Non-time-homogeneous Generalized Mehler Semigroups and Applications[*]

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Abstract

A non-time-homogeneous generalized Mehler semigroup on a real separable Hilbert space $\mathbb{H}$ is defined through

$$p_{s,t}f(x) = \int_{\mathbb{H}} f(U(t,s)x + y)\,\mu_{t,s}(dy), \quad t \geq s,\ x \in \mathbb{H},$$

for every bounded measurable function f on $\mathbb{H}$, where $(U(t,s))_{t \geq s}$ is an evolution family of bounded operators on $\mathbb{H}$ and $\mu_{t,s}$ is a family of probability measures on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$ satisfying $\mu_{t,s} = \mu_{t,r} * (\mu_{r,s} \circ U(t,r)^{-1})$ for $t \geq r \geq s$. This kind of semigroups is closely related with the "transition semigroup" of non-autonomous (possibly non-continuous) Ornstein-Uhlenbeck process driven by some proper additive process. We show the infinite divisibility and a Lévy-Khintchine type representation of $\mu_{t,s}$. We also study the corresponding evolution systems of measures (=space-time invariant measures), dimension free Harnack inequality and their applications to derive important properties of $p_{s,t}$. We also prove the Harnack inequality and show the strong Feller property for the transition semigroup of semi-linear non-autonomous Ornstein-Uhlenbeck processes driven by a Wiener process.

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1 Introduction

Generalized Mehler semigroups $(p_t)_{t \geq 0}$ on a real separable Hilbert space $\mathbb{H}$, which have been extensively studied in [BRS96, SS01, FR00, LR02, RW03, DL04, DLSS04, LR04] etc., are defined by the formula

$$p_t f(x) = \int_{\mathbb{H}} f(T_t x + y) \, \mu_t(dy), \quad t \geq 0, \ x \in \mathbb{H}, \ f \in B_b(\mathbb{H}). \qquad (1.1)$$

Here $B_b(\mathbb{H})$ is the space of all bounded Borel measurable functions on $\mathbb{H}$, and $(T_t)_{t \geq 0}$ is a strongly continuous semigroup on $\mathbb{H}$ and $(\mu_t)_{t \geq 0}$ is a family of probability measures on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$ satisfying the following skew convolution semigroup condition

$$\mu_{t+s} = \mu_s * (\mu_t \circ T_s^{-1}), \quad s, t \geq 0. \qquad (1.2)$$

Condition (1.2) is necessary and sufficient for the semigroup property of $(p_t)_{t \geq 0}$: for all $t, s \geq 0$, $p_t p_s = p_{t+s}$ on $B_b(\mathbb{H})$ (and the Markov property of the corresponding stochastic process respectively).

    The transition semigroups of Lévy driven Ornstein-Uhlenbeck processes are typical examples of generalized Mehler semigroups. See e.g. [App06, PZ07]. It is shown in [BRS96, FR00] that under some mild conditions the reverse result also holds.

Recently, much work, for instance, [DPL07, DPR08, GL08, Knä09, Woo09], has been devoted to the study of non-autonomous Ornstein-Uhlenbeck processes. In these papers, the drift is time-dependent, and the noise is modeled by a stationary process, e.g. a Wiener process or Lévy process. To get a full non-homogeneous Ornstein-Uhlenbeck process, it is natural to consider a more general noise given by non-stationary processes such as additive processes. To be more precise, let us describe our framework in detail.

Let $(A(t), \mathscr{D}(A(t)))_{t \in \mathbb{R}}$ be a family of linear operators on $\mathbb{H}$ with dense domains. Suppose that the non-autonomous Cauchy problem $dx_t = A(t)x_t dt$, $t \geq s$, with initial condition $x_s = x$, is well posed (see [Paz83]). That is, there exists an evolution family of bounded operators $(U(t, s))_{t \geq s}$ on $\mathbb{H}$ such that $x(t) = U(t, s)x$ for $x \in \mathscr{D}(A(s))$ is a classical unique solution of this Cauchy problem.

Recall that a family of bounded linear operators $(U(t, s))_{t \geq s}$ on $\mathbb{H}$ is an evolution family if

(1) For every $s \in \mathbb{R}$, $U(s, s) = I$ and for all $t \geq r \geq s$, $U(t, r)U(r, s) = U(t, s)$;

(2) For every $x \in \mathbb{H}$, $(t, s) \to U(t, s)x$ is strongly continuous on $\{(t, s) : t \geq s;\ t, s \in \mathbb{R}\}$.

An evolution family is also called evolution system, propagator etc.. For more details we refer e.g. to [Paz83].

Let $(Z_t)_{t \geq s}$ be an additive processes in $\mathbb{H}$, i.e. an $\mathbb{H}$-valued stochastic continuous stochastic process with independent increments, and $(B(t))_{t \in \mathbb{R}}$ a family of bounded linear operators on $\mathbb{H}$. Consider the following stochastic differential equation

$$\begin{cases} dX_t = A(t)X_t dt + B(t)dZ_t, \\ X_s = x. \end{cases} \qquad (1.3)$$

We call the following process a mild solution of (1.3):

$$X(t, s, x) = U(t, s)x + \int_s^t U(t, r)B(r)\, dZ_r, \qquad (1.4)$$

$t \geq s$ and $x \in \mathbb{H}$, if the stochastic convolution integral in (1.4) is well-defined for a proper additive process $(Z_t)_{t \geq 0}$ (see [Det83, Sat06]). Denote the distribution of the convolution $\int_s^t U(t, r)B(r)\, dZ_r$ by $\tilde{\mu}_{t,s}$. Then the transition semigroup of $X(t, s, x)$ is given by

$$P_{s,t}f(x) = \mathbb{E}_{\mathbb{P}}f(X(t, s, x)) = \int_{\mathbb{H}} f\left(U(t, s)x + y\right) \tilde{\mu}_{t,s}(dy). \qquad (1.5)$$

for all $t \geq s$, $x \in \mathbb{H}$.

The aim of the present paper is to adopt the axiomatic approach from [BRS96] to study this kind of non-autonomous processes through its semigroup (1.5).

Complementing the analytic study in this work, the more probabilistic paper (see [Ouy10]) containing a detailed study of non-autonomous Ornstein-Uhlenbeck processes (1.4) driven by an additive process is in preparation. For the present paper, we consider an abstract form of (1.5) given by the following non-time-homogeneous version of the generalized Mehler semigroup (1.1):

$$p_{s,t}f(x) = \int_{\mathbb{H}} f(U(t,s)x + y)\,\mu_{t,s}dy, \quad x \in \mathbb{H}, \ f \in B_b(\mathbb{H}). \quad (1.6)$$

Here $\mu_{t,s}$, $s \leq t$, is a family of probability measures on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$ satisfying a non-time-homogeneous analog of (1.2) (see (1.7) below).

The organization of this paper is as follows.

In Section 2 we shall introduce the non-time-homogeneous transition function $p_{s,t}$ of generalized Mehler type. We show the continuity and characterize the Markov property of $p_{s,t}$. We also indicate how it fits into the more general framework of non-homogeneous skew convolution semigroups.

In Section 3 we study the skew convolution equation for measures

$$\mu_{t,s} = \mu_{t,r} * \left(\mu_{r,s} \circ U(t,r)^{-1}\right) \quad (1.7)$$

which is equivalent to the flow property for $(p_{s,t})_{t \geq s}$, i.e. the Chapman-Kolmogorov equations (see Proposition 2.2 below). We prove that if $\mu_{t,s}$ is stochastically continuous in s,t (cf. Assumption 3.1 below), then for every $t \geq s$, $\mu_{t,s}$ is infinitely divisible. We then investigate the structure and representation of the measures $\mu_{t,s}$.

In Section 4 we study evolution systems of measures, i.e. space-time invariant measures, for the semigroup $(p_{s,t})_{t \geq s}$. We show some sufficient and necessary conditions for the existence and uniqueness of evolution systems of measures. The basic idea can be found in [BRS96, FR00, DPL07, Knä09, Woo09] etc.. But we are in a more general framework. Theorem 4.3 and Theorem 4.4 below are the infinite dimensional generalizations of the results in [Woo09] for finite dimensional Lévy driven non-autonomous Ornstein-Uhlenbeck processes. However, Items (3) and (4) of Theorem 4.4, which are not in [Woo09], are new also in finite dimensions and interesting by themselves since they are converse results to Theorem 4.2 below about the relationship of two evolution systems of measures. We borrow the idea to use periodic (in time) conditions to prove uniqueness (see Theorem 4.7) from [DPL07, Knä09].

In Section 5 we prove Harnack inequalities for $p_{s,t}$ using much simpler arguments than in the previous papers [RW03, Knä09, ORW09, Ouy09a] in which also Harnack inequalities for generalized Mehler semigroups or Ornstein-Uhlenbeck semigroup driven by Lévy processes were shown. The method in [Knä09] and [RW03] relies on taking the derivative of a proper functional; the method in

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[ORW09, Ouy09a] is based on coupling and Girsanov transformation. Our approach in this paper is based on a decomposition of $p_{s,t}$. As applications of the Harnack inequality, we prove that null controllability implies the strong Feller property and that for the Gaussian case, null controllability, Harnack inequality and strong Feller property are even equivalent to each other as in the time homogeneous case.

In Section 6 we apply Girsanov's theorem to study the existence of martingale solutions of semi-linear non-autonomous Ornstein-Uhlenbeck process driven by a Wiener process for possibly non-Lipschitz non-linearities. For the Lipschitz case we refer to [Ver09]. Our approach is an adaption of the standard procedure when the linear part A does not depend on time (see [DPZ92, Chapter 10]). Our main contribution here is to establish a Harnack inequality and hence show the strong Feller property for the transition semigroup (based on applying a properly adapted version of the method in [ORW09, Section 4]).

In Section 7 we append a short introduction to control theory for non-autonomous linear control systems and null controllability. This is closely related to the strong Feller property of the corresponding Ornstein-Uhlenbeck processes. The minimal energy representation also proves useful for more precise estimates of the constants in the Harnack inequalities.

2 Non-time-homogeneous generalized Mehler semigroups

Let $\mathbb{H}$ be a real separable Hilbert space with norm and inner product denoted by $|\cdot|$ and $\langle\cdot,\cdot\rangle$ respectively. Let $(U(t,s))_{t\geq s}$ be an evolution family on $\mathbb{H}$ and $(\mu_{t,s})_{t\geq s}$ a family of probability measures on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$. For every $f \in B_b(\mathbb{H})$ and $t \geq s$, define

$$p_{s,t}f(x) = \int_{\mathbb{H}} f(U(t,s)x + y)\,\mu_{t,s}(dy), \quad x \in \mathbb{H}. \tag{2.1}$$

For every $(t,s) \in \Lambda := \{(t,s) \in \mathbb{R}^2 \colon t \geq s\}$, it is clear that $p_{s,t}$ is Feller, i.e. $p_{s,t}(C_b(\mathbb{H})) \subset C_b(\mathbb{H})$, where $C_b(\mathbb{H})$ is the space of all bounded continuous functions on $\mathbb{H}$. Now we look at the continuity of the map $(t,s,x) \mapsto p_{s,t}f(x)$ for f in $C_b(\mathbb{H})$.

The following proposition is a direct generalization of [BRS96, Lemma 2.1]. The proof is quite similar to the proof in [BRS96].

Proposition 2.1. *Assume that $(s_n, t_n) \in \Lambda$, $n \in \mathbb{N}$ with $(s_n, t_n) \to (s, t)$ as $n \to \infty$ such that $\mu_{t_n, s_n} \to \mu_{t,s}$ weakly as $n \to \infty$. Then $p_{s_n, t_n}f(x_n) \to p_{s,t}f(x)$ if $x_n \to x$ in $\mathbb{H}$ as $n \to \infty$ and $f \in C_b(\mathbb{H})$.*

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Proof. Since $\mu_{t_n, s_n} \to \mu_{t,s}$ weakly, by Prohorov's theorem, for every $\epsilon > 0$, there exists a compact set $K \subset \mathbb{H}$ such that

$$\mu_{r,\sigma}(K) \geq 1 - \varepsilon, \quad \text{for all } (r, \sigma) \in \{(t,s), (t_n, s_n) \colon n \in \mathbb{N}\}. \tag{2.2}$$

For abbreviation, we set $z_n = U(t_n, s_n)x_n$ and $z = U(t,s)x$.

By the strong continuity of the evolution family $(U(t,s))_{t \geq s}$, the set $S := \{z, z_n \colon n \in \mathbb{N}\}$ is compact. Hence $S + K$ is also compact. So there exists $N \in \mathbb{N}$ such that for any $n > N$ and any $y \in K$,

$$|f(z_n + y) - f(z + y)| < \varepsilon, \tag{2.3}$$

since f is uniformly continuous on compacts.

Because the map $(t,s) \mapsto \mu_{t,s}$ on Λ is weakly continuous, (taking N larger if necessary) we have for all $n > N$

$$\left| \int_{\mathbb{H}} f(z + y)\, \mu_{t_n, s_n}(dy) - \int_{\mathbb{H}} f(z + y)\, \mu_{t,s}(dy) \right| < \varepsilon. \tag{2.4}$$

From (2.2), (2.3) and (2.4) we get

$$\left| \int_{\mathbb{H}} f(z_n + y)\, \mu_{t_n, s_n}(dy) - \int_{\mathbb{H}} f(z + y)\, \mu_{t,s}(dy) \right|$$

$$\leq \left| \int_{\mathbb{H}} f(z + y)\, \mu_{t_n, s_n}(dy) - \int_{\mathbb{H}} f(z + y)\, \mu_{t,s}(dy) \right|$$

$$+ \int_K |f(z_n + y) - f(z + y)|\, \mu_{t_n, s_n}(dy) + 2\|f\|_\infty \varepsilon$$

$$\leq 2\varepsilon(1 + \|f\|_\infty),$$

and the result is proved since ε was arbitrary. $\qquad \square$

We are interested in the case when $(p_{s,t})_{t \geq s}$ in (2.1) satisfies the Chapman-Kolmogorov equations:

Proposition 2.2. *For all $s \leq r \leq t$,*

$$p_{s,r} p_{r,t} = p_{s,t} \quad (\text{``Chapman-Kolmogorv equations''}) \tag{2.5}$$

on $B_b(\mathbb{H})$ if and only if for all $s \leq r \leq t$,

$$\mu_{t,s} = \mu_{t,r} * \left(\mu_{r,s} \circ U(t,r)^{-1} \right). \tag{2.6}$$

For the proof we refer to Example 2.6 below which is based on Proposition 2.5 below where we deal with the more general skew convolution semigroups.

