The stochastic reflection problem on an infinite dimensional convex set and BV functions in a Gelfand triple

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Abstract

In this paper, we introduce a definition of BV functions in a Gelfand triple which is an extension of the definition of BV functions in [1] by using Dirichlet form theory. By this definition, we can consider the stochastic reflection problem associated with a self-adjoint operator A and a cylindrical Wiener process on a convex set Γ in a Hilbert space H. We prove the existence and uniqueness of a strong solution of this problem when Γ is a regular convex set. The result is also extended to the non-symmetric case. Finally, we extend our results to the case when $\Gamma = K_\alpha$, where $K_\alpha = \{f \in L^2(0,1) | f \geq -\alpha\}, \alpha \geq 0$.

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1 Introduction

A definition of BV functions in abstract Wiener spaces has been given by M. Fukushima in [12], M. Fukushima and M. Hino in [13], based upon Dirichlet form theory. In this paper, we introduce BV functions in a Gelfand triple, which is an extension of BV functions in a Hilbert space defined in [1]. Here we use a version of the Riesz-Markov representation theorem in infinite dimensions proved by M. Fukushima using the quasi-regularity of the Dirichlet form (see [17]) to give a characterization of BV functions.

In this paper, we consider the Dirichlet form

$$\mathcal{E}^\rho(u,v) = \frac{1}{2}\int_H \langle Du, Dv\rangle \rho(z)\mu(dz)$$

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(where μ is a Gaussian measure in H and ρ is a BV function) and its associated process. By using BV functions, we obtain a Skorohod-type representation for the associated process, if $\rho = I_\Gamma$ and Γ is a convex set.

As a consequence of these results, we can solve the following stochastic differential inclusion in the Hilbert space H:

$$\begin{cases} dX(t) + (AX(t) + N_\Gamma(X(t)))dt \ni dW(t), \\ X(0) = x, \end{cases} \tag{1.1}$$

where our solution is strong (in the probabilistic sense), if Γ is regular. Here $A : D(A) \subset H \to H$ is a self-adjoint operator. $N_\Gamma(x)$ is the normal cone to Γ at x and $W(t)$ is a cylindrical Wiener process in H. The precise meaning of the above inclusion will be defined in Section 5.2. The solution to (1.1) is called distorted (if $\rho = I_\Gamma$, reflected) Ornslein-Uhlenbek (OU for short)-process.

(1.1) was first studied (strongly solved) in [19], when $H = L^2(0,1)$, A is the Laplace operator with Dirichlet or Neumann boundary conditions and Γ is the convex set of all nonnegative functions of $L^2(0,1)$; see also [28]. In [6] the authors study the situation when Γ is a regular convex set with nonempty interior. They get precise information about the corresponding Kolmogorov operator, but did not construct a strong solution to (1.1).

In this paper, we consider a convex set Γ. If Γ is a regular convex set, we show that I_Γ is a BV-function and thus obtain existence and uniqueness results for (1.1). By a modification of [12] and using [7], we obtain the existence of an (in the probabilistic sense) weak solution to (1.1). Then, we prove pathwise uniqueness. Thus, by a version of the Yamada-Watanabe Theorem (see [15]), we deduce that (1.1) has a unique strong solution. We also consider the case when $\Gamma = K_\alpha$, where $K_\alpha = \{f \in L^2(0,1) | f \geq -\alpha\}, \alpha \geq 0$, and prove I_Γ is a BV function. Thus our result about Skorohod-type representation applies.

This paper is organized as follows. In Section 2, we consider the Dirichlet form and its associated distorted OU-process. We introduce BV functions in Section 3, by which we can get the Skorohod type representation for the OU- process. In Section 4, we analyze the reflected OU-process. In Section 5, we get the existence and uniqueness of the solution for (1.1) if Γ is a regular convex set. We also extend these results to the non-symmetric case. In Section 6, we consider the case when $\Gamma = K_\alpha$, where $K_\alpha = \{f \in L^2(0,1) | f \geq -\alpha\}, \alpha \geq 0$.

2 The Dirichlet form and the associated distorted OU-process

Let H be a real separable Hilbert space (with scalar product $\langle \cdot, \cdot \rangle$ and norm denoted by $| \cdot |$). We denote its Borel σ-algebra by $\mathcal{B}(H)$. Assume that:

Hypothesis 2.1 $A : D(A) \subset H \to H$ is a linear self-adjoint operator on H such that $\langle Ax, x \rangle \geq \delta |x|^2$ $\forall x \in D(A)$ for some $\delta > 0$ and A^{-1} is of trace class.

Since A^{-1} is trace class, there exists an orthonormal basis $\{e_j\}$ in H consisting of eigenfunctions for A with corresponding eigenvalues $\alpha_j \in \mathbb{R}, j \in \mathbb{N}$, that is,

$$Ae_j = \alpha_j e_j, j \in \mathbb{N}.$$

Then $\alpha_j \geq \delta$ for all $j \in \mathbb{N}$.

Below $D\varphi : H \to H$ denotes the Frêchet-derivative of a function $\varphi : H \to \mathbb{R}$. By $C_b^1(H)$ we shall denote the set of all bounded differentiable functions with continuous and bounded derivatives. For $K \subset H$, the space $C_b^1(K)$ is defined as the space of restrictions of all functions in $C_b^1(H)$ to the subset K. μ will denote the Gaussian measure in H with mean 0 and covariance operator

$$Q := \frac{1}{2}A^{-1}.$$

Since A is strictly positive, μ is nondegenerate and has full topological support. Let $L^p(H, \mu), p \in [1, \infty]$, denote the corresponding real L^p-spaces equipped with the usual norms $\| \cdot \|_p$. We set

$$\lambda_j := \frac{1}{2\alpha_j} \ \forall j \in \mathbb{N},$$

so that

$$Qe_j = \lambda_j e_j \ \forall j \in \mathbb{N}.$$

For $\rho \in L^1_+(H, \mu)$ we consider

$$\mathcal{E}^\rho(u, v) = \frac{1}{2}\int_H \langle Du, Dv\rangle \rho(z)\mu(dz), u, v \in C_b^1(F),$$

where $F := Supp[\rho \cdot \mu]$ and $L^1_+(H, \mu)$ denotes the set of all non-negative elements in $L^1(H, \mu)$. Let $QR(H)$ be the set of all functions $\rho \in L^1_+(H, \mu)$ such that $(\mathcal{E}^\rho, C_b^1(F))$ is closable on $L^2(F, \rho\cdot \mu)$. Its closure is denoted by $(\mathcal{E}^\rho, \mathcal{F}^\rho)$. We denote by $\mathcal{F}^\rho_e$ the extended Dirichlet space of $(\mathcal{E}^\rho, \mathcal{F}^\rho)$, that is, $u \in \mathcal{F}^\rho_e$ if and only if $|u| < \infty \ \rho \cdot \mu - a.e.$ and there exists a sequence $\{u_n\}$ in $\mathcal{F}^\rho$ such that $\mathcal{E}^\rho(u_m - u_n, u_m - u_n) \to 0$ as $n \geq m \to \infty$ and $u_n \to u \ \ \rho \cdot \mu - a.e.$ as $n \to \infty$.

Theorem 2.2 Let $\rho \in QR(H)$. Then $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ is a quasi-regular local Dirichlet form on $L^2(F; \rho \cdot \mu)$ in the sense of [17, IV Definition 3.1].

Proof The assertion follows from the main result in [27]. □

By virtue of Theorem 2.2 and [17], there exists a diffusion process $M^\rho = (\Omega, \mathcal{M}, \{\mathcal{M}_t\}, \theta_t, X_t, P_z)$ on F associated with the Dirichlet form $(\mathcal{E}^\rho, \mathcal{F}^\rho)$. M^ρ will be called distorted OU-process on F. Since constant functions are in $\mathcal{F}^\rho$ and $\mathcal{E}^\rho(1, 1) = 0$, M^ρ is recurrent and conservative. We denote by $\mathbf{A}^\rho_+$ the set of all positive continuous additive functionals (PCAF in abbreviation) of M^ρ, and define $\mathbf{A}^\rho := \mathbf{A}^\rho_+ - \mathbf{A}^\rho_+$. For $A \in \mathbf{A}^\rho$, its total variation process is denoted by $\{A\}$. We also define $\mathbf{A}^\rho_0 := \{A \in \mathbf{A}^\rho | E_{\rho\cdot\mu}(\{A\}_t) < \infty \forall t > 0\}$. Each element in $\mathbf{A}^\rho_+$ has a corresponding positive $\mathcal{E}^\rho$-smooth measure on F by the Revuz correspondence. The set of all such measures will be denoted by S^ρ_+. Accordingly, $A_t \in \mathbf{A}^\rho$ corresponds to a $\nu \in S^\rho := S^\rho_+ - S^\rho_+$, the set of all $\mathcal{E}^\rho$-smooth signed measure in the sense that $A_t = A^1_t - A^2_t$ for $A^k_t \in \mathbf{A}^\rho_+, k = 1, 2$ whose Revuz measures are $\nu^k, k = 1, 2$ and $\nu = \nu^1 - \nu^2$ is the Hahn-Jordan decomposition of ν . The element of $\mathbf{A}^\rho$ corresponding to $\nu \in S^\rho$ will be denoted byA^ν.

Note that for each $l \in H$ the function $u(z) = \langle l, z\rangle$ belongs to the extended Dirichlet space $\mathcal{F}^\rho_e$ and

$$\mathcal{E}^\rho(l(\cdot), v) = \frac{1}{2}\int \langle l, Dv(z)\rangle \rho(z)d\mu(z) \ \forall v \in C_b^1(F). \tag{2.1}$$

On the other hand, the AF $\langle l, X_t - X_0\rangle$ of M^ρ admits a unique decomposition into a sum of a martingale AF (M_t) of finite energy and CAF (N_t) of zero energy. More precisely, for every $l \in H$,

$$\langle l, X_t - X_0\rangle = M_t^l + N_t^l \ \forall t \geq 0 \ P_z - a.s. \tag{2.2}$$

for $\mathcal{E}^\rho$-q.e. $z \in F$.

Now for $\rho \in L^1(H, \mu)$ and $l \in H$, we say that $\rho \in BV_l(H)$ if there exists a constant $C_l > 0$,

$$\left| \int \langle l, Dv(z)\rangle \rho(z) d\mu(z)\right| \leq C_l \parallel v \parallel_\infty \ \ \forall v \in C_b^1(F). \tag{2.3}$$

By the same argument as in [13, Theorem 2.1], we obtain the following:

Theorem 2.3 Let $\rho \in L^1_+$ and $l \in H$.

(1) The following two conditions are equivalent:

(i)$\rho \in BV_l(H)$

(ii) There exists a (unique) signed measure ν_l on F of finite total variation such that

$$\frac{1}{2}\int \langle l, Dv(z)\rangle \rho(z) d\mu(z) = -\int_F v(z)\nu_l(dz) \ \forall v \in C_b^1(F). \tag{2.4}$$

In this case, ν_l necessarily belongs to $S^{\rho+1}$.

Suppose further that $\rho \in QR(H)$. Then the following condition is also equivalent to the above:

(iii)$N^l \in \mathbf{A}_0^\rho$

In this case, $\nu_l \in S^\rho$, and $N^l = A^{\nu_l}$

(2) M^l is a martingale AF with quadratic variation process

$$\langle M^l\rangle_t = t|l|^2, t \geq 0. \tag{2.5}$$

Remark 2.4 Recall that the Riesz representation theorem of positive linear functionals on continuous functions by measures is not applicable to obtain Theorem 2.3, $(i) \Rightarrow (ii)$, because of the lack of local compactness. However, the quasi-regularity of the Dirichlet form provides a means to circumvent this difficulty.

