A mild Itô formula for SPDEs

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Abstract

This article introduces a new - somehow mild - Itô type formula for the solution process of a stochastic partial differential equation of evolutionary type.

1 A mild Itô formula for SPDEs

Throughout this article suppose that the following setting and the following assumptions are fulfilled. Fix $T \in (0, \infty)$ and $t_0 \in [0, T)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [t_0, T]}$ and let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be two separable $\mathbb{R}$-Hilbert spaces. In addition, let $Q : U \to U$ be a bounded nonnegative symmetric linear operator and let $(W_t)_{t \in [t_0, T]}$ be a cylindrical Q-Wiener process with respect to $(\mathcal{F}_t)_{t \in [t_0, T]}$.

Assumption 1 (Linear operator A)**.** *Let $\mathcal{I}$ be a finite or countable set and let $(\lambda_i)_{i \in \mathcal{I}} \subset \mathbb{R}$ be a family of real numbers with $\inf_{i \in \mathcal{I}} \lambda_i > -\infty$. Moreover, let $(e_i)_{i \in \mathcal{I}} \subset H$ be an orthonormal basis of H and let $A : D(A) \subset H \to H$ be a linear operator with*

$$Av = \sum_{i \in \mathcal{I}} -\lambda_i \langle e_i, v \rangle_H e_i \tag{1}$$

for all $v \in D(A)$ and with $D(A) = \left\{ w \in H \middle| \sum_{i \in \mathcal{I}} |\lambda_i|^2 |\langle e_i, w \rangle_H|^2 < \infty \right\}$.

Let $\eta \in [0, \infty)$ be a nonnegative real number with $\eta > -\inf_{i \in \mathcal{I}} \lambda_i$. By $\left(H_r := D\left((\eta - A)^r \right), \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r} \right)$ for $r \in \mathbb{R}$ we denote the $\mathbb{R}$-Hilbert spaces of domains of fractional powers of the linear operator $\eta - A : D(A) \subset H \to H$.

Assumption 2 (Drift term F)**.** *Let $\alpha, \gamma \in \mathbb{R}$ be real numbers with $\gamma - \alpha < 1$ and let $F : H_\gamma \to H_\alpha$ be globally Lipschitz continuous.*

In order to formulate the assumption on the diffusion coefficient of our SPDE, we denote by $\left(U_0, \langle\cdot,\cdot\rangle_{U_0}, \|\cdot\|_{U_0}\right)$ the separable $\mathbb{R}$-Hilbert space $U_0 := Q^{\frac{1}{2}}(U)$ with $\langle v, w\rangle_{U_0} = \left\langle Q^{-\frac{1}{2}}v, Q^{-\frac{1}{2}}w\right\rangle_U$ and $\|v\|_{U_0} = \left\|Q^{-\frac{1}{2}}v\right\|_U$ for all $v, w \in U_0$ (see, for example, Proposition 2.5.2 in Prévôt and Röckner [17]).

Assumption 3 (Diffusion term B)**.** *Let $\beta \in \mathbb{R}$ be a real number with $\gamma - \beta < \frac{1}{2}$ and let $B : H_\gamma \to HS(U_0, H_\beta)$ be globally Lipschitz continuous.*

Assumption 4 (Initial value ξ)**.** *Let $\xi : \Omega \to H_\gamma$ be $\mathcal{F}_{t_0}/\mathcal{B}(H_\gamma)$-measurable with $\mathbb{E}\|\xi\|_{H_\gamma}^p < \infty$ for all $p \in [1, \infty)$.*

Proposition 1. *Let the assumptions above be fulfilled. Then there exists an up to indistinguishability unique adapted stochastic process with continuous sample paths $X : [t_0, T] \times \Omega \to H_\gamma \in \cap_{p\in[1,\infty)} L^p(\Omega; C([t_0, T], H_\gamma))$ fulfilling*

$$X_t = e^{A(t-t_0)}\xi + \int_{t_0}^t e^{A(t-s)} F(X_s)\, ds + \int_{t_0}^t e^{A(t-s)} B(X_s)\, dW_s \tag{2}$$

for all $t \in [t_0, T]$ $\mathbb{P}$-a.s..

Let $\mathcal{J}$ be a finite or countable set and let $g_j \in U_0$, $j \in \mathcal{J}$, be an arbitrary orthonormal basis of the $\mathbb{R}$-Hilbert space $\left(U_0, \langle\cdot,\cdot\rangle_{U_0}, \|\cdot\|_{U_0}\right)$. Such a set and such an orthonormal basis exists since $\left(U_0, \langle\cdot,\cdot\rangle_{U_0}, \|\cdot\|_{U_0}\right)$ is separable.

Theorem 1 (Main result: A new - somehow mild - Itô formula for SPDEs)**.** *Let the assumptions above be fulfilled, let $(V, \langle\cdot,\cdot\rangle_V, \|\cdot\|_V)$ be a separable $\mathbb{R}$-Hilbert space and let $\varphi : H_\gamma \to V$ be a twice continuously Fréchet differentiable mapping with at most polynomially growing derivatives, i.e. suppose that there exists a real number $c \in [0, \infty)$ such that $\|\varphi''(v)\|_{L^{(2)}(H_\gamma, V)} \le c(1 + \|v\|_{H_\gamma}^c)$ for all $v \in H_\gamma$. Then we have*

$$\begin{aligned}
\varphi(X_t) &= \varphi(e^{A(t-t_0)}X_{t_0}) + \int_{t_0}^t \varphi'(e^{A(t-s)}X_s)\, e^{A(t-s)}F(X_s)\, ds \\
&\quad + \int_{t_0}^t \varphi'(e^{A(t-s)}X_s)\, e^{A(t-s)}B(X_s)\, dW_s \\
&\quad + \frac{1}{2}\sum_{j\in\mathcal{J}} \int_{t_0}^t \varphi''(e^{A(t-s)}X_s)\left(e^{A(t-s)}B(X_s)g_j, e^{A(t-s)}B(X_s)g_j\right) ds
\end{aligned} \tag{3}$$

for all $t \in [t_0, T]$ $\mathbb{P}$-a.s..