Later on, we shall always assume (2.6) to hold or equivalently the following equation holds:

$$\hat{\mu}_{t,s}(\xi) = \hat{\mu}_{t,r}(\xi)\hat{\mu}_{r,s}(U(t,r)^*\xi), \quad \xi \in \mathbb{H}. \tag{2.7}$$

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Here for every probability measure μ on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$, we denote its Fourier transform by $\hat{\mu}$, i.e.,

$$\hat{\mu}(\xi) = \int e^{i\langle x, \xi \rangle} \mu(dx), \quad \xi \in \mathbb{H}.$$

Obviously, (2.6) implies that $\mu_{t,t} = \delta_0$ for all $t \in \mathbb{R}$.

Definition 2.3. The family of probability kernels $(p_{s,t})_{t \geq s}$ defined in (2.1) with $(\mu_{t,s})_{t \geq s}$ satisfying (2.6) is called a non-time-homogeneous generalized Mehler semigroup.

Naturally there exists a Markov process associated with $(p_{s,t})_{t \geq s}$ by Kolmogorov's consistency theorem. However, this process is of interest only if one can prove certain regularity properties of its sample paths. One can mimic the idea in [BRS96] and [FR00] to construct corresponding Markov processes with càdlàg paths and even show that the process solves some stochastic equation. This will be contained in [Ouy10].

As noted by Li et al. (see [Li06] for a survey), a generalized Mehler semigroup is a special case of a so called skew convolution semigroup. In the remainder of this section we shall briefly discuss non-time-homogeneus skew convolution semigroups which constitute a more general framework than non-time-homogeneous generalized Mehler semigroups. But in the following sections of this paper we shall concentrate on the setting introduced above.

Let $(S, +)$ be a metrizable abelian semigroup, that is, S is a metrizable topological space and there is an operation $+ \colon S^2 \to S$ which is associative, commutative and continuous. Let $(u_{s,t})_{t \geq s}$ be a Borel Markov transition function on S satisfying

$$u_{s,t}(x + y, \cdot) = u_{s,t}(x, \cdot) * u_{s,t}(y, \cdot) \tag{2.8}$$

for every $t \geq s$ and $x, y \in S$.

Since $(u_{s,t})_{t \geq s}$ is a family of Markov transition functions, the Chapman-Kolmogorov equations hold, i.e. for all $x \in S$, $f \in B_b(S)$, $t \geq r \geq s$,

$$u_{s,t}f(x) = u_{s,r}(u_{r,t}f)(x),$$

or written in integral form:

$$\int_{S^2} f(z)u_{r,t}(y, dz)u_{s,r}(x, dy) = \int_S f(z)u_{s,t}(x, dz). \tag{2.9}$$

From (2.8) we see that for every $t \geq s$

$$u_{s,t}(0, \cdot) = \delta_0. \tag{2.10}$$

Here 0 denotes the neutral element in S.

For every probability measure μ on $(S, \mathscr{B}(S))$ we set

$$\mu u_{s,t}(A) = \int_S u_{s,t}(x, A)\,\mu(dx), \quad A \in B(S)$$

for every $t \geq s$. It is easy to show the following result.

Proposition 2.4. *For any two probability measures μ and ν on $(S, \mathscr{B}(S))$,*

$$(\mu * \nu)u_{s,t} = (\mu u_{s,t}) * (\nu u_{s,t})$$

for all $t \geq s$.

Now let $(\mu_{t,s})_{t \geq s}$ be a family of probability measures on $(S, \mathscr{B}(S))$. Define

$$q_{s,t}(x, \cdot) = u_{s,t}(x, \cdot) * \mu_{t,s}(\cdot)$$

for all $x \in S$ and $t \geq s$. We have the following result.

Proposition 2.5. *$(q_{s,t})_{t \geq s}$ is a family of transition functions, i.e.*

$$q_{s,t} = q_{s,r}q_{r,t}, \quad t \geq r \geq s \tag{2.11}$$

if and only if

$$\mu_{t,s} = \mu_{t,r} * (\mu_{r,s}u_{r,t}), \quad t \geq r \geq s, \tag{2.12}$$

or equivalently,

$$\hat{\mu}_{t,s}(\xi) = \hat{\mu}_{t,r}(\xi)(\widehat{\mu_{r,s}u_{r,t}})(\xi), \quad \xi \in S, \ t \geq r \geq s.$$

Proof. For every $f \in B_b(S)$, $x \in S$, we have

$$q_{s,r}q_{r,t}f(x)$$

$$= \int_S q_{r,t}f(y)q_{s,r}(x, dy)$$

$$= \int_{S^2} q_{r,t}f(y_1 + y_2)u_{s,r}(x, dy_1)\mu_{r,s}(dy_2)$$

$$= \int_{S^4} f(z)q_{r,t}(y_1 + y_2, dz)u_{s,r}(x, dy_1)\mu_{r,s}(dy_2)$$

$$= \int_{S^4} f(z_1 + z_2)u_{r,t}(y_1 + y_2, dz_1)\mu_{t,r}(dz_2)u_{s,r}(x, dy_1)\mu_{r,s}(dy_2)$$

$$= \int_{S^5} f(z_{11} + z_{12} + z_2)u_{r,t}(y_1, dz_{11})u_{r,t}(y_2, dz_{12})\mu_{t,r}(dz_2)u_{s,r}(x, dy_1)\mu_{r,s}(dy_2)$$

$$= \int_{S^4} f(z_{11} + z_{12} + z_2)u_{s,t}(x, dz_{11})u_{r,t}(y_2, dz_{12})\mu_{t,r}(dz_2)\mu_{r,s}(dy_2)$$

$$= \int_{S^3} f(z_{11} + z_{12} + z_2)u_{s,t}(x, dz_{11})(\mu_{r,s}u_{r,t})(dz_{12})\mu_{t,r}(dz_2)$$

$$= \int_S f(z)(u_{s,t}(x, \cdot) * (\mu_{r,s}u_{r,t}) * \mu_{t,r})(dz).$$

$$\tag{2.13}$$

Here to get the sixth identity we used (2.9). If (2.12) holds, then by (2.13) we obtain

$$q_{s,r}q_{r,t}f(x) = \int_S f(z)[u_{s,t}(x,\cdot) * \mu_{t,s}](dz) = q_{s,t}f(x).$$

That is, (2.11) holds.

Conversely, if (2.11) holds, then by taking $x = 0$ in (2.13) and using (2.10), we get

$$\int_S f(z)[(\mu_{r,s}u_{r,t}) * \mu_{t,r}](dz) = \int_S f(z)(\mu_{t,s})(dz)$$

for every $f \in B_b(\mathbb{H})$. This implies (2.12). $\qquad\square$

Example 2.6. When $S = \mathbb{H}$ is a real separable Hilbert space and $u_{s,t}(x,\cdot) = \delta_{U(t,s)x}$ for every $t \geq s$ and $x \in S$. Then $(q_{s,t})_{t \geq s}$ is the non-homogeneous generalized Mehler semigroup $(p_{s,t})_{t \geq s}$ defined in (2.1) and the equivalence of (2.11) and (2.12) in Proposition 2.5 is exactly the equivalence of (2.5) and (2.6) in Proposition 2.2. The latter is thus proved.

Example 2.7. Let $S = M(E)$ be the space of all finite Borel measures on a Lusin topological space E. Let $(u_{s,t})_{t \geq s}$ be the transition semigroup of some measure-valued branching process and $(\mu_{t,s})_{t \geq s}$ be a family of probability measures on $M(E)$ satisfying (2.12). Then $(q_{s,t})_{s \leq t}$ is called an immigration process in [Li02].

3 On the equation $\mu_{t,s} = \mu_{t,r} * \left(\mu_{r,s} \circ U(t,r)^{-1} \right)$

3.1 Infinite divisibility

Recall that a probability measure μ on $\mathbb{H}$ is said to be infinitely divisible if for any $n \in \mathbb{N}$ there exists a probability measure μ_n on $\mathbb{H}$ such that $\mu = \mu_n^{*n} := \mu_n * \mu_n * \cdots * \mu_n$ (n-times). We first look at equation (1.2). If $T_s \equiv I$, then it is clear that for every $t \geq 0$, μ_t is infinitely divisible. It is proved in [SS01] that in fact μ_t satisfying (1.2) is also infinitely divisible. Consider (2.6) for the case when $\mathbb{H}$ is finite and $U(t,s) \equiv I$. It is known (see [Itô06] or [Sat99, Theorem 9.1 and Theorem 9.7]) that $\mu_{t,s}$ is infinitely divisible provided Assumption 3.1 below holds. In the following, assuming Assumption 3.1, we shall prove infinite divisibility for $\mu_{t,s}$ satisfying (2.6) for the general case. That is, we generalize the results mentioned above both to the time-inhomogeneous with general $U(t,s)$ and infinite dimensional state space.

In the rest of this chapter we fix $\mu_{t,s}$, $t \geq s$ satisfying (2.6) and we shall use the following assumption.

Assumption 3.1. For every $\varepsilon, \eta > 0$, there exists $\delta > 0$ such that for every $t, s \in \mathbb{R}$ with $0 \leq t - s < \delta$,

$$\mu_{t,s}(\{x \colon |x| > \varepsilon\}) < \eta.$$

In this case, we say the family of probability measures $(\mu_{t,s})_{t \geq s}$ is stochastically continuous.

Remark 3.2. Note that measures $(\mu_t)_{t \geq 0}$ satisfying $\mu_{t+s} = \mu_t * \mu_s$ for every $t, s \geq 0$ do not necessarily fulfill Assumption 3.1. See for instance the arguments in [Bre92, Section 14.4]. We also note that for the homogeneous case, the infinite divisibility of μ_t satisfying (1.2) is proved without the above continuity assumption.

Lemma 3.3. $(U(t,s))_{t \geq s}$ *is uniformly bounded on every compact interval. That is, for every fixed $s_0 < t_0$, there exists some constant $c > 0$ such that for every $s_0 \leq s \leq t \leq t_0$,*

$$|U(t,s)x| \leq c|x|, \quad x \in \mathbb{H}, \ s_0 \leq s \leq t \leq t_0. \tag{3.1}$$

Proof. For every fixed $x \in \mathbb{H}$, $|U(t,s)x|$ is a continuous function in (t,s) on $\Lambda_{t_0,s_0} := \{(t,s) \colon s_0 \leq s \leq t \leq t_0\}$. Hence $|U(t,s)x|$ is uniformly bounded on Λ_{t_0,s_0} for every fixed $x \in \mathbb{H}$. By the Banach–Steinhaus theorem $\sup_{(t,s) \in \Lambda_{t_0,s_0}} \|U(t,s)\| < \infty$. That is, there exists some $c > 0$ such that

$$|U(t,s)x| \leq c|x|, \quad x \in \mathbb{H}, \ s_0 \leq s \leq t \leq t_0. \tag{3.2}$$

$\square$

Lemma 3.4. *Suppose that Assumption 3.1 holds. On every compact interval $[s_0, t_0]$, there exists a $\delta > 0$ such that for all $s, t \in [s_0, t_0]$ with $0 \leq t - s < \delta$,*

$$\mu_{t,s} \circ U(t_0,t)^{-1}(\{x \in \mathbb{H} \colon |x| > \varepsilon\}) < \eta. \tag{3.3}$$

Proof. Since the case $t = s$ is trivial, we shall assume $t > s$. For convenience, set $A(r) := \{x \in \mathbb{H} \colon |x| > r\}$ for every $r > 0$. By Lemma 3.3, there exists a constant $c \geq 1$ such that

$$|U(t,s)x| \leq c|x|, \quad x \in \mathbb{H}, \ s_0 \leq s < t \leq t_0.$$

By Assumption 3.1, for every $\varepsilon, \eta > 0$, $t \in [s_0, t_0]$, there exists a $\delta_t \geq 0$ such that for each $s \in (t - \delta_t, t)$,

$$\mu_{t,s}(A(\varepsilon'/2)) < \eta/2, \quad \mu_{t,s} \circ U(t_0,t)^{-1}(A(\varepsilon'/2)) < \eta/2,$$

where we set $\varepsilon' = \varepsilon/c$. For each $s \in (t, t + \delta_t)$,

$$\mu_{s,t}(A(\varepsilon'/2)) < \eta/2, \quad \mu_{s,t} \circ U(t_0,s)^{-1}(A(\varepsilon'/2)) < \eta/2.$$

For every $t \in [s_0, t_0]$, set $I_t := (t - \delta_t, t + \delta_t)$. Then $\{I_t \colon t \in [s_0, t_0]\}$ covers the interval $[s_0, t_0]$. Hence there is a finite sub–covering $\{I_{t_j} \colon j = 1, 2 \ldots, n\}$ of $[s_0, t_0]$.

Let δ be the minimum of $\{\delta_{t_j}/2\colon j = 1, 2, \ldots, n\}$. Then for every $t \in [s_0, t_0]$, we have $t \in I_{t_j}$ for some j. For every $s \in [s_0, t_0]$ satisfying $0 < t - s < \delta$, we have $|s - t_j| < \delta_{t_j}$ since $|s - t_j| \leq |s - t| + |t - t_j| < \delta + \delta_{t_j}/2 \leq \delta_{t_j}$. Now we consider the following three cases respectively: 1. $s \leq t_j < t$; 2. $s < t \leq t_j$; 3. $t_j < s < t$.