In the rest of this section, we shall introduce a special class of $\rho \in QR(H)$, which will be used in Section 4 below.

A non-negative measurable function $h(s)$ on $\mathbb{R}^1$ is said to possess the *Hamza property* if $h(s) = 0 \ ds - a.e.$ on the closed set $\mathbb{R}^1 \setminus R(h)$ where

$$R(h) = \{s \in \mathbb{R}^1 : \int_{s-\varepsilon}^{s+\varepsilon} \frac{1}{h(r)} dr < \infty \text{ for some } \varepsilon > 0\}.$$

We say that a function $\rho \in L^1_+(H, \mu)$ satisfies *the ray Hamza condition in direction* $l \in H$ ($\rho \in \mathbf{H}_l$ in notation) if there exists a non-negative function $\tilde{\rho}_l$ such that

$$\tilde{\rho}_l = \rho \ \mu - a.e. \text{ and } \tilde{\rho}_l(z + sl) \text{ has the Hamza property in } s \in \mathbb{R}^1 \text{ for each } z \in H.$$

We set $\mathbf{H} := \cap_k \mathbf{H}_{e_k}$, where e_k is as in Hypothesis 2.1. A function in the family $\mathbf{H}$ is simply said to satisfy the *ray Hamza condition.* By [5] $\mathbf{H} \subset QR(H)$, and thus we always have $\rho + 1 \in QR(H)$, since clearly $\rho + 1 \in \mathbf{H}$.

Next we will present some explicit description of the Dirichlet form $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ for $\rho \in \mathbf{H}$.

For $e_j \in H$ as in Hypothesis 2.1, we set $H_{e_j} = \{se_j : s \in \mathbb{R}^1\}$. We then have the direct sum decomposition $H = H_{e_j} \oplus E_{e_j}$ given by

$$z = se_j + x, s = \langle e_j, z \rangle .$$

Let π_j be the projection onto the space E_{e_j} and μ_{e_j} be the image measure of μ under $\pi_j : H \to E_{e_j}$ i.e $\mu_{e_j} = \mu \circ \pi_j^{-1}$. Then we see that for any $F \in L^1(H, \mu)$

$$\int_H F(z)\mu(dz) = \int_{E_{e_j}} \int_{\mathbb{R}^1} F(se_j + x)p_j(s)ds\mu_{e_j}(dx), \tag{2.6}$$

where $p_j(s) = (1/\sqrt{2\pi\lambda_j})e^{-s^2/2\lambda_j}$. Thus by [5, Theorem3.10] for all $u, v \in D(\mathcal{E}^\rho)$,

$$\mathcal{E}^\rho(u, v) = \sum_{j=1}^{\infty} \mathcal{E}^{\rho,e_j}(u, v), \tag{2.7}$$

where

$$\mathcal{E}^{\rho,e_j}(u, v) = \frac{1}{2} \int_{E_{e_j}} \int_{R(\rho(\cdot e_j + x))} \frac{d\tilde{u}_j(se_j + x)}{ds} \times \frac{d\tilde{v}_j(se_j + x)}{ds} \rho(se_j + x)p_j(s)ds\mu_{e_j}(dx), \tag{2.8}$$

and $u, \tilde{u}_j$ satisfy $\tilde{u}_j = u\ \rho\mu - a.e$ and $\tilde{u}_j(se_j + x)$ is absolutely continuous in s on $R(\rho(\cdot\, e_j + x))$ for each $x \in E_{e_j}$. v and $\tilde{v}_j$ are related in the same way.

3 BV functions and distorted OU-processes in F

As in [13], we introduce some function spaces on H. Let

$$A_{1/2}(x) := \int_0^x (\log(1 + s))^{1/2} ds, x \geq 0,$$

and let ψ be its complementary function, namely,

$$\psi(y) := \int_0^y (A'_{1/2})^{-1}(t)dt = \int_0^y (\exp(t^2) - 1)dt.$$

Define

$$L(\log L)^{1/2}(H, \mu) := \{f : H \to \mathbb{R} | f \text{ Borel measurable}, A_{1/2}(|f|) \in L^1(H, \mu)\},$$

$$L^\psi(H, \mu) := \{g : H \to \mathbb{R} | g \text{ Borel measurable}, \psi(c|g|) \in L^1(H, \mu) \text{ for some } c > 0\}.$$

From the general theory of Orlicz spaces (cf. [24]), we have the following properties.

(i) $L(\log L)^{1/2}$ and L^{ψ} are Banach spaces under the norms

$$\|f\|_{L(\log L)^{1/2}} = \inf\{\alpha > 0 | \int_H A_{1/2}(|f|/\alpha) d\mu \leq 1\},$$

$$\|g\|_{L^{\psi}} = \inf\{\alpha > 0 | \int_H \psi(|g|/\alpha) d\mu \leq 1\}.$$

(ii) For $f \in L(\log L)^{1/2}$ and $g \in L^{\psi}$, we have

$$\|fg\|_1 \leq 2\|f\|_{L(\log L)^{1/2}} \|g\|_{L^{\psi}}. \tag{3.1}$$

(iii) Since μ is Gaussian, the function $x \mapsto \langle x, l\rangle$ belongs to L^{ψ}.

Let $c_j, j \in \mathbb{N}$, be a sequence in $[1, \infty)$. Define

$$H_1 := \{x \in H | \sum_{j=1}^{\infty} \langle x, e_j\rangle^2 c_j^2 < \infty\},$$

equipped with the inner product

$$\langle x, y\rangle_{H_1} := \sum_{j=1}^{\infty} c_j^2 \langle x, e_j\rangle \langle y, e_j\rangle.$$

Then clearly $(H_1, \langle , \rangle_{H_1})$ is a Hilbert space such that $H_1 \subset H$ continuously and densely. Identifying H with its dual we obtain the continuous and dense embeddings

$$H_1 \subset H (\equiv H^*) \subset H_1^*.$$

It follows that

$${}_{H_1}\langle z, v\rangle_{H_1^*} = \langle z, v\rangle_H \forall z \in H_1, v \in H,$$

and that (H_1, H, H_1^*) is a Gelfand triple. Furthermore, $\{\frac{e_j}{c_j}\}$ and $\{c_j e_j\}$ are orthonormal bases of H_1 and H_1^*, respectively.

We also introduce a family of H-valued functions on H by

$$(C_b^1)_{D(A)\cap H_1} := \{G : G(z) = \sum_{j=1}^{m} g_j(z) l^j, z \in H, g_j \in C_b^1(H), l^j \in D(A) \cap H_1\}$$

Denote by D^* the adjoint of $D : C_b^1(H) \subset L^2(H, \mu) \to L^2(H, \mu; H)$. That is

$$Dom(D^*) := \{G \in L^2(H, \mu; H) | C_b^1 \ni u \mapsto \int \langle G, Du\rangle d\mu \text{ is continuous with respect to } L^2(H, \mu)\}.$$

Obviously, $(C_b^1)_{D(A)\cap H_1} \subset Dom(D^*)$. Then

$$\int_H D^*G(z) f(z) \mu(dz) = \int_H \langle G(z), Df(z)\rangle \mu(dz) \ \forall G \in (C_b^1)_{D(A)\cap H_1}, f \in C_b^1(H). \tag{3.2}$$

For $\rho \in L(\log L)^{1/2}(H,\mu)$, we set

$$V(\rho) := \sup_{G\in (C_b^1)_{D(A)\cap H_1}, \|G\|_{H_1}\le 1} \int_H D^*G(z)\rho(z)\mu(dz).$$

A function ρ on H is called a BV function in the Gelfand triple $(H_1, H, H_1^*)(\rho \in BV(H,H_1)$ in notation), if $\rho \in L(\log L)^{1/2}(H,\mu)$ and $V(\rho)$ is finite. When $H_1 = H = H_1^*$, this coincides with the definition of BV functions defined in [1] and clearly $BV(H,H) \subset BV(H,H_1)$. We can prove the following theorem by a modification of the proof of [12, Theorem 3.1].

Theorem 3.1 (i) $BV(H,H_1) \subset \bigcap_{l\in D(A)\cap H_1} BV_l(H)$.

(ii) Suppose $\rho \in BV(H,H_1) \cap L^1_+(H,\mu)$, then there exist a positive finite measure $\|d\rho\|$ on H and a Borel-measurable map $\sigma_\rho : H \to H_1^*$ such that $\|\sigma_\rho(z)\|_{H_1^*} = 1$ $\|d\rho\|-a.e$, $\|d\rho\|(H) = V(\rho)$,

$$\int_H D^*G(z)\rho(z)\mu(dz) = \int_H {}_{H_1}\langle G(z), \sigma_\rho(z)\rangle_{H_1^*} \|d\rho\|(dz) \ \forall G \in (C_b^1)_{D(A)\cap H_1} \tag{3.3}$$

and $\|d\rho\| \in S^{\rho+1}$.

Furthermore, if $\rho \in QR(H)$, $\|d\rho\|$ is $\mathcal{E}^\rho$-smooth in the sense that it charges no set of zero $\mathcal{E}_1^\rho$-capacity. In particular, the domain of integration H on both sides of (3.3) can be replaced by F, the topological support of $\rho\mu$.

Also, σ_ρ and $\|d\rho\|$ are uniquely determined, that is, if there are σ'_ρ and $\|d\rho\|'$ satisfying relation (3.3), then $\|d\rho\| = \|d\rho\|'$ and $\sigma_\rho(z) = \sigma'_\rho(z)$ for $\|d\rho\| - a.e.z$

(iii) Conversely, if Eq.(3.3) holds for $\rho \in L(\log L)^{1/2}(H,\mu)$ and for some positive finite measure $\|d\rho\|$ and a map σ_ρ with the stated properties, then $\rho \in BV(H,H_1)$ and $V(\rho) = \|d\rho\|(H)$.

(iv) Let $W^{1,1}(H)$ be the domain of the closure of $(D, C_b^1(H))$ with norm

$$\|f\| := \int_H (|f(z)| + |Df(z)|)\mu(dz).$$

Then $W^{1,1}(H) \subset BV(H,H)$ and Eq.(3.3) is satisfied for each $\rho \in W^{1,1}(H)$. Furthermore,

$$\|d\rho\| = |D\rho|\cdot\mu, V(\rho) = \int_H |D\rho|\mu(dz), \sigma_\rho = \frac{1}{|D\rho|} D\rho I_{\{|D\rho|>0\}}.$$

Proof (i) Let $\rho \in BV(H,H_1)$ and $l \in D(A)\cap H_1$. Take $G \in (C_b^1)_{D(A)\cap H_1}$ of the type

$$G(z) = g(z)l, z \in H, g \in C_b^1(H). \tag{3.4}$$

By (3.2)

$$\begin{aligned}\int_H D^*G(z)f(z)\mu(dz) &= \int_H \langle G(z), Df(z)\rangle\mu(dz)\\ &= -\int_H \langle l, Dg(z)\rangle f(z)\mu(dz) + 2\int_H \langle Al, z\rangle g(z)f(z)\mu(dz) \ \forall f \in C_b^1(H);\end{aligned}$$

consequently,

$$D^*G(z) = -\langle l, Dg(z)\rangle + 2g(z)\langle Al, z\rangle. \tag{3.5}$$

Accordingly,

$$\int_H \langle l, Dg(z)\rangle \rho(z)\mu(dz) = -\int_H D^*G(z)\rho(z)\mu(dz) + 2\int_H \langle Al, z\rangle g(z)\rho(z)\mu(dz). \tag{3.6}$$

For any $g \in C_b^1(H)$, satisfying $\|g\|_\infty \le 1$, by (3.1) the right hand side is dominated by

$$V(\rho)\|l\|_{H_1} + 4\|\rho\|_{L(\log L)^{1/2}}\|\langle Al, \cdot\rangle\|_{L^\psi} < \infty,$$

hence, $\rho \in BV_l(H)$.