We remark that the possibly infinite sum and all integrals in (3) are well defined under the assumptions of Theorem 1 (see Section 2 for details). Other kinds of Itô formulas for SPDEs can be found in [1, 3, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

Example 1 (Identity)**.** *Let $V = H_\gamma$ and let $\langle v, w\rangle_V = \langle v, w\rangle_{H_\gamma}$ and $\|v\|_V = \|v\|_{H_\gamma}$ for all $v, w \in V = H_\gamma$. Moreover, let $\varphi : H_\gamma \to H_\gamma$ be the identity on H_γ, i.e. $\varphi(v) = v$ for all $v \in H_\gamma$. Theorem 1 then shows*

$$X_t = e^{A(t-t_0)} X_{t_0} + \int_{t_0}^{t} e^{A(t-s)} F(X_s)\, ds + \int_{t_0}^{t} e^{A(t-s)} B(X_s)\, dW_s$$

for all $t \in [t_0, T]$ $\mathbb{P}$-a.s. which is nothing else than the mild formulation of the SPDE (2)*. In this sense the formula* (3) *is somehow a mild Itô formula for SPDEs.*

Example 2 (Squared norm)**.** *Let $V = \mathbb{R}$ and let $\langle v, w\rangle_V = v \cdot w$ and $\|v\|_V = |v|$ for all $v \in V = \mathbb{R}$. Moreover, assume $\gamma \leq 0$ and let $\varphi : H_\gamma \to V$ be given by $\varphi(v) = \|v\|_H^2$ for all $v \in H$. Theorem 1 then shows*

$$\begin{aligned}\|X_t\|_H^2 &= \left\|e^{A(t-t_0)} X_{t_0}\right\|_H^2 + 2 \int_{t_0}^{t} \left\langle e^{A(t-s)} X_s, e^{A(t-s)} F(X_s)\right\rangle_H ds \\ &\quad + 2 \int_{t_0}^{t} \left\langle e^{A(t-s)} X_s, e^{A(t-s)} B(X_s)\, dW_s\right\rangle_H \\ &\quad + \int_{t_0}^{t} \left\|e^{A(t-s)} B(X_s)\right\|_{HS(U_0,H)}^2 ds\end{aligned}$$

for all $t \in [t_0, T]$ $\mathbb{P}$-a.s.. We refer to [6, 7, 10, 14, 15, 17, 18] for other Itô type formulas with the particular test function $\varphi(v) = \|v\|_H^2$, $v \in H_\gamma$.

Example 3 (Deterministic case)**.** *Let $\Omega = \{\emptyset\}$. Proposition 1 then implies the existence of a unique continuous function $x : [t_0, T] \to H_\gamma$ fulfilling*

$$x_t = e^{A(t-t_0)} \xi + \int_{t_0}^{t} e^{A(t-s)} F(x_s)\, ds \tag{4}$$

for all $t \in [t_0, T]$. Moreover, Theorem 1 shows

$$\varphi(x_t) = \varphi(e^{A(t-t_0)} x_{t_0}) + \int_{t_0}^{t} \varphi'(e^{A(t-s)} x_s)\, e^{A(t-s)} F(x_s)\, ds \tag{5}$$

for all $t \in [t_0, T]$. Equation (5) *is somehow a mild chain rule for the PDE* (4)*.*

Example 4 (Nonautonomous case)**.** *In this example let $(\tilde{H}, \langle\cdot,\cdot\rangle_{\tilde{H}}, \|\cdot\|_{\tilde{H}})$ be a separable $\mathbb{R}$-Hilbert spaces, let $\tilde{\mathcal{I}}$ be a finite or countable set and let $\tilde{A} : D(\tilde{A}) \subset \tilde{H} \to \tilde{H}$ be a linear operator with eigenvectors $\tilde{e}_i$, $i \in \tilde{\mathcal{I}}$, and eigenvalues $-\tilde{\lambda}_i \in \mathbb{R}$, $i \in \tilde{\mathcal{I}}$, satisfying Assumption 1. Moreover, let $\tilde{\eta} \in [0, \infty)$ with $\tilde{\eta} > -\inf_{i \in \tilde{\mathcal{I}}} \tilde{\lambda}_i$ and let $(\tilde{H}_r := D((\tilde{\eta} - \tilde{A})^r), \langle\cdot,\cdot\rangle_{\tilde{H}_r}, \|\cdot\|_{\tilde{H}_r})$ for*

$r \in \mathbb{R}$ be the $\mathbb{R}$-Hilbert spaces of domains of fractional powers of the linear operator $\tilde{\eta} - \tilde{A} : D(\tilde{A}) \subset \tilde{H} \to \tilde{H}$. In addition, let $\tilde{F} : \mathbb{R} \times \tilde{H}_\gamma \to \tilde{H}_\alpha$ and $\tilde{B} : \mathbb{R} \times \tilde{H}_\gamma \to \tilde{H}_\beta$ be globally Lipschitz continuous and let $\tilde{\xi} : \Omega \to \tilde{H}_\gamma$ be $\mathcal{F}_{t_0}/\mathcal{B}(\tilde{H}_\gamma)$-measurable with $\mathbb{E}\|\tilde{\xi}\|^p_{\tilde{H}_\gamma} < \infty$ for all $p \in [1, \infty)$. Then let $H = \mathbb{R} \times \tilde{H}$ and $\langle (s, v), (t, w) \rangle_H = s \cdot t + \langle v, w \rangle_{\tilde{H}}$ for all $s, t \in \mathbb{R}$ and all $v, w \in \tilde{H}$. Moreover, let $D(A) = \mathbb{R} \times D(\tilde{A})$ and let $A(s, v) = (0, \tilde{A}v)$ for all $s \in \mathbb{R}$ and all $v \in D(\tilde{A})$. In particular, we obtain $H_r = \mathbb{R} \times \tilde{H}_r$ for all $r \in \mathbb{R}$. Furthermore, let $F(s, v) = (1, \tilde{F}(s, v))$ for all $s \in \mathbb{R}$ and all $v \in \tilde{H}_\gamma$, let $B(s, v)u = (0, \tilde{B}(s, v)u)$ for all $s \in \mathbb{R}$, $v \in \tilde{H}_\gamma$ and all $u \in U_0$ and let $\xi(\omega) = (t_0, \tilde{\xi}(\omega))$ for all $\omega \in \Omega$. We now consider the consequences of Proposition 1 and Theorem 1 in this particular example. More precisely, Proposition 1 implies the existence of an up to indistinguishability unique adapted stochastic process with continuous sample paths $\tilde{X} : [t_0, T] \times \Omega \to \tilde{H}_\gamma \in \cap_{p \in [1,\infty)} L^p(\Omega; C([t_0, T], \tilde{H}_\gamma))$ fulfilling

$$\tilde{X}_t = e^{\tilde{A}(t-t_0)}\tilde{\xi} + \int_{t_0}^t e^{\tilde{A}(t-s)}\tilde{F}(s, \tilde{X}_s)\, ds + \int_{t_0}^t e^{\tilde{A}(t-s)}\tilde{B}(s, \tilde{X}_s)\, dW_s$$

for all $t \in [t_0, T]$ $\mathbb{P}$-a.s.. Finally, let $\varphi : \mathbb{R} \times \tilde{H}_\gamma \to V$ be a twice continuously Fréchet differentiable mapping with at most polynomially growing derivatives. Theorem 1 then shows

