Random attractors for a class of stochastic partial differential equations driven by general additive noise *

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Abstract

The existence of random attractors for a large class of stochastic partial differential equations (SPDE) driven by general additive noise is established. The main results are applied to various types of SPDE, as e.g. stochastic reaction-diffusion equations, the stochastic p-Laplace equation and stochastic porous media equations. Besides classical Brownian motion, we also include space-time fractional Brownian Motion and space-time Lévy noise as admissible random perturbations. Moreover, cases where the attractor consists of a single point are considered and bounds for the speed of attraction are obtained.

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1 Introduction

Since the foundational work in [16, 17, 44] the long time behaviour of several examples of SPDE perturbed by additive noise has been extensively investigated by means of proving the existence of a global random attractor (cf. e.g. [8, 10, 11, 12, 19, 20, 31, 46, 47]). However, these results address only some specific examples of SPDE of semilinear type. To the best of our knowledge the only result concerning a non-semilinear SPDE, namely stochastic generalized porous media equations is given in [9]. In this work we provide a

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general result yielding the existence of a (unique) random attractor for a large class of SPDE perturbed by general additive noise. In particular, the result is applicable also to quasilinear equations like stochastic porous media equations and the stochastic p-Laplace equation. The existence of the random attractor for the stochastic porous medium equation (SPME) as obtained in [9] is contained as a special case (at least if the noise is regular enough, cf. Remark 3.4). We also would like to point out that we include the well-studied case of stochastic reaction-diffusion equations, even in the case of high order growth of the nonlinearity by reducing it to the deterministic case and then applying our general results (cf. Remark 3.2 for details and comparison with previous results). Apart from allowing a large class of admissible drifts, we also formulate our results for general additive perturbations, thus containing the case of Brownian motion and fractional Brownian motion (cf. [21, 37]). We emphasize, however, that the continuity of the noise in time is not necessary. Our techniques are designed so that they also apply to cádlág noise. In particular, Lévy-type noises are included (cf. Section 3). Under a further condition on the drift, we prove that the random attractor consists of a single point, i.e. the existence of a random fixed point. Hence the existence of a unique stationary solution is also obtained.

Our results are based on the variational approach to (S)PDE. The variational approach has been used intensively in recent years to analyze SPDE driven by an infinite dimensional Wiener process. For general results on the existence and uniqueness of variational solutions to SPDE we refer to [22, 27, 36, 38, 41, 49]. As a typical example of an SPDE in this framework stochastic porous media equations have been intensively investigated in [4, 5, 6, 7, 18, 26, 33, 35, 43].

Let us now describe our framework, conditions and main results. Let

$$V \subseteq H \equiv H^* \subseteq V^*$$

be a Gelfand triple, *i.e.* $(H, \langle \cdot, \cdot \rangle_H)$ is a separable Hilbert space and is identified with its dual space H^* by the Riesz isomorphism $i : H \to H^*$, V is a reflexive Banach space such that it is continuously and densely embedded into H. ${}_{V^*}\langle \cdot, \cdot \rangle_V$ denotes the dualization between V and its dual space V^*. Let $A : V \to V^*$ be measurable, $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space and $(N_t)_{t\in\mathbb{R}}$ be a V-valued adapted stochastic process. For $[s,t] \subseteq \mathbb{R}$ we consider the following stochastic evolution equation

$$\begin{aligned} \mathrm{d}X_r &= A(X_r)\mathrm{d}r + dN_r,\ r \in [s,t], \\ X_s &= x \in H. \end{aligned} \tag{1.1}$$

If A satisfies the standard monotonicity and coercivity conditions (cf. $(H1) - (H4)$ below) we shall prove the existence and uniqueness of solutions to (1.1) in the sense of Definition 1.1.

Suppose that there exists $\alpha > 1$ and constants $\delta > 0$, $K, C \in \mathbb{R}$ such that the following conditions hold for all $v, v_1, v_2 \in V$ and $\omega \in \Omega$:

$(H1)$ (Hemicontinuity) The map $s \mapsto {}_{V^*}\langle A(v_1 + sv_2), v\rangle_V$ is continuous on $\mathbb{R}$.