(ii) Suppose $\rho \in L_+^1(H,\mu)\bigcap BV(H,H_1)$. By (i) and Theorem 2.3 for each $l \in D(A)\cap H_1$, there exists a finite signed measure ν_l on H for which Eq.(2.4) holds. Define

$$D_l^A\rho(dz) := 2\nu_l(dz) + 2\langle Al, z\rangle\rho(z)\mu(dz).$$

In view of (3.6), for any G of type (3.4), we have

$$\int_H D^*G(z)\rho(z)\mu(dz) = \int_H g(z)D_l^A\rho(dz), \tag{3.7}$$

which in turn implies

$$V(D_l^A\rho)(H) = \sup_{g\in C_b^1(H), \|g\|_\infty\le 1} \int_H g(z)D_l^A\rho(dz) \le V(\rho)\|l\|_{H_1}, \tag{3.8}$$

where $V(D_l^A\rho)$ denotes the total variation measure of the signed measure $D_l^A\rho$.

For the orthonormal basis $\{\frac{e_j}{c_j}\}$ of H_1, we set

$$\gamma_\rho^A := \Sigma_{j=1}^\infty 2^{-j}V(D_{\frac{e_j}{c_j}}^A\rho),\ v_j(z) := \frac{dD_{\frac{e_j}{c_j}}^A\rho(z)}{d\gamma_\rho^A(z)}, z \in H, j \in \mathbb{N}. \tag{3.9}$$

γ_ρ^A is a positive finite measure with $\gamma_\rho^A(H) \le V(\rho)$ and v_j is Borel-measurable. Since $D_{\frac{e_j}{c_j}}^A\rho$ belongs to $S^{\rho+1}$, so does γ_ρ^A . Then for

$$G_n := \sum_{j=1}^n g_j\frac{e_j}{c_j} \in (C_b^1)_{D(A)\cap H_1}, n \in \mathbb{N}, \tag{3.10}$$

by (3.7) the following equation holds

$$\int_H D^*G_n(z)\rho(z)\mu(dz) = \sum_{j=1}^n \int_H g_j(z)v_j(z)\gamma_\rho^A(dz). \tag{3.11}$$

Since $|v_j(z)| \le 2^j$ γ_ρ^A-a.e. and $C_b^1(H)$ is dense in $L^1(H,\gamma_\rho^A)$, we can find $v_{j,m} \in C_b^1(H)$ such that

$$\lim_{m\to\infty} v_{j,m} = v_j\ \gamma_\rho^A - a.e.,$$

Substituting

$$g_{j,m}(z) := \frac{v_{j,m}(z)}{\sqrt{\sum_{k=1}^n v_{k,m}(z)^2 + 1/m}}, \tag{3.12}$$

for $g_j(z)$ in (3.10) and (3.11) we get a bound

$$\sum_{j=1}^n \int_H g_{j,m}(z) v_j(z) \gamma_\rho^A(dz) \le V(\rho),$$

because $\|G_n(z)\|_{H_1}^2 = \sum_{j=1}^n g_{j,m}(z)^2 \le 1 \ \forall z \in H$. By letting $m \to \infty$, we obtain

$$\int_H \sqrt{\sum_{j=1}^n v_j(z)^2} \gamma_\rho^A(dz) \le V(\rho) \ \forall n \in \mathbb{N}.$$

Now we define

$$\|d\rho\| := \sqrt{\sum_{j=1}^\infty v_j(z)^2} \gamma_\rho^A(dz) \tag{3.13}$$

and $\sigma_\rho : H \to H_1^*$ by

$$\sigma_\rho(z) = \begin{cases} \sum_{j=1}^\infty \frac{v_j(z)}{\sqrt{\sum_{k=1}^\infty v_k(z)^2}} \cdot c_j e_j, & \text{if } z \in \{\sum_{k=1}^\infty v_k(z)^2 > 0\} \\ 0 & \text{otherwise.} \end{cases} \tag{3.14}$$

Then

$$\|d\rho\|(H) \le V(\rho), \ \|\sigma_\rho(z)\|_{H_1^*} = 1 \ \|d\rho\| - a.e., \tag{3.15}$$

$\|d\rho\|$ is $S^{\rho+1}$-smooth and σ_ρ is Borel-measurable. By (3.11) we see that the desired equation (3.3) holds for $G = G_n$ as in (3.10). It remains to prove (3.3) for any G of type (3.4), i.e. $G = g \cdot l, g \in C_b^1(H), l \in D(A) \cap H_1$. In view of (3.6), Eq.(3.3) then reads

$$-\int_H \langle l, Dg(z)\rangle \rho(z)\mu(dz) + 2\int_H g(z)\langle Al, z\rangle \rho(z)\mu(dz) = \int_H g(z)_{H_1}\langle l, \sigma_\rho(z)\rangle_{H_1^*} \|d\rho\|(dz). \tag{3.16}$$

We set

$$k_n := \sum_{j=1}^n \langle l, e_j\rangle e_j = \sum_{j=1}^n \langle l, \frac{e_j}{c_j}\rangle_{H_1} \frac{e_j}{c_j}, G_n(z) := g(z) k_n.$$

Thus $k_n \to l$ in H_1 and $Ak_n \to Al$ in H as $n \to \infty$. But then also

$$\lim_{n\to\infty} \int_H \langle Dg, k_n\rangle \rho d\mu = \int_H \langle Dg, l\rangle \rho d\mu,$$

and

$$\begin{aligned} &|\int_H g(z)\langle Ak_n, z\rangle \rho(z)\mu(dz) - \int_H g(z)\langle Al, z\rangle \rho(z)\mu(dz)| \\ &\le 2\|g\|_\infty \|\rho\|_{L(\log L)^{1/2}} \|\langle Ak_n - Al, \cdot\rangle\|_{L^\psi}. \end{aligned}$$

Furthermore,

$$\lim_{n\to\infty}\int_H g(z)_{H_1}\langle k_n,\sigma_\rho(z)\rangle_{H_1^*}\|d\rho\|(dz)=\int_H g(z)_{H_1}\langle l,\sigma_\rho(z)\rangle_{H_1^*}\|d\rho\|(dz).$$

So letting $n\to\infty$ yields (3.16).

If $\rho\in QR(H)$, we can get the claimed result by the same arguments as above.

Uniqueness follows by the same argument as [13, Theorem 3.9].

(iii) Suppose $\rho\in L(\log)^{1/2}(H,\mu)$ and that Eq.(3.3) holds for some positive finite measure $\|d\rho\|$ and some map σ_ρ with the properties stated in (ii). Then clearly

$$V(\rho)\le\|d\rho\|(H)$$

and hence $\rho\in BV(H,H_1)$. To obtain the converse inequality, set

$$\sigma_j(z):=\langle c_je_j,\sigma_\rho(z)\rangle_{H_1^*}={}_{H_1}\langle\frac{e_j}{c_j},\sigma_\rho(z)\rangle_{H_1^*},j\in\mathbb{N}.$$

Fix an arbitrary n. As in the proof of (ii) we can find functions

$$v_{j,m}\in C_b^1(H),\qquad\lim_{m\to\infty}v_{j,m}(z)=\sigma_j(z)\ \|d\rho\|-a.e.$$

Define $g_{j,m}(z)$ by (3.12). Substituting $G_{n,m}(z):=\sum_{j=1}^n g_{j,m}(z)\frac{e_j}{c_j}$ for $G(z)$ in (3.3) then yields

$$\sum_{j=1}^n\int_H g_{j,m}(z)\sigma_j(z)\|d\rho\|(dz)\le V(\rho).$$

By letting $m\to\infty$, we get

$$\int_H\sqrt{\sum_{j=1}^n\sigma_j(z)^2}\|d\rho\|(dz)\le V(\rho)\ \forall n\in\mathbb{N}.$$

We finally let $n\to\infty$ to obtain $\|d\rho\|(H)\le V(\rho)$.

(iv) Obviously the duality relation (3.2) extends to $\rho\in W^{1,1}(H)$ replacing $f\in C_b^1(H)$. By defining $\|d\rho\|$ and $\sigma_\rho(z)$ in the stated way, the extended relation (3.2) is exactly (3.3). □

Theorem 3.2 Let $\rho\in QR(H)\cap BV(H,H_1)$ and consider the measure $\|d\rho\|$ and σ_ρ from Theorem 3.1(ii). Then there is an $\mathcal{E}^\rho$-exceptional set $S\subset F$ such that $\forall z\in F\backslash S$ under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated distorted OU-process M^ρ on F satisfy the following: for $l\in D(A)\cap H_1$

$$\langle l,X_t-X_0\rangle=\int_0^t\langle l,dW_s^z\rangle+\frac12\int_0^t{}_{H_1}\langle l,\sigma_\rho(X_s)\rangle_{H_1^*}dL_s^{\|d\rho\|}-\int_0^t\langle Al,X_s\rangle ds\ \forall t\ge0\ P_z\text{−a.s..}\quad(3.17)$$

Here $L_t^{\|d\rho\|}$ is the real valued PCAF associated with $\|d\rho\|$ by the Revuz correspondence.

In particular, if $\rho\in BV(H,H)$, then $\forall z\in F\backslash S$, $l\in D(A)\cap H$

$$\langle l,X_t-X_0\rangle=\int_0^t\langle l,dW_s^z\rangle+\frac12\int_0^t\langle l,\sigma_\rho(X_s)\rangle dL_s^{\|d\rho\|}-\int_0^t\langle Al,X_s\rangle ds\ \forall t\ge0\ P_z\text{−a.s..}$$

Proof Let $\{e_j\}$ be the orthonormal basis of H introduced above. Define for all $k \in \mathbb{N}$

$$W_k^z(t) := \langle e_k, X_t - z\rangle - \frac{1}{2}\int_0^t {}_{H_1}\langle e_k, \sigma_\rho(X_s)\rangle_{H_1^*} dL_s^{\|d\rho\|} + \int_0^t \langle Ae_k, X_s\rangle ds. \tag{3.18}$$

By (2.1) and (3.16) we get for all $k \in \mathbb{N}$

$$\mathcal{E}^\rho(e_k(\cdot), g) = \int_H g(z)\langle Ae_k, z\rangle\rho(z)\mu(dz) - \frac{1}{2}\int_H g(z)_{H_1}\langle e_k, \sigma_\rho(z)\rangle_{H_1^*}\|d\rho\|(dz)\ \forall g \in C_b^1(H).$$

By Theorem 2.3 it follows that for all $k \in \mathbb{N}$

$$N_t^{e_k} = \frac{1}{2}\int_0^t {}_{H_1}\langle e_k, \sigma_\rho(X_s)\rangle_{H_1^*} dL_s^{\|d\rho\|} - \int_0^t \langle Ae_k, X_s\rangle ds. \tag{3.19}$$

Here we get from (3.18), (3.19) and the uniqueness of decomposition (2.2) that for $\mathcal{E}^\rho$-q.e. $z \in F$,

$$W_k^z(t) = M_t^{e_k}\ \forall t \geq 0\ P_z\text{−a.s.},$$

where the $\mathcal{E}^\rho$-exceptional set and the zero measure set does not depend on e_k. Indeed, we can choose the capacity zero set $S = \cup_{j=1}^\infty S_j$, where S_j is the $\mathcal{E}^\rho$-exceptional set for e_j, and for $z \in F\backslash S$, we can use the same method to get a zero measure set independent of e_k. By Dirichlet form theory we get $\langle M^{e_i}, M^{e_j}\rangle_t = t\delta_{ij}$. So for $z \in F \setminus S$, W_k^z is an $\mathcal{M}_t$-Wiener process under P_z. Thus, with W^z being an $\mathcal{M}_t$- cylindrical Wiener process given by $W^z(t) = (W_k^z(t)e_k)_{k\in\mathbb{N}}$, (3.17) is satisfied for $P_z - a.e.$, where $z \in F \setminus S$. $\square$

4 Reflected OU-processes

In this section we consider the situation where $\rho = I_\Gamma \in BV(H, H_1)$, where $\Gamma \subset H$ and

$$I_\Gamma(x) = \begin{cases} 1, & \text{if } x \in \Gamma, \\ 0 & \text{if } x \in \Gamma^c. \end{cases}$$

Denote the corresponding objects σ_ρ, $\|dI_\Gamma\|$ in Theorem 3.1(ii) by $-\mathbf{n}_\Gamma$, $\|\partial\Gamma\|$ respectively. Then formula (3.3) reads

$$\int_\Gamma D^*G(z)\mu(dz) = -\int_F {}_{H_1}\langle G(z), \mathbf{n}_\Gamma\rangle_{H_1^*}\|\partial\Gamma\|(dz)\ \forall G \in (C_b^1)_{D(A)\cap H_1},$$

where the domain of integration F on the right hand side is the topological support of $I_\Gamma \cdot \mu$. F is contained in $\bar{\Gamma}$, but we shall show that the domain of integration on the right hand side can be restricted to $\partial\Gamma$. We need to use the associated distorted OU-process M^{I_Γ} on F, which will be called reflected OU-process on Γ.

