ one has:

$$\mathbb{E}\|J_4\|^2_{W^{1,2}(0,T;H)}\le C(T,M)\big[1+\mathbb{E}|\zeta|^4_H+\bar\nu_0\mathbb{E}\|\zeta\|^4_H\big]. \tag{5.9}$$

Furthermore, Hölder's inequality and condition **(C7)** imply that for $\nu\in(0,\bar\nu_0]$ and $p\in[4,\infty)$:

$$\int_0^T|J_4(t)|^p_H dt\le M^{\frac p2}C\big[1+\sup_{s\le T}|u^\nu_{h_\nu}(s)|^p_H+\bar\nu_0^{p/2}\sup_{s\le T}\|u^\nu_{h_\nu}(s)\|^p\big].$$

Let $\alpha\in(0,\frac12)$; then using again Minkowski's and Hölder's inequalities, condition **(C7)** and Fubini's theorem, we deduce that for $\nu\in(0,\bar\nu_0]$:

$$\begin{aligned}&\int_0^T\int_0^T\frac{|J_4(t)-J_4(s)|^p_H}{(t-s)^{1+\alpha p}}dsdt\\ &\le 2\int_0^T dt\int_0^t ds(t-s)^{-1-\alpha p}\Big|\int_s^t|\sigma_\nu(r,u^\nu_{h_\nu}(r))|_{L_Q}|h_\nu(r)|_0dr\Big|^p\\ &\le CM^{\frac p2}\big[1+\sup_{r\le T}|u^\nu_{h_\nu}(r)|^p_H+\bar\nu_0^{p/2}\sup_{r\le T}\|u^\nu_{h_\nu}(r)\|^p\big]\int_0^T dt\int_0^t(t-s)^{-1+(1/2-\alpha)p}ds.\end{aligned}$$

The two above estimates, (2.17) and (2.23) imply that for $\alpha\in(0,\frac12)$, $p\in[4,\infty)$ and $\nu\in(0,\bar\nu_0]$:

$$\mathbb{E}\|J_4\|^p_{W^{\alpha,p}(0,T;H)}\le C(p,\alpha,T,M)\big[1+\mathbb{E}|\zeta|^p_H+\bar\nu_0^{p/2}\mathbb{E}\|\zeta\|^p\big]. \tag{5.10}$$

The Burkholder-Davis-Gundy and Hölder inequalities imply

$$\begin{aligned}\mathbb{E}\int_0^T |J_3(t)|_H^p dt &\leq C_p \nu^{p/2}\int_0^T \mathbb{E}\Big(\int_0^T |\sigma_\nu(s,u_{h_\nu}^\nu(s))|_{L_Q}^2 ds\Big)^{p/2} dt\\ &\leq C_p T^{p/2-1}\nu^{p/2}\int_0^T \mathbb{E}|\sigma_\nu(s,u_{h_\nu}^\nu(t))|_{L_Q}^p dt.\end{aligned}$$

Let $p\in[4,\infty)$, $\alpha\in(0,\frac12)$ and for $t\in[0,T]$ set $\phi(t):=\int_0^t |\sigma_\nu(s,u_{h_\nu}^\nu(s))|_{L_Q}^2 ds$; then the Burkholder-Davis-Gundy and Hölder inequalities imply

$$\begin{aligned}\mathbb{E}\int_0^T\int_0^T \frac{|J_3(t)-J_3(s)|_H^p}{|t-s|^{1+p\alpha}}dtds &= \nu^{p/2}\int_0^T dt\int_0^T ds\frac{\mathbb{E}|\int_{s\wedge t}^{s\vee t}\sigma_\nu(r,u_{h_\nu}^\nu(r))dW(r)|_H^p}{|t-s|^{1+p\alpha}}\\ &\leq C_p\nu^{p/2}\int_0^T\int_0^T \mathbb{E}\left|\int_{s\wedge t}^{s\vee t}|\sigma_\nu(r,u_{h_\nu}^\nu(r))|_{L_Q}^2 dr\right|^{p/2}|t-s|^{-(1+p\alpha)}dtds\\ &\leq C_p\nu^{p/2}\mathbb{E}\int_0^T\int_0^T |\phi(t)-\phi(s)|^{p/2}|t-s|^{-(1+p\alpha)}dtds\\ &\leq C_p\nu^{p/2}\mathbb{E}\|\phi\|_{W^{2\alpha,p/2}(0,T;\mathbb{R})}^{\frac p2}\\ &\leq C_p\nu^{p/2}\mathbb{E}\|\phi\|_{W^{1,p/2}(0,T;\mathbb{R})}^{\frac p2}\\ &\leq C_pC(T)\nu^{p/2}\mathbb{E}\int_0^T|\sigma_\nu(s,u_{h_\nu}^\nu(s))|_{L_Q}^p ds.\end{aligned}$$

