Cores for generators of some Markov semigroups *

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Abstract

We consider a stochastic differential equation on $\mathbb{R}^d$ with Lipschitz coefficients. We find a core for the infinitesimal generator of the corresponding Markov process. Some applications, in particular, to well-posedness of Fokker–Planck equations are given.

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1 Introduction

We are here concerned with a stochastic differential equation in $H := \mathbb{R}^d$,

$$\begin{cases} dX = b(X)dt + \sigma(X)dW(t), \\ X(0) = x, \end{cases} \tag{1.1}$$

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where $b : H \to H$ and $\sigma : H \to L(H)$ are Lipschitz continuous. It is well known that equation (1.1) has a unique solution $X(\cdot, x)$.

Moreover, the transition semigroup

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \quad \varphi \in C_b(H), \tag{1.2}$$

is Feller. ($C_b(H)$ is the Banach space of all mappings $\varphi : H \to \mathbb{R}$ which are uniformly continuous and bounded, endowed with the norm $\|\varphi\|_0 := \sup_{x\in H} |\varphi(x)|$.)

If b and σ are not bounded $P_t,\ t \geq 0$, is not strongly continuous in $C_b(H)$ but only pointwise continuous. We call it a π-semigroup, see later for a precise definition. Though the Hille–Yosida theory cannot be applied to P_t, one can define an infinitesimal generator $\mathscr{K}$ following [1] or [5]. Then the problem arises to show the relationship between $\mathscr{K}$ and the Kolmogorov operator

$$\mathscr{K}_0\varphi := \frac{1}{2}\,\mathrm{Tr}\,[a(x)D^2\varphi] + \langle b(x), D\varphi(x)\rangle, \quad \forall\, \varphi \in C_b^2(H), \tag{1.3}$$

where

$$a(x) = \sigma(x)\sigma^*(x), \quad \forall\, x \in H.$$

The main result of this paper is that if in addition b and σ are of class C^2 with bounded second derivatives, then

$$\mathscr{K}\varphi = \mathscr{K}_0\varphi, \quad \forall\, \varphi \in C_b^2(H)$$

and the space $C_b^2(H)$ is a core for $\mathscr{K}$.

This result seems to be new when a is not uniformly elliptic, see the monograph [3] and references therein for the case of uniformly elliptic a.

2 Notations and preliminaries

Let us precise our assumptions.

Hypothesis 2.1 *(i) $b : H \to H$ and there is $K_1 > 0$ such that*

$$|b(x) - b(y)| \leq K_1|x - y|, \quad \forall\, x \in H. \tag{2.1}$$

(ii) $\sigma : H \to L(H)$ and there is $K_2 > 0$ such that

$$\|\sigma(x) - \sigma(y)\|_{HS} \leq K_2|x - y|, \quad \forall\, x \in H, \tag{2.2}$$

where the sub-index HS means the Hilbert–Schmidt norm.

We note that by (2.1) and (2.2) it follows that

$$|b(x)| \le K_1|x| + |b(0)|, \quad \forall\, x \in H, \tag{2.3}$$

and

$$\|\sigma(x)\|_{HS} \le K_2|x| + \|\sigma(0)\|_{HS}, \quad \forall\, x \in H, \tag{2.4}$$

respectively.

Sometimes we shall need the following more stringent assumptions.

Hypothesis 2.2 *(i) b and σ are of class C^2 and fulfill Hypothesis 2.1.*

(ii) There is $K_3 > 0$ such that

$$|b''(x)| + \|\sigma''(x)\|_{HS} \le K_3, \quad \forall\, x \in H. \tag{2.5}$$

The following two propositions are well known.