First we consider a μ-measurable set $\Gamma \subset H$ satisfying

$$I_\Gamma \in BV(H, H_1) \cap \mathbf{H}. \tag{4.1}$$

Remark 4.1 We emphasize that if Γ is a convex closed set in H, then obviously $I_\Gamma \in \mathbf{H}$. Indeed, for each $z, l \in H$ the set $\{s \in \mathbb{R}|z + sl \in \Gamma\}$ is a closed interval in $\mathbb{R}$, whose indicator function hence trivially has the Hamza property. Hence, in particular, $I_\Gamma \in QR(H)$.

By a modification of [12, Theorem 4.2], we can prove the following theorem.

Theorem 4.2 Let $\Gamma \subset H$ be μ-measurable satisfying condition (4.1). Then the support of $\|\partial\Gamma\|$ is contained in the boundary $\partial\Gamma$ of Γ, and the following generalized Gauss formula holds:

$$\int_\Gamma D^* G(z)\mu(dz) = -\int_{\partial\Gamma} {}_{H_1}\langle G(z), \mathbf{n}_\Gamma\rangle_{H_1^*} \|\partial\Gamma\|(dz)\ \forall G \in (C_b^1)_{D(A)\cap H_1}. \tag{4.2}$$

Proof For any G of type (3.4) we have from (2.1), (3.5) and (3.7) that

$$\mathcal{E}^{I_\Gamma}(l(\cdot), g) - \int_\Gamma g(z)\langle Al, z\rangle\mu(dz) = -\frac{1}{2}\int_F g(z) D_l^A I_\Gamma(dz). \tag{4.3}$$

Since the finite signed measure $D_l^A I_\Gamma$ charges no set of zero $\mathcal{E}_1^{I_\Gamma}$-capacity, Eq.(4.3) readily extends to any $\mathcal{E}^{I_\Gamma}$-quasicontinuous function $g \in \mathcal{F}_b^{I_\Gamma} := \mathcal{F}^{I_\Gamma} \cap L^\infty(\Gamma, \mu)$.

Denote by Γ^0 the interior of Γ. Then $\Gamma^0 \subset F \subset \bar{\Gamma}$. In view of the construction of the measure $\|dI_\Gamma\|$ in Theorem 3.1, it suffices to show that for $\frac{e_j}{c_j} \in D(A) \cap H_1$

$$V(D^A_{\frac{e_j}{c_j}} I_\Gamma)(\Gamma^0) = 0.$$

By linearity and since positive constants interchange with sup, it suffices to show that,

$$V(D^A_{e_j} I_\Gamma)(\Gamma^0) = 0. \tag{4.4}$$

Take an arbitrary $\varepsilon > 0$ and set

$$U := \{z \in H : d(z, H\backslash\Gamma^0) > \varepsilon\}, V := \{z \in H : d(z, H\backslash\Gamma^0) \geq \varepsilon\},$$

where d is the metric distance of the Hilbert space H. Then $\bar{U} \subset V$ and V is a closed set contained in the open set Γ^0. We define a function h by

$$h(z) := 1 - E_z(e^{-\tau_V}), z \in F, \tag{4.5}$$

where τ_V denotes the first exit time of M^{I_Γ} from the set V. The nonnegative function h is in the space $\mathcal{F}_b^{I_\Gamma}$ and furthermore it is $\mathcal{E}^{I_\Gamma}$-quasicontinuous because it is M^{I_Γ} finely continuous.

Moreover,

$$h(z) > 0\ \forall z \in U, \qquad h(z) = 0\ \forall z \in F\backslash V. \tag{4.6}$$

Set

$$\nu_j(dz) := h(z) D^A_{e_j} I_\Gamma(dz) \tag{4.7}$$

and

$$I_g^j := \mathcal{E}^{I_\Gamma}(e_j(\cdot), gh) - \int_\Gamma g(z)h(z)\langle Ae_j, z\rangle\mu(dz). \tag{4.8}$$

Then Eq.(4.3) with the $\mathcal{E}^{I_\Gamma}$-quasicontinuous function $gh \in \mathcal{F}_b^{I_\Gamma}$ replacing g implies

$$I_g^j = -\frac{1}{2}\int_F g(z)\nu_j(dz).$$

In order to prove (4.4), it is enough to show that $I_g^j = 0$ for any function $g(z)$ of the type

$$g(z) = f(\langle e_j, z\rangle, \langle l_2, z\rangle, ..., \langle l_m, z\rangle); l_2, ..., l_m \in H, f \in C_0^1(R^m), \tag{4.9}$$

for we have then $\nu_j = 0$.

On account of (2.8) we have the expression

$$\mathcal{E}^{I_\Gamma}(e_j(\cdot), gh) = \mathcal{E}^{I_\Gamma, e_j}(e_j(\cdot), gh) = \frac{1}{2}\int_{E_{e_j}}\int_{R_x} \frac{d(g\tilde{h})(se_j + x)}{ds} p_j(s)ds\mu_{e_j}(dx), \tag{4.10}$$

where $R_x = R(I_\Gamma(\cdot e_j + x)), F_x := \{s : se_j + x \in F\}$ for $x \in E_{e_j}$ and $\tilde{h}$ is a $I_\Gamma \cdot \mu$-version of h appearing in the description of (2.8). For $x \in E_{e_j}$ set

$$V_x := \{s : se_j + x \in V\}, \Gamma_x^0 := \{s : se_j + x \in \Gamma^0\}.$$

We then have the inclusion $V_x \subset \Gamma_x^0 \subset R_x \cap F_x$. By (4.6), $h(se_j + x) = 0$ for any $x \in E_{e_j}$ and for any $s \in R_x \setminus V_x$. On the other hand, there exists a Borel set $N \subset E_{e_j}$ with $\mu_{e_j}(N) = 0$ such that for each $x \in E_{e_j} \backslash N$,

$$h(se_j + x) = \tilde{h}(se_j + x) \ ds - a.e.$$

Here we set $h \equiv 0$ on $H \backslash F$. Since $\tilde{h}(\cdot e_j + x)$ is absolutely continuous in s, we can conclude that

$$\tilde{h}(se_j + x) = 0 \ \forall x \in E_{e_j} \backslash N, \ \forall s \in R_x \backslash V_x.$$

Fix $x \in E_{e_j} \backslash N$ and let I be any connected component of the one dimensional open set R_x. Furthermore, for any function g of type (4.9) we denote the support of $g(\cdot e_j + x)$ by K_x (which is a compact set) and choose a bounded open interval J containing K_x. Then $I \cap V_x \cap K_x$ is a closed set contained in the bounded open interval $I \cap J$ and

$$g\tilde{h}(se_j + x) = 0 \ \forall s \in (I \cap J) \backslash (I \cap V_x \cap K_x).$$

Therefore, an integration by part gives

$$\int_{I \cap J} \frac{d(g\tilde{h})(se_j + x)}{ds} p_j(s)ds = \int_{I \cap J} \frac{1}{\lambda_j}(g\tilde{h})(se_j + x)sp_j(s)ds.$$

Combining this with (4.8) and (4.10), we arrive at

$$I_g^j = \int_{E_{e_j}}\int_{R_x} \frac{1}{2\lambda_j}(g\tilde{h})(se_j + x)sp_j(s)ds\mu_{e_j}(dx) - \int_H g(z)h(z)\langle Ae_j, z\rangle I_\Gamma(z)\mu(dz) = 0.$$

□

Now we state Theorem 3.2 for $\rho = I_\Gamma$.

Theorem 4.3 Suppose $\Gamma \subset H$ is a μ-measurable set satisfying condition (4.1). Then there is an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F \backslash S$, under P_z there exists an $\mathcal{M}_t$- cylindrical

Wiener process W^z, such that the sample paths of the associated reflected OU-process M^ρ on F with $\rho = I_\Gamma$ satisfy the following: for $l \in D(A) \cap H_1$

$$\langle l, X_t - X_0 \rangle = \int_0^t \langle l, dW_s^z \rangle - \frac{1}{2} \int_0^t {}_{H_1}\langle l, \mathbf{n}_\Gamma(X_s) \rangle_{H_1^*} dL_s^{\|\partial\Gamma\|} - \int_0^t \langle Al, X_s \rangle ds \ P_z\text{−a.s.}. \quad (4.11)$$

Here, $L_t^{\|\partial\Gamma\|}$ is the real valued PCAF associated with $\|\partial\Gamma\|$ by the Revuz correspondence, which has the following additional property: $\forall z \in F \backslash S$

$$I_{\partial\Gamma}(X_s) dL_s^{\|\partial\Gamma\|} = dL_s^{\|\partial\Gamma\|} \ P_z - a.s.. \quad (4.12)$$

In particular, if $\rho \in BV(H, H)$, then $\forall z \in F \backslash S, l \in D(A) \cap H$

$$\langle l, X_t - X_0 \rangle = \int_0^t \langle l, dW_s^z \rangle - \frac{1}{2} \int_0^t \langle l, \mathbf{n}_\Gamma(X_s) \rangle dL_s^{\|\partial\Gamma\|} - \int_0^t \langle Al, X_s \rangle ds \ \forall t \geq 0 \ P_z - a.s..$$

Proof All assertions except for (4.12) follow from Theorem 3.2 for $\rho := I_\Gamma$. (4.12) follows by Theorem 4.2 and [10, Theorem 5.1.3]. □

5 Stochastic reflection problem on a regular convex set

In this section, we consider Γ satisfying [6] Hypothesis 1.1 (ii) with $K := \Gamma$, that is:

Hypothesis 5.1 There exists a convex C^∞ function $g : H \to \mathbb{R}$ with $g(0) = 0, g'(0) = 0$, and $D^2 g$ strictly positive definite, that is,$\langle D^2 g(x) h, h \rangle \geq \gamma |h|^2 \ \forall h \in H$ for some $\gamma > 0$, such that

$$\Gamma = \{x \in H : g(x) \leq 1\}, \partial\Gamma = \{x \in H : g(x) = 1\}$$

Moreover, we also suppose that $D^2 g$ is bounded on Γ and g and all its derivatives grow at infinity at most polynomially.

Remark 5.2 By [6, Lemma 1.2], Γ is convex and closed and there exists some constant $\delta > 0$ such that $|Dg(x)| \leq \delta \ \forall x \in \Gamma$.