$(H2)$ (Monotonicity)

$$2\,{}_{V^*}\langle A(v_1) - A(v_2), v_1 - v_2\rangle_V \leq C\|v_1 - v_2\|_H^2.$$

($H3$) (Coercivity)

$$2_{V^*}\langle A(v), v\rangle_V + \delta\|v\|_V^\alpha \leq C + K\|v\|_H^2.$$

($H4$) (Growth)

$$\|A(v)\|_{V^*} \leq C(1 + \|v\|_V^{\alpha-1}).$$

We can now define the notion of a solution to (1.1).

Definition 1.1. An H-valued, $(\mathcal{F}_t)$-adapted process $\{X_r\}_{r\in[s,t]}$ is called a solution of (1.1) if $X_\cdot(\omega) \in L^\alpha([s,t];V) \cap L^2([s,t];H)$ and

$$X_r(\omega) = x + \int_s^r A(X_u(\omega))\mathrm{d}u + N_r(\omega) - N_s(\omega)$$

holds for all $r \in [s,t]$ and all $\omega \in \Omega$.

Since the solution to (1.1) will be constructed via a transformation of (1.1) into a deterministic equation (parametrized by ω) we can allow very general additive stochastic perturbations. In particular, we do not have to assume the noise to be a martingale or a Markov process.

Since the noise is not required to be Markovian, the solutions to the SPDE cannot be expected to define a Markov process. Therefore, the approach to study long-time behaviour of solutions to SPDE via invariant measures and ergodicity of the associated semigroup is not an option here. In particular, the results from [28] cannot be applied to prove that the attractor consists of a single point. Consequently, our analysis is instead based on the framework of random dynamical systems (RDS), which more or less requires the driving process to have stationary increments (cf. Lemma 3.1).

Let $((\Omega,\mathcal{F},\mathbb{P}),(\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system, i.e. $(t,\omega)\mapsto\theta_t(\omega)$ is $\mathcal{B}(\mathbb{R})\otimes\mathcal{F}/\mathcal{F}$ measurable, $\theta_0 = \mathrm{id}$, $\theta_{t+s} = \theta_t\circ\theta_s$ and θ_t is $\mathbb{P}$-preserving, for all $s,t\in\mathbb{R}$.

($S1$) (Strictly stationary increments) For all $t,s\in\mathbb{R}$, $\omega\in\Omega$:

$$N_t(\omega) - N_s(\omega) = N_{t-s}(\theta_s\omega) - N_0(\theta_s\omega).$$

($S2$) (Regularity) For each $\omega\in\Omega$,

$$N_\cdot(\omega) \in L^\alpha_{loc}(\mathbb{R};V)\cap L^2_{loc}(\mathbb{R};H)$$

(with the same $\alpha > 1$ as in ($H3$)).

($S3$) (Joint measurability) $N:\mathbb{R}\times\Omega\to V$ is $\mathcal{B}(\mathbb{R})\otimes\mathcal{F}/\mathcal{B}(V)$ measurable.

Remark 1.1. Although we do not explicitly assume N_t to have cádlág paths, in the applications the underlying metric dynamical system $((\Omega,\mathcal{F},\mathbb{P}),(\theta_t)_{t\in\mathbb{R}})$ is usually defined as the space of all cádlág functions endowed with a topology making the Wiener shift $\theta:\mathbb{R}\times\Omega\to\Omega;\ \theta_t(\omega) = \omega(\cdot+t)-\omega(t)$ measurable and the probability measure $\mathbb{P}$ is given by the distribution of the noise N_t. Thus, in the applications we will always require N_t to have cádlág paths.

We now recall the notion of a random dynamical system. For more details concerning the theory of random dynamical systems we refer to [16, 17].

Definition 1.2. Let (H, d) be a complete and separable metric space.

(i) A random dynamical system (RDS) over θ_t is a measurable map

$$\varphi : \mathbb{R}_+ \times H \times \Omega \to H;\ (t, x, \omega) \mapsto \varphi(t, \omega)x$$

such that $\varphi(0, \omega) = \mathrm{id}$ and

$$\varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega),$$

for all $t, s \in \mathbb{R}_+$ and $\omega \in \Omega$. φ is said to be a continuous RDS if $x \mapsto \varphi(t, \omega)x$ is continuous for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$.