$$\begin{aligned}
\varphi(t, \tilde{X}_t) &= \varphi(t_0, e^{\tilde{A}(t-t_0)}\tilde{X}_{t_0}) \\
&\quad + \int_{t_0}^t \left[\varphi_t(s, e^{\tilde{A}(t-s)}\tilde{X}_s) + \varphi_x(s, e^{\tilde{A}(t-s)}\tilde{X}_s)\, e^{\tilde{A}(t-s)}\tilde{F}(s, \tilde{X}_s) \right] ds \\
&\quad + \int_{t_0}^t \varphi_x(s, e^{\tilde{A}(t-s)}\tilde{X}_s)\, e^{\tilde{A}(t-s)}\tilde{B}(s, \tilde{X}_s)\, dW_s \\
&\quad + \frac{1}{2}\sum_{j \in \mathcal{J}} \int_{t_0}^t \varphi_{xx}(s, e^{\tilde{A}(t-s)}\tilde{X}_s) \left(e^{\tilde{A}(t-s)}\tilde{B}(s, \tilde{X}_s)g_j, e^{\tilde{A}(t-s)}\tilde{B}(s, \tilde{X}_s)g_j \right) ds
\end{aligned} \tag{6}$$

for all $t \in [t_0, T]$ $\mathbb{P}$-a.s. where $\varphi_t : \mathbb{R} \times \tilde{H}_\gamma \to V$, $\varphi_x : \mathbb{R} \times \tilde{H}_\gamma \to L(\tilde{H}_\gamma, V)$ and $\varphi_{xx} : \mathbb{R} \times \tilde{H}_\gamma \to L^{(2)}(\tilde{H}_\gamma, V)$ denote appropriate partial derivatives of φ. Formula (6) *is nothing else but the nonautonomous counterpart of* (3).

2 Proofs

2.1 Proof of Proposition 1

Proof of Proposition 1. First of all, we assume $\alpha \leq \gamma$ and $\beta \leq \gamma$ w.l.o.g.. Moreover, the real number $R \in (0, \infty)$ defined by

$$R := 1 + \left\| (\eta - A)^{-1} \right\|_{L(H)} + \|F(0)\|_{H_\alpha} + \sup_{\substack{v,w \in H_\gamma \\ v \neq w}} \frac{\|F(v) - F(w)\|_{H_\alpha}}{\|v - w\|_{H_\gamma}}$$
$$+ \|B(0)\|_{HS(U_0,H_\beta)} + \sup_{\substack{v,w \in H_\gamma \\ v \neq w}} \frac{\|B(v) - B(w)\|_{HS(U_0,H_\beta)}}{\|v - w\|_{H_\gamma}} < \infty$$

is used throughout this proof. Due to Assumptions 1-3 the number R is indeed finite. Moreover, let $\mathcal{H}_p$ for $p \in [1, \infty)$ be the $\mathbb{R}$-vector space of equivalence classes of H_γ-valued predictable stochastic processes $Y : [t_0, T] \times \Omega \to H_\gamma$ that satisfy

$$\sup_{t \in [t_0,T]} \mathbb{E} \|Y_t\|_{H_\gamma}^p < \infty \tag{7}$$

where two stochastic processes lie in one equivalence class if and only if they are modifications of each other. As usual we do not distinguish between a predictable stochastic process $Y : [t_0, T] \times \Omega \to H_\gamma$ satisfying (7) and its equivalence class in $\mathcal{H}_p$ for $p \in [1, \infty)$. In the next step we equip the vector spaces $\mathcal{H}_p$, $p \in [1, \infty)$, with the norms

$$\|Y\|_{\mathcal{H}_p,r} := \sup_{t \in [t_0,T]} \left(e^{rt} \|Y_t\|_{L^p(\Omega;H_\gamma)} \right)$$

for all $Y \in \mathcal{H}_p$, $r \in \mathbb{R}$ and all $p \in [1, \infty)$. Note that the pair $\left(\mathcal{H}_p, \|\cdot\|_{\mathcal{H}_p,r}\right)$ is an $\mathbb{R}$-Banach space for every $r \in \mathbb{R}$ and every $p \in [1, \infty)$. Moreover, consider the mappings $\Phi_p : \mathcal{H}_p \to \mathcal{H}_p$, $p \in [2, \infty)$, defined by

$$(\Phi_p Y)_t := e^{A(t-t_0)}\xi + \int_{t_0}^t e^{A(t-s)} F(Y_s)\, ds + \int_{t_0}^t e^{A(t-s)} B(Y_s)\, dW_s \tag{8}$$

$\mathbb{P}$-a.s. for all $t \in [t_0, T]$ and all $p \in [2, \infty)$. In the following we show that the mappings $\Phi_p : \mathcal{H}_p \to \mathcal{H}_p$, $p \in [2, \infty)$, given by (8) are well defined.

To this end note that Assumptions 1 and 4 yield that $e^{A(t-t_0)}\xi$, $t \in [t_0, T]$, is an H_γ-valued adapted stochastic process with continuous sample paths. In particular, $e^{A(t-t_0)}\xi$, $t \in [t_0, T]$, is an H_γ-valued predictable stochastic process (see Proposition 3.6 (ii) in Da Prato and Zabczyk [3]). Additionally, we have

$$\sup_{t\in[t_0,T]} \mathbb{E}\left\|e^{A(t-t_0)}\xi\right\|_{H_\gamma}^p \\ \leq \sup_{t\in[t_0,T]} \left(\left\|e^{A(t-t_0)}\right\|_{L(H)}^p \mathbb{E}\left\|\xi\right\|_{H_\gamma}^p\right) \leq e^{p\eta T}\cdot\mathbb{E}\left\|\xi\right\|_{H_\gamma}^p < \infty \quad (9)$$

for all $p\in[1,\infty)$ which shows that the stochastic process $e^{A(t-t_0)}\xi$, $t\in[t_0,T]$, is indeed in $\cap_{p\in[1,\infty)}\mathcal{H}_p$.

We now concentrate on the second summand on the right hand side of (8). To this end note that Lemma 1 in [8] yields

$$\begin{aligned}
&\int_{t_0}^t \mathbb{E}\left\|e^{A(t-s)}F(Y_s)\right\|_{H_\gamma} ds \\
&\leq \int_{t_0}^t \left\|(\eta-A)^{(\gamma-\alpha)}\, e^{A(t-s)}\right\|_{L(H)} \mathbb{E}\left\|F(Y_s)\right\|_{H_\alpha} ds \\
&\leq \int_{t_0}^t \left\|(\eta-A)^{(\gamma-\alpha)}\, e^{(A-\eta)(t-s)}\right\|_{L(H)} e^{\eta T} R\left(1+\mathbb{E}\left\|Y_s\right\|_{H_\gamma}\right) ds \\
&\leq Re^{\eta T}\left(\int_{t_0}^t (t-s)^{(\alpha-\gamma)}\, ds\right)\left(1+\sup_{s\in[t_0,T]}\mathbb{E}\left\|Y_s\right\|_{H_\gamma}\right)
\end{aligned}$$