Case 1. ($s \leq t_j < t$) By (2.6),

$$
\begin{aligned}
\mu_{t,s}(A(\varepsilon')) &= \mu_{t,t_j} * (\mu_{t_j,s} \circ U(t, t_j)^{-1})(A(\varepsilon')) \\
&= \int_{\mathbb{H}} \int_{\mathbb{H}} \mathbb{1}_{A(\varepsilon')}(x + y)\, \mu_{t,t_j}(dx)(\mu_{t_j,s} \circ U(t, t_j)^{-1})(dy) \\
&\leq \int_{\mathbb{H}} \int_{\mathbb{H}} (\mathbb{1}_{A(\varepsilon'/2)}(x) + \mathbb{1}_{A(\varepsilon'/2)}(y))\, \mu_{t,t_j}(dx)(\mu_{t_j,s} \circ U(t, t_j)^{-1})(dy) \\
&= \mu_{t,t_j}(A(\varepsilon'/2)) + (\mu_{t_j,s} \circ U(t, t_j)^{-1})(A(\varepsilon'/2)) \\
&< \frac{\eta}{2} + \frac{\eta}{2} = \eta.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mu_{t,s} \circ U(t_0, t)^{-1}(\{x \in \mathbb{H}\colon |x| > \varepsilon\}) &= \mu_{t,s}(\{x \in \mathbb{H}\colon |U(t_0, t)x| > \varepsilon\}) \\
&\leq \mu_{t,s}(\{x \in \mathbb{H}\colon |x| > \varepsilon/c\}) < \eta.
\end{aligned}
$$

Case 2. ($s < t \leq t_j$) We first show $(\mu_{t,s} \circ U(t_j, t)^{-1})(A(\varepsilon')) < \eta$ by contradiction. If otherwise, we have

$$
(\mu_{t,s} \circ U(t_j, t)^{-1})(A(\varepsilon')) \geq \eta. \tag{3.4}
$$

Then by (2.6), we have

$$
\begin{aligned}
\frac{\eta}{2} > \mu_{t_j,s}(A(\varepsilon'/2)) &= \mu_{t_j,t} * (\mu_{t,s} \circ U(t_j, t)^{-1})(A(\varepsilon'/2)) \\
&= \int_{\mathbb{H}} \int_{\mathbb{H}} \mathbb{1}_{A(\varepsilon'/2)}(x + y)\, \mu_{t_j,t}(dx)(\mu_{t,s} \circ U(t_j, t)^{-1})(dy) \\
&\geq \int_{\mathbb{H}} \int_{\mathbb{H}} \mathbb{1}_{A(\varepsilon'/2)^c}(x) \cdot \mathbb{1}_{A(\varepsilon')}(y))\, \mu_{t_j,t}(dx)(\mu_{t,s} \circ U(t_j, t)^{-1})(dy) \\
&= \mu_{t_j,t}(A(\varepsilon'/2)^c) \cdot (\mu_{t,s} \circ U(t_j, t)^{-1})(A(\varepsilon')) \\
&> \eta\left(1 - \frac{\eta}{2}\right) = \eta - \frac{\eta^2}{2}.
\end{aligned}
$$

Here we used the fact that if $|y| > \varepsilon'$ and $|x| < \varepsilon'/2$, then $|x + y| \geq |y| - |x| > \frac{\varepsilon'}{2}$. The inequality obtained above shows $\frac{\eta}{2} > \eta - \frac{\eta^2}{2}$. Consequently, $\eta > 1$ which contradicts (3.4).

Then

$$
\begin{aligned}
\mu_{t,s} \circ U(t_0, t)^{-1}(\{x \in \mathbb{H}\colon |x| > \varepsilon\}) &= \mu_{t,s}(\{x \in \mathbb{H}\colon |U(t_0, t_j)U(t_j, t)x| > \varepsilon\}) \\
&\leq \mu_{t,s}(\{x \in \mathbb{H}\colon |U(t_j, t)x| > \varepsilon/c\}) < \eta.
\end{aligned}
$$

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Case 3. $(t_j < s < t)$ Similar to Case 1 we only need to show $\mu_{t,s}(A(\varepsilon')) < \eta$ whose proof is similar to the proof in Case 2. Indeed, if $\mu_{t,s}(A(\varepsilon')) \geq \eta$ then

$$
\begin{aligned}
\frac{\eta}{2} &> \mu_{t,t_j}(A(\varepsilon'/2)) = \mu_{t,s} * (\mu_{s,t_j} \circ U(t,s)^{-1})(A(\varepsilon'/2)) \\
&= \int_{\mathbb{H}} \int_{\mathbb{H}} \mathbb{1}_{A(\varepsilon'/2)}(x+y) \, \mu_{t,s}(dx)(\mu_{s,t_j} \circ U(t,s)^{-1})(dy) \\
&\geq \mu_{t,s}(A(\varepsilon')) \cdot (\mu_{s,t_j} \circ U(t,s)^{-1})(A(\varepsilon'/2)^c) \\
&\geq \eta \left(1 - \frac{\eta}{2}\right).
\end{aligned}
$$

This implies $\eta > 1$ which contradicts the assumption. $\qquad\square$

Remark 3.5. From the proof of Lemma 3.4 (or (3.3)) we obtain the following result. Suppose that Assumption 3.1 holds and for every fixed $s_0 < t_0$, there exists some constant $c > 0$ such that for every $s_0 \leq s \leq t \leq t_0$ (cf. (3.1)),

$$
1/c|x| \leq |U(t,s)x| \leq c|x|, \quad x \in \mathbb{H}, \ s_0 \leq s \leq t \leq t_0.
$$

Then $(\mu_{t,s})_{t,s}$ is uniformly stochastically continuous on compact intervals. That is, for every $s_0 < t_0$ and every $\varepsilon, \eta > 0$, there exists a $\delta > 0$ such that for all $s,t \in [s_0, t_0]$ with $s \leq t$ and $t - s < \delta$, we have $\mu_{t,s}(\{x \in \mathbb{H} \colon |x| > \varepsilon\}) < \eta$.

Now we can prove the following theorem.

Theorem 3.6. *The measures $(\mu_{t,s})_{t \geq s}$ satisfying (2.6) and Assumption 3.1 are infinitely divisible.*

Proof. For simplicity we only show that $\mu_{1,0}$ is infinitely divisible since the proof for arbitray $s \leq t$ is similar. By (2.6), we can write for every $m \in \mathbb{N}$,

$$
\mu_{1,0} = \Pi_{j=0}^{*(2^m - 1)} \mu_{\frac{j+1}{2^m}, \frac{j}{2^m}} \circ U\left(1, \frac{j+1}{2^m}\right)^{-1}.
$$

Here we use Π^* to denote the convolution product. By Lemma 3.4 we know that $\mu_{1,0}$ is the limit of an infinitesimal triangular array. Then by [Par67, Corollary VI.6.2] we know that $\mu_{1,0}$ is an infinitely divisible distribution. $\qquad\square$

By the Lévy-Khintchine theorem [Par67, Theorem VI.4.10], for every $t \geq s$, there exists a negative definite measurable function $\psi_{t,s}$ on $\mathbb{H}$ such that

$$
\mu_{t,s} \exp(i\langle \cdot, \xi \rangle) = \exp(-\psi_{t,s}(\xi)), \quad \xi \in \mathbb{H}
$$

and $\psi_{t,s}$ has the following form

$$
\psi_{t,s}(\xi) = -i\langle a_{t,s}, \xi \rangle + \frac{1}{2}\langle \xi, R_{t,s}\xi \rangle - \int_{\mathbb{H}} \left(e^{i\langle \xi, x \rangle} - 1 - \frac{i\langle \xi, x \rangle}{1 + |x|^2}\right) m_{t,s}(dx), \quad (3.5)
$$

where $a_{t,s} \in \mathbb{H}$, $R_{t,s}$ is a trace class operator on $\mathbb{H}$ and $\mathrm{m}_{t,s}$ is a Lévy measure on $\mathbb{H}$.

Then condition (2.6) turns to

$$\psi_{t,s}(\xi) = \psi_{t,r}(\xi) + \psi_{r,s}(U(t,r)^*\xi) \tag{3.6}$$

for every $t \geq r \geq s$ and $\xi \in \mathbb{H}$.

For simplicity, we shall denote the infinite divisible measure with characteristic function $\exp(-\psi_{t,s})$ given as in (3.5) by $D[a_{t,s}, R_{t,s}, \mathrm{m}_{t,s}]$. That is,

$$\widehat{D[a_{t,s}, R_{t,s}, \mathrm{m}_{t,s}]}(\xi) = \exp(-\psi_{t,s}(\xi)), \quad \xi \in \mathbb{H}.$$

By (2.6) and the uniqueness of the canonical representation of infinite divisible distributions (see also the proof of [Ouy09a, Corollary 1.4.11]), we have the following identities

$$
\begin{aligned}
R_{t,s} &= R_{t,r} + U(t,r)R_{r,s}U(t,r)^*, \\
\mathrm{m}_{t,s} &= m_{t,r} + \mathrm{m}_{r,s} \circ U(t,r)^{-1}, \\
a_{t,s} &= a_{t,r} + U(t,r)a_{r,s} \\
&\quad + \int_{\mathbb{H}\backslash\{0\}} U(t,r)x \left[\frac{1}{1+|U(t,r)x|^2} - \frac{1}{1+|x|^2} \right] \mathrm{m}_{r,s}(dx)
\end{aligned}
\tag{3.7}
$$

for every $t \geq r \geq s$.

3.2 Representation of $\mu_{t,s}$

The following proposition shows a typical form of the measure $\mu_{t,s}$ which satisfies the equation (2.6).

Proposition 3.7. *Let $(\lambda_{t,s})_{t>s}$ be a family of negative definite Sazonov continuous functions on $\mathbb{H}$ satisfying $\lambda_{t,s}(0) = 1$ for every $t > s$ such that the function $s \mapsto \lambda_{t,s}(\xi)$, $\xi \in \mathbb{H}$, $t > s$, is locally integrable. Assume for every $t > r > s$,*

$$\lambda_{t,s}(\xi) = \lambda_{r,s}(U(t,r)^*\xi), \quad \xi \in \mathbb{H}. \tag{3.8}$$

Let π be a σ-finite measure on $\mathbb{R}$. Then

$$\hat{\mu}_{t,s}(\xi) = \exp\left(-\int_s^t \lambda_{t,\sigma}(\xi)\,\pi(d\sigma) \right), \quad \xi \in \mathbb{H}, \ t > s \tag{3.9}$$

with $\hat{\mu}_{t,t} \equiv 1$ defines a family of probability measures $(\mu_{t,s})_{t \geq s}$ such that (2.6) holds. If the setting for $(\lambda_{t,s})_{t>s}$ above extends to $(\lambda_{t,s})_{t \geq s}$ then we have

$$\hat{\mu}_{t,s}(\xi) = \exp\left(-\int_s^t \lambda_\sigma(U(t,\sigma)^*\xi)\,\pi(d\sigma) \right), \quad \xi \in \mathbb{H}, \ t > s, \tag{3.10}$$

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where we set $\lambda_s := \lambda_{s,s}$ for every $s \in \mathbb{R}$.

Conversely, let $\lambda_r \colon \mathbb{H} \to \mathbb{C}$ be a negative definite Sazonov continuous function for every $r \in \mathbb{R}$ with $\lambda_r(0) = 0$. Then $\lambda_{t,s}(\xi) := \lambda_s\big(U(t,s)^*\xi\big)$, $t \geq s$, satisfies (3.8).

Proof. By the assumptions, it is easy to see that for every $t > s$, $\int_s^t \lambda_{t,\sigma}(\xi)\,\pi(d\sigma)$ is also negative definite and Sazonov continuous. Hence by [BF75, Theorem 7.8] we know the right hand side of (3.9) is positive definite and Sazonov continuous. By Bochner's theorem (see e.g. [VTC87, Chapter VI; Proposition 3.2(c)]), we know $\mu_{t,s}$ is well defined through (3.9). It is a probability measure by the fact that $\lambda_{t,s}(0) = 1$ for every $t \geq s$.

Now we verify (2.6). We only need to consider the case when $t > r > s$. For every $\xi \in \mathbb{H}$,

$$
\hat{\mu}_{t,r}(\xi)\hat{\mu}_{r,s}\big(U(t,r)^*\xi\big) = \exp\left(-\int_r^t \lambda_{t,\sigma}(\xi)\,\pi(d\sigma) - \int_s^r \lambda_{r,\sigma}(U(t,r)^*\xi)\right)
$$

$$
= \exp\left(-\int_s^t \lambda_{t,\sigma}(\xi)\,\pi(d\sigma)\right) = \hat{\mu}_{t,s}(\xi).
$$

Now we show the last assertion. Suppose that for every $t \geq s$, $\lambda_{t,s} := \lambda_s\big(U(t,s)^*\xi\big)$. Then for every $t \geq r \geq s$,

$$
\lambda_{r,s}\big(U(t,r)^*\xi\big) = \lambda_s\big(U(r,s)^*U(t,r)^*\xi\big)
$$

$$
= \lambda_s\big((U(t,r)U(r,s))^*\xi\big) = \lambda_s\big(U(t,s)^*\xi\big) = \lambda_{t,s}(\xi).
$$

$\square$

Remark 3.8. For $t > s$, let $\nu_{t,s}$ be the measure on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$ with Fourier transformation $\hat{\nu}_{t,s} = \exp(-\lambda_{t,s}(\xi))$ for every $\xi \in \mathbb{H}$. Fix s_0 and set $\nu_t = \nu_{t,s_0}$. Define a transition semigroup $u_{s,t}$ by $u_{s,t}(x,\cdot) = \delta_{U(t,s)x}(\cdot)$ for every $x \in \mathbb{H}$. Then (3.8) implies that $(\nu_t)_{t>s>s_0}$ is an entrance law (or an evolution system of measures, see Section 4) for $u_{s,t}$.

Remark 3.9. Let $\lambda_r \colon \mathbb{H} \to \mathbb{C}$ be a negative definite Sazonov continuous function for every $r \in \mathbb{R}$ with $\lambda_r(0) = 0$. Then for every $r \in \mathbb{R}$, $\exp(-\lambda_r)$ is the characteristic function of an infinitely divisible probability measure on $\mathbb{H}$. By the Lévy-Khintchine Theorem [Par67, Theorem VI.4.10], for every $r \in \mathbb{R}$ the symbol λ_r can be written in the form

$$
\lambda_r(\xi) = -i\langle a_r, \xi\rangle + \frac{1}{2}\langle \xi, R_r\xi\rangle - \int_{\mathbb{H}} \left(e^{i\langle \xi, x\rangle} - 1 - \frac{i\langle \xi, x\rangle}{1 + |x|^2}\right) \mathrm{m}_r(dx), \quad \xi \in \mathbb{H},
$$
(3.11)

where $a_r \in \mathbb{H}$, R_r is a trace class operator on $\mathbb{H}$ and m_r is a Lévy measure on $\mathbb{H}$.