(ii) A stochastic flow is a family of mappings $S(t, s; \omega) : H \to H$, $-\infty < s \leq t < \infty$, parametrized by ω such that

$$(t, s, x, \omega) \mapsto S(t, s; \omega)x$$

is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(H) \otimes \mathcal{F}/\mathcal{B}(H)$-measurable and

$$\begin{aligned} S(t, r; \omega)S(r, s; \omega)x &= S(t, s; \omega)x, \\ S(t, s; \omega)x &= S(t - s, 0; \theta_s\omega)x, \end{aligned}$$

for all $s \leq r \leq t$ and all $\omega \in \Omega$. S is said to be a continuous stochastic flow if $x \mapsto S(t, s; \omega)x$ is continuous for all $s \leq t$ and $\omega \in \Omega$.

In order to apply the theory of RDS and in particular to apply Proposition 1.2 below, we first need to define the RDS associated with (1.1). For this we consider the unique ω-wise solution (denoted by $Z(\cdot, s; \omega)x$) of

$$Z_t = x - N_s(\omega) + \int_s^t A(Z_r + N_r(\omega))\mathrm{d}r,\ t \geq s, \tag{1.2}$$

and then define

$$S(t, s; \omega)x := Z(t, s; \omega)x + N_t(\omega), \tag{1.3}$$

$$\varphi(t, \omega)x := S(t, 0; \omega)x = Z(t, 0; \omega)x + N_t(\omega). \tag{1.4}$$

Note that $S(\cdot, s; \omega)$ satisfies

$$S(t, s; \omega)x = x + \int_s^t A(S(r, s; \omega)x)\mathrm{d}r + N_t(\omega) - N_s(\omega),$$

for each fixed $\omega \in \Omega$ and all $t \geq s$. Hence $S(t, s; \omega)x$ solves (1.1) in the sense of Definition 1.1.

Theorem 1.1. *Under the assumptions* $(H1)$-$(H4)$ *and* $(S1)$-$(S3)$, $S(t,s;\omega)$ *defined in* (1.3) *is a continuous stochastic flow and* φ *defined in* (1.4) *is a continuous random dynamical system.*

For the proof of Theorem 1.1 as well as the other theorems in this section we refer to the next section.

With the notion of an RDS above we can now recall the stochastic generalization of notions of absorption, attraction and Ω-limit sets (cf. [16, 17]).

Definition 1.3. (i) A set-valued map $K : \Omega \to 2^H$ is measurable if for all $x \in H$ the map $\omega \mapsto d(x, K(\omega))$ is measurable, where for nonempty sets $A, B \in 2^H$ we set

$$d(A,B) = \sup_{x\in A} \inf_{y\in B} d(x,y)$$

and $d(x,B) = d(\{x\},B)$. A measurable set-valued map is also called a random set.

(ii) Let A, B be random sets. A is said to absorb B if $\mathbb{P}$-a.s. there exists an absorption time $t_B(\omega)$ such that for all $t \geq t_B(\omega)$

$$\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subseteq A(\omega).$$

A is said to attract B if

$$d(\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega), A(\omega)) \xrightarrow[t\to\infty]{} 0, \ \mathbb{P}\text{-a.s. .}$$

(iii) For a random set A we define the Ω-limit set to be

$$\Omega_A(\omega) = \Omega(A,\omega) = \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \varphi(t,\theta_{-t}\omega)A(\theta_{-t}\omega)}.$$

Definition 1.4. A random attractor for an RDS φ is a compact random set A satisfying $\mathbb{P}$-a.s.:

(i) A is invariant, i.e. $\varphi(t,\omega)A(\omega) = A(\theta_t\omega)$ for all $t > 0$.

(ii) A attracts all deterministic bounded sets $B \subseteq H$.

Note that by [14] the random attractor for an RDS is uniquely determined.

The following proposition yields a sufficient criterion for the existence of a random attractor of an RDS.

Proposition 1.2. *(cf. [17, Theorem 3.11]) Let* φ *be an RDS and assume the existence of a compact random set* K *absorbing every deterministic bounded set* $B \subseteq H$. *Then there exists a random attractor* A, *given by*

$$A(\omega) = \overline{\bigcup_{B\subseteq H,\ B \text{ bounded}} \Omega_B(\omega)}.$$

We aim to apply Proposition 1.2 to prove the existence of a random attractor for the RDS associated with (1.1). Thus, we need to prove the existence of a compact globally absorbing random set K. To show the existence of such a set for (1.1), we require some additional assumptions to derive an a priori estimate of the solution in a norm $\|\cdot\|_S$, which is stronger than the norm $\|\cdot\|_H$.