Using the assumption **(C7)** and the two above upper estimates of J_3, we deduce that

$$\mathbb{E}\|J_3\|_{W^{\alpha,p}(0,T;H)}^p \leq C(p,T)\nu^{p/2}\big[1+\sup_{0\leq t\leq T}\mathbb{E}|u_{h_\nu}^\nu(t)|_H^p+\nu^{p/2}\sup_{0\leq t\leq T}\mathbb{E}\|u_{h_\nu}^\nu(t)\|^p\big].$$

Finally, the upper estimates (2.17) and (2.23) yield for $\nu\in(0,\bar\nu_0]$ and $p\in[4,\infty)$:

$$\mathbb{E}\|J_3\|_{W^{\alpha,p}(0,T;H)}^p\leq C(p,T)\bar\nu_0^{p/2}\big[1+\mathbb{E}|\zeta|_H^p+\bar\nu_0^{p/2}\mathbb{E}\|\zeta\|^p\big]. \qquad (5.11)$$

Collecting all the estimates (5.6)-(5.11) we deduce that for $p\in[4,\infty)$, $\alpha\in(0,1/2)$ and $\nu\in(0,\bar\nu_0]$, there exists a positive constant $C(p,M,T)$ such that

$$\mathbb{E}\|u_{h_\nu}^\nu\|_{W^{\alpha,2}(0,T;H)}^2+\mathbb{E}\|u_{h_\nu}^\nu\|_{W^{\alpha,p}(0,T;V')}^p\leq C_{p,M,T}. \qquad (5.12)$$

Step 2: The upper estimates (2.23) and (5.12) show that the process $(u_{h_\nu}^\nu,\nu\in(0,\bar\nu_0])$ is bounded in probability in

$$W^{\alpha,2}(0,T;H)\bigcap L^2(0,T;V)\bigcap W^{\alpha,p}(0,T;V').$$

Thanks to the compactness theorem given in [27], Chapter 1, Section 5. the space $W^{\alpha,2}(0,T;H)\bigcap L^2(0,T;V)$ is compactly embedded in $L^2(0,T;\mathcal{H})$. For $p\alpha>1$, thanks to Theorem 2.2 given in [23], see also [10] and the references therein, the space $W^{\alpha,p}(0,T;V')$ is compactly embedded in $C([0,T];D(A^{-\beta}))$ with $2\beta>1$.

On the other hand, the family (h_ν) is included in $\mathcal{A}_M$. Set $F_\nu(t)=\int_0^t h_\nu(s)ds$; since H_0 is compactly embedded in H, we can again use the above compact embedding theorem and deduce that $W^{1,2}(0,T;H_0)$ is compactly embedded in $C([0,T];H)$. Furthermore, by assumption $h_\nu\to h$ in distribution in $L^2(0,T;H_0)$ endowed with the weak topology. This yields that $F_\nu\to F$ in distribution in the weak topology of $W^{1,2}(0,T;H_0)$ (denoted by $W^{1,2}(0,T;H_0)_w$), where $F(t):=\int_0^t h(s)ds$.

Hence, by the Prokhorov theorem, the family of distributions $(\mathcal{L}(h_\nu, u^\nu_{h_\nu}, \nu \in (0,\bar\nu_0])$ of the process $(F_\nu, u^\nu_{h_\nu}, \nu \in (0,\bar\nu_0])$ is tight in

$$\mathcal{Z} = \big[W^{1,2}(0,T;H_0)_w \cap C([0,T],H)\big] \times \big[L^2(0,T;\mathcal{H}) \cap C([0,T];D(A^{-\beta}))\big].$$

Let $(\nu_n, n\geq 0)$ be a sequence in $(0,\bar\nu_0]$ such that $\nu_n\to 0$. Thus, we can extract a subsequence, still denoted by $(F_{\nu_n}, u^{\nu_n}_{h_{\nu_n}})$, that converges in distribution in $\mathcal{Z}$ to a pair $(\bar F, \bar u)$ as $n\to\infty$. Note that by assumption, $\bar F = F$.

Step 3: By Skorohod's theorem, there exists a stochastic basis $(\Omega^1, \mathcal{F}^1, (\mathcal{F}^1_t), \mathbb{P}^1)$ and on this basis, $\mathcal{Z}$- valued random variables $(F^1 = \int_0^\cdot h^1(s)ds, u^1)$ and for $n\geq 0$ $(F^{\nu_n,1} = \int_0^\cdot h^{\nu_n,1}(s)ds, u^{\nu_n,1}_{h^{\nu_n,1}})$, such that the pairs (F^1,u^1) and $(\bar F, \bar u)$ have the same distribution, for $n\geq 0$ the pairs $(F^{\nu_n,1}, u^{\nu_n,1}_{h^{\nu_n,1}})$ and $(F_{\nu_n}, u^{\nu_n}_{h_{\nu_n}})$ have the same distribution on $\mathcal{Z}$, and as $n\to\infty$, $(F^{\nu_n,1}, u^{\nu_n,1}_{h^{\nu_n,1}}) \longrightarrow (F^1,u^1)$ in $\mathcal{Z}$. To ease notations in the sequel, we will skip the upper index 1 and the index n of the subsequence and still denote $F^{1,\nu}$ by F_ν, $h^{1,\nu}$ by h_ν, $u^{1,\nu}_{h^{1,\nu}}$ by $u^\nu_{h_\nu}$, F^1 by F, h^1 by h and u^1 by $\bar u$. Let again ζ denote the initial condition $u^{\nu,1}_{h^{1,\nu}}(0)$.
Moreover, by (2.17), (2.23) and (4.1) we deduce the existence of constants C_i such that for $\nu\geq 0$ and for $q\in[2,\infty)$:

$$\mathbb{E}_1\Big(\sup_{0\leq t\leq T} |u^\nu_{h_\nu}(t)|^2_H\Big) \leq C_1, \quad \mathbb{E}_1\int_0^T \|u^\nu_{h_\nu}(t)\|^2 dt \leq C_2, \quad \mathbb{E}_1\Big(\sup_{0\leq t\leq T} \|u^\nu_{h_\nu}(t)\|^q_{H^{1,q}(D)}\Big) \leq C_3.$$

Therefore, we deduce as $n\to\infty$

$$\bar u \in C([0,T];H) \bigcap L^\infty\big(0,T; V\bigcap H^{1,q}\big) \;\; \mathbb{P}^1 - \text{a.s.} \tag{5.13}$$

and that $u^{\nu_n}_{h_{\nu_n}} \longrightarrow \bar u$ weakly in $L^2(\Omega^1\times(0,T);V)\bigcap L^q(\Omega^1\times(0,T);H^{1,q})$ as $n\to\infty$.

Step 4: (Identification of the limit.) We have to prove that the limit $\bar u$ is solution of the equation

$$d\bar u(t) + B(\bar u(t),\bar u(t))\,dt = \sigma(t,\bar u(t))\,h(t)\,dt\;, \quad \bar u(0)=\zeta. \tag{5.14}$$

Let $\varphi\in D(A^\beta)$ with $2\beta>1$; then

$$(u^\nu_{h_\nu}(t)-\zeta,\varphi) + \int_0^t \langle B(\bar u(s),\bar u(s)) - \sigma(s,\bar u(s))h(s),\varphi\rangle ds = \sum_{1\leq i\leq 6} I_i, \tag{5.15}$$

where

$$\begin{aligned}
I_1 &= -\nu\int_0^t \big(Au^\nu_{h_\nu}(s),\varphi\big)\,ds, \quad I_2 = \sqrt{\nu}\int_0^t \big(\sigma_\nu(s,u^\nu_{h_\nu}(s))dW(s),\varphi\big)\,,\\
I_3 &= -\int_0^t\big[\,\big\langle B(u^\nu_{h_\nu}(s)-\bar u(s), u^\nu_{h_\nu}(s)),\varphi\big\rangle + \big\langle B(u^\nu_{h_\nu}(s)-\bar u(s),\bar u(s)),\varphi\big\rangle\,\big]ds,\\
I_4 &= \sqrt{\nu}\int_0^t \big(\bar\sigma_\nu(s,u^\nu_{h_\nu}(s))h_\nu(s),\varphi\big)\,ds,\\
I_5 &= \int_0^t \big(\big[\sigma(s,u^\nu_{h_\nu}(s)) - \sigma(s,\bar u(s))\big]h_\nu(s),\varphi\big)\,ds,\\
I_6 &= \int_0^t \left(\sigma(s,\bar u(s))\left[h_\nu(s)-h(s)\right],\varphi\right)ds.
\end{aligned}$$

Since $\beta \ge 1/2$ implies that $Dom(A^\beta) \subset V$, using Cauchy-Schwarz's inequality and (2.23), we deduce for $t\in[0,T]$ and $\nu\in(0,\bar\nu_0]$:

$$\begin{aligned}\mathbb{E}_1|I_1| &\le \nu \mathbb{E}_1\int_0^t \|u^\nu_{h_\nu}(s)\|\|\varphi\| ds \le \nu\sqrt{t}\|\varphi\|\Big(\mathbb{E}_1\int_0^t \|u^\nu_{h_\nu}(s)\|^2 ds\Big)^{1/2}\\ &\le \bar\nu_0 C(T,M)\|\varphi\|\big[1+\mathbb{E}\|\zeta\|^4\big]^{1/2}. \end{aligned}\tag{5.16}$$

The Itô isometry, the Cauchy-Schwarz's inequality, condition **(C7)** and (2.17) yield

$$\begin{aligned}\mathbb{E}_1|I_2| &\le \sqrt{\nu}\mathbb{E}_1\Big(\int_0^t |\sigma_\nu(s,u^\nu_{h_\nu}(s))|^2_{L_Q}\|\varphi\|^2\Big)^{1/2}\\ &\le \sqrt{\nu}\|\varphi\|\Big(\mathbb{E}_1\int_0^t |\sigma_\nu(s,u^\nu_{h_\nu}(s))|^2_{L_Q}ds\Big)^{1/2}\\ &\le \sqrt{\nu}\|\varphi\| C(T,M)\big[1+\mathbb{E}|\zeta|^4_H\big]^{1/2}.\end{aligned}\tag{5.17}$$