By Proposition 3.7,

$$\hat{\mu}_{t,s}(\xi) := \exp\left(-\int_s^t \lambda_r(U(t,r)^*\xi)\,\pi(dr)\right) \tag{3.12}$$

defines a family of measures $(\mu_{t,s})_{t \geq s}$ satisfying (2.6). Then $\mu_{t,s}$ an infinite divisible measure with triplet $(a_{t,s}, R_{t,s}, \nu_{t,s})$ given by

$$a_{t,s} = \int_s^t U(t,r)^* a_r\,\pi(dr)$$
$$+ \int_s^t \int_{\mathbb{H}} U(t,r)x \left[\frac{1}{1+|U(t,r)x|^2} - \frac{1}{1+|x|^2}\right] m_r(dx)dr$$
$$R_{t,s} = \int_s^t U(t,r)R_r U(t,r)^*\,\pi(dr),$$
$$m_{t,s}(\{0\}) = 0 \quad \text{and} \quad m_{t,s}(A) = \int_s^t m_r(U(t,r)^{-1}(A))\,\pi(dr), \quad A \in \mathscr{B}(\mathbb{H} \setminus \{0\}).$$

Remark 3.10. For the case when $\mathbb{H}$ is a finite dimensional Euclidean space, it is shown in [Sat06] that a natural additive process Z_t admits a factorization and hence the distribution of the convolution integral $\int_s^t U(t,r)B(r)\,dZ_r$ has the form (3.12). An extension to the infinite dimensional case is in preparation [Ouy10].

The following result is a converse to Proposition 3.7. Recall that we set $\psi_{t,s}(\xi) := -\log \hat{\mu}_{t,s}(\xi)$ for every $\xi \in \mathbb{H}$ and $r \geq s$.

Proposition 3.11. *Assume that (2.6) (equivalently (3.6)) holds. Let $\xi \in \mathbb{H}$ and $t \in \mathbb{R}$. If $\psi_{t,\cdot}(\xi)$ is of bounded variation on $(-\infty, t]$, then there is a unique signed measure F_ξ^t on $((-\infty, t], \mathscr{B}((-\infty, t]))$ such that*

$$F_\xi^t([s,t]) = \psi_{t,s}(\xi), \quad s \leq t. \tag{3.13}$$

Let π be a σ-finite measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Suppose that for any $t \in \mathbb{R}$ and $\xi \in \mathbb{H}$, F_ξ^t is absolutely continuous with respect to π on $((-\infty, t], \mathscr{B}((-\infty, t]))$ with Randon–Nikodým derivative $\lambda_{t,s}(\xi) := \frac{dF_\xi^t}{d\pi}(s)$, $s \leq t$. Then $\lambda_{t,s}(\cdot)$ is negative definite and Sazonov continuous on $\mathbb{H}$, (3.9) holds and (3.8) holds for π almost every s with $t \geq r > s$.

Proof. Define $F_\xi^t([s,r]) = \psi_{r,s}(U(t,r)^*\xi)$ for every $s \leq r \leq t$. By (3.6) we have the following additive property: for every $s \leq \sigma \leq r \leq t$,

$$F_\xi^t([\sigma,r]) + F_\xi^t([s,\sigma]) = \psi_{r,\sigma}(U(t,r)^*\xi) + \psi_{\sigma,s}(U(t,\sigma)^*\xi)$$
$$= \psi_{r,\sigma}(U(t,r)^*\xi) + \psi_{\sigma,s}(U(r,\sigma)^*U(t,r)^*\xi)$$
$$= \psi_{r,s}(U(t,r)^*\xi) = F_\xi^t([s,r]).$$

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Note also that $s \mapsto \psi_{t,s}(\xi)$ is left continuous at t. Hence by the standard extension procedure of additive set-functions on rings, we know F_ξ^t can be uniquely extended to a signed measure on $((-\infty, t], \mathscr{B}((-\infty, t]))$ which is denoted still by F_ξ^t.

Since $\lambda_{t,\cdot}(\xi)$ is the density of F_ξ^t with respect to π, we have

$$\int_s^t \lambda_{t,\sigma}(\xi)\, \pi(d\sigma) = F_\xi^t([s,t]) = \psi_{t,s}(\xi).$$

Hence

$$\hat{\mu}_{t,s}(\xi) = \exp(-\psi_{t,s}(\xi)) = \exp\left(-\int_s^t \lambda_{t,\sigma}(\xi)\, \pi(d\sigma)\right).$$

That is, (3.9) holds.

Let $t \geq r$. For any $r_1 < r_2 \leq r$, we have

$$\int_{r_1}^{r_2} \lambda_{t,\sigma}(\xi)\, \pi(d\sigma) = F_\xi^t([r_1, r_2]) = \psi_{r_2,r_1}(U(t,r_2)^*\xi) = \psi_{r_2,r_1}(U(r,r_2)^*U(t,r)^*\xi)$$

$$= F_{U(t,r)^*\xi}^r([r_1, r_2]) = \int_{r_1}^{r_2} \lambda_{r,\sigma}(U(t,r)^*\xi)\, \pi(d\sigma).$$

This implies that for π almost every s, $s < r \leq t$,

$$\lambda_{t,s}(\xi) = \lambda_{r,s}(U(t,r)^*\xi)$$

This proves (3.8). The negative definiteness and Sazonov continuity of $\lambda_{t,s}(\cdot)$ are easy to show by (3.9) and the corresponding properties of $\psi_{t,s}$. $\qquad \square$

We shall consider the special case where π is Lebesgue measure. We need the following fact. For the proof we refer to [MV86, Theorem 1] (or the references therein, e.g. [Hob57, Page 365 (3rd Ed.) or Page 341 (2nd Ed.)]).

Lemma 3.12. *Let f be a continuous function on $[a,b]$. If for each $x \in (a,b)$ either the left derivative or the right derivative vanishes, then f is constant.*

Proposition 3.13. *Assume that (2.6) (equivalently (3.6)) hold and for every $\xi \in \mathbb{H}$ and $t \geq s$,*

(1) *the function $s \mapsto \psi_{t,s}(\xi)$ is continuous and left differentiable at $s = t$. Denote the left derivative by $-\lambda_t(\xi)$, i.e.*

$$-\lambda_t(\xi) := \left.\frac{d^-}{ds}\psi_{t,s}(\xi)\right|_{s=t} = \lim_{r\uparrow t}\frac{\psi_{t,r}(\xi)}{r-t}; \tag{3.14}$$

(2) *the function $s \mapsto \lambda_s\big(U(t,s)^*\xi\big)$ is continuous.*

Then for every $t \in \mathbb{R}$, $\lambda_t(\cdot)$ is negative definite and Sazonov continuous on $\mathbb{H}$, and for every $\xi \in \mathbb{H}$, $t \geq s$,

$$\hat{\mu}_{t,s}(\xi) = \exp\left(-\int_s^t \lambda_r\big(U(t,r)^*\xi\big)\,dr\right). \tag{3.15}$$

Proof. For every $\xi \in \mathbb{H}$ and $r \leq t$, by (3.6) we get

$$\frac{d^-}{dr}\psi_{t,r}(\xi) = \lim_{r'\uparrow r}\frac{\psi_{t,r'}(\xi) - \psi_{t,r}(\xi)}{r' - r} = \lim_{r'\uparrow r}\frac{\psi_{t,r}(\xi) + \psi_{r,r'}(U(t,r)^*\xi) - \psi_{t,r}(\xi)}{r' - r}$$

$$= \lim_{r'\uparrow r}\frac{\psi_{r,r'}(U(t,r)^*\xi)}{r' - r} = -\lambda_r\big(U(t,r)^*\xi\big).$$

By our assumption, for every $\xi \in \mathbb{H}$, $r \mapsto \lambda_r(U(t,r)^*\xi)$, $r \leq t$, is continuous. Hence we see that

$$\frac{d^-}{dr}\Phi_{t,r}(\xi) = 0, \quad r \leq t,\ \xi \in \mathbb{H}, \tag{3.16}$$

where

$$\Phi_{t,r}(\xi) := \psi_{t,r}(\xi) - \int_r^t \lambda_u(U(t,u)^*\xi)\,du, \quad r \leq t,\ \xi \in \mathbb{H}.$$

By Lemma 3.12 we know $\Phi_{t,r}(\xi)$ is constant for every $r \in [s,t]$. But $\Phi_{t,t}(\xi) = 0$, hence $\Phi_{t,s}(\xi) = 0$ also. This implies

$$\psi_{t,s}(\xi) = \int_s^t \lambda_r(U(t,r)^*\xi)\,dr.$$

Since $\psi_{t,s}(\xi) = -\log\hat{\mu}_{t,s}(\xi)$, we obtain (3.15).

From the negative definiteness and Sazonov continuity of $\psi_{t,s}(\cdot)$ we get the corresponding property of $\lambda_t(\cdot)$. $\square$

Remark 3.14. The assumption that $s \mapsto \lambda_s(U(t,s)^*\xi)$ is continuous for $s \leq t$ and $\xi \in \mathbb{H}$, is used to ensure that the map $s \mapsto \int_s^t \lambda_u(U(t,u)^*\xi)\,du$ is continuous and has (left)-derivative $-\lambda_s(U(t,s)^*\xi)$. This continuity assumption on $\lambda_\cdot(U(t,\cdot)^*\xi)$ holds if we assume that for every $\varepsilon > 0$, $s \leq t$, and for every bounded set $B \subset \mathbb{H}$, there exists a $\delta > 0$ such that $\sup_{x \in B}|\lambda_{s+h}(x) - \lambda_s(x)| < \varepsilon$ provided $|h| < \delta$ and $h < t - s$. Indeed, note that

$$|\lambda_{s+h}(U(t,s+h)^*\xi) - \lambda_s(U(t,s)^*\xi)|$$
$$\leq |\lambda_{s+h}(U(t,s+h)^*\xi) - \lambda_s(U(t,s+h)^*\xi)| + |\lambda_s(U(t,s+h)^*\xi) - \lambda_s(U(t,s)^*\xi)|.$$

Hence $|\lambda_{s+h}(U(t,s+h)^*\xi) - \lambda_s(U(t,s)^*\xi)|$ can be made arbitrarily small, since the first term on the right hand side of the inequality above can be made small by the assumption that $s \mapsto \lambda_s(x)$ is continuous uniformly in x on bounded sets; the second term can be made samll by the strong continuity of $U(t,\cdot)$.

Remark 3.15. Proposition 3.13 generalize [BRS96, Lemma 2.6] which dealt with homogeneous generalized Mehler semigroups (see (1.1)) using differentiability condition. For the homogeneous case, there are some generalizations of [BRS96, Lemma 2.6]. Neerven [vN00] relaxed the differentiability condition for general Gaussian Mehler semigroups on Banach space. Dawson et al. [DL04, Theorem 2.1] (see also [DLSS04, Theorem 2.3]) used entrance laws to characterize μ_t and hence dropped the differentiability condition for homogeneous generalized Mehler semigroups on Hilbert spaces. For measure-valued skew convolution semigroups, the sufficiency and necessity of the representation were proved in [Li96, Theorem 2] and [Li02, Theorem 3.1] respectively, for the homogeneous case and the non-homogeneous case by using entrance laws. Proposition 3.11 can be seen as an attempt to use entrance laws to characterize $\mu_{s,t}$. But we do not know how to find a natural measure π.

4   Evolution systems of measures

Let $(p_{s,t})_{t \geq s}$ be defined as in (2.1) on a separable Hilbert space $\mathbb{H}$ with $(U(t,s))_{t \geq s}$ and $(\mu_{t,s})_{t \geq s}$ satisfying (2.6).

Generally, for a family of non-autonomous operators $(p_{s,t})_{t \geq s}$ on $\mathbb{H}$, we cannot expect to have a stationary invariant measure for them. But we can try to look for a family of probability measures $(\nu_t)_{t \in \mathbb{R}}$ on $\mathbb{H}$ such that

$$\int p_{s,t} f(x)\, \nu_s(dx) = \int f(x)\, \nu_t(dx), \quad s \leq t \tag{4.1}$$

for all $f \in B_b(\mathbb{H})$. Such a family of probability measures is called an evolution system of measures for $(p_{s,t})_{t \geq s}$ (see [DPR08]). Evolution systems of measures are also called entrance law in [Dyn89].

Lemma 4.1. *A family of probability measures $(\nu_t)_{t \in \mathbb{R}}$ on $\mathbb{H}$ is an evolution system of measures for $(p_{s,t})_{t \geq s}$ if and only if for every $t \geq s$,*

$$\hat{\mu}_{t,s}(\xi)\hat{\nu}_s\big(U(t,s)^*\xi\big) = \hat{\nu}_t(\xi), \quad \xi \in \mathbb{H}. \tag{4.2}$$

Proof. Identity (4.2) comes from (4.1) for functions f of the form $\exp(i\langle \xi, x \rangle)$, $\xi \in \mathbb{H}$, which is enough to ensure (4.1) for all bounded measurable functions. $\square$

Theorem 4.2. *Suppose that $(\nu_t^{(1)})_{t \in \mathbb{R}}$ is an evolution system of measures for $(p_{s,t})_{t \geq s}$. Let $(\nu_t^{(2)})_{t \in \mathbb{R}}$ be another system of probability measures and assume that there exists a family of probability measures $(\sigma_t)_{t \in \mathbb{R}}$ on $\mathbb{H}$ such that*

$$\nu_t^{(2)} = \nu_t^{(1)} * \sigma_t \quad and \quad \sigma_s \circ U(t,s)^{-1} = \sigma_t.$$

Then $(\nu_t^{(2)})_{t \in \mathbb{R}}$ is also an evolution system of measures for $(p_{s,t})_{t \geq s}$.