5.1 Reflected OU processes on regular convex sets

Under Hypothesis 5.1, by [7, Lemma A.1] we can prove that $I_\Gamma \in BV(H, H) \cap QR(H)$:

Theorem 5.3 Assume that Hypothesis 5.1 holds. Then $I_\Gamma \in BV(H, H) \cap QR(H)$.

Proof We first note that trivially by Remark 4.1 we have that $I_\Gamma \in QR(H)$. Let

$$\rho_\varepsilon(x) := \exp(-\frac{(g(x) - 1)^2}{\varepsilon} 1_{\{g \geq 1\}}), x \in H.$$

Thus,

$$\lim_{\varepsilon \to 0} \rho_\varepsilon = I_\Gamma.$$

Moreover,

$$D\rho_\varepsilon = -\frac{2}{\varepsilon}\rho_\varepsilon 1_{\{g\geq 1\}} Dg(g-1)\ \mu - a.e..$$

By [7, Lemma A.1] we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_H \varphi(x)1_{\{g(x)\geq 1\}}(g(x)-1)\langle Dg(x), z\rangle\rho_\varepsilon(x)\mu(dx) = \frac{1}{2}\int_{\partial\Gamma}\varphi(y)\langle n(y), z\rangle\mu_{\partial\Gamma}(dy)\ \forall z\in H, \varphi\in C_b^1(H),$$

where $n := Dg/|Dg|$ is the exterior normal to $\partial\Gamma$ at y and $\mu_{\partial\Gamma}$ is the surface measure on $\partial\Gamma$ induced by μ (cf. [6], [7], [16]), whereas by (3.2) for any $\varphi\in C_b^1(H)$ and $z\in D(A)$

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_H \varphi(x)1_{\{g(x)\geq 1\}}(g(x)-1)\langle Dg(x), z\rangle\rho_\varepsilon(x)\mu(dx)\\
=& -\lim_{\varepsilon\to 0}\frac{1}{2}\int_H \langle D\rho_\varepsilon(x), \varphi(x)z\rangle\mu(dx)\\
=& -\frac{1}{2}\lim_{\varepsilon\to 0}\int_H \rho_\varepsilon(x)D^*(\varphi z)(x)\mu(dx)\\
=& -\frac{1}{2}\int_H 1_\Gamma(x)D^*(\varphi z)(x)\mu(dx).
\end{aligned}$$

Thus,

$$\int_H 1_\Gamma(x)D^*(\varphi z)(x)\mu(dx) = -\int_{\partial\Gamma}\varphi(x)\langle n(x), z\rangle\mu_{\partial\Gamma}(dx)\ \forall z\in D(A), \varphi\in C_b^1. \tag{5.1}$$

By the proof of [7, Lemma A.1], we get that g is a non-degenerate map. So we can use the co-area formula (see [16, Theorem 6.3.1, Ch. V] or [7, (A.4)]):

$$\int_H f\mu(dx) = \int_0^\infty [\int_{g=r} f(y)\frac{1}{|Dg(y)|}\mu_{\Sigma_r}(dy)]dr.$$

By [16, Theorem 6.2, Ch. V] the surface measure is defined for all $r\geq 0$, moreover [16, Theorem 1.1, Corollary 6.3.2, Ch. V] imply that $r\mapsto\mu_{\Sigma_r}$ is continuous in the topology induced by $D_r^p(H)$ for some $p\in(1,\infty), r\in(0,\infty)$(cf [16]) on the measures on $(H,\mathcal{B}(H))$. Take $f\equiv 1$ in the co-area formula, then by the continuity property of the surface measure with respect to r we have that $\frac{1}{|Dg(y)|}\mu_{\Sigma_r}(dy)$ is a finite measure supported in $\{g=r\}$. By Remark 5.2 and since $\mu_{\partial\Gamma}=\mu_{\Sigma_1}$, we have that $\mu_{\partial\Gamma}$ is a finite measure. And hence by Theorem 3.1 (iii), we get $I_\Gamma\in BV(H,H)$.

□

Thus by Theorem 4.3 we immediately get the following.

Theorem 5.4 Assume Hypothesis 5.1. Then there exists an $\mathcal{E}^\rho$-exceptional set $S\subset F$ such that $\forall z\in F\backslash S$, under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated reflected OU-process M^ρ on F with $\rho = I_\Gamma$ satisfy the following: for $l\in D(A)\cap H_1$

$$\langle l, X_t - X_0\rangle = \int_0^t\langle l, dW_s^z\rangle - \frac{1}{2}\int_0^t\langle l, \mathbf{n}_\Gamma(X_s)dL_s^{\|\partial\Gamma\|}\rangle - \int_0^t\langle Al, X_s\rangle ds\ \forall t\geq 0\ P_z - a.e.$$

where $\mathbf{n}_\Gamma := \frac{Dg}{|Dg|}$ is the exterior normal to Γ and

$$\|\partial\Gamma\| = \mu_{\partial\Gamma},$$

where $\mu_{\partial\Gamma}$ is the surface measure induced by μ (c.f [6], [7], [16]).

Remark 5.5 It can be shown that for $x \in \partial\Gamma$, $\mathbf{n}_\Gamma(x) = \frac{Dg}{|Dg|}$ is the exterior normal to Γ, i.e the unique element in H of unit length such that

$$\langle \mathbf{n}_\Gamma(x), y - x\rangle \leq 0 \ \forall y \in \Gamma.$$

5.2 Existence and uniqueness of solutions

Let $\Gamma \subset H$ and our linear operator A satisfy Hypothesis 5.1 and Hypothesis 2.1, respectively. Consider the following stochastic differential inclusion in the Hilbert space H,

$$\begin{cases} dX(t) + (AX(t) + N_\Gamma(X(t)))dt \ni dW(t), \\ X(0) = x, \end{cases} \tag{5.2}$$

where $W(t)$ is a cylindrical Wiener process in H on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and $N_\Gamma(x)$ is the normal cone to Γ at x, i.e.

$$N_\Gamma(x) = \{z \in H : \langle z, y - x\rangle \leq 0 \ \forall y \in \Gamma\}.$$

Definition 5.6 A pair of continuous $H \times \mathbb{R}$-valued and $\mathcal{F}_t$-adapted processes $(X(t), L(t)), t \in [0, T]$, is called a solution of (5.2) if the following conditions hold.
(i) $X(t) \in \Gamma$ for all $t \in [0, T]$ $P - a.s.$;
(ii) L is an increasing process with the property that

$$I_{\partial\Gamma}(X_s)dL_s = dL_s \ P - a.s.$$

and for any $l \in D(A)$ we have

$$\langle l, X_t - x\rangle = \int_0^t \langle l, dW_s\rangle - \int_0^t \langle l, \mathbf{n}_\Gamma(X_s)dL_s\rangle - \int_0^t \langle Al, X_s\rangle ds \ \forall t \geq 0 \ P - a.s.$$

where $\mathbf{n}_\Gamma$ is the exterior normal to Γ.

Remark 5.7 By Remark 5.5 we know that $\mathbf{n}_\Gamma(x) \in N_\Gamma(x)$ for all $x \in \Gamma$. Hence by Definition 5.6 (ii) it follows that Definition 5.6 is appropriate to define a solution for the multi-valued equation (5.2).

We denote the semigroup with the infinitesimal generator $-A$ by $S(t)$, $t \geq 0$.

Definition 5.8 A pair of continuous $H \times \mathbb{R}$ valued and $\mathcal{F}_t$-adapted processes $(X(t), L(t)), t \in [0, T]$ is called a mild solution of (5.2) if
(i) $X(t) \in \Gamma$ for all $t \in [0, T]$ $P - a.s.$;
(ii) L is an increasing process with the property

$$I_{\partial\Gamma}(X_s)dL_s = dL_s \ P - a.s.$$

and

$$X_t = S(t)x + \int_0^t S(t-s)dW_s - \int_0^t S(t-s)\mathbf{n}_\Gamma(X_s)dL_s \; \forall t \in [0,T] \; P-a.s.$$

where $\mathbf{n}_\Gamma$ is the exterior normal to Γ. In particular, the appearing integrals have to be well defined.

Lemma 5.9 The process given by

$$\int_0^t S(t-s)\mathbf{n}_\Gamma(X_s)dL_s$$

is P-a.s. continuous and adapted to $\mathcal{F}_t, t \in [0,T]$. This especially implies that it is predictable.
Proof As $|S(t-s)\mathbf{n}_\Gamma(X_s)| \leq M_T|\mathbf{n}_\Gamma(X_s)|, s \in [0,T]$, the integrals $\int_0^t S(t-s)\mathbf{n}_\Gamma(X_s)dL_s, t \in [0,T]$, are well defined. For $0 \leq s \leq t \leq T$,

$$\begin{aligned}
&|\int_0^s S(s-u)\mathbf{n}_\Gamma(X_u)dL_u - \int_0^t S(t-u)\mathbf{n}_\Gamma(X_u)dL_u| \\
\leq &|\int_0^s [S(s-u) - S(t-u)]\mathbf{n}_\Gamma(X_u)dL_u| + |\int_s^t S(t-u)\mathbf{n}_\Gamma(X_u)dL_u| \\
\leq &\int_0^s |[S(s-u) - S(t-u)]\mathbf{n}_\Gamma(X_u)|dL_u + \int_s^t |S(t-u)\mathbf{n}_\Gamma(X_u)|dL_u,
\end{aligned}$$

where the first summand converges to zero as $s \uparrow t$ or $t \downarrow s$, because

$$|1_{[0,s)}(u)[S(s-u) - S(t-u)]\mathbf{n}_\Gamma(X_u)| \to 0 \qquad \text{as } s \uparrow t \text{ or } t \downarrow s.$$

For the second summand we have

$$\int_s^t |S(t-u)\mathbf{n}_\Gamma(X_u)|dL_u \leq M_T(L_t - L_s) \to 0 \qquad \text{as } s \uparrow t \text{ or } t \downarrow s.$$

By the same arguments as in [25, Lemma 5.1.9] we conclude that the integral is adapted to $\mathcal{F}_t, t \in [0,T]$. □

Theorem 5.10 $(X(t), L_t), t \in [0,T]$, is a solution of (5.2) if and only if it is a mild solution.
Proof (⇒) First, we prove that for arbitrary $\zeta \in C^1([0,T], D(A))$ the following equation holds:

$$\langle X_t, \zeta_t\rangle = \langle x, \zeta_0\rangle + \int_0^t \langle \zeta_s, dW_s\rangle - \int_0^t \langle \mathbf{n}_\Gamma(X_s), \zeta_s\rangle dL_s + \int_0^t \langle X_s, -A\zeta_s + \zeta_s'\rangle ds \; \forall t \geq 0 \; P-a.s.. \tag{5.3}$$

If $\zeta_s = \eta f_s$ for $f \in C^1([0,T])$ and $\eta \in D(A)$, by Itô's formula we have the above relation for such ζ. Then by [25, Lemma G.0.10] and the same arguments as the proof of Proposition G.0.11 we obtain the above formula for all $\zeta \in C^1([0,T], D(A))$. As in [25, Proposition G.0.11], for the resolvent $R_n := (n+A)^{-1} : H \to D(A)$ and $t \in [0,T]$ choosing $\zeta_s := S(t-s)nR_n\eta, \eta \in H$, we deduce from (5.3) that

$$\begin{aligned}\langle X_t, nR_n\eta\rangle =&\langle x, S(t)nR_n\eta\rangle + \int_0^t \langle S(t-s)nR_n\eta, dW_s\rangle - \int_0^t \langle \mathbf{n}_\Gamma(X_s), S(t-s)nR_n\eta\rangle dL_s \\ &+ \int_0^t \langle X_s, AS(t-s)nR_n\eta\rangle + \langle X_s, -AS(t-s)nR_n\eta\rangle ds \\ =&\langle S(t)x + \int_0^t S(t-s)dW_s + \int_0^t S(t-s)\mathbf{n}_\Gamma(X_s)dL_s, nR_n\eta\rangle \ \forall t\in[0,T] \ P-a.s..\end{aligned}$$

Letting $n\to\infty$, we conclude that $(X(t), L_t), t\in[0,T]$, is a mild solution.