Using (2.8), the Cauchy-Schwarz inequality and (2.23) we get

$$\begin{aligned}\mathbb{E}_1|I_3| &\le C\mathbb{E}\int_0^t \|u^\nu_{h_\nu}(s)-\bar u(s)\|_{\mathcal H}\big(\|u^\nu_{h_\nu}(s)\|_{\mathcal H}+\|\bar u(s)\|_{\mathcal H}\big)\|\varphi\| ds\\ &\le C\|\varphi\|\Big(\mathbb{E}_1\int_0^t \|u^\nu_{h_\nu}(s)-\bar u(s)\|^2_{\mathcal H}ds\Big)^{1/2}\Big(\mathbb{E}_1\int_0^t\big[\|u^\nu_{h_\nu}(s)\|^2_V+\|\bar u(s)\|^2_V\big]ds\Big)^{1/2}\\ &\le C(T,M)\|\varphi\|\big[1+\mathbb{E}\|\zeta\|^4\big]^{1/2}\Big(\mathbb{E}_1\int_0^t \|u^\nu_{h_\nu}(s)-\bar u(s)\|^2_{\mathcal H}ds\Big)^{1/2}\end{aligned}\tag{5.18}$$

Using assumption **(C7)**, the Cauchy Schwarz inequality and (2.23), we obtain

$$\begin{aligned}\mathbb{E}_1|I_4| &\le \sqrt{\nu}\,\mathbb{E}_1\int_0^t \big|\bar\sigma_\nu(s,u^\nu_{h_\nu}(s))\big|_{L(H_0,H)}|h_\nu(s)|_0\,|\varphi|_H ds\\ &\le \sqrt{\nu}\,|\varphi|_H\,\sqrt{MT}\,\Big(\mathbb{E}_1\int_0^t \big|\bar\sigma_\nu(s,u^\nu_{h_\nu}(s))\big|^2_{L_Q}ds\Big)^{1/2}\\ &\le \sqrt{\nu}|\varphi|_H\,\sqrt{MT}\,\Big(\mathbb{E}_1\int_0^t \big[\bar K_0+\bar K_5\|u^\nu_{h_\nu}(s)\|^2 ds\Big)^{1/2}\\ &\le \sqrt{\nu}\,|\varphi|_H\,C(T,M)\,\big[1+\mathbb{E}\|\zeta\|^4\big]^{1/2}.\end{aligned}\tag{5.19}$$

Condition **(C7)** and the Cauchy Schwarz inequality yield

$$\begin{aligned}\mathbb{E}_1|I_5| &\le \mathbb{E}_1\int_0^t \big|\sigma(s,u^\nu_{h_\nu})-\sigma(s,\bar u(s))\big|_{L(H_0,H)}\,|h_\nu(s)|_0|\varphi|_H ds\\ &\le |\varphi|_H\,\sqrt{MT}\Big(\mathbb{E}_1\int_0^t |\sigma(s,u^\nu_{h_\nu}(s))-\sigma(s,\bar u(s))|^2_{L_Q}ds\Big)^{1/2}\\ &\le |\varphi|_H\sqrt{MT}\sqrt{\bar L_1}\,\Big(\mathbb{E}_1\int_0^t |u^\nu_{h_\nu}(s)-\bar u(s)|^2_H ds\Big)^{1/2}.\end{aligned}\tag{5.20}$$

Finally, we have that

$$\mathbb{E}_1|I_6| = \mathbb{E}_1\Big|\int_0^t\Big(\big[h_\nu(s)-h(s)\big],\sigma^*(s,\bar u(s))\varphi\Big)ds\Big|.\tag{5.21}$$

Using the upper estimates (5.16), (5.17) and (5.19) we deduce that $\mathbb{E}_1|I_i|\to 0$ for $i=1,2,4$ as $n\to\infty$. Furthermore, by construction, we have $\mathbb{P}^1$ a.s. $u^{\nu_n}_{h_{\nu_n}}-\bar u\to 0$ in $L^2(0,T;H^{1,4})$ and hence in $L^2(0,T;\mathcal H)$ and in $L^2(0,T;H)$ as $n\to 0$. Furthermore, the estimates (2.23), (3.3) prove that $\int_0^T \|u^\nu_{h_\nu}(s)-\bar{u}(s)\|^2 ds$ is bounded in $L^2(\mathbb{P}^1)$ and hence is uniformly integrable. Therefore, the dominated convergence theorem, (5.18) and (5.20) prove that $\mathbb{E}_1|I_i| \to 0$ for $i=3,5$. Finally, condition **(C7)** shows that