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Proof. For every $\xi \in \mathbb{H}$,

$$\hat{\nu}_t^{(2)}(\xi) = \hat{\nu}_t^{(1)}(\xi)\hat{\sigma}_t(\xi) = \hat{\mu}_{t,s}(\xi)\hat{\nu}_s^{(1)}\big(U(t,s)^*\xi\big)\hat{\sigma}_t(\xi)$$
$$= \hat{\mu}_{t,s}(\xi)\hat{\nu}_s^{(1)}\big(U(t,s)^*\xi\big)\hat{\sigma}_s(U(t,s)^*\xi) = \hat{\mu}_{t,s}(\xi)\hat{\nu}_s^{(2)}\big(U(t,s)^*\xi\big).$$

Hence the assertion follows by Lemma 4.1. $\qquad\square$

Assume that for every $t \geq s$, $\mu_{t,s}$ is infinitely divisible and has the form $\mu_{t,s} = D[a_{t,s}, R_{t,s}, \mathrm{m}_{t,s}]$, where $a_{t,s} \in \mathbb{H}$, $R_{t,s}$ is a trace class operator on $\mathbb{H}$, and $\mathrm{m}_{t,s}$ is a Lévy measure on $\mathbb{H}$.

By (3.7), we know that for every fixed $t \in \mathbb{R}$, $(\mathrm{m}_{t,s})_{s \leq t}$ is a decreasing family of Lévy measures. This allows us to define $\mathrm{m}_{t,-\infty}$ for every $t \in \mathbb{R}$ by setting $\mathrm{m}_{t,-\infty}(\{0\}) = 0$ and

$$\mathrm{m}_{t,-\infty}(A) = \lim_{s \to -\infty} \mathrm{m}_{t,s}(A), \quad A \in \mathscr{B}(\mathbb{H} \setminus \{0\}).$$

From (3.7) we also see for every $x \in \mathbb{H}$ and $t \in \mathbb{R}$, $\langle R_{t,s}x, x\rangle$ is decreasing in s. Hence the limit $\lim_{s \to -\infty}\langle R_{t,s}x, x\rangle$ exists for every $x \in \mathbb{H}$. By the polarization identity, we see that for every $t \in \mathbb{R}$, $x, y \in \mathbb{H}$, the limit $\lim_{s \to -\infty}\langle R_{t,s}x, y\rangle$ exists. Fixing $x \in \mathbb{H}$ and letting $y \in \mathbb{H}$ vary, we get a functional $\lim_{s \to -\infty}\langle R_{t,s}x, \cdot\rangle$. We shall assume $\sup_{s \leq t} \mathrm{tr}\, R_{t,s} < \infty$. Then for every $t \in \mathbb{R}$, there exists $C_t > 0$ such that $\sup_{s < t}\langle R_{t,s}x, x\rangle \leq C_t|x|^2$ for every $x \in \mathbb{H}$. So we can apply Riesz's representation theorem and see that for every $x \in \mathbb{H}$, there exists an element $x_t^* \in \mathbb{H}$ such that for every $t \in \mathbb{R}$ and all $y \in \mathbb{H}$,

$$\lim_{s \to -\infty} \langle R_{t,s}x, y\rangle = \langle x_t^*, y\rangle.$$

By the property of $R_{t,s}$, we see that the mapping from x to x_t^* is a trace class operator and we denote it by $R_{t,-\infty}$. That is, for every $t \in \mathbb{R}$ there is a trace class operator $R_{t,-\infty}$ on $\mathbb{H}$ such that

$$\langle R_{t,-\infty}x, y\rangle = \lim_{s \to -\infty} \langle R_{t,s}x, y\rangle, \quad x, y \in \mathbb{H}.$$

Consequently for every $t \in \mathbb{R}$, the central Gaussian measure with covariance operator $R_{t,-\infty}$ is well defined.

Theorem 4.3. *Suppose that for $t \in \mathbb{R}$,*

(H1) $\sup_{s \leq t} \mathrm{tr}\, R_{t,s} < \infty$;

(H2) $\sup_{s \leq t} \int_{\mathbb{H}} (1 \wedge |x|^2)\, \mathrm{m}_{t,s}(dx) < \infty$;

(H3) $a_{t,-\infty} := \lim_{s \to -\infty} a_{t,s}$.

Then for every $t \in \mathbb{R}$, $\mathrm{m}_{t,-\infty}$ is a Lévy measure, $R_{t,-\infty} < \infty$ is a trace class operator and the system of measures $(\nu_t)_{t\in\mathbb{R}}$ given by $\nu_t = D[a_{t,-\infty}, R_{t,-\infty}, \mathrm{m}_{t,-\infty}]$, $t \in \mathbb{R}$, is an evolution system of measures for $(p_{s,t})_{t\geq s}$.

Proof. Suppose that (H1), (H2) and (H3) hold. For every $t \in \mathbb{R}$, we note that $\operatorname{tr} R_{t,-\infty} = \sup_{s\leq t} R_{t,s} < \infty$. So $R_{t,-\infty}$ is a trace class operator. Moreover, for each $t \in \mathbb{R}$,

$$\int_{\mathbb{H}} (1 \wedge |y|^2) \, \mathrm{m}_{t,-\infty}(dy) = \lim_{s\to-\infty} \int_{\mathbb{H}} (1 \wedge |x|^2) \, \mathrm{m}_{t,s}(dx)dr < \infty.$$

This shows that $\mathrm{m}_{t,-\infty}$ is a Lévy measure.

Now we show that $(\nu_t)_{t\in\mathbb{R}}$ is an evolution system of measures. By (2.6), for every $t \geq s \geq r$, we have

$$\mu_{t,s} * \left(\mu_{s,r} \circ U(t,s)^{-1} \right) = \mu_{t,r}. \tag{4.3}$$

Note that $\mu_{t,s} = D[a_{t,s}, R_{t,s}, \mathrm{m}_{t,s}]$ converge weakly to $D[a_{t,-\infty}, R_{t,-\infty}, \mathrm{m}_{t,-\infty}] = \nu_t$ as $s \to -\infty$ (cf. [FR00, Lemma 3.4]). Hence letting $r \to -\infty$ in (4.3) we obtain $\left(\nu_s \circ U(t,s)^{-1} \right) * \mu_{t,s} = \nu_t$. This proves that $(\nu_t)_{t\in\mathbb{R}}$ is an evolution system of measures for $(p_{s,t})_{s\leq t}$ by Lemma 4.1. $\qquad\square$

The following theorem is the converse to Theorem 4.3.

Theorem 4.4. *Let $(\tilde{\nu}_t)_{t\in\mathbb{R}}$ be an evolution system of measures for $(p_{s,t})_{t\geq s}$. Then*

(1) *Conditions (H1) and (H2) hold.*

(2) *For every $t \in \mathbb{R}$ there exist $x_{t,s} \in \mathbb{H}$, $s \leq t$, such that $\delta_{x_{t,s}} * \delta_{a_{t,s}} * \left(\tilde{\nu}_s \circ U(t,s)^{-1} \right)$ is relatively compact.*

(3) *There exists some probability measure $\tilde{\sigma}_t$ such that $\delta_{a_{t,s}} * \left(\tilde{\nu}_s \circ U(t,s)^{-1} \right) \to \tilde{\sigma}_t$ weakly as $s \to -\infty$. Moreover $\tilde{\nu}_t = D[0, R_{t,-\infty}, \mathrm{m}_{t,-\infty}] * \tilde{\sigma}_t, t \in \mathbb{R}$.*

(4) *Assume in addition that the following condition holds*

   (H4) *For every $t \in \mathbb{R}$, $\tilde{\nu}_s \circ U(t,s)^{-1} \to \sigma_t$ weakly as $s \to -\infty$.*

   Then the limit in (H3) exists and

$$\tilde{\nu}_t = \nu_t * \sigma_t, \quad t \in \mathbb{R}. \tag{4.4}$$

   Moreover

$$\sigma_t = \sigma_s \circ U(t,s)^{-1}, \quad t \geq s. \tag{4.5}$$

   Especially, if $\sigma_t \equiv \delta_0$, then $\tilde{\nu}_t = \nu_t, t \in \mathbb{R}$.

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(5) If the limit in (H3) exists, then the limit in (H4) exists, and hence (4.4), (4.5) hold.

Proof. Since $(\tilde{\nu}_t)_{t\in\mathbb{R}}$ is an evolution system of measures for $(p_{s,t})_{t\geq s}$, by Lemma 4.1 we have for every $t \geq s$,

$$\tilde{\nu}_t = \mu_{t,s} * \left(\tilde{\nu}_s \circ U(t,s)^{-1}\right) = D[a_{t,s}, R_{t,s}, \mathrm{m}_{t,s}] * \left(\tilde{\nu}_s \circ U(t,s)^{-1}\right)$$
$$= \delta_{a_{t,s}} * N_{R_{t,s}} * M_{t,s} * \left(\tilde{\nu}_s \circ U(t,s)^{-1}\right). \tag{4.6}$$

Here we set $N_{R_{t,s}} := D[0, N_{R_{t,s}}, 0]$ and $M_{t,s} = D[0, 0, \mathrm{m}_{t,s}]$. Consider $s = -n$, $n \in \mathbb{N}$, for (4.6). The sequence $\delta_{a_{t,-n}} * N_{R_{t,-n}} * M_{t,-n}$, $n \in \mathbb{N}$, is right shift relatively compact by [Par67, Theorem III.2.2], i.e. there exist $y_{t,-n} \in \mathbb{H}$, $t \in \mathbb{R}$, $n \in \mathbb{N}$ such that the sequence

$$\delta_{y_{t,-n}} * \left(\delta_{a_{t,-n}} * N_{R_{t,-n}} * M_{t,-n}\right) = D[y_{t,-n} + a_{t,-n}, R_{t,-n}, \mathrm{m}_{t,-n}]$$

is weakly relatively compact. This implies (see [Par67, Theorem VI.5.3]) that

$$\sup_n \mathrm{m}_{t,-n}(\{|x| \geq 1\}) < \infty.$$

and

$$\sup_n \left(\operatorname{tr} R_{t,-n} + \int_{|x|<1} |x|^2 \, \mathrm{m}_{t,-n}(dx)\right) < \infty.$$

Therefore, we can define naturally a Lévy measure $\mathrm{m}_{t,-\infty}$ and trace class operator $R_{t,-\infty}$ for each $t \in \mathbb{R}$. It is easy to show (H1), (H2) by a slightly modified argument from [FR00, Lemma 3.4]. This proves (1).

Similarly, from (4.6) and by applying [Par67, Theorem III.2.2] we get (2). From (4.6) we also get (3) by applying [Par67, Theorem III.2.1] since $N_{R_{t,s}} * M_{t,s}$ converge weakly to $N_{R_{t,-\infty}} * M_{t,-\infty}$. Here we set $M_{t,-\infty} := D[0, 0, \mathrm{m}_{t,-\infty}]$.

Suppose that in addition (H4) hold. Then $N_{R_{t,s}} * M_{t,s} * \left(\tilde{\nu}_s \circ U(t,s)^{-1}\right)$ converges weakly to $N_{R_{t,-\infty}} * M_{t,-\infty} * \sigma_t$ as $s \to -\infty$. Hence it follows from (4.6) by [Par67, Theorem III.2.1] that $(\delta_{a_{t,-n}})$ is relatively compact. It is easy to see that this implies (H3). Moreover, the statement $\tilde{\nu}_t = \nu_t * \sigma_t$ also follows easily.

Now we show (4.5). For every $r \leq s \leq t$ and $\xi \in \mathbb{H}$, we have

$$\hat{\tilde{\nu}}_r(U(t,r)^*\xi) = \hat{\tilde{\nu}}_r(U(s,r)^*U(t,s)^*\xi). \tag{4.7}$$

Letting $r \to -\infty$ in (4.7) above, we get $\hat{\sigma}_t(\xi) = \hat{\sigma}_s\left(U(t,s)^*\xi\right)$. This completes the proof of (4).

The proof of the last assertion (5) uses the same arguments as in the proof of (4). $\qquad\square$

Remark 4.5. Similarly, the invariant measure for p_t defined in (1.1) is of the form $\nu * \mu_\infty$, where ν is a measure on $\mathbb{H}$ that is invariant under the action of the semigroup T_t and μ_∞ is the centered Gaussian measure with variance Q_∞ which is the proper limit of the variance of μ_t. We refer to [Hai09, Theorem 5.22] for details.

Remark 4.6. Condition (H4) holds if the following conditions (H5) and (H6) hold (see [Woo09, Lemma 3.7], which also holds for the infinite dimensional case):

(H5) For every $t \in \mathbb{R}$ and $x \in \mathbb{H}$, $U(t, s)x \to 0$ as $s \to -\infty$.

(H6) There exists some $t_0 \in \mathbb{R}$ such that $(\nu_t)_{t < t_0}$ is uniformly tight.