and Jensen's inequality therefore implies

$$\begin{aligned}
&\int_{t_0}^t \mathbb{E}\left\|e^{A(t-s)}F(Y_s)\right\|_{H_\gamma} ds \\
&\leq Re^{\eta T}\left(\int_0^T s^{(\alpha-\gamma)}\, ds\right)\left(1+\sup_{s\in[t_0,T]}\left\|Y_s\right\|_{L^p(\Omega;H_\gamma)}\right) \\
&= \frac{Re^{\eta T}T^{(1+\alpha-\gamma)}}{(1+\alpha-\gamma)}\left(1+\sup_{s\in[t_0,T]}\left\|Y_s\right\|_{L^p(\Omega;H_\gamma)}\right) < \infty
\end{aligned}$$

for all $t\in[t_0,T]$, $Y\in\mathcal{H}_p$ and all $p\in[1,\infty)$. (We used here that $1+\alpha-\gamma>0$ due to Assumption 2.) This shows that $\int_{t_0}^t e^{A(t-s)}F(Y_s)\,ds$, $t\in[t_0,T]$, is a well defined H_γ-valued adapted stochastic process for every $Y\in\cup_{p\in[1,\infty)}\mathcal{H}_p=\mathcal{H}_1$.

Moreover, we have

$$
\begin{aligned}
&\left\| \int_{t_0}^{t_2} e^{A(t_2-s)} F(Y_s)\, ds - \int_{t_0}^{t_1} e^{A(t_1-s)} F(Y_s)\, ds \right\|_{L^p(\Omega;H_\gamma)} \\
&\leq \int_{t_1}^{t_2} \left\| e^{A(t_2-s)} F(Y_s) \right\|_{L^p(\Omega;H_\gamma)} ds \\
&\quad + \int_{t_0}^{t_1} \left\| \left(e^{A(t_2-s)} - e^{A(t_1-s)} \right) F(Y_s) \right\|_{L^p(\Omega;H_\gamma)} ds \\
&\leq e^{\eta T} \int_{t_1}^{t_2} \left\| (\eta - A)^{(\gamma-\alpha)} e^{(A-\eta)(t_2-s)} \right\|_{L(H)} \| F(Y_s) \|_{L^p(\Omega;H_\alpha)} ds \\
&\quad + e^{\eta T} \left\| (\eta - A)^{-\varepsilon} \left(e^{A(t_2-t_1)} - I \right) \right\|_{L(H)} \\
&\qquad \cdot \int_{t_0}^{t_1} \left\| (\eta - A)^{(\gamma+\varepsilon-\alpha)} e^{(A-\eta)(t_1-s)} \right\|_{L(H)} \| F(Y_s) \|_{L^p(\Omega;H_\alpha)} ds
\end{aligned}
$$

and Lemmas 1 and 2 in [8] then show

$$
\begin{aligned}
&\left\| \int_{t_0}^{t_2} e^{A(t_2-s)} F(Y_s)\, ds - \int_{t_0}^{t_1} e^{A(t_1-s)} F(Y_s)\, ds \right\|_{L^p(\Omega;H_\gamma)} \\
&\leq R e^{\eta T} \int_{t_1}^{t_2} (t_2 - s)^{(\alpha-\gamma)} \left(1 + \| Y_s \|_{L^p(\Omega;H_\gamma)} \right) ds \\
&+ R^2 e^{2\eta T} (\eta+1)(T+1)(t_2-t_1)^{\varepsilon} \int_{t_0}^{t_1} (t_1 - s)^{(\alpha-\gamma-\varepsilon)} \left(1 + \| Y_s \|_{L^p(\Omega;H_\gamma)} \right) ds \\
&\leq \frac{R e^{\eta T} (t_2-t_1)^{(1+\alpha-\gamma)}}{(1+\alpha-\gamma)} \left(1 + \sup_{s\in[t_0,T]} \| Y_s \|_{L^p(\Omega;H_\gamma)} \right) \\
&\quad + \frac{R^2 e^{2\eta T} (\eta+1)(T+1)^2 (t_2-t_1)^{\varepsilon}}{(1+\alpha-\gamma-\varepsilon)} \left(1 + \sup_{s\in[t_0,T]} \| Y_s \|_{L^p(\Omega;H_\gamma)} \right) \\
&\leq \frac{2 R^2 e^{2\eta T} (\eta+1)(T+1)^2}{(1+\alpha-\gamma-\varepsilon)} \left(1 + \sup_{s\in[t_0,T]} \| Y_s \|_{L^p(\Omega;H_\gamma)} \right) (t_2-t_1)^{\varepsilon} < \infty
\end{aligned}
\tag{10}
$$

for all $t_1, t_2 \in [t_0, T]$ with $t_1 \leq t_2$, $\varepsilon \in [0, 1+\alpha-\gamma)$, $Y \in \mathcal{H}_p$ and all $p \in [1, \infty)$. Proposition 3.6 (ii) in Da Prato and Zabczyk [3] therefore yields that the stochastic process $\int_{t_0}^{t} e^{A(t-s)} F(Y_s)\, ds$, $t \in [t_0, T]$, has a modification in $\mathcal{H}_p$ for every $Y \in \mathcal{H}_p$ and every $p \in [1, \infty)$.

It remains to analyze the stochastic integral in (8). To this end observe that Lemma 1 in [8] gives

$$\begin{aligned}
&\int_{t_0}^{t} \mathbb{E} \left\| e^{A(t-s)} B(Y_s) \right\|^2_{HS(U_0,H_\gamma)} ds \\
&\leq e^{2\eta T} \int_{t_0}^{t} \left\| (\eta - A)^{(\gamma-\beta)} \, e^{(A-\eta)(t-s)} \right\|^2_{L(H)} \mathbb{E} \left\| B(Y_s) \right\|^2_{HS(U_0,H_\beta)} ds \\
&\leq 2R^2 e^{2\eta T} \int_{t_0}^{t} (t-s)^{2(\beta-\gamma)} \left(1 + \mathbb{E} \left\| Y_s \right\|^2_{H_\gamma} \right) ds \\
&\leq \frac{2R^2 e^{2\eta T} \, (T+1)}{(1+2\beta-2\gamma)} \left(1 + \sup_{s\in[t_0,T]} \mathbb{E} \left\| Y_s \right\|^2_{H_\gamma} \right) < \infty
\end{aligned}$$

for all $t \in [t_0, T]$ and all $Y \in \cup_{p\in[2,\infty)} \mathcal{H}_p = \mathcal{H}_2$. (We used here that $1 + 2\beta - 2\gamma = 2(\frac{1}{2} + \beta - \gamma) > 0$ due to Assumption 3.) Remark 1 in [8] hence shows that $\int_{t_0}^{t} e^{A(t-s)} B(Y_s) \, dW_s$, $t \in [t_0, T]$, is a well defined H_γ-valued adapted stochastic process for every $Y \in \cup_{p\in[2,\infty)} \mathcal{H}_p = \mathcal{H}_2$. Additionally, the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato and Zabczyk [3] gives