$$\int_0^T |\sigma^*(s,\bar{u}(s))\varphi|_0^2 ds \le |\varphi|_H^2 \int_0^T \big[\bar{K}_0 + \bar{K}_1 |\bar{u}(s)|_H^2\big] ds$$

and by assumption, as $n\to\infty$, we have $h_{\nu_n} - h \to 0$ in $L^2(0,T;H_0)$ for the weak topology $\mathbb{P}^1$ a.s. Hence $\mathbb{P}^1$ a.s., $\int_0^t \left([h_{\nu_n}(s)-h(s)] , \sigma^*(s,\bar{u}(s))\varphi \right) ds$ converges to 0 as $n\to\infty$. Furthermore, the upper estimate (3.3) proves that this family is bounded in $L^2(\mathbb{P}^1)$; using once more the dominated convergence theorem, (5.21) proves that $\mathbb{E}_1|I_6|\to 0$ as $n\to\infty$. Thus, (5.15) shows that as $n\to\infty$, for any $t\in[0,T]$,

$$\mathbb{E}_1\Big[(u^{\nu_n}_{h_{\nu_n}}(t),\varphi) - \int_0^t \big\langle -B(\bar{u}(s),\bar{u}(s)) + \sigma(s,\bar{u}(s))h(s), \varphi\big\rangle ds\Big] \to 0. \tag{5.22}$$

On the other hand, by construction, since $\varphi\in Dom(A^\beta)$, we have $\mathbb{P}^1$ a.s.

$$\sup_{t\in[0,T]} |(u^{\nu_n}_{h_{\nu_n}}(t) - \bar{u}(t),\varphi)| \to 0 \quad \mathbb{P}^1 \text{ a.s. as } \nu\to 0.$$

Using again (2.23), (3.3) and the dominated convergence theorem, we deduce that as $n\to\infty$,

$$\mathbb{E}_1\Big(\sup_{t\in[0,T]} |(u^{\nu_n}_{h_{\nu_n}}(t)-\bar{u}(t),\varphi)|\Big) \to 0. \tag{5.23}$$

Since $\mathbb{P}^1$ a.s. $\bar{u}\in C([0,T],H)$, this identity holds a.s. for all $t\in[0,T]$ and $\bar{u}$ is a solution to the inviscid evolution equation (3.1).

Hence, from any sequence $\nu_n\to 0$, one can extract a subsequence $(\nu_{n_k}, k\ge 0)$ such that $u^{\nu_{n_k}}_{h_{\nu_{n_k}}} \to u^0_h$ in distribution in $\mathcal{X}$. This implies that the family $u^\nu_{h_\nu}$ converges to u^0_h in distribution in $\mathcal{H}$, which concludes the proof. □

The following compactness result is the second ingredient which allows to transfer the LDP from $\sqrt{\nu}W$ to u^ν.

Proposition 5.4. *Suppose that σ satisfies condition* **(C7)** *and let $\tilde{\sigma}_0=\sigma$. Fix $M>0$, $\zeta\in\mathcal{Y}$ and let $K_M=\{u^0_h : h\in S_M\}$, where u^0_h is the unique solution in $\mathcal{X}$ of the deterministic control equation* (3.1). *Then K_M is a compact subset of $\mathcal{X}$.*

Proof. To simply the notation, we skip the superscript 0 which refers to the inviscid case. By Theorems 3.1 and 3.2, $K_M\subset\mathcal{X}$. Let $(u_n, n\ge 1)$ be a sequence in K_M, corresponding to solutions of (3.1) with controls $(h_n, n\ge 1)$ in S_M:

$$du_n(t) + B(u_n(t),u_n(t))dt = \sigma(t,u_n(t))h_n(t)dt,\ \ u_n(0)=\zeta.$$

Since S_M is a bounded closed subset of the Hilbert space $L^2(0,T;H_0)$, it is weakly compact. So there exists a subsequence of (h_n), still denoted as (h_n), which converges weakly to a limit $h\in L^2(0,T;H_0)$. Note that in fact $h\in S_M$ as S_M is closed.

We at first prove that (u_n) is bounded in $W^{1,2}(0,T;L^q)\cap W^{\alpha,p}(0,T;L^q)\cap L^2(0,T;H^{1,q})$ for any $p,q>2$ and $\alpha<\frac{1}{2}$. Indeed, $u_n(t)=\zeta+J_1(t)+J_2(t)$, where

$$J_1(t) = -\int_0^t B(u_n(s),u_n(s))ds, \quad J_2(t)=\int_0^t \sigma(s,u_n(s))h_n(s)ds.$$

Hölder's inequality, (6.8) and (3.11) yield

$$\|J_1\|_{W^{1,q}(0,T,L^q)} \le C(T) \sup_{t\in[0,T]} \|u_n(t)\|^{2q}_{H^{1,q}} \le q^{2q} C(T,M)[1+\|\zeta\| + \|\text{curl }\zeta\|_q]^{2q}. \tag{5.24}$$