Proposition 2.3 *Assume that Hypothesis 2.1 is fulfilled. Then for any $x \in H$ and any $T > 0$ there is a unique solution $X(\cdot, x) \in L^2_W(\Omega; C([0,T]; H))$ of equation (1.1). Moreover, for all $m \in \mathbb{N}$, $X(\cdot, x) \in L^m_W(\Omega; C([0,T]; H))$ and there is $C_{T,m} > 0$ such that*

$$\mathbb{E}\left(|X(t,x)|^m\right) \le C_{T,m}(1 + |x|^m), \quad \forall\, x \in H,\ t \in [0,T]. \tag{2.6}$$

Finally, there is $C'_T > 0$ such that

$$\mathbb{E}|X(t,x) - X(t,y)| \le C'_T|x - y|, \quad \forall\, x, y \in H,\ t \in [0,T]. \tag{2.7}$$

By $L^m_W(\Omega; C([0,T]; H))$ we mean the space of all adapted continuous stochastic processes F such that

$$\mathbb{E}\left(\sup_{t\in[0,T]} |F(t)|^m\right) < +\infty.$$

Proposition 2.4 *Assume that Hypothesis 2.2 is fulfilled and let $X(\cdot, x)$ be the solution to (1.1). Then the following statements hold.*

(i) $X(t,x)$ is continuously differentiable in any direction $h \in H$ and setting $\eta^h(t,x) = X_x(t,x) \cdot h$ we have

$$\begin{cases} d\eta^h = b'(X)\eta^h dt + \sigma'(X)(\eta^h, dW(t)), \\ \\ \eta^h(0) = h. \end{cases} \tag{2.8}$$

Moreover, there exists $\omega_1 \in \mathbb{R}$ such that

$$\mathbb{E}|\eta^h(t,x)|^2 \le e^{2\omega_1 t}|h|^2, \quad \forall\, h \in H,\ t > 0. \tag{2.9}$$

(ii) $X(t,x)$ *is twice continuously differentiable in any couple of directions* $h,k\in H$ *and setting* $\zeta^{h,k}(t,x)=X_x(t,x)(h,k)$ *we have*

$$\begin{cases} d\zeta^{h,k} &= b'(X)\zeta^{h,k}dt + b''(X)(\eta^h,\eta^k)dt \\ & \quad +\sigma'(X)(\zeta^{h,k},dW(t)) + \sigma''(X)(\eta^h,\eta^k,dW(t)), \\ \zeta^{h,k}(0) &= 0. \end{cases} \tag{2.10}$$

Moreover, there exists $\omega_2\in\mathbb{R}$ *and* $C_2>0$ *such that*

$$\mathbb{E}|\zeta^{h,k}(t,x)|^2 \le C_2 e^{2\omega_2 t}|h|^2|k|^2, \quad \forall\, h,k\in H,\ t>0. \tag{2.11}$$

2.1 Transition semigroup

Let us introduce some notations. For any $k\in\mathbb{N}$ by $C_b^k(H)$ we denote the space of all mappings $\varphi: H\to\mathbb{R}$ which are uniformly continuous and bounded together with their derivatives of order lesser than k. $C_b^k(H)$, endowed with the norm

$$\|\varphi\|_k := \|\varphi\|_0 + \sum_{j=1}^k \|D^j\varphi\|_0,$$

is a Banach space.

Moreover, for any $m\in\mathbb{N}$ by $C_{b,m}(H)$ we denote the space of all mappings $\varphi: H\to\mathbb{R}$ such that the mapping

$$H\to\mathbb{R}, \quad x\to\frac{|\varphi(x)|}{1+|x|^m}$$

belongs to $C_b(H)$. $C_{b,m}(H)$, endowed with the norm

$$\|\varphi\|_{b,m} := \sup_{x\in H}\frac{|\varphi(x)|}{1+|x|^m},$$

is a Banach space.

Taking into account (2.6) we can define the transition semigroup

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \quad \varphi\in C_{b,m}(H). \tag{2.12}$$

We know that $P_t\varphi\in C_b(H)$ for all $t\ge 0$ and all $\varphi\in C_b(H)$. It follows from (2.6) that also $P_t\varphi\in C_{b,m}(H)$ for all $\varphi\in C_{b,m}(H)$. Furthermore, $P_t,\ t\ge 0$, is a semigroup.