$$\begin{aligned}
&\left\| \int_{t_0}^{t_2} e^{A(t_2-s)} B(Y_s) \, dW_s - \int_{t_0}^{t_1} e^{A(t_1-s)} B(Y_s) \, dW_s \right\|_{L^p(\Omega;H_\gamma)} \\
&\leq p \left(\int_{t_1}^{t_2} \left\| e^{A(t_2-s)} B(Y_s) \right\|^2_{L^p(\Omega;HS(U_0,H_\gamma))} ds \right)^{\frac{1}{2}} \\
&\quad + p \left(\int_{t_0}^{t_1} \left\| \left(e^{A(t_2-s)} - e^{A(t_1-s)} \right) B(Y_s) \right\|^2_{L^p(\Omega;HS(U_0,H_\gamma))} ds \right)^{\frac{1}{2}} \\
&\leq e^{\eta T} p \left(\int_{t_1}^{t_2} \left\| (\eta - A)^{(\gamma-\beta)} \, e^{(A-\eta)(t_2-s)} \right\|^2_{L(H)} \left\| B(Y_s) \right\|^2_{L^p(\Omega;HS(U_0,H_\beta))} ds \right)^{\frac{1}{2}} \\
&\quad + e^{\eta T} p \left\| (\eta - A)^{-\varepsilon} \left(e^{A(t_2-t_1)} - I \right) \right\|_{L(H)} \\
&\qquad \cdot \left(\int_{t_0}^{t_1} \left\| (\eta - A)^{(\gamma+\varepsilon-\beta)} \, e^{(A-\eta)(t_1-s)} \right\|^2_{L(H)} \left\| B(Y_s) \right\|^2_{L^p(\Omega;HS(U_0,H_\beta))} ds \right)^{\frac{1}{2}}
\end{aligned}$$

and Lemmas 1 and 2 in [8] hence yield

$$
\begin{aligned}
&\left\| \int_{t_0}^{t_2} e^{A(t_2-s)} B(Y_s)\, dW_s - \int_{t_0}^{t_1} e^{A(t_1-s)} B(Y_s)\, dW_s \right\|_{L^p(\Omega;H_\gamma)} \\
&\leq e^{\eta T} p \left(\sup_{s\in[t_0,T]} \|B(Y_s)\|_{L^p(\Omega;HS(U_0,H_\beta))} \right) \frac{(t_2-t_1)^{\left(\frac{1}{2}+\beta-\gamma\right)}}{(1+2\beta-2\gamma)^{\frac{1}{2}}} \\
&\quad + R e^{2\eta T} p \left(\sup_{s\in[t_0,T]} \|B(Y_s)\|_{L^p(\Omega;HS(U_0,H_\beta))} \right) \frac{(\eta+1)\,(T+1)^2\,(t_2-t_1)^{\varepsilon}}{(1+2\beta-2\gamma-2\varepsilon)^{\frac{1}{2}}} \\
&\leq \frac{R^2 e^{2\eta T} p\,(\eta+1)\,(T+1)^2}{\left(\frac{1}{2}+\beta-\gamma-\varepsilon\right)} \left(1 + \sup_{s\in[t_0,T]} \|Y_s\|_{L^p(\Omega;H_\gamma)} \right) (t_2-t_1)^{\varepsilon} < \infty
\end{aligned}
\tag{11}
$$

for all $t_1, t_2 \in [t_0, T]$ with $t_1 \leq t_2$, $\varepsilon \in [0, \frac{1}{2} + \beta - \gamma)$, $Y \in \mathcal{H}_p$ and all $p \in [2, \infty)$. Proposition 3.6 (ii) in Da Prato and Zabczyk [3] therefore yields that the stochastic process $\int_{t_0}^{t} e^{A(t-s)} B(Y_s)\, dW_s$, $t \in [t_0, T]$, has a modification in $\mathcal{H}_p$ for every $p \in [2, \infty)$ and this finally shows the well definedness of the mappings $\Phi_p : \mathcal{H}_p \to \mathcal{H}_p$, $p \in [2, \infty)$, given by (8) (see (9), (10) and (11)).

In the next step we show that the mappings $\Phi_p : \mathcal{H}_p \to \mathcal{H}_p$, $p \in [2, \infty)$, are contractions with respect to appropriate norms. More formally, Lemma 7.7 in Da Prato and Zabczyk [3] and Lemma 1 in [8] give

$$
\begin{aligned}
&\left\| (\Phi_p Y)_t - (\Phi_p Z)_t \right\|_{L^p(\Omega;H_\gamma)} \\
&\leq \int_{t_0}^{t} \left\| e^{A(t-s)} \left(F(Y_s) - F(Z_s) \right) \right\|_{L^p(\Omega;H_\gamma)} ds \\
&\quad + p \left(\int_{t_0}^{t} \left\| e^{A(t-s)} \left(B(Y_s) - B(Z_s) \right) \right\|^2_{L^p(\Omega;HS(U_0,H_\gamma))} ds \right)^{\frac{1}{2}} \\
&\leq e^{\eta T} \int_{t_0}^{t} (t-s)^{(\alpha-\gamma)} \|F(Y_s) - F(Z_s)\|_{L^p(\Omega;H_\alpha)}\, ds \\
&\quad + e^{\eta T} p \left(\int_{t_0}^{t} (t-s)^{2(\beta-\gamma)} \|B(Y_s) - B(Z_s)\|^2_{L^p(\Omega;HS(U_0,H_\beta))}\, ds \right)^{\frac{1}{2}} \\
&\leq R e^{\eta T} \int_{t_0}^{t} (t-s)^{(\alpha-\gamma)} \|Y_s - Z_s\|_{L^p(\Omega;H_\gamma)}\, ds \\
&\quad + R e^{\eta T} p \left(\int_{t_0}^{t} (t-s)^{2(\beta-\gamma)} \|Y_s - Z_s\|^2_{L^p(\Omega;H_\gamma)}\, ds \right)^{\frac{1}{2}}
\end{aligned}
$$

and hence

$$
\begin{aligned}
&e^{rt}\left\|(\Phi_p Y)_t - (\Phi_p Z)_t\right\|_{L^p(\Omega;H_\gamma)} \\
&\le R e^{\eta T}\left(\int_{t_0}^{t} e^{r(t-s)}\,(t-s)^{(\alpha-\gamma)}\,ds\right)\|Y-Z\|_{\mathcal{H}_p,r} \\
&\quad + R e^{\eta T} p\left(\int_{t_0}^{t} e^{2r(t-s)}\,(t-s)^{2(\beta-\gamma)}\,ds\right)^{\frac{1}{2}}\|Y-Z\|_{\mathcal{H}_p,r} \\
&\le R e^{\eta T} p\left(\int_{0}^{(t-t_0)} e^{rs}\,s^{(\alpha-\gamma)}\,ds + \left[\int_{0}^{(t-t_0)} e^{2rs}\,s^{2(\beta-\gamma)}\,ds\right]^{\frac{1}{2}}\right)\|Y-Z\|_{\mathcal{H}_p,r}
\end{aligned}
$$