Furthermore, Minkowski's inequality, the Sobolev embedding theorem (see (6.1)), (6.3), condition **(C7)** and (3.3) yield

$$\begin{aligned}
\|J_2\|^2_{W^{1,2}(0,T;L^q)} &\leq C(T,q)\int_0^T \|\sigma(t,u_n(t))h_n(t)\|_q^2 dt \leq C(T,q)\int_0^T \|\sigma(t,u_n(t))h_n(t)\|^2 dt\\
&\leq C(T,q)\int_0^T |\text{curl } \sigma(t,u_n(t))|^2_{L_Q} |h_n(t)|_0^2 dt \leq C(T,q) M\big[\bar{K}_0 + \bar{K}_1 \sup_{t\in[0,T]}\|u(t)\|^2\big]\\
&\leq C(T,q,M)[1+\|\zeta\|^2]. \qquad (5.25)
\end{aligned}$$

The Minkowski and Hölder inequalities, the Sobolev embedding theorem, (6.3), condition **(C7)** and (3.3) imply,

$$\begin{aligned}
\int_0^T \|J_2(t)\|_q^p dt &\leq \int_0^T \Big|\int_0^t \|\sigma(s,u_n(s))h_n(s)\|_q ds\Big|^p dt \leq C(q)\int_0^T \Big|\int_0^t \|\sigma(s,u_n(s))h_n(s)\| ds\Big|^p dt\\
&\leq C(q)(MT)^{p/2} \sup_{t\in[0,T]} |\sigma(t,u_n(t))|^p_{L_Q} \leq C(q,T,M)\big[1+\sup_{t\in[0,T]}\|u(t)\|^p\big]\\
&\leq C(q,T,M)[1+\|\zeta\|^p]. \qquad (5.26)
\end{aligned}$$

Finally, similar arguments imply that for $\alpha\in(0,\frac{1}{2})$, we have

$$\begin{aligned}
&\int_0^T\int_0^T \frac{\|J_2(t)-J_2(s)\|_q^p}{(t-s)^{1+\alpha p}} ds dt\\
&\quad\leq 2C(q)\int_0^T dt\int_0^t ds (t-s)^{-1-\alpha p}\Big|\int_s^t \|\sigma(r,u_n(r))h_n(r)\| ds\Big|^p\\
&\quad\leq 2C(q)(TM)^{\frac{p}{2}}\, C\,\big[1+\sup_{r\in[0,T]}\|u_n(r)\|^p\big]\int_0^T dt\int_0^t (t-s)^{-1+(1/2-\alpha)p} ds\\
&\quad\leq C(q,T,M)\big[1+\|\zeta\|^p\big]. \qquad (5.27)
\end{aligned}$$

As in the proof of Proposition 5.3, Step 3, using [27] we deduce from the upper estimates (5.24)-(5.27) that the sequence (u_n) is relatively compact in $L^2(0,T;\mathcal{H})\cap C([0,T],Dom(A^{-\beta}))$ with $2\beta>1$. Hence there exists a subsequence, still denoted (u_n), which converges in $L^2(0,T;\mathcal{H})\cap C([0,T],Dom(A^{-\beta}))$ to some element u. It remains to check that u is the solution to the evolution equation

$$du(t)+B(u(t),u(t))dt = \sigma(t,u(t))h(t)dt, \quad u(0)=\zeta.$$

The proof, which is similar to that of Step 4 in Proposition 5.3 and easier, is briefly sketched. Only (deterministic) terms similar to I_i for $i=3,5$ and 6 have to be dealt with. As in the proof of Proposition 5.3, these terms are estimated replacing the upper estimate (2.23) by (3.3). This concludes the proof of the Proposition. □

The proof of Theorem 5.2 is a straightforward consequence of Propositions 5.3 and 5.4, as shown in [12].

6. Appendix

6.1. **Properties of the bilinear operator.** Let us at first recall the following classical Sobolev embeddings which hold since D is a bounded domain of $\mathbb{R}^2$ which satisfies the cone condition (see e.g. [1]):

$$\|u\|_q \leq C(q)\|u\|_{W^{1,2}} \quad \text{for } u\in W^{1,2} \quad \text{and } 1\leq q<+\infty, \qquad (6.1)$$

$$W^{2,1}\subset \mathcal{C}^0_B(D),\ \ W^{1,q}\subset\mathcal{C}^0_B(D) \text{ for } q\in(2,\infty). \qquad (6.2)$$

Furthermore, recall the following result proved in [24] (see also [10] and [36] for the way the constant depends on q). Given $q \in [2,\infty)$ there exists a constant C such that for every $u \in H^{1,q}$ one has:

$$\|\nabla u\|_q \le Cq \|\text{curl } u\|_q \text{ for } q \in [2,\infty). \tag{6.3}$$