$(H5)$ Suppose there is a subspace $(S, \|\cdot\|_S)$ of H such that the embedding $V \subseteq S$ is continuous and $S \subseteq H$ is compact. Let T_n be positive definite self-adjoint operators on H such that

$$\langle x, y\rangle_n := \langle x, T_n y\rangle_H,\ x, y \in H, n \geq 1,$$

define a sequence of new inner products on H. Suppose that the induced norms $\|\cdot\|_n$ are all equivalent to $\|\cdot\|_H$ and for all $x \in S$ we have

$$\|x\|_n \uparrow \|x\|_S \text{ as } n \to \infty.$$

Moreover, we assume that $T_n : V \to V,\ n \geq 1$, are continuous and that there exists a constant $C > 0$ such that

$$2_{V^*}\langle A(v), T_n v\rangle_V \leq C(\|v\|_n^2 + 1),\ v \in V, \tag{1.5}$$

and

$$\sup_{n\in\mathbb{N}} \int_{-1}^{0} \|T_n N_t\|_V^{\alpha} dt \leq C. \tag{1.6}$$

Remark 1.2. (1) Assumption $(H5)$ looks quite abstract at first glance. But it is applicable to a large class of SPDE within the variational framework, as e.g. stochastic reaction diffusion equations, stochastic porous media equations and the stochastic p-Laplace equation (see Section 3 for more examples).

(2) Under assumption (1.5) the following regularity property of solutions to general SPDE driven by a Wiener process was established in [34]:

$$\mathbb{E} \sup_{s\in[0,t]} \|X_s\|_S^2 < \infty, \text{ for all } t > 0.$$

In order to prove the existence of a random attractor, we need to assume some growth condition on the paths of the noise.

$(S4)$ (Subexponential growth) For $\mathbb{P}$-a.a. $\omega \in \Omega$ and $|t| \to \infty$, $N_t(\omega)$ is of subexponential growth, i.e. $\|N_t(\omega)\|_V = o(e^{\lambda|t|})$ for every $\lambda > 0$.

Theorem 1.3. *Suppose* $(H1)$-$(H5)$ *hold for* $\alpha = 2, K = 0$ *or for* $\alpha > 2$, *and that* $(S1)$-$(S4)$ *are satisfied. Then the RDS* φ *associated with SPDE* (1.1) *has a compact random attractor.*

Remark 1.3. $(H1)$-$(H4)$ are the classical monotonicity and coercivity conditions for the existence and uniqueness of solutions to (1.1). It can be replaced by some much weaker assumptions (e.g. local monotonicity) according to a recent result in [36]. The existence of random attractors for SPDE with locally monotone coefficients in [36] will be the subject for future investigation.

In order to make the proof easier to follow, we first give a quick outline. By Proposition 1.2 we only need to prove the existence of a compact globally absorbing random set K. This set will be chosen as

$$K(\omega) := \overline{B_S(0, r(\omega))}^H,$$

where $B_S(0,r)$ denotes the ball with center 0 and radius r (depending on ω) in S. Since $S \subseteq H$ is a compact embedding, K is a compact random set in H. Note that

$$\varphi(t, \theta_{-t}\omega) = S(t, 0; \theta_{-t}\omega) = S(0, -t; \omega).$$

Hence we need pathwise bounds on $S_0(= S(0, -t; \omega))$ in the S-norm. In order to get such estimates we consider the norms $\|\cdot\|_n$ on H for which we can apply Itô's formula.