Proposition 2.5 *Assume that Hypothesis 2.2 is fulfilled. Then $C^1_b(H)$ and $C^2_b(H)$ are stable for $P_t,\ t\ge 0$. Moreover there exist positive constants K_1 and K_2 such that*

$$\|P_t\varphi\|_j \le K_j\|\varphi\|_j,\quad \forall\,\varphi\in C^j_b(H),\ j=1,2,\ t\ge 0, \tag{2.13}$$

where $\|\cdot\|_j$ denotes the norm in $C^j_b(H)$.

Proof. The assertions follow from Propositions 2.4 and the identities

$$\langle DP_t\varphi(x),h\rangle = \mathbb{E}[\langle D\varphi(X(t,x)),\eta^h(t,x)\rangle],\quad t\ge 0,\ h,x\in H,$$

and

$$\begin{aligned}\langle D^2P_t\varphi(x)h,k\rangle &= \mathbb{E}[\langle D\varphi(X(t,x)),\zeta^{h,k}(t,x)\rangle]\\ &\quad+\mathbb{E}[D^2\varphi(X(t,x))(\eta^h(t,x),\eta^k(t,x))],\quad t\ge 0,\ h,k,x\in H.\end{aligned}$$

□

2.2 Itô's formula

Let us consider the Kolmogorov operator (1.3). By Hypothesis 2.1 it follows that there exists $M>0$ such that for all $\varphi\in C^2_b(H)$ we have

$$|\mathscr{K}_0\varphi(x)|\le M(1+|x|^2),\quad \forall\,x\in H, \tag{2.14}$$

so that $\mathscr{K}_0\varphi\in C_{b,2}(H)$.

Proposition 2.6 (Itô's formula) *Assume that Hypothesis 2.1 is fulfilled. Then for all $\varphi\in C^2_b(H)$ we have*

$$\mathbb{E}[\varphi(X(t,x))]=\varphi(x)+\int_0^t\mathbb{E}[\mathscr{K}_0\varphi(X(s,x))]ds. \tag{2.15}$$

Proof. Write

$$X(t,x)=x+\int_0^t b(X(s,x))ds+\int_0^t\sigma(X(s,x))dW(s).$$

By (2.3) and (2.6) we see that

$$\mathbb{E}|b(X(t,x))|\le C_{T,1}K_1(1+|x|)+|b(0)|.$$

so that

$$b(X(\cdot,x))\in C_W([0,T];L^1(\Omega,\mathbb{R}^d)).$$

Moreover, by (2.4) and (2.6)

$$\mathbb{E}\|\sigma(X(t,x))\|_{HS}^2 \le 2K_2(1+|x|)^2 + 2\|\sigma(0)\|_{HS}^2,$$

so that,

$$\sigma(X(\cdot,x)) \in C_W([0,T];L^2(\Omega,\mathbb{R}^d)).$$

Then we may apply Itô's formula and the conclusion follows. □

Note that if $\varphi \in C_b^2(H)$ we have $\mathscr{K}_0\varphi \in C_{b,2}(H)$ by (2.14). Therefore,

$$\frac{d}{dt}\,P_t\varphi = P_t\mathscr{K}_0\varphi, \quad \forall \varphi \in C_b^2(H). \tag{2.16}$$

Proposition 2.7 *Assume that Hypothesis 2.2 is fulfilled. Then*

$$\frac{d}{dt}\,P_t\varphi = \mathscr{K}_0 P_t\varphi, \quad \forall \varphi \in C_b^2(H). \tag{2.17}$$

Proof. By the semigroup property, Proposition 2.5 and (2.16) we have for all $\varphi \in C_b^2(H)$

$$\frac{d}{dt}\,P_t\varphi = \frac{d}{d\epsilon}\,P_{t+\epsilon}\varphi\big|_{\epsilon=0} = \frac{d}{d\epsilon}\,P_\epsilon(P_t\varphi)\big|_{\epsilon=0} = \mathscr{K}_0 P_t\varphi$$

□

3 The infinitesimal generator of P_t

Let us introduce some notations. Given a sequence $(\varphi_n) \subset C_b(H)$ and $\varphi \in C_b(H)$, we say that (φ_n) is π-convergent to φ and write $\varphi_n \stackrel{\pi}{\to} \varphi$, if the following conditions are fulfilled.