Furthermore, given $q \in [2,\infty)$ and $r > 0$, the operator B has a unique extension to a continuous bilinear operator from $H^{1,q} \times H^{1,q}$ to $H^{-r,q}$ and the following estimates are satisfied for some constant C and all $u, v \in H^{1,q}$ resp. $\varphi, \psi \in D(A)$:

$$\|B(u,v)\|_{H^{-r,q}} \le C\, \|u\|_{H^{1,q}}\, \|v\|_{H^{1,q}}, \tag{6.4}$$

$$\langle B(u,v)\,,\, v\rangle = 0, \tag{6.5}$$

$$\langle \text{curl } B(\varphi,\varphi), \psi\rangle = \langle \varphi \cdot \nabla (\text{curl } \varphi), \psi\rangle = \langle B(\varphi, \text{curl } \varphi), \psi\rangle, \tag{6.6}$$

$$\langle \text{curl } B(u,v)\,,\, \text{curl } v |\text{curl } v|^{q-2}\rangle = 0 \quad \text{for all } u,v \in H^{2,q} \bigcap D(A). \tag{6.7}$$

Finally, if $q > 2$, there exists a constant $C > 0$ such that for all $u, v \in H^{1,q}$

$$|B(u,v)| \le C\, \|u\|_{H^{1,q}}\, \|v\|_{H^{1,2}} \quad \text{and} \quad \|B(u,v)\|_q \le C\, \|u\|_{H^{1,q}}\, \|v\|_{H^{1,q}}. \tag{6.8}$$

6.2. **Radonifying operators and stochastic calculus in $W^{k,q}$ spaces.** In this section, we recall the basic definitions and results of stochastic calculus on non Hilbert Sobolev spaces used in this paper. Their proofs can be found in references [9], [10], [19], [32] and [33].

Let $\mathcal{E}$ be a Banach space, such as the Sobolev spaces $W^{k,q}$ for $k \ge 0$ and $q \in [1,\infty)$, and let H_0 be a Hilbert space. The following notion extends that of Hilbert Schmidt operator from H_0 to $\mathcal{E}$ when $\mathcal{E}$ is not a Hilbert space. Let (e_k) denote an orthonormal basis of H_0 and (β_k) be a sequence of independent standard Gaussian random variables on some probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde P)$.

Definition 6.1. *A linear operator $K : H_0 \to \mathcal{E}$ is Radonifying if the series $\sum_k \beta_k K e_k$ converges in $L^2(\tilde\Omega\ \mathcal{E})$. Let $R(H_0,\mathcal{E})$ denote the set of Radonifying operators, and given $K \in R(H_0,\mathcal{E})$, set*

$$\|K\|_{R(H_0,\mathcal{E})} = \Big(\tilde{\mathbb{E}} \Big| \sum_k \beta_k K e_k \Big|_{\mathcal{E}}^2 \Big)^{\frac12}. \tag{6.9}$$

Then $(R(H_0,\mathcal{E}), \|K\|_{R(H_0,\mathcal{E})})$ is a separable Banach space and $\|K\|_{R(H_0,\mathcal{E})}$ does not depend on the choice of (e_k) and (β_k).

We now suppose that H_0 is the RKHS of the H-valued Wiener process $(W(t), t \ge 0)$ and fix some orthonormal basis (e_k) of H_0. Simple $R(H_0,\mathcal{E})$- valued processes σ on $[0,T]$ are defined as follows. Given integers $m,n \ge 1$, $0 \le t_1 < t_2 < \cdots < t_{m+1} \le T$, and $\big(\sigma_j \in L^2(\Omega, \mathcal{F}_{t_j}; R(H_0,\mathcal{E})), j = 0, \cdots, m\big)$ set

$$\sigma(t,\omega) := \sum_{0\le j\le m} \sigma_j(\omega) 1_{(t_j,t_{j+1}]}(t).$$

For such a simple process σ, and $t \in (0,T)$, set

$$\int_0^t \sigma(s) dW_s := \sum_{0\le j\le m} \sigma_j(\omega) Q^{\frac12} \big(W(t_{j+1}\wedge t) - W(t_j \wedge t)\big).$$

The extension of stochastic integrals to predictable square integrable processes cannot be done for any Banach space $\mathcal{E}$. Fix $k \in [0,\infty)$ and $q \in [2,\infty)$ and let $\mathcal{E} = W^{k,q}$ (with the convention $L^q = W^{0,q}$). The stochastic integral can be extended uniquely as a linear bounded operator from the set of predictable processes in $L^2(0,T; R(H_0, H^{k,q}))$ to the set of $(\mathcal{F}_t)$ adapted random variables in $L^2(\Omega, H^{k,q})$. Moreover, the following Burkholder-Davies-Gundy inequality holds (see e.g. [33], section 5): For any $p \in [1,\infty)$, there exists a constant $C_p > 0$ such that for any predictable process $\sigma \in L^2(0,T; R(H_0, H^{k,q}))$,

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} \Big| \int_0^t \sigma(s) dW_s \Big|^p_{H^{k,q}} \Big) \leq C_p\, \mathbb{E}\Big(\int_0^T \|\sigma(s)\|^2_{R(H_0,H^{k,q})}\, ds\Big)^{\frac{p}{2}} \tag{6.10}$$