Under the following stronger monotonicity condition we prove that the random attractor consists of a single point:

$(H2')$ There exist constants $\beta \geq 2$ and $\lambda > 0$ such that

$$2_{V^*}\langle A(v_1) - A(v_2), v_1 - v_2\rangle_V \leq -\lambda \|v_1 - v_2\|_H^\beta, \ \forall v_1, v_2 \in V.$$

Theorem 1.4. *Suppose that $(H1)$,$(H2')$,$(H3)$,$(H4)$ and $(S1)$-$(S3)$ hold. If $\beta = 2$ also suppose $(S4)$ holds. Then the RDS φ associated with SPDE (1.1) has a compact random attractor $\mathcal{A}(\omega)$ consisting of a single point:*

$$\mathcal{A}(\omega) = \{\eta_0(\omega)\}.$$

In particular, there is a unique random fixed point $\eta_0(\omega)$ and a unique invariant random measure $\mu_\cdot \in \mathcal{P}_\Omega(H)$ which is given by

$$\mu_\omega = \delta_{\eta_0(\omega)}, \quad \mathbb{P}\text{-}a.s. \ .$$

Moreover,

(i) if $\beta > 2$, then the speed of convergence is polynomial, more precisely,

$$\|S(t, s; \omega)x - \eta_0(\theta_t\omega)\|_H^2 \leq \left\{\frac{\lambda}{2}(\beta - 2)(t - s)\right\}^{-\frac{2}{\beta-2}}, \ \forall x \in H.$$

(ii) if $\beta = 2$, then the speed of convergence is exponential. More precisely, for every $\eta \in (0, \lambda)$ there is a random variable K_η such that

$$\|S(t, s; \omega)x - \eta_0(\theta_t\omega)\|_H^2 \leq 2\left(K_\eta(\omega) + \|x\|_H^2\right) e^{(\lambda-\eta)s} e^{-\lambda t}, \ \forall x \in H.$$

Remark 1.4. (1) In case $\beta > 2$ we recover the optimal rate of convergence found in the deterministic case in [3] for the porous media equation.

(2) Note that $(H5)$ and for $\beta > 2$ the growth condition for the noise $(S4)$ are not required in Theorem 1.4.

The paper is organized as follows. The proofs of main theorems are given in the next section. In Section 3 we apply the main results to various examples of SPDE such as stochastic reaction-diffusion equations, the stochastic p-Laplace equation and stochastic porous medium equations. As the examples of admissible random perturbation (noise), we also show that assumptions $(S1)-(S4)$ hold not only for Brownian motion, but also for fractional Brownian motion and Lévy processes.

2 Proofs of main theorems

2.1 Proof of Theorem 1.1

We need to show that the solution to (1.1) generates a random dynamical system. In order to verify the cocycle property, we use the standard transformation to rewrite the SPDE (1.1) as a PDE with a random parameter. This is the reason why we need to restrict N_t to take values in V instead of H. For simplicity, in the proof the generic constant C may change from line to line.

Proof. Consider the PDE (1.2) with random parameter $\omega \in \Omega$ and let

$$\tilde{A}_\omega(t, v) := A(v + N_t(\omega)),$$

which is a well defined operator from V to V^* since $N_t(\omega) \in V$. To obtain the existence and uniqueness of solutions to (1.2) we check the assumptions of [40, Theorem 4.2.4]. Since $N_\cdot(\omega)$ is measurable, $A_\omega(t, v)$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(V)$ measurable. It is obvious that hemicontinuity and (weak) monotonicity hold for $\tilde{A}_\omega$. For the coercivity, using $(H3)$, $(H4)$ and Young's inequality we have

$$\begin{aligned}
&{}_{V^*}\langle \tilde{A}_\omega(t, v), v\rangle_V = {}_{V^*}\langle A(v + N_t(\omega)), v + N_t(\omega) - N_t(\omega)\rangle_V \\
\leq& -\delta\|v + N_t(\omega)\|_V^\alpha + K\|v + N_t(\omega)\|_H^2 + C - {}_{V^*}\langle A(v + N_t(\omega)), N_t(\omega)\rangle_V \\
\leq& -\delta\|v + N_t(\omega)\|_V^\alpha + K\|v + N_t(\omega)\|_H^2 + C + C\left(1 + \|v + N_t(\omega)\|_V^{\alpha-1}\right)\|N_t(\omega)\|_V \\
\leq& -\frac{\delta}{2}\|v + N_t(\omega)\|_V^\alpha + K\|v + N_t(\omega)\|_H^2 + C\left(1 + \|N_t(\omega)\|_V^\alpha\right) \\
\leq& -2^{-\alpha}\delta\|v\|_V^\alpha + 2K\|v\|_H^2 + f_t,
\end{aligned} \tag{2.1}$$

where $f_t = 2K\|N_t(\omega)\|_H^2 + C + C\|N_t(\omega)\|_V^\alpha \in L^1_{loc}(\mathbb{R})$ by $(S2)$.