(i) For each $x \in H$ we have

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x).$$

(ii) $\sup_{n\in\mathbb{N}} \|\varphi_n\|_0 < +\infty$.

Proposition 3.1 *Assume that Hypothesis 2.1 is fulfilled and let* $(\varphi_n) \subset C_b(H)$, $\varphi \in C_b(H)$ *such that* $\varphi_n \stackrel{\pi}{\to} \varphi$. *Then for all* $t \ge 0$ *we have* $P_t\varphi_n \stackrel{\pi}{\to} P_t\varphi$.

We say that P_t is a π-semigroup.
Proof of Proposition 3.1. For any $x \in H$ we have

$$\lim_{n\to\infty} P_t\varphi_n(x) = \lim_{n\to\infty} \mathbb{E}[\varphi_n(X(t,x))] = P_t\varphi(x),$$

thank's to the dominated convergence theorem. Moreover

$$\|P_t\varphi_n\|_0 \le \|\varphi_n\|_0 \le \sup_{n\in\mathbb{N}} \|\varphi_n\|_0 < \infty.$$

□

We follow here [5].

Definition 3.2 *We say that φ belongs to the domain of the infinitesimal generator $\mathscr{K}$ of P_t in $C_b(H)$ if*

(i) For each $x \in H$ there exists the limit

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\,(P_\epsilon\varphi(x) - \varphi(x)) =: \mathscr{K}\varphi(x)$$

and $\mathscr{K}\varphi \in C_b(H)$.

(ii) $\displaystyle\sup_{\epsilon\in(0,1]} \frac{1}{\epsilon}\,\|P_\epsilon\varphi - \varphi\|_0 < +\infty.$

$\mathscr{K}$ *is called the* infinitesimal generator *of P_t.*

In the following we set

$$\Delta_\epsilon := \frac{1}{\epsilon}\,(P_\epsilon - 1).$$

Proposition 3.3 *Let $\varphi \in D(\mathscr{K})$ and let $t \ge 0$. Then $P_t\varphi \in \mathscr{K}$ and we have*

$$\mathscr{K} P_t\varphi(x) = P_t\mathscr{K}\varphi(x), \quad \forall\, x \in H. \tag{3.1}$$

Moreover, $P_t\varphi(x)$ is differentiable in t and

$$\frac{d}{dt}\,P_t\varphi(x) = \mathscr{K} P_t\varphi(x) = P_t\mathscr{K}\varphi(x), \quad \forall\, x \in H. \tag{3.2}$$

Proof. Let $\varphi \in \mathscr{K}$. Then we have

$$\Delta_\epsilon P_t\varphi(x) = P_t\Delta_\epsilon\varphi(x).$$

Since $\Delta_\epsilon \varphi \xrightarrow{\pi} \mathscr{K}\varphi$, by Proposition 3.1 it follows that

$$\Delta_\epsilon P_t\varphi \xrightarrow{\pi} P_t\mathscr{K}\varphi.$$

So, $P_t\varphi \in D(\mathscr{K})$ and $\mathscr{K}P_t\varphi = P_t\mathscr{K}\varphi$. On the other hand,

$$D^+P_t\varphi(x) = \lim_{\epsilon\to 0} P_t\Delta_\epsilon\varphi(x) = P_t\mathscr{K}\varphi(x)$$

and (3.1) follows because $\mathscr{K}\varphi$ is continuous. □

We shall denote by $\rho(\mathscr{K})$ the resolvent set of $\mathscr{K}$ and by $R(\lambda, \mathscr{K})$ its resolvent.

The following result is proved in [5] under slightly different assumptions. We sketch here the proof for the reader's convenience.