Finally, given $2 \leq q \leq p < \infty$, some predictable processes $\sigma \in L^2(0,T; R(H_0, H^{0,q}))$ and $f \in L^1(0,T; H^{0,q})$, we state a particular case of the Itô formula applied to the function $\Psi_{q,p}(.) = \|.\|_q^p$ on $H^{0,q}$ and the $H^{0,q}$-valued process $(Z_t, t \in [0,T])$ defined by

$$Z(t) = Z(0) + \int_0^t \sigma(s) dW(s) + \int_0^t f(s) ds.$$

With the above notations, if $\langle F, G\rangle$ denotes the duality between $F \in L^q$ and $G \in L^{q*}$ with $q* = \frac{q}{q-1}$, we have:

$$\begin{aligned} \|Z(t)\|_q^p &= \|Z(0)\|_q^p + p \int_0^t \|Z(s)\|_q^{p-q} (|Z(s)|^{q-2} Z(s)\,,\, f(s)) ds \\ &+ p \int_0^t \|Z(s)\|_q^{p-q} (|Z(s)|^{q-2} Z(s)\,,\, \sigma(s) dW(s)) + \frac{1}{2} \int_0^t \mathrm{tr}_{\sigma(s)} \Psi''_{q,p}(Z(s)) ds, \end{aligned} \tag{6.11}$$

and for every $u \in H^{0,q}$,

$$0 \leq \mathrm{tr}_{\sigma(s)} \Psi''_{q,p}(u) \leq p(p-1)\, \|u\|_q^{p-2}\, \|\sigma(s)\|^2_{R(H_0,H^{0,q})}. \tag{6.12}$$

6.3. **Nemytski operators.** In this section we will show that assumptions **(C5q)** and **(C6q)** are satisfied by Nemytski operators.

Definition 6.2. *Let $q \in [2,\infty)$. A mapping $g : [0,T] \times D \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ belongs to the class $U(D,q)$ if and only if $g(t,x,y) = g^1(t,x) + g^2(t,x,y)$, $t \in [0,T]$, $x \in D$, $y \in \mathbb{R}^2$, where:*

(1) *g^1 and g^2 are measurable, and for any $t \in [0,T]$, $g^1(t,\cdot) \in H^{1,2} \cap H^{1,q}$ and $g^2(t,\cdot,\cdot)$ is differentiable,*
(2) *there are a constant $c > 0$ and $\phi \in L^2(D) \cap L^q(D)$ such that all $t \in [0,T]$, and $x \in D$, $y \in \mathbb{R}^2$,*

$$|g^1(t,\cdot)|_{H^{1,2}} + |g^2(t,\cdot)|_{H^{1,q}} \leq c,$$
$$|g^2(t,x,y)| + \sum_{i=1,2} |\partial_{x_i} g^2(t,x,y)| \leq c(\phi(x) + |y|), \quad \sum_{i=1,2} |\partial_{y_i} g^2(t,x,y)| \leq c.$$

We say that $g : [0,T] \times D \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ belongs to the class $U(D,\infty)$ if and only if it is differentiable with respect to the second and third variables, and there is a constant $c > 0$ such that for all $t \in [0,T],\ x \in D,\ y \in \mathbb{R}^2$:

$$|g(t,x,y)| + \sum_{i=1,2} |\partial_{x_i} g(t,x,y)| + \sum_{i=1,2} |\partial_{y_i} g(t,x,y)| \leq c.$$

Let g_i, $i = 1, \cdots, m$ and $\tilde{g}$ be in $U(D,q)$ and define the Nemytski operators

$$\tilde{\sigma}(t,u)(x) = \tilde{g}(t,x,u(x)), \quad \text{and} \quad \sigma(t,u)\psi(x) = \sum_{1\leq i\leq m} g_i(t,x,u(x))\psi_i(x), \tag{6.13}$$

where $\psi_i \in H_0$, $i = 1, \cdots, m$. These operators satisfy the assumptions **(C5q)** and **(C6q)** (see e.g. [10]). The condition $U(D,\infty)$ obviously implies $U(D,q)$ for every $q \in [2,\infty)$. Therefore, if the coefficients $\tilde{g}$ and g_i belong to the class $U(D,\infty)$, then σ and $\tilde{\sigma}$ satisfy the conditions **(C5q)** and **(C6q)** for all $q \in [2,\infty)$.

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University of Wyoming, Department of Mathematics, Dept. 3036, 1000 East University Avenue, Laramie WY 82071, United States
E-mail address: bessaih@uwyo.edu

SAMM, EA 4543, Université Paris 1 Panthéon Sorbonne, 90 Rue de Tolbiac, 75634 Paris Cedex France *and* Laboratoire de Probabilités et Modèles Aléatoires, Universités Paris 6-Paris 7, Boîte Courrier 188, 4 place Jussieu, 75252 Paris Cedex 05, France
E-mail address: annie.millet@univ-paris1.fr *and* annie.millet@upmc.fr