The growth condition also holds for $\tilde{A}_\omega$ since

$$\begin{aligned}
\|\tilde{A}_\omega(t, v)\|_{V^*} &= \|A(v + N_t(\omega))\|_{V^*} \\
&\leq C(1 + \|v + N_t(\omega)\|_V^{\alpha-1}) \\
&\leq f_t^{(\alpha-1)/\alpha} + C\|v\|_V^{\alpha-1}.
\end{aligned}$$

Therefore, according to the classical results in [27, 40] (applied to the deterministic case), (1.2) has a unique solution

$$Z(\cdot, s; \omega)x \in L^\alpha_{loc}([s, \infty); V) \cap C([s, \infty), H)$$

and $x \mapsto Z(t, s; \omega)x$ is continuous in H for all $s \leq t$ and $\omega \in \Omega$.

Now we define $S(t, s; \omega)x$ by (1.3) and $\varphi(t, \omega)x$ by (1.4). For fixed s, ω, x we abbreviate $S(t, s; \omega)x$ by S_t and $Z(t, s; \omega)x$ by Z_t. By the pathwise uniqueness of the solution to equation (1.2) and $(S1)$ we have

$$\begin{aligned}
S(t, s; \omega) &= S(t, r; \omega)S(r, s; \omega), \\
S(t, s; \omega) &= S(t - s, 0; \theta_s\omega),
\end{aligned} \tag{2.2}$$

for all $r, s, t \in \mathbb{R}$ and all $\omega \in \Omega$.

It remains to prove the measurability of $\varphi : \mathbb{R} \times H \times \Omega \to H$. By (2.2) this also implies the measurability of $(t, s, x, \omega) \mapsto S(t, s; \omega)x$. Since $\varphi(t, \omega)x = Z(t, 0; \omega)x + N_t(\omega)$ and by $(S3)$ it is sufficient to show the measurability of $(t, x, \omega) \mapsto Z(t, 0; \omega)x$. Note that the maps $t \mapsto Z(t, 0; \omega)x$ and $x \mapsto Z(t, 0; \omega)x$ are continuous, thus we only need to prove the measurability of $\omega \mapsto Z(t, 0; \omega)x$.

Let $x \in H$ and $t \in \mathbb{R}$ be arbitrary, fix and choose some interval $[s_0, t_0] \subseteq \mathbb{R}$ such that $t \in (s_0, t_0)$. By the proof of the existence and uniqueness of solutions to (1.2) we know that $Z(t, 0; \omega)x$ is the weak limit of a subsequence of the Galerkin approximations $Z^n(t, 0; \omega)x$ in $L^\alpha([s_0, t_0]; V)$. Since every subsequence of $Z^n(t, 0; \omega)x$ has a subsequence weakly converging to $Z(t, 0; \omega)x$, this implies that the whole sequence of Galerkin approximants $Z^n(t, 0; \omega)x$ weakly converges to $Z(t, 0; \omega)x$ in $L^\alpha([s_0, t_0]; V)$.

Let $\varphi_k \in C_0^\infty(\mathbb{R})$ be a Dirac sequence with $supp(\varphi_k) \subseteq B_{\frac{1}{k}}(0)$. Then $(\varphi_k * Z^n(\cdot, 0; \omega)x)(t)$ is well-defined for k large enough. For each such $k \in \mathbb{N}$ and $h \in H$ we have

$$(\varphi_k * \langle Z^n(\cdot, 0; \omega)x, h\rangle_H)(t) \to (\varphi_k * \langle Z(\cdot, 0; \omega)x, h\rangle_H)(t), \ n \to \infty.$$

Since $\omega \mapsto Z^n(\cdot, 0; \omega)x \in L^\alpha([s_0, t_0]; V)$ is measurable, so is $\omega \mapsto (\varphi_k * Z^n(\cdot, 0; \omega)x)(t)$. Consequently, $\omega \mapsto (\varphi_k * \langle Z(\cdot, 0; \omega)x, h\rangle_H)(t)$ is measurable as it is the ω-wise limit of $(\varphi_k * \langle Z^n(\cdot, 0; \omega)x, h\rangle_H)(t)$. We know that $r \mapsto Z(r, 0; \omega)x$ is continuous in H. Therefore, $(\varphi_k * \langle Z(\cdot, 0; \omega)x, h\rangle_H)(t) \to \langle Z(t, 0; \omega)x, h\rangle_H$ and the measurability of $\omega \mapsto (\varphi_k * \langle Z(\cdot, 0; \omega)x, h\rangle_H)(t)$ implies the measurability of $\omega \mapsto \langle Z(t, 0; \omega)x, h\rangle_H$.