Proposition 3.4 $\rho(\mathscr{K}) \supset (0, +\infty)$. *Moreover, for any* $\lambda > 0$ *and any* $f \in C_b(H)$ *we have*

$$R(\lambda, \mathscr{K})f(x) = \int_0^\infty e^{-\lambda t} P_t f(x)dt, \quad x \in H. \tag{3.3}$$

Proof. Set

$$\Delta_\epsilon := \frac{1}{\epsilon}\,(P_\epsilon - 1).$$

Let $f \in C_b(H)$ and for any $\lambda > 0$, $x \in H$ set

$$F(\lambda)f(x) = \int_0^{+\infty} e^{-\lambda t} P_t f(x)dt.$$

Then it is not difficult to see that $F(\lambda)f \in C_b(H)$. Let us show that $\lambda \in \rho(\mathscr{K})$. Since

$$\begin{aligned} \Delta_\epsilon F(\lambda)f(x) &= \frac{1}{\epsilon}\, e^{\lambda\epsilon}\left[\int_\epsilon^\infty e^{-\lambda t}P_t f(x)dt - \int_0^\infty e^{-\lambda t}P_t f(x)dt\right] \\ &= \frac{1}{\epsilon}\,(e^{\lambda\epsilon}-1)\int_\epsilon^\infty e^{-\lambda t}P_t f(x)dt - \frac{1}{\epsilon}\, e^{\lambda\epsilon}\int_0^\epsilon e^{-\lambda t}P_t f(x)dt, \end{aligned}$$

we easily deduce that $F(\lambda)f \in D(\mathscr{K})$ and

$$\lim_{\epsilon\to 0} \Delta_\epsilon F(\lambda)f(x) = \lambda F(\lambda)f(x) - f(x). \tag{3.4}$$

Therefore

$$\mathscr{K}F(\lambda)f = \lambda F(\lambda) - f. \tag{3.5}$$

It remains to show that

$$F(\lambda)(\lambda - \mathscr{K})\varphi = \varphi, \quad \forall\, \varphi \in D(\mathscr{K}). \tag{3.6}$$

This will complete the proof that $\lambda \in \rho(\mathscr{K})$. To prove (3.6) choose $\varphi \in D(\mathscr{K})$, then, taking into account Proposition 3.3, we have

$$\int_0^\infty e^{-\lambda t} P_t \mathscr{K}\varphi(x)dt = \int_0^\infty e^{-\lambda t} \frac{d}{dt}\, P_t\varphi(x)dt$$

$$= -\varphi(x) + \lambda F(\lambda)\varphi(x), \quad \forall\, x \in H.$$

which implies (3.6). □

Remark 3.5 By (3.3) it follows that

$$\|R(\lambda, \mathscr{K})f\|_0 \le \frac{1}{\lambda}\, \|f\|_0, \quad \forall\, f \in C_b(H).$$

Therefore $\mathscr{K}$ is m–dissipative in $C_b(H)$.

We are going to investigate the relationship between $\mathscr{K}$ and $\mathscr{K}_0$.

Proposition 3.6 *Assume that Hypothesis 2.1 is fulfilled. If $\varphi \in D(\mathscr{K}) \cap C_b^2(H)$ then $\mathscr{K}\varphi = \mathscr{K}_0\varphi$.*

Proof. By Proposition 2.6 we have

$$\frac{1}{t}\, (P_t\varphi(x) - \varphi(x)) = \int_0^t \mathbb{E}[\mathscr{K}_0\varphi(X(s, x))]ds.$$

As $t \to 0$ we find

$$\mathscr{K}\varphi(x) = \mathscr{K}_0\varphi(x), \quad \forall\, x \in H.$$

□

Now we show that

$$\mathscr{D} := \{\varphi \in C_b^2(H) : \ \mathscr{K}_0\varphi \in C_b(H)\},$$

is a core for $\mathscr{K}$. Let us first show that $\mathscr{D} \subset D(\mathscr{K})$.