In the following we consider periodicity condition in time. We shall assume:

(HT) The function $U(t, s), \mu_{t,s}$ on $\Lambda = \{(t, s) \colon t \geq s\}$ are T-periodic for some $T > 0$. That is, for every $(t, s) \in \Lambda$,

$$U(t + T, s + T) = U(t, s), \quad \mu_{t+T, s+T} = \mu_{t,s}.$$

Theorem 4.7. *Suppose that (HT), (H1),(H2), H(3) hold and that for every $t \geq s$, there exist some some $M, \omega > 0$ such that $\|U(t, s)\| \leq M\,\mathrm{e}^{-\omega(t-s)}$. Then $\nu_t = D[a_{t,-\infty}, R_{t,-\infty}, m_{t,-\infty}]$ is the unique evolution system of measures with period T for $p_{s,t}$.*

Proof. It has already been shown in Theorem 4.3 that $(\nu_t)_{t \in \mathbb{R}}$ is an evolution system of measures. It remains to show the uniqueness. Let $\tilde{\nu}_t$ be a T-periodic evolution system of measures for $p_{s,t}$. Then for every $t \in \mathbb{R}$,

$$\hat{\tilde{\nu}}_{t+T}(\xi) = \hat{\mu}_{t+T,t}(\xi)\hat{\tilde{\nu}}_t(V(t)^*\xi).$$

Here we set $V(t) := U(t+T, t)$. From (2.6) we see that for every $t \in \mathbb{R}$ and $\xi \in \mathbb{H}$

$$\hat{\mu}_{t+T,-\infty}(\xi) = \hat{\mu}_{t+T,t}(\xi)\hat{\mu}_{t,-\infty}(V(t)^*\xi).$$

So, by the T-periodicity we get

$$\frac{\hat{\tilde{\nu}}_t(\xi)}{\hat{\mu}_{t,-\infty}(\xi)} = \frac{\hat{\tilde{\nu}}_{t+T}(\xi)}{\hat{\mu}_{t+T,-\infty}(\xi)} = \frac{\hat{\tilde{\nu}}_t(V(t)^*\xi)}{\hat{\mu}_{t,-\infty}(V(t)^*\xi)}.$$

Iterating the identity above, for any $k \in \mathbb{N}$, we get

$$\frac{\hat{\tilde{\nu}}_t(\xi)}{\hat{\mu}_{t,-\infty}(\xi)} = \frac{\hat{\tilde{\nu}}_t((V(t)^*)^k\xi)}{\hat{\mu}_{t,-\infty}((V(t)^*)^k\xi)}.$$

By assumption, for any $x \in \mathbb{H}$ we see that $(U(t, s))^k x$ converges to 0 as $k \to \infty$. This is enough to see that the right hand side of the identity above goes to 1 as $k \to \infty$. Therefore, we obtain that $\hat{\tilde{\nu}}_t(\xi) = \hat{\mu}_{t,-\infty}(\xi)$. $\square$

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Remark 4.8. Let $\mu_{t,s}$ be the distribution of the convolution integral $\int_s^t T_{t-r}\, dZ_r$ of a one-parameter C_0-semigroup T_t with respect to a semi-Lévy process Z_r (see [MS03]). Then $\mu_{t,s}$ is automatically periodic. Assume $\int_{-\infty}^t T_{t-r}\, dZ_r$ exists with distribution ν_t. Then it is shown in [MS03] that for the finite dimensional case (which can obviously be extended to the infinite dimensional case), ν_0 is semi-self-decomposable. Moreover, this is closely related to semi-self similar processes etc.. We refer to [MS03] for more details.

5 Harnack inequalities and applications

Let $(p_{s,t})_{t\geq s}$ be as in (2.1), that is $p_{s,t}f(x) = (\mu_{t,s} * \delta_{U(t,s)x})f$ for every $x \in \mathbb{H}$ and $f \in B_b(\mathbb{H})$. Suppose that for every $t \geq s$, $\mu_{t,s} = D[a_{t,s}, R_{t,s}, \mathrm{m}_{t,s}]$ is an infinite divisible measure on $(\mathbb{H}, \mathscr{B}(\mathbb{H}))$ satisfying (2.6).

For each $t \geq s$, set

$$\mu_{t,s}^g = D[0, R_{t,s}, 0], \quad \mu_{t,s}^j = D[a_{t,s}, 0, \mathrm{m}_{t,s}],$$

and for every $f \in B_b(\mathbb{H})$, $x \in \mathbb{H}$,

$$p_{s,t}^g f(x) := (\mu_{t,s}^g * \delta_{U(t,s)x})f = \int_{\mathbb{H}} f(U(t,s)x + y)\, \mu_{t,s}^g(dy),$$

$$p_{s,t}^j f(x) := (\mu_{t,s}^j * \delta_x)f = \int_{\mathbb{H}} f(x + y)\, \mu_{t,s}^j(dy).$$

With these notations, we have the following decomposition for $p_{s,t}$.

Proposition 5.1. *For every $t \geq s$, $x \in \mathbb{H}$ and $f \in B_b(\mathbb{H})$, $p_{s,t}f(x) = p_{s,t}^g(p_{s,t}^j)f(x)$.*

Proof. Note that $\mu_{t,s} = \mu_{t,s}^g * \mu_{t,s}^j$. Hence we get

$$p_{s,t}f(x) = (\mu_{t,s} * \delta_{U(t,s)x})f = (\mu_{t,s}^g * \mu_{t,s}^j * \delta_{U(t,s)x})f$$

$$= \left((\mu_{t,s}^g * \delta_{U(t,s)x}) * (\mu_{t,s}^j)\right)f = \int_{\mathbb{H}} \mu_{t,s}^g * \delta_{U(t,s)x}(dy) \int_{\mathbb{H}} f(y + z)\, \mu_{t,s}^j(dz)$$

$$= (\mu_{t,s}^g * \delta_{U(t,s)x})(p_{s,t}^j f) = p_{s,t}^g(p_{s,t}^j f)(x).$$

$\square$

Define for every $t \geq s$,
$$\Gamma_{t,s} = R_{t,s}^{-1/2} U(t,s) \tag{5.1}$$

with domain $\mathscr{D}(\Gamma_{t,s}) = \{x \in \mathbb{H}\colon U(t,s)x \in R_{t,s}^{1/2}(\mathbb{H})\}$. If $x \notin \mathscr{D}(\Gamma_{t,s})$ then set $|\Gamma_{t,s}x| := \infty$. Let $B_b^+(\mathbb{H})$ denote the space of all bounded positive measurable functions on $\mathbb{H}$.

Theorem 5.2. *For every $\alpha > 1$, $t \geq s$ and $f \in B_b^+(\mathbb{H})$*

$$(p_{s,t}f(x))^\alpha \leq \exp\left(\frac{\alpha|\Gamma_{t,s}(x-y)|^2}{2(\alpha-1)}\right) p_{s,t}f^\alpha(y), \quad x, y \in \mathbb{H}. \qquad (5.2)$$

Proof. It is sufficient to consider the case $U(t,s)(\mathbb{H}) \in R_{t,s}^{1/2}(\mathbb{H})$, since otherwise the right hand side of (5.3) is infinite by the definition of $|\Gamma_{t,s}(\cdot)|$ and the inequality (5.3) becomes trivial.

We claim that we only need to show the following Harnack inequality for $p_{s,t}^g$

$$(p_{s,t}^g f(x))^\alpha \leq \exp\left(\frac{\alpha|\Gamma_{t,s}(x-y)|^2}{2(\alpha-1)}\right) p_{s,t}^g f^\alpha(y), \quad x, y \in \mathbb{H}. \qquad (5.3)$$

Indeed, by Proposition 5.1, we know $p_{s,t} = p_{s,t}^g p_{s,t}^j$. If (5.3) holds, then by applying inequality (5.3) to $p_{s,t}^g$ and Jensen's inequality to $p_{s,t}^j$ we see

$$\left(p_{s,t}f(x)\right)^\alpha = \left(p_{s,t}^g(p_{s,t}^j f)(x)\right)^\alpha \leq \exp\left(\frac{\alpha|\Gamma_{t,s}(x-y)|^2}{2(\alpha-1)}\right) \left(p_{s,t}^g(p_{s,t}^j f)^\alpha\right)(y)$$

$$\leq \exp\left(\frac{\alpha|\Gamma_{t,s}(x-y)|^2}{2(\alpha-1)}\right) \left(p_{s,t}^g(p_{s,t}^j f^\alpha)\right)(y) = \exp\left(\frac{\alpha|\Gamma_{t,s}(x-y)|^2}{2(\alpha-1)}\right) \left(p_{s,t}f^\alpha\right)(y).$$

Applying the Cameron-Martin formula for Gaussian measures (see [DPZ92, Theorem 2.21]) we see

$$\begin{aligned}
\rho_{t,s}(x-y,z) &= \frac{d\,N(U(t,s)(x-y), R_{t,s})}{d\,N(0, R_{t,s})}(z) \\
&= \exp\left(\left\langle R_{t,s}^{-1/2}U(t,s)(x-y), R_{t,s}^{-1/2}z\right\rangle - \frac{1}{2}|R_{t,s}^{-1/2}U(t,s)(x-y)|^2\right).
\end{aligned} \qquad (5.4)$$

By changing variables and using Hölder's inequality we obtain

$$\begin{aligned}
&p_{s,t}^g f(x) \\
&= \int f(U(t,s)x + z)\,\mu_{t,s}(dz) \\
&= \int f(U(t,s)y + z)\rho_{t,s}(x-y,z)\,\mu_{t,s}(dz) \\
&\leq \exp\left(-\frac{1}{2}|\Gamma_{t,s}(x-y)|^2\right)\left(\int f^\alpha(U(t,s)y+z)\,\mu_{t,s}(dz)\right)^{1/\alpha} \cdot \\
&\quad \left(\int \exp\left(\frac{\alpha}{\alpha-1}\langle R_{t,s}^{-1/2}U(t,s)(x-y), R_{t,s}^{-1/2}z\rangle\right)\mu_{t,s}(dz)\right)^{(\alpha-1)/\alpha} \\
&= \exp\left(\frac{1}{2(\alpha-1)}|\Gamma_{t,s}(x-y)|^2\right)\left(p_{s,t}^g f^\alpha(y)\right)^{1/\alpha}.
\end{aligned}$$

$\square$

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Applying the previous theorem, we have the following result.

Theorem 5.3. *Fix $t \geq s$. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) of the following statements hold.*

(1)
$$U(t,s)(\mathbb{H}) \subset Q_{t,s}^{1/2}(\mathbb{H}), \tag{5.5}$$

(2) $\|\Gamma_{t,s}\| < \infty$ *and for every $\alpha > 1$ and $f \in B_b^+(\mathbb{H})$,*

$$(p_{s,t}f(x))^\alpha \leq \exp\left[\frac{\alpha(\|\Gamma_{t,s}\| \cdot |x-y|)^2}{2(\alpha-1)}\right] p_{s,t}f^\alpha(y), \quad x,y \in \mathbb{H}; \tag{5.6}$$

(3) $\|\Gamma_{t,s}\| < \infty$ *and there exists $\alpha > 1$ such that (5.6) holds for all $f \in B_b^+(\mathbb{H})$;*

(4) $\|\Gamma_{t,s}\| < \infty$ *and for every $f \in B_b^+(\mathbb{H})$ with $f > 1$,*

$$p_{s,t}\log f(x) \leq \log p_{s,t}f(y) + \frac{\|\Gamma_{t,s}\|^2}{2}|x-y|^2, \quad x,y \in \mathbb{H}; \tag{5.7}$$

(5) $p_{s,t}$ *is strong Feller.*

In particular, if $m_{t,s} \equiv 0$, then these statements are equivalent to each other.

Proof. If (5.5) hold, then $\|\Gamma_{t,s}\|$ is bounded. Hence by Theorem 5.2, we get (2) from (1). That (2) implies (3) is trivial. The implications (3)$\Rightarrow$(4)$\Rightarrow$(5) are consequences of Harnack inequalities, as proved in [Wan09].

It remains to show that (5) implies (4) in the case $m_{t,s} \equiv 0$. Note that

$$p_{s,t}f(x) = \int_{\mathbb{H}} f(y)\, N(U(t,s)x, R_{t,s})(dy).$$

If (5.5) doesn't hold, then there exists $x_0 \in \mathbb{H}$ such that $U(t,s)x_0 \notin R_{t,s}^{1/2}(\mathbb{H})$. Take $x_n = \frac{1}{n}x_0 \in \mathbb{H}$, $n = 1,2,\cdots$. By the Cameron-Martin theorem (see e.g. [DPZ92]), we know that for each $n = 1,2,\cdots$, the Gaussian measure $\mu_n := N(U(t,s)x_n, R_{t,s})$ is orthogonal to $\mu_0 := N(0, R_{t,s})$ since $U(t,s)x_n \notin R_{t,s}^{1/2}(\mathbb{H})$. That is, there exists $A_n \in \mathscr{B}(H)$ such that $\mu_n(A_n) = 1$, $\mu_0(A_n) = 0$. Set $A := \cup_{n \geq 1} A_n$. Then $\mu_0(A) = 0$, $\mu_n(A) = 1$ since $\mu_0(A) \leq \sum_n \mu_0(A_n) = 0$ and $\mu_n(A) \geq \mu_n(A_n) = 1$.

Take $f = \mathbb{1}_A$. We get $p_{s,t}f(x_n) = 1$, $p_{s,t}f(0) = 0$. This contradicts the fact that $p_{s,t}$ is strong Feller, since it is obvious that $p_{s,t}f(x_n)$ does not converge to $p_{s,t}f(0)$ as x_n tends to 0. $\square$

Remark 5.4. If $R_{t,s}$ has the form (7.2), then (5.5) is equivalent to the null controllability of a non-autonomous control system (7.1) (see Section 7 for details). For this reason, condition (5.5) is also called null-controllability condition. This gives an equivalent description of the strong Feller property.

Remark 5.5. In [DP95] the fact that the null controllability implies the strong Feller property was proved for autonomous Ornstein-Uhlenbeck processes driven by a Wiener process and with deterministic perturbation. Our result generalizes this result.

In fact (5.5) implies more. Denote the space of all infinitely Fréchet differentiable functions with uniform continuous derivatives on $\mathbb{H}$ by $UC^\infty(\mathbb{H})$.

Proposition 5.6. *Suppose* (5.5) *holds. Then for every* $f \in B_b(\mathbb{H})$ *and every* $t > s$, $p_{s,t}f \in UC^\infty(\mathbb{H})$.

Proof. In view of the decomposition $p_{s,t} = p_{s,t}^g p_{s,t}^j$ shown in Proposition 5.1, we only need to show that $p_{s,t}^g \in UC^\infty(\mathbb{H})$ for every $g \in B_b(\mathbb{H})$. The rest of the proof is as in [DPZ02, Theorem 6.2.2]. □

We have the following quantitative estimate for the strong Feller property. This result is shown in [ORW09] for Lévy driven Ornstein-Uhlenbeck process by a coupling method.