Since this is true for all $h \in H$ and $\mathcal{B}(H)$ is generated by $\sigma(\{\langle h, \cdot\rangle_H |\ h \in H\})$, this implies the measurability of $\omega \mapsto Z(t, 0; \omega)x$. This finishes the proof that φ defines a continuous RDS and consequently, that S defines a continuous stochastic flow.

Note that adaptedness of S_t to $\mathcal{F}_t$ can be shown in the same way as the measurability of φ. □

2.2 Proof of Theorem 1.3

Since in Theorem 1.1 we have proved that φ defines an RDS, we can apply Proposition 1.2 to show the existence of a random attractor for φ. For this we follow the procedure outlined in the introduction. First we prove the absorption of $Z(t, s; \omega)x$ in H at time $t = -1$.

Lemma 2.1. *Suppose $(H1)$-$(H4)$ hold for $\alpha = 2, K = 0$ or for $\alpha > 2$ and that $(S1)$-$(S4)$ are satisfied. Then there exists a random radius $r_1(\omega) > 0$ such that for all $\rho > 0$, there exists $\bar{s} \leq -1$ in such a way that $\mathbb{P}-$a.s. we have*

$$\|Z(-1, s; \omega)x\|_H^2 \leq r_1^2(\omega),$$

which holds for all $s \leq \bar{s}$ and all $x \in H$ with $\|x\|_H \leq \rho$.

Proof. By the coercivity of $\tilde{A}_\omega$ proved in the previous section (see (2.1)) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Z_t\|_H^2 = 2_{V^*}\langle \tilde{A}_\omega(t, Z_t), Z_t\rangle_V \leq -\delta_0\|Z_t\|_V^\alpha + 2K\|Z_t\|_H^2 + f_t, \tag{2.3}$$

where $\delta_0 = 2^{-\alpha}\delta > 0$ and $f_t = 2K\|N_t(\omega)\|_H^2 + C(\|N_t(\omega)\|_V^\alpha + 1)$.

If $\alpha > 2$ or $\alpha = 2, K = 0$, then there exist constants $\lambda > 0$ and C such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Z_t\|_H^2 + \frac{\delta_0}{2}\|Z_t\|_V^\alpha \leq -\lambda\|Z_t\|_H^2 + f_t + C. \tag{2.4}$$

By Gronwall's Lemma for all $s \leq -1$ we have,

$$\begin{aligned}\|Z_{-1}\|_H^2 &\leq e^{-\lambda(-1-s)}\|Z_s\|_H^2 + \int_s^{-1} e^{-\lambda(-1-r)}(f_r + C)\mathrm{d}r \\ &\leq 2e^{-\lambda(-1-s)}\|x\|_H^2 + 2e^{-\lambda(-1-s)}\|N_s(\omega)\|_H^2 + \int_{-\infty}^{-1} e^{-\lambda(-1-r)}(f_r + C)\mathrm{d}r.\end{aligned} \tag{2.5}$$

By $(S4)$, i.e. the subexponential growth of $N_t(\omega)$ for $t \to -\infty$ we know that the following quantity is finite for all $\omega \in \Omega$,

$$r_1^2(\omega) = 2 + 2\sup_{r\leq -1} e^{-\lambda(-1-r)}\|N_r(\omega)\|_H^2 + \int_{-\infty}^{-1} e^{-\lambda(-1-r)}(f_r(\omega) + C)\mathrm{d}r.$$

Applying $(S3)$, i.e. the joint measurability of N in (t, ω), $r_1(\omega)$ is measurable and then the assertion follows by taking some $\bar{s} < -1$ such that $e^{-\lambda(-1-\bar{s})}\rho^2 \leq 1$. □