for all $t \in [t_0, T]$, $Y, Z \in \mathcal{H}_p$, $p \in [2, \infty)$ and all $r \in \mathbb{R}$. Finally, we obtain

$$
\begin{aligned}
&\|\Phi_p(Y) - \Phi_p(Z)\|_{\mathcal{H}_p,r} \\
&\qquad \le R e^{\eta T} p\left(\int_{0}^{T} e^{rs}\,s^{(\alpha-\gamma)}\,ds + \left[\int_{0}^{T} e^{2rs}\,s^{2(\beta-\gamma)}\,ds\right]^{\frac{1}{2}}\right)\|Y-Z\|_{\mathcal{H}_p,r}
\end{aligned}
$$

for all $Y, Z \in \mathcal{H}_p$, $p \in [2, \infty)$ and all $r \in \mathbb{R}$. This shows that $\Phi_p : \mathcal{H}_p \to \mathcal{H}_p$ is a contraction with respect to $\|\cdot\|_{\mathcal{H}_p,r}$ for a sufficiently small $r \in (-\infty, 0)$ and every $p \in [2, \infty)$. Hence, there exists a unique $Y : [t_0, T] \times \Omega \to H_\gamma \in \cap_{p\in[1,\infty)} \mathcal{H}_p$ with $Y = \Phi_2(Y)$, i.e.

$$
Y_t = e^{A(t-t_0)}\xi + \int_{t_0}^{t} e^{A(t-s)} F(Y_s)\,ds + \int_{t_0}^{t} e^{A(t-s)} B(Y_s)\,dW_s \tag{12}
$$

$\mathbb{P}$-a.s. for all $t \in [t_0, T]$. Moreover, the Kolmogorov-Chentsov theorem (see, e.g., Theorem 21.6 and Remark 21.7 in Klenke [9]), (10) and (11) show that $Y : [t_0, T] \times \Omega \to H_\gamma$ has an up to indistinguishability unique modification $X : [t_0, T] \times \Omega \to H_\gamma$ with continuous sample paths. In particular, (12) gives

$$
X_t = e^{A(t-t_0)}\xi + \int_{t_0}^{t} e^{A(t-s)} F(X_s)\,ds + \int_{t_0}^{t} e^{A(t-s)} B(X_s)\,dW_s \tag{13}
$$

$\mathbb{P}$-a.s. for all $t \in [t_0, T]$. Additionally, the Garsia-Rodemich-Rumsey lemma (see [4]) and again (10) and (11) yield that $X : [t_0, T] \times \Omega \to H_\gamma$ even lies in $\cap_{p\in[1,\infty)} L^p(\Omega; C([t_0, T], H_\gamma))$.

In order to complete the proof of Proposition 1, it remains to show that X_t, $t \in [t_0, T]$, is even indistinguishable from the right hand side of (13). For this, it is sufficient to check that the H_γ-valued adapted stochastic process $\int_{t_0}^{t} e^{A(t-s)} F(X_s)\,ds$, $t \in [t_0, T]$, has continuous sample paths. Indeed, Lemma 1 in [8] gives

$$\begin{aligned}
&\left\| \int_{t_0}^{t_2} e^{A(t_2-s)} F(X_s)\, ds - \int_{t_0}^{t_1} e^{A(t_1-s)} F(X_s)\, ds \right\|_{H_\gamma} \\
&\leq e^{\eta T} \int_{t_1}^{t_2} (t_2-s)^{(\alpha-\gamma)} \|F(X_s)\|_{H_\alpha} \, ds \\
&+ e^{\eta T} \left\| (\eta - A)^{-\varepsilon} \left(e^{A(t_2-t_1)} - I \right) \right\|_{L(H)} \int_{t_0}^{t_1} (t_1-s)^{(\alpha-\gamma-\varepsilon)} \|F(X_s)\|_{H_\alpha} \, ds \\
&\leq e^{\eta T} \left(\sup_{s\in[t_0,T]} \|F(X_s)\|_{H_\alpha} \right) \frac{(t_2-t_1)^{(1+\alpha-\gamma)}}{(1+\alpha-\gamma)} \\
&+ e^{\eta T} \left(\sup_{s\in[t_0,T]} \|F(X_s)\|_{H_\alpha} \right) \frac{\left\| (\eta - A)^{-\varepsilon} \left(e^{A(t_2-t_1)} - I \right) \right\|_{L(H)} (T+1)}{(1+\alpha-\gamma-\varepsilon)}
\end{aligned}$$

for all $t_1, t_2 \in [t_0, T]$ with $t_1 \leq t_2$ and all $\varepsilon \in [0, 1+\alpha-\gamma)$ and Lemma 2 in [8] hence shows that the H_γ-valued adapted stochastic process $\int_{t_0}^{t} e^{A(t-s)} F(X_s)\, ds$, $t \in [t_0, T]$, has continuous sample paths. This completes the proof of Proposition 1. □

2.2 Proof of Theorem 1

Proof of Theorem 1. A central difficulty in order to establish an Itô formula for the solution process $X : [t_0, T] \times \Omega \to H_\gamma$ of the SPDE (2) is that this solution process is, in general, not a semimartingale anymore to which Itô's formula in infinite dimensions (see Theorem 4.17 in Section 4.5 in Da Prato and Zabczyk [3]) could be applied. The solution process $X : [t_0, T] \times \Omega \to H_\gamma$ of (2) is, in general, not a semimartingale since the mild SPDE (2) is an Itô-Volterra type equation and the integrand processes $e^{A(t-s)} F(X_s)$, $s \in [t_0, t]$, and $e^{A(t-s)} B(X_s)$, $s \in [t_0, t]$, in (2) depend on $t \in [t_0, T]$ too. In order to overcome this difficulty we fix the time variable $t \in [t_0, t]$ within these integrands and then apply classical stochastic calculus to the resulting semimartingal processes. This well known trick has already been used in proofs of Davis-Burkholder-Gundy type inequalities for SPDEs (see, e.g., Proposition 7.8 in Da Prato and Zabczyk [3]) and has also been intensively exploited in the work of Conus [2] (see Theorem 5.1 and Theorem 7.2 in [2]). More formally, let $Y^t : [t_0, t] \times \Omega \to H_\gamma \in \cap_{p\in[1,\infty)} L^p(\Omega; C([t_0, t], H_\gamma))$, $t \in [t_0, T]$, be a family of stochastic processes given by

$$Y_u^t = e^{A(t-t_0)} X_{t_0} + \int_{t_0}^{u} e^{A(t-s)} F(X_s)\, ds + \int_{t_0}^{u} e^{A(t-s)} B(X_s)\, dW_s$$