Proposition 3.7 *Assume that Hypothesis 2.1 is fulfilled and let $\varphi \in \mathscr{D}$. Then $\varphi \in D(\mathscr{K})$ and we have $\mathscr{K}\varphi = \mathscr{K}_0\varphi$.*

Proof. Let $\varphi \in \mathscr{D}$. By Itô's formula (2.15) we have

$$\frac{1}{t}\,(P_t\varphi(x) - \varphi(x)) = \int_0^t \mathbb{E}[\mathscr{K}_0\varphi(X(s,x))]ds.$$

Therefore,

$$\lim_{t\to 0}\frac{1}{t}\,(P_t\varphi(x) - \varphi(x)) = \mathscr{K}_0\varphi(x), \quad \forall\, x \in H,$$

and

$$\left\|\frac{1}{t}\,(P_t\varphi - \varphi)\right\|_0 \le \|\mathscr{K}_0\varphi\|_0,$$

which implies that $\varphi \in D(\mathscr{K})$ and $\mathscr{K}\varphi = \mathscr{K}_0\varphi$. $\square$

Theorem 3.8 *Assume that Hypothesis 2.2 is fulfilled. Then $\mathscr{D}$ is a core for $\mathscr{K}$, that is, for any $\varphi \in D(\mathscr{K})$ there exists a sequence $(\varphi_n) \subset \mathscr{D}$ such that*

$$\varphi_n \to \varphi, \quad \mathscr{K}_0\varphi_n \to \mathscr{K}\varphi, \quad \text{in } C_b(H).$$

Proof. Let $\varphi \in D(\mathscr{K})$. Fix $\lambda > 0$ and set $f := \lambda\varphi - \mathscr{K}\varphi$. Since $C^2_b(H)$ is dense in $C_b(H)$ there exists a sequence $(f_n) \subset C^2_b(H)$ such that $f_n \to f$ in $C_b(H)$. Set

$$\varphi_n := (\lambda - \mathscr{K})^{-1} f_n, \quad n \in \mathbb{N}.$$

By Proposition 3.4 it is clear that $\varphi_n \in D(\mathscr{K})$ and

$$\lim_{n\to\infty}\varphi_n = \varphi, \quad \lim_{n\to\infty}\mathscr{K}\varphi_n = \mathscr{K}\varphi, \quad \text{in } C_b(H).$$

It remains to show that $\varphi_n \in \mathscr{D}$ for any $n \in \mathbb{N}$. In fact, taking into account Proposition 2.5, we see that $\varphi_n \in D(\mathscr{K}) \cap C^2_b(H)$, so that by Proposition 3.6 we have $\mathscr{K}_0\varphi_n = \mathscr{K}\varphi_n \in C_b(H)$, and hence $\varphi_n \in \mathscr{D}$ for any $n \in \mathbb{N}$. So, $\mathscr{D}$ is a core as claimed. $\square$

Remark 3.9 The above result can be generalized. In fact Hypothesis 2.2 and also the global Lipschitz assumption in Hypothesis 2.1 are too strong. E. g. assumptions as Hypotheses 1.1 and 1.2 in [2] are sufficient. Details will be the subject of future work.

4 Applications

We start with an important identity

$$\mathscr{K}_0(\varphi^2) = 2\varphi\mathscr{K}_0\varphi + |\sigma^* D\varphi|^2, \quad \forall\varphi \in C^2_b(H), \tag{4.1}$$

whose proof is straightforward.

Proposition 4.1 *Assume, besides Hypothesis* 2.2*, that* σ *is bounded. Then for any* $\varphi \in \mathscr{D}$ *we have* $\varphi^2 \in \mathscr{D}$ *and*

$$\mathscr{K}(\varphi^2) = 2\varphi \mathscr{K}\varphi + |\sigma^* D\varphi|^2. \tag{4.2}$$

Proof. Let $\varphi \in \mathscr{D}$. Then $\varphi^2 \in C_b^2(H)$ and by (4.1) it follows that $\mathscr{K}_0(\varphi^2) \in C^b(H)$. □

Corollary 4.2 *Let* $\varphi \in \mathscr{D}$*. Given* $\lambda > 0$ *set* $f = \lambda\varphi - \mathscr{K}\varphi$*. Then we have*

$$\varphi^2 = (2\lambda - \mathscr{K})^{-1}(2\varphi f - |\sigma^* D\varphi|^2). \tag{4.3}$$

4.1 Invariant measures

In this subsection we assume that Hypothesis 2.2 holds.