($\Leftarrow$) By Lemma 5.9 and [25, Theorem 5.1.3], we have

$$\int_0^t S(t-s)\mathbf{n}_\Gamma(X_s)dL_s \qquad \text{and} \qquad \int_0^t \mathrm{S(t-s)dW_s, t\in[0,T]},$$

have predictable versions. And we use the same notation for the predictable versions of the respective processes. As (X_t, L_t) is a mild solution, for all $\eta\in D(A)$ we get

$$\begin{aligned}\int_0^t \langle X_s, A\eta\rangle ds = \int_0^t \langle S(s)x, A\eta\rangle ds - \int_0^t \langle \int_0^s S(s-u)\mathbf{n}_\Gamma(X_u)dL_u, A\eta\rangle ds \\ + \int_0^t \langle \int_0^s S(s-u)dW_u, A\eta\rangle ds \ \forall t\in[0,T] \ P-a.s..\end{aligned}$$

The assertion that $(X(t), L_t), t\in[0,T]$, is a solution of (5.2) now follows as in the proof of [25, Proposition G.0.9] because

$$\begin{aligned}\int_0^t \langle \int_0^s S(s-u)\mathbf{n}_\Gamma(X_u)dL_u, A\eta\rangle ds =& \int_0^t \int_0^s \langle \mathbf{n}_\Gamma(X_u), -\frac{d}{ds}S(s-u)\eta\rangle dL_u ds \\ =& -\langle \int_0^t S(t-s)\mathbf{n}_\Gamma(X_s)dL_s, \eta\rangle + \langle \int_0^t \mathbf{n}_\Gamma(X_s)dL_s, \eta\rangle.\end{aligned}$$

□

Below, we prove (5.2) has a unique solution in the sense of Definition 5.6.

Theorem 5.11 Let $\Gamma\subset H$ satisfy Hypothesis 5.1. Then the stochastic inclusion (5.2) admits at most one solution in the sense of Definition 5.6.

Proof Let (u, L^1) and (v, L^2) be two solutions of (5.2), and let $\{e_k\}_{k\in N}$ be the eigenbasis of A from above. We then have

$$\langle e_k, u(t)-v(t)\rangle + \int_0^t \langle \alpha_k e_k, u(s)-v(s)\rangle ds + \int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle dL_s^1 - \int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle dL_s^2 = 0$$

Setting $\phi_k(t) := \langle e_k, u(t) - v(t)\rangle$, we obtain

$$\begin{aligned}
\phi_k^2(t) =& 2\int_0^t \phi_k(s) d\phi_k(s) \\
=& -2\Big(\int_0^t \langle \alpha_k e_k, u(s) - v(s)\rangle\langle e_k, u(s) - v(s)\rangle ds + \int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle\langle e_k, u(s) - v(s)\rangle dL_s^1 \\
& - \int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle\langle e_k, u(s) - v(s)\rangle dL_s^2\Big) \\
\le& -2\int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle\langle e_k, u(s) - v(s)\rangle dL_s^1 + 2\int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle\langle e_k, u(s) - v(s)\rangle dL_s^2.
\end{aligned} \tag{5.4}$$

By dominated convergence theorem for all $t \ge 0$ we have $P - a.s$:

$$\sum_{k\le N}\int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle\langle e_k, u(s) - v(s)\rangle dL_s^1$$

$$\to \int_0^t \langle \mathbf{n}_\Gamma(u(s)), u(s) - v(s)\rangle dL_s^1 \text{ as } N \to \infty,$$

and

$$\sum_{k\le N}\int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle\langle e_k, u(s) - v(s)\rangle dL_s^2$$

$$\to \int_0^t \langle \mathbf{n}_\Gamma(v(s)), u(s) - v(s)\rangle dL_s^2 \text{ as } N \to \infty.$$

Summing over $k \le N$ in (5.4) and letting $N \to \infty$ yield that for all $t \ge 0$ $P - a.s$

$$|u(t) - v(t)|^2 \le 2\int_0^t \langle \mathbf{n}_\Gamma(u(s)), v(s) - u(s)\rangle dL_s^1 + 2\int_0^t \langle \mathbf{n}_\Gamma(v(s)), u(s) - v(s)\rangle dL_s^2$$

By Remark 5.5 it follows that

$$|u(t) - v(t)|^2 \le 0,$$

which implies

$$u(t) = v(t),$$

and thus

$$L^1(t) = L^2(t).$$

□

Combining Theorem 5.4 and 5.11 with the Yamada-Watanabe Theorem, we now obtain the following:

Theorem 5.12 If Γ satisfies Hypothesis 5.1, then there exists a Borel set $M \subset H$ with $I_\Gamma \cdot \mu(M) = 1$ such that for every $x \in M$, (5.2) has a pathwise unique continuous strong solution in the sense that for every probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with an $\mathcal{F}_t$-Wiener process W, there exists a unique pair of $\mathcal{F}_t$-adapted processes (X, L) satisfying Definition 5.6 and $P(X_0 = x) = 1$. Moreover $X(t) \in M$ for all $t \ge 0$ P-a.s.

Proof By Theorem 5.4 and Theorem 5.11, one sees that [15, Theorem 3.14] a) is satisfied for the solution (X, L). So, the assertion follows from [15, Theorem 3.14] b). $\square$

Remark 5.13 Following the same arguments as in the proof of [26, Theorem 2.1], we can give an alternative proof of Theorem 5.12 for a stronger notion of strong solutions (see e.g. [26]). Also, because of Theorem 5.10, by a modification of [20, Theorem 12.1], we can prove the Yamada Watanabe Theorem for the mild solution in Definition 5.8, and then also a corresponding version of Theorem 5.12 for mild solutions for (5.2). This will be contained in forthcoming work.

5.3 The non-symmetric case

In this section, we extend our results to the non-symmetric case. For $\Gamma \subset H$ satisfying Hypothesis 5.1, we consider the non-symmetric Dirichlet form,

$$\mathcal{E}^\Gamma(u,v) = \int_\Gamma (\frac{1}{2}\langle Du(z), Dv(z)\rangle + \langle B(z), Du(z)\rangle v(z))\mu(dz), u, v \in C_b^1(\Gamma),$$

where B is a map from H to H such that

$$B \in L^\infty(\Gamma \to \Gamma, \mu), \int_\Gamma \langle B, Du\rangle d\mu \geq 0 \text{ for all } u \in C_b^1(\Gamma), u \geq 0. \tag{5.5}$$

Then $(\mathcal{E}, C_b^1(\Gamma))$ is a densely defined bilinear form on $L^2(\Gamma;\mu)$ which is positive definite, since for all $u \in C_b^1(\Gamma)$

$$\mathcal{E}^\Gamma(u,u) = \int_\Gamma \frac{1}{2}(\langle Du(z), Du(z)\rangle + \langle B(z), Du^2(z)\rangle(z))\mu(dz) \geq 0.$$

Furthermore, by the same argument as [17, II.3.e] we have $(\mathcal{E}, C_b^1(\Gamma))$ is closable on $L^2(\Gamma,\mu)$ and its closure $(\mathcal{E}^\Gamma, \mathcal{F}^\Gamma)$ is a Dirichlet form on $L^2(\Gamma,\mu)$. We denote the extended Dirichlet space of $(\mathcal{E}^\Gamma, \mathcal{F}^\Gamma)$ by $\mathcal{F}_e^\Gamma$: Recall that $u \in \mathcal{F}_e^\Gamma$ if and only if $|u| < \infty$ $I_\Gamma \cdot \mu - a.e.$ and there exists a sequence $\{u_n\}$ in $\mathcal{F}^\Gamma$ such that $\mathcal{E}^\Gamma(u_m - u_n, u_m - u_n) \to 0$ as $n \geq m \to \infty$ and $u_n \to u$ $I_\Gamma \cdot \mu - a.e.$ as $n \to \infty$. This Dirichlet form satisfies the weak sector condition

$$|\mathcal{E}_1^\Gamma(u,v)| \leq K\mathcal{E}_1^\Gamma(u,u)^{1/2}\mathcal{E}_1^\Gamma(v,v)^{1/2}.$$

Furthermore, we have:

Theorem 5.14 Suppose $\Gamma \subset H$ satisfies Hypothesis 5.1. Then $(\mathcal{E}^\Gamma, \mathcal{F}^\Gamma)$ is a quasi-regular local Dirichlet form on $L^2(\Gamma;\mu)$.

Proof The assertion follows by [17 IV,4b] and [28]. $\square$

By virtue of Theorem 5.14 and [17], there exists a diffusion process $M^\Gamma = (X_t, P_z)$ on Γ associated with the Dirichlet form $(\mathcal{E}^\Gamma, \mathcal{F}^\Gamma)$. Since constant functions are in $\mathcal{F}^\Gamma$ and $\mathcal{E}^\Gamma(1,1) = 0$, M^Γ is recurrent and conservative. We denote by $\mathbf{A}_+^\Gamma$ the set of all positive continuous additive functionals (PCAF in abbreviation) of M^Γ, and define $\mathbf{A}^\Gamma = \mathbf{A}_+^\Gamma - \mathbf{A}_+^\Gamma$. For $A \in \mathbf{A}^\Gamma$, its total variation process is denoted by $\{A\}$. We also define $\mathbf{A}_0^\Gamma = \{A \in \mathbf{A}^\Gamma | E_{I_\Gamma\cdot\mu}(\{A\}_t) <$

$\infty\ \forall t>0\}$. Each element in $\mathbf{A}_+^\Gamma$ has a corresponding positive $\mathcal{E}^\Gamma$-smooth measure on Γ by the Revuz correspondence. The totality of such measures will be denoted by S_+^Γ. Accordingly, $\mathbf{A}^\Gamma$ corresponds to $S^\Gamma=S_+^\Gamma-S_+^\Gamma$, the set of all $\mathcal{E}^\Gamma$-smooth signed measure in the sense that $A_t=A_t^1-A_t^2$ for $A_t^k\in\mathbf{A}_+^\rho, k=1,2$ whose Revuz measures are $\nu^k, k=1,2$ and $\nu=\nu^1-\nu^2$ is the Hahn-Jordan decomposition of ν. The element of $\mathbf{A}$ corresponding to $\nu\in S$ will be denoted byA^ν.

Note that for each $l\in H$ the function $u(z)=\langle l,z\rangle$ belongs to the extended Dirichlet space $\mathcal{F}_e^\Gamma$ and

$$\mathcal{E}^\Gamma(l(\cdot),v)=\int_\Gamma(\frac{1}{2}\langle l,Dv(z)\rangle+\langle B(z),l\rangle v(z))\mu(dz)\ \forall v\in C_b^1(\Gamma). \tag{5.6}$$

On the other hand, the AF $\langle l,X_t-X_0\rangle$ of M^Γ admits a decomposition into a sum of a martingale AF (M_t) of finite energy and CAF (N_t) of zero energy. More precisely, for every $l\in H$

$$\langle l,X_t-X_0\rangle=M_t^l+N_t^l\ \forall t\geq 0\ P_z-a.s. \tag{5.7}$$

for $\mathcal{E}^\rho$-q.e. $z\in\Gamma$.

Then we have the following:

Theorem 5.15 Suppose $\Gamma\subset H$ satisfies Hypothesis 5.1.

(1) The next three conditions are equivalent:

(i)$N^l\in A_0$.

(ii)$|\mathcal{E}^\Gamma(l(\cdot),v)|\leq C\|v\|_\infty\ \forall v\in C_b^1(\Gamma)$.