for all $u \in [t_0, t]$ $\mathbb{P}$-a.s. and all $t \in [t_0, t]$. Note that $Y^t : [t_0, t] \times \Omega \to H_\gamma$ is well defined and indeed in $\cap_{p \in [1,\infty)} L^p(\Omega; C([t_0, t], H_\gamma))$ for all $t \in [t_0, t]$. Itô's formula in infinite dimensions (see Theorem 4.17 in Section 4.5 in Da Prato and Zabczyk [3]) then gives

$$\varphi(Y_u^t) = \varphi(Y_{t_0}^t) + \int_{t_0}^{u} \varphi'(Y_s^t)\, e^{A(t-s)} F(X_s)\, ds + \int_{t_0}^{u} \varphi'(Y_s^t)\, e^{A(t-s)} B(X_s)\, dW_s + \frac{1}{2} \sum_{j \in \mathcal{J}} \int_{t_0}^{u} \varphi''(Y_s^t) \left(e^{A(t-s)} B(X_s)\, g_j, e^{A(t-s)} B(X_s)\, g_j \right) ds$$

for all $u \in [t_0, t]$ $\mathbb{P}$-a.s. and all $t \in [t_0, T]$. Exploiting the facts $Y_t^t = X_t$ and $Y_{t_0}^t = e^{A(t-t_0)} X_{t_0}$ $\mathbb{P}$-a.s. for all $t \in [t_0, T]$ then gives

$$\begin{aligned} \varphi(X_t) = \varphi(Y_t^t) &= \varphi(e^{A(t-t_0)} X_{t_0}) + \int_{t_0}^{t} \varphi'(Y_s^t)\, e^{A(t-s)} F(X_s)\, ds \\ &\quad + \int_{t_0}^{t} \varphi'(Y_s^t)\, e^{A(t-s)} B(X_s)\, dW_s \\ &\quad + \frac{1}{2} \sum_{j \in \mathcal{J}} \int_{t_0}^{t} \varphi''(Y_s^t) \left(e^{A(t-s)} B(X_s)\, g_j, e^{A(t-s)} B(X_s)\, g_j \right) ds \end{aligned} \tag{14}$$

$\mathbb{P}$-a.s. for all $t \in [t_0, T]$. Equation (14) is an expansion formula for the stochastic process $\varphi(X_t)$, $t \in [t_0, T]$. Nevertheless, this formula seems to be of limited use since the integrands in (14) contain the stochastic processes Y_s^t, $s \in [t_0, t]$, $t \in [t_0, T]$, instead of the solution process X_s, $s \in [t_0, T]$, of (2) only. However, a key observation here is to exploit the elementary identity

$$\begin{aligned} Y_s^t &= e^{A(t-t_0)} X_{t_0} + \int_{t_0}^{s} e^{A(t-u)} F(X_u)\, du + \int_{t_0}^{s} e^{A(t-u)} B(X_u)\, dW_u \\ &= e^{A(t-s)} \left(e^{A(s-t_0)} X_{t_0} + \int_{t_0}^{s} e^{A(s-u)} F(X_u)\, du + \int_{t_0}^{s} e^{A(s-u)} B(X_u)\, dW_u \right) \\ &= e^{A(t-s)} X_s \end{aligned} \tag{15}$$

for all $s \in [t_0, t]$ $\mathbb{P}$-a.s. and all $t \in [t_0, T]$ in equation (14). This enables us to obtain a closed formula for the stochastic process $\varphi(X_t)$, $t \in [t_0, T]$. More precisely, putting (15) into (14) gives

$$\begin{aligned} \varphi(X_t) &= \varphi(e^{A(t-t_0)} X_{t_0}) + \int_{t_0}^{t} \varphi'(e^{A(t-s)} X_s)\, e^{A(t-s)} F(X_s)\, ds \\ &\quad + \int_{t_0}^{t} \varphi'(e^{A(t-s)} X_s)\, e^{A(t-s)} B(X_s)\, dW_s \\ &\quad + \frac{1}{2} \sum_{j \in \mathcal{J}} \int_{t_0}^{t} \varphi''(e^{A(t-s)} X_s) \left(e^{A(t-s)} B(X_s) g_j, e^{A(t-s)} B(X_s) g_j \right) ds \end{aligned} \tag{16}$$

$\mathbb{P}$-a.s. for all $t \in [t_0, T]$. The stochastic process $\varphi(X_t)$, $t \in [t_0, T]$, is thus a modification of the stochastic process on the right hand side of (16). In order to complete the proof of Theorem 1, it remains to show that $\varphi(X_t)$, $t \in [t_0, T]$, is even indistinguishable from the right hand side of (16). For this, it is sufficient to verify that the first, the second and the fourth summand on the right hand side of (16) are V-valued adapted stochastic processes with continuous sample paths.

To this end note that $\varphi(e^{A(t-t_0)} X_{t_0})$, $t \in [t_0, T]$, is a V-valued adapted stochastic process with continuous sample paths since $\varphi : H_\gamma \to V$ is continuous and since $e^{At} \in L(H_\gamma)$, $t \in [0, \infty)$, is a strongly continuous semigroup on H_γ.

In the next step we concentrate on the second summand on the right hand side of (16). More formally, Lemma 1 in [8] gives

$$\begin{aligned}
&\left\| \int_{t_0}^{t_2} \varphi'\big(e^{A(t_2-s)} X_s\big)\, e^{A(t_2-s)} F(X_s)\, ds - \int_{t_0}^{t_1} \varphi'\big(e^{A(t_1-s)} X_s\big)\, e^{A(t_1-s)} F(X_s)\, ds \right\|_V \\
&\leq e^{\eta T} \int_{t_1}^{t_2} \left\| \varphi'\big(e^{A(t_2-s)} X_s\big) \right\|_{L(H_\gamma, V)} (t_2 - s)^{(\alpha-\gamma)} \left\| F(X_s) \right\|_{H_\alpha} ds \\
&+ e^{\eta T} \int_{t_0}^{t_1} \left\| \varphi'\big(e^{A(t_2-s)} X_s\big) - \varphi'\big(e^{A(t_1-s)} X_s\big) \right\|_{L(H_\gamma, V)} (t_2 - s)^{(\alpha-\gamma)} \left\| F(X_s) \right\|_{H_\alpha} ds \\
&+ e^{\eta T} \int_{t_0}^{t_1} \left\| \varphi'\big(e^{A(t_1-s)} X_s\big) \right\|_{L(H_\gamma, V)} \left\| (\eta - A)^{-\varepsilon} \left(e^{A(t_2-t_1)} - I\right) \right\|_{L(H)} \\
&\quad \cdot \left\| (\eta - A)^{(\gamma+\varepsilon-\alpha)}\, e^{(A-\eta)(t_1-s)} \right\|_{L(H)} \left\| F(X_s) \right\|_{H_\alpha} ds
\end{aligned}$$