Proposition 5.7. *Let* $t > s$ *and* $x, y \in \mathbb{H}$. *Then*

$$
\begin{aligned}
&|p_{s,t}f(x) - p_{s,t}f(y)|^2 \\
&\leq \left(e^{|\Gamma_{t,s}(x-y)|^2} - 1 \right) \min \left\{ p_{s,t}f^2(z) - (p_{s,t}f(z))^2 : z = x, y \right\}.
\end{aligned}
\tag{5.8}
$$

Proof. Let $h = p_{s,t}^j f$. Then by Proposition 5.1 we see that $p_{s,t}f = p_{s,t}^g h$. So, for every $z \in \mathbb{H}$, we have

$$
\begin{aligned}
&p_{s,t}^g h^2(z) - \left(p_{s,t}^g h(z) \right)^2 \\
&\leq p_{s,t}^g p_{s,t}^j f^2(z) - \left(p_{s,t}^g p_{s,t}^j f(z) \right)^2 = p_{s,t}f^2(z) - (p_{s,t}f(z))^2.
\end{aligned}
\tag{5.9}
$$

Note also that x, y play the same role in (5.8). So, according to (5.9) we only need to show the following inequality

$$
|p_{s,t}^g h(x) - p_{s,t}^g h(y)|^2 \leq \left(e^{|\Gamma_{t,s}(x-y)|^2} - 1 \right) \left(p_{s,t}^g h^2(y) - (p_{s,t}^g h(y))^2 \right).
\tag{5.10}
$$

Recalling formula (5.4) for $\rho_{t,s}(x - y, z)$, we see

$$
p_{s,t}^g h(x) = \int_{\mathbb{H}} h(U(t,s)x + z)\, \mu_{t,s}^g(dz) = \int_{\mathbb{H}} \rho_{t,s}(x - y, z) h(U(t,s)y + z)\, \mu_{t,s}^g(dz).
$$

So,

$$|p_{s,t}^g h(x) - p_{s,t}^g h(y)|^2$$

$$= \left(\int_{\mathbb{H}} [\rho_{t,s}(x-y,z) - 1] \cdot [h(U(t,s)y+z) - p_{s,t}^g h(y)] \, \mu_{t,s}^g(dz) \right)^2$$

$$\leq \int_{\mathbb{H}} (\rho_{t,s}(x-y,z) - 1)^2 \, \mu_{t,s}^g(dz) \int_{\mathbb{H}} \left[h(U(t,s)y+z) - p_{s,t}^g h(y) \right]^2 \mu_{t,s}^g(dz)$$

$$= \left(\int_{\mathbb{H}} \rho_{t,s}^2(x-y,z) \, \mu_{t,s}^g(dz) - 1 \right) \cdot \left(\int_{\mathbb{H}} h^2(U(t,s)y+z) \, \mu_{t,s}^g(dz) - (p_{s,t}^g h(y))^2 \right)$$

$$= \left(e^{|\Gamma_{t,s}(x-y)|^2} - 1 \right) \left(p_{s,t}^g h^2(y) - (p_{s,t}^g h(y))^2 \right).$$

$\square$

Now we apply the Harnack inequality (5.2) to study the hyperboundedness of the transition function $p_{s,t}$. In [GL08] hypercontractivity is studied for the Gaussian case via log-Soboblev inequality.

Theorem 5.8. *Let $(\nu_t)_{t\in\mathbb{R}}$ be an evolution system of measures for $p_{s,t}$. For every $s \leq t$, $\alpha > 1$, and $\varepsilon > 0$, let*

$$C_{s,t}(\alpha,\varepsilon) := \int_{\mathbb{H}} \left[\int_{\mathbb{H}} \exp\left(-\frac{\alpha|\Gamma_{t,s}(x-y)|^2}{2(\alpha-1)} \right) \nu_s(dy) \right]^{-(1+\varepsilon)} \nu_s(dx).$$

Then

$$\|p_{s,t}f\|_{L^{\alpha(1+\varepsilon)}(\mathbb{H},\nu_s)} \leq C_{s,t}(\alpha,\varepsilon)^{-\alpha(1+\varepsilon)} \|f\|_{L^\alpha(\mathbb{H},\nu_t)}. \tag{5.11}$$

Proof. From the Harnack inequality (5.2) we have

$$(p_{s,t}f(x))^\alpha \exp\left[-\frac{\alpha|\Gamma_{t,s}(x-y)|^2}{2(\alpha-1)} \right] \leq p_{s,t}f^\alpha(y), \quad x,y \in \mathbb{H}.$$

Integrating both sides of the inequality above with respect to $\nu_s(dy)$ and using the fact that $(\nu_t)_{t\in\mathbb{R}}$ is an evolution system of measures, we get

$$(p_{s,t}|f|)^\alpha(x) \int_{\mathbb{H}} \exp\left(-\frac{\alpha|\Gamma_{t,s}(x-y)|^2}{2(\alpha-1)} \right) \nu_s(dy) \leq \int_{\mathbb{H}} |f|^\alpha \, \nu_t(dy).$$

Hence

$$(p_{s,t}|f|)^{\alpha(1+\varepsilon)}(x) \leq \left[\int_{\mathbb{H}} \exp\left(-\frac{\alpha|\Gamma_{t,s}(x-y)|^2}{2(\alpha-1)} \right) \nu_s(dy) \right]^{-(1+\varepsilon)} \|f\|_{L^\alpha(\mathbb{H},\nu_t)}^{\alpha(1+\varepsilon)}.$$

Integrating both sides of the equation above with respect to $\nu_s(dx)$, we get (5.11).
$\square$

6   Semi-linear equations

Fix $s \in \mathbb{R}$ and consider the following equation for $t \geq s$,

$$
\begin{cases}
dX(t,s,x) = A(t)X(t,s,x)\,dt + F(t,X(t,s,x))dt + R^{1/2}dW_t, \\
X(s,s,x) = x \in \mathbb{H},
\end{cases}
\tag{6.1}
$$

where

(1) $(A(t))_{t \in \mathbb{R}}$ is a family of operators on $\mathbb{H}$ associated with an evolution family $(U(t,s)_{t \geq s})$ (See Section 1);

(2) R is a trace class operator on $\mathbb{H}$;

(3) $(W_t)_{t \in \mathbb{R}}$ is a cylindrical Wiener process on $\mathbb{H}$ on some filtered probability space $(\Omega, (\mathscr{F}_t)_{t \in \mathbb{R}}, \mathscr{F}, \mathbb{P})$;

(4) F is a measurable map from $[s, +\infty) \times \mathbb{H}$ to $R^{1/2}(\mathbb{H})$ satisfying

$$
|R^{-1/2}F(t,x)|^2 \leq k_1 + k_2|x|^2, \quad t \in \mathbb{R}, \; x \in \mathbb{H}
\tag{6.2}
$$

for some constants $k_1, k_2 > 0$.

Proposition 6.1. *Equation* (6.1) *has a martingale solution.*

Proof. For every $r \in [s,t]$, set $\tilde{X}(r,s,x) := U(r,s)x + W_U(r,s)$, where

$$
W_U(r,s) := \int_s^r U(r,\sigma)R^{1/2}\,dW_\sigma.
$$

For every $r \in [s,t]$, $[s',t'] \subset [s,t]$, define

$$
\psi_x(r,s) := R^{-1/2}F(r,\tilde{X}(r,s,x)) = R^{-1/2}F(r,U(r,s)x + W_U(r,s)),
$$

$$
\tilde{W}_r^x = W_r - \int_s^r \psi_x(\sigma,s)\,d\sigma,
$$

$$
M_{t',s'}^x = \exp\left(\int_{s'}^{t'} \langle \psi_x(\sigma,s), dW_\sigma \rangle - \frac{1}{2}\int_{s'}^{t'} |\psi_x(\sigma,s)|^2\,d\sigma \right).
$$

We first show that $\mathbb{E}M_{t,s}^x = 1$. By (6.2), for every $r \in [s,t]$,

$$
|\psi_x(r,s)|^2 \leq k_1 + 2k_2(|U(r,s)x|^2 + |W_U(r,s)|^2).
$$

Hence,

$$
\mathbb{E}\exp\left(\frac{1}{2}\int_s^t |\psi_x(\sigma,s)|^2\,d\sigma \right)
$$

$$
\leq \mathbb{E}\exp\left(\frac{k_1 \vee k_2}{2}\int_s^t (1 + 2|U(\sigma,s)x|^2)\,d\sigma \right) \mathbb{E}\exp\left(\frac{1}{2}\int_s^t |W_U(\sigma,s)|^2\,d\sigma \right).
$$

Since $\int_s^t |W_U(\sigma,s)|^2\,d\sigma$ is Gaussian distributed, applying Fernique's Theorem, for a fine partition $s = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$, we have

$$\mathbb{E}\exp\left(\frac{1}{2}\int_{t_{i-1}}^{t_i} |\psi_x(\sigma,s)|^2\,d\sigma\right) < +\infty.$$

This implies that for each $i = 1, 2, \cdots, n$, $M_{t_{i-1},t}^x$ for $t \in [t_{i-1}, t_i]$, is a martingale. Noting that $M_{t,s}^x = M_{t_n,t_{n-1}}^x \cdots M_{t_1,t_0}^x$, we get $\mathbb{E}M_{t,s}^x = 1$.

Consequently, we can define a new probability measure $\mathbb{Q}_x := M_{t,s}^x \mathbb{P}$ on $(\Omega, \mathscr{F}_t)$. By [DPZ92, Theorem 10.14], $\tilde{W}_r^x$ is also a Wiener process with respect to $\mathbb{Q}_x$. Hence

$$\tilde{X}(t,s,x) = U(t,s)x + \int_s^t U(t,r)R^{1/2}\,dW_r$$

$$= U(t,s)x + \int_s^t U(t,r)F(r, \tilde{X}(r,s,x))\,dr + \int_s^t U(t,r)R^{1/2}\,d\tilde{W}_r.$$

This shows that $\tilde{X}(t,s,x)$ is a martingale solution of (6.1) on $(\Omega, (\mathscr{F}_t)_{t\geq s}, \mathscr{F}, \mathbb{Q}_x)$.

$\square$

We shall need the following fact.

Lemma 6.2. *Let $s \in \mathbb{R}$. Set*

$$\lambda := \operatorname{tr}\int_s^{s+1} U(s+1,\sigma)RU(s+1,\sigma)^*\,d\sigma.$$

Then

$$C_0 := \sup_{r\in[s,s+1]} \mathbb{E}\exp\left(|W_U(r,s)|^2/4\lambda\right) < \infty$$

and for every $\kappa > 0$ and $t \in [s, s + (1 \wedge (4\lambda\kappa)^{-1})]$,

$$\mathbb{E}\exp\left(\kappa \int_s^t |W_U(r,s)|^2\,dr\right) < C_0^{4\lambda\kappa(t-s)}. \tag{6.3}$$

Proof. Note that the covariance operator of $W_U(r,s) = \int_s^r U(r,\sigma)R^{1/2}\,dW_\sigma$ is given by $\int_s^r U(r,\sigma)RU(r,\sigma)^*\,d\sigma$. By Fernique's Theorem (see [DPZ92, Propostion 2.16]), it follows that $C_0 < \infty$. Moreover,

$$\mathbb{E}\exp\left(\kappa \int_s^t |W_U(r,s)|^2\,dr\right) = \mathbb{E}\exp\left(\frac{1}{t-s}\int_s^t \kappa(t-s)|W_U(r,s)|\,dr\right)$$

$$\leq \frac{1}{t-s}\int_s^t \mathbb{E}\exp\left(\kappa(t-s)|W_U(r,s)|^2\right)\,dr$$

$$\leq \frac{1}{t-s}\int_s^t \left[\mathbb{E}\exp\left(|W_U(r,s)|^2/4\lambda\right)\right]^{4\lambda\kappa(t-s)}\,dr \leq C_0^{4\lambda\kappa(t-s)}.$$

$\square$

From Lemma 6.2 we see that for every $p > 0$, there exists $t_p > 0$ such that for every $t \in [s, s + t_p]$,

$$C_{p,k_2}(t, s) := \mathbb{E} \exp \left(2p(2p + 1)k_2 \int_s^t |W_U(r, s)|^2 \, ds \right) < \infty.$$

In particular, if $k_2 = 0$ then $C_{p,0}(t, s) = 1$ for all $t \geq s$.

Lemma 6.3. *For any $t > s$, $p > 1$, $\delta > 0$ and $x \in \mathbb{H}$,*

$$\mathbb{E}(M_{t,s}^x)^p \leq (C_{p,k_2}(t, s))^{1/2} \exp \left(\frac{p(2p - 1)}{2} \int_s^t (k_1 + 2k_2 |U(r, s)x|^2) \, dr \right)$$

$$\mathbb{E}(M_{t,s}^x)^{-\delta} \leq (C_{\delta,k_2}(t, s))^{1/2} \exp \left(\frac{\delta(2\delta + 1)}{2} \int_s^t (k_1 + 2k_2 |U(r, s)x|^2) \, dr \right).$$

Proof. From the proof of Proposition 6.1 we see that for every $\kappa \in \mathbb{R}$

$$t \mapsto \exp \left(\kappa \int_s^t \langle \psi_x(r, s), dW_r \rangle - \frac{\kappa^2}{2} \int_s^t |\psi_x(r, s)|^2 \, dr \right)$$

is a martingale. Therefore,

$$
\begin{aligned}
\mathbb{E}(M_{t,s}^x)^p =\ & \mathbb{E} \exp \left(p \int_s^t \langle \psi_x(r, s), dW_r \rangle - p^2 \int_s^t |\psi_x(r, s)|^2 \, dr \right) \\
& \cdot \exp \left(\frac{p(2p - 1)}{2} \int_s^t |\psi_x(r, s)|^2 \, dr \right) \\
\leq\ & \left[\mathbb{E} \exp \left(2p \int_s^t \langle \psi_x(r, s), dW_r \rangle - 2p^2 \int_s^t |\psi_x(r, s)|^2 \, ds \right) \right]^{1/2} \\
& \cdot \left[\mathbb{E} \exp \left(p(2p - 1) \int_s^t |\psi_x(r, s)|^2 \, ds \right) \right]^{1/2} \\
=\ & \left[\mathbb{E} \exp \left(p(2p - 1) \int_s^t |\psi_x(r, s)|^2 \, ds \right) \right]^{1/2}.
\end{aligned}
$$

This implies the first inequality, since by (6.2)

$$|\psi_x(r, s)|^2 \leq k_1 + 2k_2 |W_U(r, s)|^2 + 2k_2 |U(r, s)x|^2.$$

Similarly, the second inequality follows by

$$\mathbb{E}(M_{t,s}^x)^{-\delta} = \mathbb{E} \exp\left(-\delta \int_s^t \langle \psi_x(r,s), dW_r \rangle - \delta^2 \int_s^t |\psi_x(r,s)|^2 \, dr\right)$$
$$\cdot \exp\left(\frac{\delta(2\delta+1)}{2} \int_s^t |\psi_x(r,s)|^2 \, dr\right)$$
$$\leq \left[\mathbb{E} \exp\left(-2\delta \int_s^t \langle \psi_x(r,s), dW_r \rangle - 2\delta^2 \int_s^t |\psi_x(r,s)|^2 \, ds\right)\right]^{1/2}$$
$$\cdot \left[\mathbb{E} \exp\left(\delta(2\delta+1) \int_s^t |\psi_x(r,s)|^2 \, dr\right)\right]^{1/2}$$
$$= \left[\mathbb{E} \exp\left(\delta(2\delta+1) \int_s^t |\psi_x(r,s)|^2 \, dr\right)\right]^{1/2}.$$

$\square$

By the proof of Proposition 6.1, we see that $\tilde{X}(t,s,x)$ is a solution of (6.1). Hence we define the "transition semigroup" of $X(t,s,x)$ by

$$P_{s,t}^F f(x) = \mathbb{E}_{\mathbb{Q}_x} f(\tilde{X}(t,s,x)), \quad f \in B_b(\mathbb{H}). \tag{6.4}$$

We have the following result.