Let $\mu \in \mathscr{P}(H)$. We say that μ is *invariant* if

$$\int_H P_t\varphi d\mu = \int_H \varphi d\mu, \quad \forall\, \varphi \in C_b(H). \tag{4.4}$$

Proposition 4.3 μ *is invariant for* P_t *if and only if*

$$\int_H \mathscr{K}_0\varphi d\mu = 0, \quad \forall\, \varphi \in \mathscr{D}. \tag{4.5}$$

Proof. Since the "only if part" is obvious, let us show the "if part". Assume that (4.5) is fulfilled. Then

$$\int_H \mathscr{K}\varphi d\mu = 0, \quad \forall\, \varphi \in D(\mathscr{K}).$$

Then if $\varphi \in D(\mathscr{K})$ we have by Proposition 3.3, that $P_s\varphi \in D(\mathscr{K})$, for all $s \geq 0$ and

$$P_t\varphi(x) - \varphi(x) = \int_0^t \mathscr{K} P_s\varphi(x) ds.$$

Integrating with respect to μ, yields

$$\int_H (P_t\varphi - \varphi) d\mu = 0, \quad \forall\, \varphi \in D(\mathscr{K}).$$

Therefore, μ is invariant. □

Proposition 4.4 *Assume that μ is an invariant measure for P_t such that*

$$\int_H |x|^2 \mu(dx) < +\infty. \tag{4.6}$$

Then we have

$$\int_H \varphi \mathscr{K}_0 \varphi d\mu = -\frac{1}{2} \int_H |\sigma^* D\varphi|^2 d\mu, \quad \forall\, \varphi \in C_b^2(H). \tag{4.7}$$

Proof. **Step 1** We have

$$\int_H \mathscr{K}_0 \varphi d\mu = 0, \quad \forall\, \varphi \in C_b^2(H). \tag{4.8}$$

The proof follows by (2.14), (2.17), (4.6) and Lebesgues' dominated convergence theorem

Step 2 Conclusion.
By (4.1) we have

$$0 = \int_H \mathscr{K}_0(\varphi^2) d\mu = 2 \int_H \varphi \mathscr{K}_0 \varphi d\mu + \int_H |\sigma^* D\varphi|^2 d\mu, \quad \varphi \in C_b^2(H),$$

so, the conclusion follows. □

4.2 Application to Fokker–Planck

In this subsection we assume that Hypothesis 2.2 holds. Let $\mathscr{B}(H)$ be the Borel σ-algebra of H. We recall that a measure $\mu(dt, dx) = \mu_t(dx)dt$ on $\mathscr{B}([0,T]) \otimes \mathscr{B}(H)$, where μ_t are probability measures on $\mathscr{B}(H)$, measurable in $t \in [0,T]$, is called a solution to the Fokker–Planck equation for $(\mathscr{K}_0, \mathscr{D})$ if

$$\int_H \varphi\, d\mu_t = \int_H \varphi\, d\mu_0 + \int_0^t \int_H \mathscr{K}_0 \varphi\, d\mu_s ds, \quad \text{a.e. } t \geq 0,\ \varphi \in \mathscr{D}. \tag{4.9}$$

Theorem 4.5 *For every probability measure ζ on $\mathscr{B}(H)$ there exists a unique measure $\mu(dt, dx) = \mu_t(dx)dt$ (as above) solving* (4.9) *with $\mu_0 = \zeta$.*

Proof. Existence is trivial. Just define the measure μ_t by

$$\int_H \varphi(x)\, \mu_t(dx) := \int_H \mathbb{E}[\varphi(X(t,x))] \zeta(dx), \quad \text{a.e. } t \geq 0,$$

$\varphi : H \to \mathbb{R}_+$, $\mathscr{B}(H)$-measurable. Then (4.9) follows by Proposition 2.6 (i.e., by Itô's formula). To show uniqueness we note that Theorem 3.8 implies that (4.9) with $\mu_0 = \zeta$ holds for all $\varphi \in \mathscr{K}$. Hence uniqueness follows from [4, Theorem 2.12]. □

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