and hence

$$\begin{aligned}
&\left\| \int_{t_0}^{t_2} \varphi'\big(e^{A(t_2-s)} X_s\big)\, e^{A(t_2-s)} F(X_s)\, ds - \int_{t_0}^{t_1} \varphi'\big(e^{A(t_1-s)} X_s\big)\, e^{A(t_1-s)} F(X_s)\, ds \right\|_V \\
&\leq e^{\eta T} \left(\sup_{u,s \in [t_0,T]} \left\| \varphi'\big(e^{A(u-t_0)} X_s\big) \right\|_{L(H_\gamma, V)} \right) \left(\sup_{s \in [t_0,T]} \left\| F(X_s) \right\|_{H_\alpha} \right) \\
&\cdot \left(\frac{(t_2 - t_1)^{(1+\alpha-\gamma)}}{(1 + \alpha - \gamma)} + \frac{(T+1) \left\| (\eta - A)^{-\varepsilon} \left(e^{A(t_2-t_1)} - I\right) \right\|_{L(H)}}{(1 + \alpha - \gamma - \varepsilon)} \right) \\
&+ e^{\eta T} \left(\sup_{s \in [t_0,T]} \left\| F(X_s) \right\|_{H_\alpha} \right) \\
&\cdot \left(\int_{t_0}^{T} \mathbb{1}_{[t_0,t_1]}(s) \left\| \varphi'\big(e^{A(t_2-s)} X_s\big) - \varphi'\big(e^{A(t_1-s)} X_s\big) \right\|_{L(H_\gamma, V)} (t_2 - s)^{(\alpha-\gamma)}\, ds \right)
\end{aligned}$$

for all $t_1, t_2 \in [t_0, T]$ with $t_1 \leq t_2$ and all $\varepsilon \in [0, 1 + \alpha - \gamma)$. Lebesgue's theorem of dominated convergence and Lemma 2 in [8] therefore show that the V-valued adapted stochastic process $\int_{t_0}^{t} \varphi'(e^{A(t-s)} X_s)\, e^{A(t-s)} F(X_s)\, ds$, $t \in [t_0, T]$, has continuous sample paths.

Finally, we analyze the fourth summand on the right hand side of (16). Here we have

$$
\begin{aligned}
&\Bigg\| \sum_{j \in \mathcal{J}} \int_{t_0}^{t_2} \varphi''\big(e^{A(t_2-s)} X_s\big) \left(e^{A(t_2-s)} B(X_s) g_j, e^{A(t_2-s)} B(X_s) g_j\right) ds \\
&\quad - \sum_{j \in \mathcal{J}} \int_{t_0}^{t_1} \varphi''\big(e^{A(t_1-s)} X_s\big) \left(e^{A(t_1-s)} B(X_s) g_j, e^{A(t_1-s)} B(X_s) g_j\right) ds \Bigg\|_V \\
&\leq \int_{t_1}^{t_2} \left\|\varphi''\big(e^{A(t_2-s)} X_s\big)\right\|_{L^{(2)}(H_\gamma, V)} \left\|e^{A(t_2-s)} B(X_s)\right\|_{HS(U_0, H_\gamma)}^2 ds \\
&+ \int_{t_0}^{t_1} \left\|\varphi''\big(e^{A(t_2-s)} X_s\big) - \varphi''\big(e^{A(t_1-s)} X_s\big)\right\|_{L^{(2)}(H_\gamma, V)} \left\|e^{A(t_2-s)} B(X_s)\right\|_{HS(U_0, H_\gamma)}^2 ds \\
&+ \sum_{j \in \mathcal{J}} \int_{t_0}^{t_1} \left\|\varphi''\big(e^{A(t_1-s)} X_s\big)\right\|_{L^{(2)}(H_\gamma, V)} \Bigg(\left\|\left(e^{A(t_2-s)} - e^{A(t_1-s)}\right) B(X_s) g_j\right\|_{H_\gamma} \\
&\cdot \left(\left\|e^{A(t_2-s)} B(X_s) g_j\right\|_{H_\gamma} + \left\|e^{A(t_1-s)} B(X_s) g_j\right\|_{H_\gamma}\right) \Bigg) ds
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\Bigg\| \sum_{j\in\mathcal{J}} \int_{t_0}^{t_2} \varphi''\big(e^{A(t_2-s)}X_s\big) \left(e^{A(t_2-s)}B(X_s)g_j, e^{A(t_2-s)}B(X_s)g_j\right) ds \\
&\quad - \sum_{j\in\mathcal{J}} \int_{t_0}^{t_1} \varphi''\big(e^{A(t_1-s)}X_s\big) \left(e^{A(t_1-s)}B(X_s)g_j, e^{A(t_1-s)}B(X_s)g_j\right) ds \Bigg\|_V \\
&\leq \left(\sup_{u,s\in[t_0,T]} \left\|\varphi''\big(e^{A(u-t_0)}X_s\big)\right\|_{L^{(2)}(H_\gamma,V)} \right) \left(\sup_{s\in[t_0,T]} \|B(X_s)\|^2_{HS(U_0,H_\beta)} \right) \\
&\cdot \frac{e^{2\eta T}\,(t_2-t_1)^{(1+2\beta-2\gamma)}}{(1+2\beta-2\gamma)} + e^{2\eta T} \left(\sup_{s\in[t_0,T]} \|B(X_s)\|^2_{HS(U_0,H_\beta)} \right) \\
&\cdot \int_{t_0}^{T} \mathbb{1}_{[t_0,t_1]}(s) \left\|\varphi''\big(e^{A(t_2-s)}X_s\big) - \varphi''\big(e^{A(t_1-s)}X_s\big)\right\|_{L^{(2)}(H_\gamma,V)} (t_2-s)^{2(\beta-\gamma)}\, ds \\
&+ \left(\sup_{u,s\in[t_0,T]} \left\|\varphi''\big(e^{A(u-t_0)}X_s\big)\right\|_{L^{(2)}(H_\gamma,V)} \right) \left\|(\eta-A)^{-\varepsilon}\left(e^{A(t_2-t_1)}-I\right)\right\|_{L(H)} \\
&\cdot 2e^{2\eta T} \int_{t_0}^{t_1} (t_1-s)^{(2\beta-2\gamma-\varepsilon)} \|B(X_s)\|_{HS(U_0,H_\beta)}\, ds
\end{aligned}
$$

for all $t_1, t_2 \in [t_0, T]$ with $t_1 \leq t_2$ and all $\varepsilon \in [0, 1+2\beta-2\gamma)$ due to Lemma 1 in [8]. Lebesgue's theorem of dominated convergence and Lemma 2 in [8] hence yield that the V-valued adapted stochastic process

$$
\sum_{j\in\mathcal{J}} \int_{t_0}^{t} \varphi''(e^{A(t-s)}X_s)\,(e^{A(t-s)}B(X_s)g_j, e^{A(t-s)}B(X_s)g_j)\, ds
$$

for $t \in [t_0, T]$ has continuous sample paths and this finally completes the proof of Theorem 1. □

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