Theorem 6.4. *For any $t > 0$, $\alpha > 1$, $x, y \in \mathbb{H}$, $p, q > 1$ with $\alpha/(pq) > 1$, and $f \in B_b^+(\mathbb{H})$*

$$(P_{s,t}^F f)^\alpha(x) \leq N P_{s,t}^F f^\alpha(y). \tag{6.5}$$

Here we set $\Gamma_{t,s}^F := R^{-1/2} U(t,s)$ and

$$N := \left(C_{\frac{p}{p-1}, k_2}(t,s)\right)^{\alpha p/(2(p-1))} \cdot \left(C_{\frac{1}{q-1}, k_2}(t,s)\right)^{\alpha q/(2(q-1))} \cdot \exp\left(\frac{\alpha q |\Gamma_{t,s}^F(x-y)|^2}{2(\alpha-q)}\right)$$
$$+ \alpha \left[\frac{p+1}{p-1} + \frac{q+1}{q(q-1)}\right] \int_s^t \left[k_1 + k_2(|U(r,s)x|^2 + |U(r,s)y|^2)\right] \, dr\right).$$

Assume that for every $s \leq r \leq t$, $P_{s,t}^F = P_{s,r}^F P_{r,t}^F$. If $\|\Gamma_{t,s}^F\| < \infty$ for every $t \geq s$, then $P_{s,t}^F$ is strong Feller.

Proof. Recall that $\tilde{X}(t,s,x)$ is a mild solution to

$$d\tilde{X}(t,s,x) = A(t)\tilde{X}(t,s,x)dt + R^{1/2}dW_t, \quad \tilde{X}(s,s,x) = x.$$

Let $P_{s,t}^0$ be the semigroup of $\tilde{X}(t,s,x)$ under $\mathbb{P}$. Then by Theorem 5.2 we have

$$(P_{s,t}^0 f)^\alpha(x) \leq P_{s,t}^0 f^\alpha(y) \exp\left(\frac{\alpha |\Gamma_{t,s}^F(x-y)|^2}{2(\alpha-1)}\right), \quad f \in B_b^+(\mathbb{H}), \tag{6.6}$$

For simplicity, we set $p' := \frac{p}{p-1}$, $q' := \frac{q}{q-1}$, $\theta = \alpha/(pq)$. By (6.6) we have

$$
\begin{aligned}
P_{s,t}^F f(x) &= \mathbb{E}_{\mathbb{Q}_x} f(\tilde{X}(t,s,x)) = \mathbb{E} M_{t,s}^x f(\tilde{X}(t,s,x)) \\
&\leq (\mathbb{E} f^p(\tilde{X}(t,s,x)))^{1/p} (\mathbb{E}(M_{t,s}^x)^{p'})^{1/p'} = (P_{s,t}^0 f^p(x))^{1/p} (\mathbb{E}(M_{t,s}^x)^{p'})^{1/p'} \\
&\leq \left[P_{s,t}^0 f^{\theta p}(y) \exp\left(\frac{\theta |\Gamma_{t,s}^F(x-y)|^2}{2(\theta-1)} \right) \right]^{1/(\theta p)} (\mathbb{E}(M_{t,s}^x)^{p'})^{1/p'}.
\end{aligned}
$$

On the other hand, for every $g \in B_b^+(\mathbb{H})$,

$$
\begin{aligned}
P_{s,t}^0 g(y) &\leq \mathbb{E}_{\mathbb{P}} g(\tilde{X}(t,s,y)) = \mathbb{E}_{\mathbb{Q}_y} g(\tilde{X}(t,s,y))(M_{t,s}^y)^{-1} \\
&\leq (P_{s,t}^F g^q(y))^{1/q} (\mathbb{E}(M_{t,s}^y)^{1-q'})^{1/q'}.
\end{aligned}
$$

So, taking $g = f^{\theta p}$ we obtain

$$
(P_{s,t}^F f)^\alpha(x) \leq P_{s,t}^F f^\alpha(y) \exp\left(\frac{\alpha |\Gamma_{t,s}^F(x-y)|^2}{2p(\theta-1)} \right) (\mathbb{E}(M_{t,s}^x)^{p'})^{\alpha/p'} (\mathbb{E}(M_{t,s}^y)^{1-q'})^{\alpha/q'}.
$$

This implies the desired Harnack inequality according to Lemma 6.3.

Now we show that $P_{s,t}^F$ is strongly Feller. Let $f \in B_b^+(\mathbb{H})$. By (6.3) and (6.5), for any $\alpha > 1$ there exist constants $t_\alpha, c_\alpha > 0$ and a positive function $H_\alpha(r,s)$, $r \in (s, s+t_\alpha)$ such that

$$
P_{s,r}^F f(x) \leq (P_{s,r}^F f^\alpha(y))^{1/\alpha} e^{c_\alpha(r-s)+|x-y|^2 H_\alpha(r,s)}, \quad r \in (s, s+t_\alpha). \tag{6.7}
$$

We take $t_\alpha < t - s$. Then, using the assumption that $P_{s,t}^F$ is a semigroup, for every $r \in (s, s+t_\alpha)$, we get

$$
\begin{aligned}
\overline{\lim_{x \to y}} P_{s,t}^F f(x) &= \overline{\lim_{x \to y}} P_{s,r}^F P_{r,t}^F f(x) \\
&\leq \overline{\lim_{\alpha \to 1}} \, \overline{\lim_{r \to s}} \, \overline{\lim_{x \to y}} \left[P_{s,r}^F (P_{r,t}^F f)^\alpha(y) \right]^{1/\alpha} e^{c_\alpha(r-s)+|x-y|^2 H_\alpha(r,s)} \\
&\leq \overline{\lim_{\alpha \to 1}} \, \overline{\lim_{r \to s}} \, \overline{\lim_{x \to y}} \left[P_{s,t}^F f^\alpha(y) \right]^{1/\alpha} e^{c_\alpha(r-s)+|x-y|^2 H_\alpha(r,s)} = P_{s,t}^F f(y).
\end{aligned} \tag{6.8}
$$

On the other hand, (6.7) also implies for every $r \in (s, s+t_\alpha)$

$$
\begin{aligned}
P_{s,t}^F f(x) &\geq \left[P_{s,r}^F (P_{r,t}^F f)^{1/\alpha}(y) \right]^\alpha e^{-\alpha c_\alpha(r-s)-\alpha H_\alpha(r,s)|x-y|^2} \\
&\geq \left[P_{s,t}^F f^{1/\alpha}(y) \right]^\alpha e^{-\alpha c_\alpha(r-s)-\alpha H_\alpha(r,s)|x-y|^2}.
\end{aligned}
$$

So, first letting $x \to y$ then $r \to s$ and finally $\alpha \to 1$, we arrive at

$$
\lim_{x \to y} P_{s,t}^F f(x) \geq P_{s,t}^F f(y). \tag{6.9}
$$

From (6.8) and (6.9) we see $P_{s,t}^F f$ is continuous. So, $P_{s,t}^F$ is strongly Feller. $\qquad \square$

7 Appendix: Null controllability

Consider the following non-autonomous linear control system

$$\begin{cases} dz(t) = A(t)z(t)dt + C(t)u(t)\,dt, \\ z(s) = x, \end{cases} \tag{7.1}$$

where $(A(t))_{t\in\mathbb{R}}$ is a family of linear operators on $\mathbb{H}$ with dense domains and $(C(t))_{t\in\mathbb{R}}$ is a family of bounded linear operators on $\mathbb{H}$. Let $(U(t,s))_{t\geq s}$ be an evolution family on $\mathbb{H}$ associated with $(A(t))_{t\in\mathbb{R}}$. Consider the mild solution of (7.1)

$$z(t,s,x) = U(t,s)x + \int_s^t U(t,r)C(r)u(r)\,dr. \quad x \in \mathbb{H},\ t \geq s.$$

$z(t,s,x)$ is interpreted as the state of the system and u as a strategy to control the system. If there exists $u \in L^2([s,t],\mathbb{H})$ such that $z(t,s,x) = 0$, then we say the system (7.1) can be transferred to 0 at time t from initial state $x \in \mathbb{H}$ at time s. If for every initial state $x \in \mathbb{H}$ the system (7.1) can be transferred to 0 then we say the system (7.1) is null controllable at time t. We refer to [Zab08] (see also [DPZ92, Appendix B]) for details on the null controllability of autonomous control systems.

Set for every $t \geq s$

$$\Pi_{t,s}x := \int_s^t U(t,r)C(r)C(r)^*U(t,r)^*\,dr, \quad x \in \mathbb{H}. \tag{7.2}$$

Proposition 7.1. *Let $x \in \mathbb{H}$ and $t \geq s$. The system (7.1) can be transferred to 0 at time t from x if and only if $U(t,s)x \in \Pi_{t,s}^{1/2}(\mathbb{H})$. Moreover, the minimal energy among all strategies transferring x to 0 at time t is given by $|\Pi_{t,s}^{-1/2}U(t,s)x|^2$, i.e.*

$$\begin{aligned} &|\Pi_{t,s}^{-1/2}U(t,s)x|^2 \\ &= \inf\left\{\int_s^t |u(r)|^2\,dr\colon z(t,s,x) = 0, z(s,s,x) = x, u \in L^2([s,t],\mathbb{H})\right\}. \end{aligned} \tag{7.3}$$

Proof. For every $t \geq s$ define a linear operator

$$L_{t,s}\colon L^2([s,t],\mathbb{H}) \to \mathbb{H}, \quad u \mapsto L_{t,s}u := \int_s^t U(t,r)C(r)u(r)\,dr.$$

The adjoint $L_{t,s}^*$ of $L_{t,s}$ is given by

$$(L_{t,s}^*x)(r) = C^*(r)U(t,r)^*x, \quad x \in \mathbb{H},\ r \in [s,t].$$

It is easy to check that $\Pi_{t,s} = L_{t,s}L_{t,s}^*$. Then by [DPZ92, Corollary B.4], we know that $L_{t,s}(L^2([s,t],\mathbb{H}) = \Pi_{t,s}(\mathbb{H})$. Hence the first assertion of the theorem is proved

since the initial state x can be transferred to 0 if and only if $U(t,s)x$ is contained in the image space of $L_{t,s}$ due to the fact that $z(t,s,x) = U(t,s)x + L_{t,s}u$.

By [DPZ92, Corollary B.4] we also get

$$|\Pi_{t,s}^{-1/2}y| = |L_{t,s}^{-1}y|, \quad y \in L_{t,s}(L^2([s,t], \mathbb{H})). \tag{7.4}$$

Here the inverse is understood as a pseudo–inverse. Taking $y = U(t,s)x$ in (7.4), we obtain (7.3). $\qquad\square$

From Proposition 7.1, we get the following corollary.

Corollary 7.2. *The system* (7.1) *is null controllable at time t if and only if*

$$U(t,s)(\mathbb{H}) \subset \Pi_{t,s}^{1/2}(\mathbb{H}). \tag{7.5}$$

From (7.3), it is easy to get upper bounds of $|\Pi_{t,s}^{-1/2}U(t,s)x|^2$ by choosing proper null control functions u. The following proposition is analogous to [ORW09, Proposition 2.1].

Proposition 7.3. *Let $t > s$. Assume that for every $r \in [s,t]$, the operator $C(r)$ is invertible. Then for every strictly positive function $\xi \in C([s,t])$,*

$$|\Pi_t^{-1/2}U(t,s)x|^2 \leq \frac{\int_s^t |C(r)^{-1}U(r,s)x|^2 \, \xi_r^2 \, dr}{\left(\int_s^t \xi_r \, dr \right)^2}, \quad x \in \mathbb{H}. \tag{7.6}$$

Especially if $C(r) \equiv C$ and $|C^{-1}U(r,s)x|^2 \leq h(r)|C^{-1}x|^2$ for every $x \in \mathbb{H}$, then

$$|\Pi_t^{-1/2}U(t,s)x|^2 \leq \frac{|C^{-1}x|^2}{\int_s^t h(r)^{-1} \, dr}, \quad x \in \mathbb{H}. \tag{7.7}$$

Proof. We only need to consider the case where $U(t,s)x \in \Pi_{t,s}^{1/2}(\mathbb{H})$ and the function $[s,t] \ni r \mapsto \xi_r C(r)^{-1}U(r,s)x$ belongs to $L^2([0,t], \mathbb{H})$. Then the following function

$$u(r) := -\frac{\xi_r}{\int_s^t \xi_r \, dr} C(r)^{-1}U(r,s)x, \quad r \in [s,t],$$

is a null control of the system (7.1). And hence the estimate (7.6) follows from (7.3). The second estimate (7.7) follows by taking $\xi(r) = h(r)^{-1}$ for all $r \in [s,t]$. $\qquad\square$

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