(iii) There exists a finite (unique) signed measure ν_l on Γ such that

$$\mathcal{E}^\Gamma(l(\cdot),v)=-\int_\Gamma v(z)\nu_l(dz)\ \forall v\in C_b^1(\Gamma). \tag{5.8}$$

In this case, ν_l is automatically smooth, and

$$N^l=A^{\nu_l}.$$

(2) M^l is a martingale AF with quadratic variation process

$$\langle M^l\rangle_t=t|l|^2, t\geq 0. \tag{5.9}$$

Proof (1) By [21, Theorem 5.2.7] and the same arguments as in [11], we can extend Theorem 6.2 in [11] to our nonsymmetric case to prove the assertions.

(2)Since

$$\mathcal{E}^\Gamma(u,v)=\int_\Gamma(\frac{1}{2}\langle Du(z),Dv(z)\rangle+\langle B(z),Du(z)\rangle v(z))\mu(dz),u,v\in\mathcal{F}^\Gamma,$$

by [21 Theorem 5.1.5] for $u\in C_b^1(\Gamma)$, $f\in\mathcal{F}^\Gamma$ bounded we have

$$\begin{aligned}\int\tilde{f}(x)\mu_{\langle M^{[u]}\rangle}(dx)=&2\mathcal{E}^\Gamma(u,uf)-\mathcal{E}^\Gamma(u^2,f)\\ =&2\int_\Gamma(\frac{1}{2}\langle Du(z),D(u\tilde{f})(z)\rangle+\langle B(z),Du(z)\rangle u(z)\tilde{f}(z))\mu(dz)\\ &-\int_\Gamma(\frac{1}{2}\langle D(u(z)^2),D\tilde{f}(z)\rangle+\langle B(z),D(u^2)(z)\rangle\tilde{f}(z))\mu(dz)\\ =&\int_\Gamma\langle Du(z),Du(z)\rangle\tilde{f}(z)\mu(dz).\end{aligned}$$

Here $\tilde{f}$ denotes the $\mathcal{E}^\Gamma$-quasi-continuous version of f, $\mu_{\langle M^{[u]}\rangle}$ is the Reuvz measure for $\langle M^{[u]}\rangle$ and $M^{[u]}$ is the martingale additive functional in the Fukushima decomposition for $u(X_t)$. Hence we have

$$\mu_{\langle M^{[u]}\rangle}(dz) = I_\Gamma \langle Du(z), Du(z)\rangle \cdot \mu(dz).$$

By [21, (5.1.3)] we also have

$$e(\langle M^l\rangle) = e(M^l) = \int_\Gamma \frac{1}{2}\langle l, l\rangle \mu(dz)$$

where $e(M^l)$ is the energy of M^l. Then (5.9) easily follows. □

By Theorem 3.1 we can now prove the following:

Theorem 5.16 Suppose $\Gamma \subset H$ satisfies Hypothesis 5.1. Then there is an $\mathcal{E}^\Gamma$-exceptional set $S \subset \Gamma$ such that $\forall z \in \Gamma\backslash S$, under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated OU-process M^Γ on Γ satisfy the following: for $l \in D(A)\cap H_1$

$$\langle l, X_t - X_0\rangle = \int_0^t \langle l, dW_s^z\rangle - \frac{1}{2}\int_0^t {}_{H_1}\langle l, \mathbf{n}_\Gamma(X_s)\rangle_{H_1^*} dL_s^{\|\partial\Gamma\|} - \int_0^t \langle Al, X_s\rangle ds - \langle l, \int_0^t B(X_s)\rangle ds \ P_z\text{−a.s.} \tag{5.11}$$

Here, $L_t^{\|\partial\Gamma\|}$ is the real valued PCAF associated with $\|\partial\Gamma\|$ by the Revuz correspondence, which has the following additional property: $\forall z \in \Gamma\backslash S$

$$I_{\partial\Gamma}(X_s) dL_s^{\|\partial\Gamma\|} = dL_s^{\|\partial\Gamma\|} \ P_z - a.s.. \tag{5.12}$$

Here $\mathbf{n}_\Gamma := \frac{Dg}{|Dg|}$ is the exterior normal to Γ, and

$$\|\partial\Gamma\| = \mu_{\partial\Gamma},$$

where $\mu_{\partial\Gamma}$ the surface measure induced by μ.

Proof By (5.6) and (3.16) we have

$$\mathcal{E}^\Gamma(l(\cdot), v) = \int_\Gamma \frac{1}{2}\langle l, Dv(z)\rangle + \langle B(z), l\rangle v(z)\mu(dz)$$

$$= \int_\Gamma \langle B(z), l\rangle v(z)\mu(dz) + \int_\Gamma v(z)\langle Al, z\rangle \mu(dz) + \frac{1}{2}\int_{\partial\Gamma} v(z)\langle l, \mathbf{n}_\Gamma(z)\rangle \|\partial\Gamma\|(dz).$$

Thus, by Theorem 5.15

$$N_t^l = -\langle Al, \int_0^t X_s(\omega)ds\rangle - \langle l, \int_0^t B(X_s(\omega))ds\rangle - \frac{1}{2}\langle l, \int_0^t \mathbf{n}_\Gamma(X_s(\omega)) dL_s^{\|\partial\Gamma\|}(\omega)\rangle.$$

By Theorem 5.15 and the same method as in Theorem 3.2 one then proves the first assertion, and the last assertion follows by Theorem 5.3 and 5.4. □

Let $\Gamma \subset H$ and our linear operator A satisfy Hypothesis 5.1 and Hypothesis 2.1, respectively. As in Section 5.2 we shall now prove the existence and uniqueness of a solution of the following stochastic differential inclusion on the Hilbert space H,

$$\begin{cases} dX(t) + (AX(t) + B(X(t)) + N_\Gamma(X(t)))dt \ni dW(t), \\ X(0) = x, \end{cases} \tag{5.13}$$

where B satisfies condition (5.5), $W(t)$ is a cylindrical Wiener process in H on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and $N_\Gamma(x)$ is the normal cone to Γ at x, i.e.

$$N_\Gamma(x) = \{z \in H : \langle z, y - x\rangle \le 0 \ \forall y \in \Gamma\}.$$

Definition 5.17 A pair of continuous $H \times \mathbb{R}$-valued and $\mathcal{F}_t$-adapted processes $(X(t), L(t)), t \in [0, T]$, is called a solution of (5.13) if the following conditions hold.

(i) $X(t) \in \Gamma$ for all $t \in [0, T]$ P-a.s;

(ii) L is an increasing process with the property that

$$I_{\partial\Gamma}(X_s)dL_s = dL_s \ P - a.s,$$

and for any $l \in D(A)$ we have

$$\langle l, X_t - x\rangle = \int_0^t \langle l, dW_s\rangle - \int_0^t \langle l, \mathbf{n}_\Gamma(X_s)\rangle dL_s - \int_0^t \langle l, B(X_s)\rangle ds - \int_0^t \langle Al, X_s\rangle ds \ \forall t \ge 0 \ P - a.s.,$$

where $\mathbf{n}_\Gamma$ is the exterior normal to Γ.

Below we prove (5.13) has a unique solution in the sense of Definition 5.17.

Theorem 5.18 Let $\Gamma \subset H$ satisfy Hypothesis 5.1 and B satisfy the monotonicity condition

$$\langle B(u) - B(v), u - v\rangle \ge -\alpha|u - v|^2 \tag{5.14}$$

for all $u, v \in dom(G)$, for some $\alpha \in [0, \infty)$ independent of u, v. The stochastic inclusion (5.13) admits at most one solution in the sense of Definition 5.17.

Proof Let (u, L^1) and (v, L^2) be two solutions of (5.13), and let $\{e_k\}_{k \in N}$ be the eigenbasis of A from above. We then have

$$\langle e_k, u(t) - v(t)\rangle + \int_0^t \langle \alpha_k e_k, u(s) - v(s)\rangle ds + \int_0^t \langle e_k, B(u(s)) - B(v(s))\rangle ds$$

$$+ \int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle dL_s^1 - \int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle dL_s^2 = 0.$$

Setting $\phi_k(t) := \langle e_k, u(t) - v(t)\rangle$, and we have

$$\begin{aligned}
\phi_k^2(t) =& 2\int_0^t \phi_k(s) d\phi_k(s) \\
=& -2\Big(\int_0^t \langle \alpha_k e_k, u(s) - v(s)\rangle \langle e_k, u(s) - v(s)\rangle ds + \int_0^t \langle e_k, B(u(s)) - B(v(s))\rangle \langle e_k, u(s) - v(s)\rangle ds \\
&+ \int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle \langle e_k, u(s) - v(s)\rangle dL_s^1 - \int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle \langle e_k, u(s) - v(s)\rangle dL_s^2\Big) \\
\leq& -2\int_0^t \langle e_k, B(u(s)) - B(v(s))\rangle \langle e_k, u(s) - v(s)\rangle ds \\
&-2\int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle \langle e_k, u(s) - v(s)\rangle dL_s^1 + 2\int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle \langle e_k, u(s) - v(s)\rangle dL_s^2.
\end{aligned} \tag{5.15}$$

By the same argument as Theorem 5.11, we have the following $P - a.s$:

$$\sum_{k\leq N} \int_0^t \langle e_k, B(u(s)) - B(v(s))\rangle \langle e_k, u(s) - v(s)\rangle ds$$

$$\to \int_0^t \langle B(u(s)) - B(v(s)), u(s) - v(s)\rangle ds \text{ as } N \to \infty,$$

$$\sum_{k\leq N} \int_0^t \langle e_k, \mathbf{n}_\Gamma(u(s))\rangle \langle e_k, u(s) - v(s)\rangle dL_s^1$$

$$\to \int_0^t \langle \mathbf{n}_\Gamma(u(s)), u(s) - v(s)\rangle dL_s^1 \text{ as } N \to \infty,$$

and

$$\sum_{k\leq N} \int_0^t \langle e_k, \mathbf{n}_\Gamma(v(s))\rangle \langle e_k, u(s) - v(s)\rangle dL_s^2$$

$$\to \int_0^t \langle \mathbf{n}_\Gamma(v(s)), u(s) - v(s)\rangle dL_s^2 \text{ as } N \to \infty.$$

Summing over $k \leq N$ in (5.15) and letting $N \to \infty$ yield that for all $t \geq 0, P - a.s$

$$\begin{aligned}
&|u(t) - v(t)|^2 + 2\int_0^t \langle B(u(s)) - B(v(s)), u(s) - v(s)\rangle ds \\
&\quad \leq 2\int_0^t \langle \mathbf{n}_\Gamma(u(s)), v(s) - u(s)\rangle dL_s^1 + 2\int_0^t \langle \mathbf{n}_\Gamma(v(s)), u(s) - v(s)\rangle dL_s^2.
\end{aligned}$$

By Remark 5.4 it follows that

$$|u(t) - v(t)|^2 + 2\int_0^t \langle B(u(s)) - B(v(s)), u(s) - v(s)\rangle ds \leq 0.$$

By (5.14) and Gronwall's Lemma it follows that

$$u(t) = v(t),$$

and thus

$$L^1(t) = L^2(t).$$

□

Combining Theorem 5.16 and 5.18 with the Yamada-Watanabe Theorem, we obtain the following:

Theorem 5.19 If Γ satisfies Hypothesis 5.1 and B in (5.13) satisfies (5.14), then there exists a Borel set $M \subset H$ with $I_\Gamma \cdot \mu(M) = 1$ such that for every $x \in M$, (5.13) has a pathwise unique continuous strong solution in the sense that for every probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with an $\mathcal{F}_t$-Wiener process W there exists a unique pair of $\mathcal{F}_t$-adapted processes (X, L) satisfying Definition 5.17 and $P(X_0 = x) = 1$. Moreover $X(t) \in M$ for all $t \geq 0$ P-a.s.

Proof The proof is completely analogous to that of Theorem 5.12. □

6 Reflected OU-processeses on a class of convex sets

Below for a topological space X we denote its Borel σ-algebra by $\mathcal{B}(X)$. In this section, we consider the case where $H := L^2(0,1), \rho = I_{K_\alpha}$, where $K_\alpha := \{f \in H | f \geq -\alpha\}, \alpha \geq 0$, and $A = -\frac{1}{2}\frac{d^2}{dr^2}$ with Dirichlet boundary conditions on [0,1]. So in this case $e_j = \sqrt{2}\sin(j\pi r),\ j \in \mathbb{N}$, is the corresponding eigenbases. We recall that (cf [28]) we have $\mu(C_0([0,1])) = 1$. In [28], L.Zambotti proved the following integration by parts formulae in this situation:

$$\int_{K_\alpha} \langle l, D\varphi\rangle d\mu = -\int_{K_\alpha} \varphi(x)\langle x, l''\rangle \mu(dx) - \int_0^1 drl(r)\int \varphi(x)\sigma_\alpha(r,dx),\ \forall l \in D(A), \varphi \in C_b^1(H),$$

where $\sigma_\alpha(r,dx) = \sigma_\alpha(r)\mu_\alpha(r,dx)$, and for $\alpha > 0$, σ_α is a positive bounded function, and for $\alpha = 0$, $\sigma_0(r) = \frac{1}{\sqrt{2\pi r^3(1-r)^3}}$, where $\mu_\alpha(r,dx), \alpha \geq 0$, are probability kernels from $(H, \mathcal{B}(H))$ to $([0,1], \mathcal{B}([0,1]))$.

Remark 6.1 Since each l in $D(A)$ has a second derivative in L^2, its first derivative is bounded, hence l goes faster than linear to zero at any point where l is zero, in particular at the boundary points $r = 0$ and $r = 1$. Hence the second integral in the right hand side of the above equality is well-defined.

We know by (3.5) that for all $l \in D(A)$

$$D^*(\varphi(\cdot)l) = -\langle l, D\varphi\rangle - \varphi\langle l'', \cdot\rangle.$$

Hence

$$\int_{K_\alpha} D^*(\varphi(\cdot)l)d\mu = \int_0^1 l(r)\int \varphi(x)\sigma_\alpha(r,dx)dr\ \forall l \in D(A), \varphi \in C_b^1(H). \tag{6.1}$$

Now take

$$c_j := \begin{cases} (j\pi)^{\frac{1}{2}+\varepsilon}, & \text{if } \alpha > 0 \\ (j\pi)^\beta, & \text{if } \alpha = 0, \end{cases} \tag{6.2}$$

where $\varepsilon \in (0, \frac{3}{2}]$ and $\beta \in (\frac{3}{2}, 2]$ respectively, and define

$$H_1 := \{x \in H | \sum_{j=1}^{\infty} \langle x, e_j \rangle^2 c_j^2 < \infty\},$$

equipped with the inner product

$$\langle x, y \rangle_{H_1} := \sum_{j=1}^{\infty} c_j^2 \langle x, e_j \rangle \langle y, e_j \rangle.$$

We note that $D(A) \subset H_1$ continuously for all $\alpha \geq 0$, since $\varepsilon \leq \frac{3}{2}, \beta \leq 2$. Furthermore, $(H_1, \langle , \rangle_{H_1})$ is a Hilbert space such that $H_1 \subset H$ continuously and densely. Identifying H with its dual we obtain the continuous and dense embeddings

$$H_1 \subset H (\equiv H^*) \subset H_1^*.$$

It follows that

$${}_{H_1}\langle z, v \rangle_{H_1^*} = \langle z, v \rangle_H \forall z \in H_1, v \in H,$$

and that (H_1, H, H_1^*) is a Gelfand triple.

The following is the main result of this section.

Theorem 6.2 $I_{K_\alpha} \in BV(H, H_1) \cap \mathbf{H}$.

Proof First for σ_α as in (6.1) we show that for each $B \in \mathcal{B}(H)$ the function $r \mapsto \sigma_\alpha(r, B)$ is in H_1^* and that the map $B \mapsto \sigma_\alpha(\cdot, B)$ is in fact an H_1^*-valued measure of bounded variation, i.e

$$\sup\{\sum_{n=1}^{\infty} \|\sigma_\alpha(\cdot, B_n)\|_{H_1^*} : B_n \in \mathcal{B}(H), n \in \mathbb{N}, H = \dot{\cup}_{n=1}^{\infty} B_n\} < \infty,$$

that is,

$$\sup\{\sum_{n=1}^{\infty} (\sum_{j=1}^{\infty} c_j^{-2} (\int_0^1 \sigma_\alpha(r, B_n) \sin(j\pi r) dr)^2)^{1/2} : B_n \in \mathcal{B}(H), n \in \mathbb{N}, H = \dot{\cup}_{n=1}^{\infty} B_n\} < \infty,$$

where $\dot{\cup}_{n=1}^{\infty} B_n$ means disjoint union.

For $\alpha > 0$ we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} (\sum_{j=1}^{\infty} c_j^{-2} (\int_0^1 \sigma_\alpha(r, B_n) \sin(j\pi r) dr)^2)^{1/2} \\
\leq &\sum_{n=1}^{\infty} (\sum_{j=1}^{\infty} c_j^{-2} (\int_0^1 \sigma_\alpha(r, B_n) dr)^2)^{1/2} \\
\leq &C \sum_{n=1}^{\infty} \int_0^1 \sigma_\alpha(r, B_n) dr \\
= &C \int_0^1 \sigma_\alpha(r) dr < \infty.
\end{aligned}$$

For $\alpha = 0$ using that $|\sin(j\pi r)| \leq 2j\pi r(1-r)\ \forall r \in [0,1]$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}(\sum_{j=1}^{\infty} c_j^{-2}(\int_0^1 \sigma_0(r,B_n)\sin(j\pi r)dr)^2)^{1/2}\\
\leq&\sum_{n=1}^{\infty}(\sum_{j=1}^{\infty} c_j^{-2}(\int_0^1 \sigma_0(r,B_n)2j\pi r(1-r)dr)^2)^{1/2}\\
\leq&C\sum_{n=1}^{\infty}\int_0^1 \sigma_0(r,B_n)r(1-r)dr\\
=&C\int_0^1 \sigma_0(r)r(1-r)dr < \infty
\end{aligned}$$

Thus σ_α in (6.1) is of bounded variation as an H_1^*-valued measure. Hence by the theory of vector-valued measures (cf [2, Section 2.1]), there is a unit vector field $n_\alpha : H \to H_1^*$, such that $\sigma_\alpha = n_\alpha \|\sigma_\alpha\|$, where $\|\sigma_\alpha\|(B) := \sup\{\sum_{n=1}^{\infty} \|\sigma_\alpha(\cdot, B_n)\|_{H_1^*} : B_n \in \mathcal{B}(H), n \in \mathbb{N}, B = \dot{\cup}_{n=1}^{\infty} B_n\}$ is a nonnegative measure, which is finite by the above proof. So (6.1) becomes

$$\int_{K_\alpha} D^*(\varphi(\cdot)l)d\mu = \int {}_{H_1}\langle \varphi(x)l, n_\alpha(x)\rangle_{H_1^*} \|\sigma_\alpha\|(dx)\ \forall l \in D(A), \varphi \in C_b^1(H),$$

which by linearity extends to all $G \in (C_b^1)_{D(A)\cap H_1}$. Thus by Theorem 3.1(iii), we get that $I_{K_\alpha} \in BV(H, H_1)$.

$I_{K_\alpha} \in QR(H)$ follows by Remark 4.1. $\square$

Remark 6.3 It has been proved by Guan Qingyang that I_{K_α} is not in $BV(H,H)$.

Thus we have Theorem 3.2 in this situation. More precisely:

Theorem 6.4 Let $\rho := I_{K_\alpha}$ and consider the measure $|\sigma_\alpha|$ and n_α appearing in Theorem 6.1. Then there is an $\mathcal{E}^\rho$-exceptional set $S \subset F$ such that $\forall z \in F\backslash S$, under P_z there exists an $\mathcal{M}_t$- cylindrical Wiener process W^z, such that the sample paths of the associated distorted OU-process M^ρ on F satisfy the following: for $l \in D(A) \cap H_1$

$$\langle l, X_t - X_0\rangle = \int_0^t \langle l, dW_s\rangle + \frac{1}{2}\int_0^t {}_{H_1}\langle l, n_\alpha(X_s)\rangle_{H_1^*} dL_s^{|\sigma_\alpha|} - \int_0^t \langle Al, X_s\rangle ds\ P_z - a.e.$$

Here $L_t^{|\sigma_\alpha|}$ is the real valued PCAF associated with $|\sigma_\alpha|$ by the Revuz correspondence, satisfying

$$I_{\{X_s+\alpha\neq 0\}} dL_s^{|\sigma_\alpha|} = 0, \tag{6.3}$$

and for $l \in H_1$ with $l(r) \geq 0$ we have

$$\int_0^t {}_{H_1}\langle l, n_\alpha(X_s)\rangle_{H_1^*} dL_s^{|\sigma_\alpha|} \geq 0. \tag{6.4}$$

Furthermore, for all $z \in F$

$$P_z[X_t \in C_0[0,1] \text{ for a.e. } t \in [0,\infty)] = 1. \tag{6.5}$$

Proof The first part of the assertion follows by Theorem 3.2 and the uniqueness part of Theorem 3.1 (ii). (6.3) and (6.4) follow by the property of σ_α in [28]. By [22, p.135 Theorem 2.4], we have $C_0[0,1]$ is a Borel subset of $L^2[0,1]$. By [10, (5.1.13)], we have

$$E_{\rho\mu}[\int_{k-1}^{k} 1_{F\setminus C_0[0,1]}(X_s)ds] = \rho\mu(F \setminus C_0[0,1]) = 0 \ \forall k \in \mathbb{N},$$

hence

$$E_{\rho\mu}[\int_0^\infty 1_{F\setminus C_0[0,1]}(X_s)ds] = 0.$$

Since $E_x[\int_0^\infty 1_{F\setminus C_0[0,1]}(X_s)ds]$ is a 0-excessive function in $x \in K_\alpha$, it is finely continuous with respect to the process X. Then for $\mathcal{E}^\rho -$ q.e. $z \in F$,

$$E_z[\int_0^\infty 1_{F\setminus C_0[0,1]}(X_s)ds] = 0,$$

thus, for $\mathcal{E}^\rho -$ q.e. $z \in F$,

$$P_z[\int_0^\infty 1_{F\setminus C_0[0,1]}(X_s)ds = 0] = 1.$$

As a consequence, we have that $\Lambda_0 := \{X_t \in C_0[0,1] \text{ for a.e. } t \in [0,\infty)\}$ is measurable and for $\mathcal{E}^\rho -$ q.e. $z \in F$

$$P_z(\Lambda_0) = 1.$$

As $\Lambda_0 = \cap_{t\in\mathbb{Q},t>0}\theta_t^{-1}\Lambda_0$ and since by [4] we have that the semigroup associated with X_t is strong Feller, by the Markov property as in [8, Lemma 7.1], we obtain that for any $z \in F, t \in \mathbb{Q}, t > 0$,

$$P_z(\theta_t^{-1}\Lambda_0) = 1.$$

Hence for any $z \in F$ we have

$$P_z[X_t \in C_0[0,1] \text{ for a.e. } t \in [0,\infty)] = 1.$$

□

Remark 6.5 From the above theorem, it follows that the solution in [19, Theorem 1.3] is the strong solution to an infinite-dimensional Skorohod problem.

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