Remark 2.1. (2.4) also implies the following estimate for the V-norm

$$\frac{\delta_0}{2}\int_{-1}^{0}\|Z_r\|_V^\alpha \mathrm{d}r \leq \|Z_{-1}\|_H^2 + \int_{-1}^{0}(f_r + C)\mathrm{d}r. \tag{2.6}$$

The next step is to show compact absorption of $Z(t, s; \omega)$ at time $t = 0$. We proceed by using the approximation scheme indicated in the outline of proof. By defining $H_n := (H, \langle\cdot,\cdot\rangle_n)$ (see $(H5)$) we obtain a sequence of new Gelfand triples

$$V \subseteq H_n \equiv H_n^* \subseteq V^*.$$

Note that we use different Riesz maps $i_n : H_n \to H_n^*$ to identify $H_n \equiv H_n^*$ in these Gelfand triples. Let i denote the Riesz map for $H \equiv H^*$. Now we recall the following Lemma, which is proved in [34].

Lemma 2.2. *If $T_n : V \to V$ is continuous, then $i_n \circ i^{-1} : H^* \to H_n^*$ is continuous w.r.t. $\|\cdot\|_{V^*}$. Therefore, there exists a unique continuous extension I_n of $i_n \circ i^{-1}$ to all of V^* such that*

$${}_{V^*}\langle I_n f, v\rangle_V = {}_{V^*}\langle f, T_n v\rangle_V, \quad f \in V^*,\ v \in V. \tag{2.7}$$

Lemma 2.3. *Suppose the assumptions of Theorem 1.3 hold. Then there exists a random radius $r_2(\omega) > 0$ such that for all $\rho > 0$, there exists $\bar{s} \leq -1$ in such a way that $\mathbb{P} - a.s.$ we have*

$$\|Z(0, s; \omega)x\|_S^2 \leq r_2^2(\omega),$$

which holds for all $s \leq \bar{s}$ and all $x \in H$ with $\|x\|_H \leq \rho$.

Proof. Using the operator $I_n : V^* \to H_n^*$ we consider the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t} Z_t = I_n A(Z_t + N_t),$$

which is well defined on the new Gelfand triple

$$V \subseteq H_n \equiv H_n^* \subseteq V^*.$$

By Lemma 2.2, (1.5) and $(H4)$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|Z_t\|_n^2 &= 2_{V^*}\langle I_n A(Z_t + N_t), Z_t\rangle_V \\
&= 2_{V^*}\langle A(Z_t + N_t), T_n Z_t\rangle_V \\
&\le C(\|Z_t + N_t\|_n^2 + 1) - 2_{V^*}\langle A(Z_t + N_t), T_n N_t\rangle_V \\
&\le C(\|Z_t + N_t\|_n^2 + 1) + 2\left(\frac{\alpha - 1}{\alpha}\|A(Z_t + N_t)\|_{V^*}^{\frac{\alpha}{\alpha-1}} + \frac{1}{\alpha}\|T_n N_t\|_V^\alpha\right) \\
&\le C\left(\|Z_t\|_n^2 + \|Z_t\|_V^\alpha\right) + C\left(1 + \|N_t\|_n^2 + \|N_t\|_V^\alpha + \|T_n N_t\|_V^\alpha\right) \\
&\le C\left(\|Z_t\|_n^2 + \|Z_t\|_V^\alpha\right) + g_t^{(n)},
\end{aligned}$$

where C is some positive constant and

$$g_t^{(n)} := C\left(1 + \|N_t\|_S^2 + \|N_t\|_V^\alpha + \|T_n N_t\|_V^\alpha\right).$$

Then Gronwall's Lemma implies that for all $s \le 0$,

$$\|Z_0\|_n^2 \le e^{-Cs}\|Z_s\|_n^2 + C\int_s^0 e^{-Cr}\|Z_r\|_V^\alpha \mathrm{d}r + \int_s^0 e^{-Cr} g_r^{(n)} \mathrm{d}r.$$

Integrating on s over $[-1, 0]$ and using (1.6) we have

$$\begin{aligned}
\|Z_0\|_n^2 &\le \int_{-1}^0 \left(e^{-Cr}\|Z_r\|_S^2 + Ce^{-Cr}\|Z_r\|_V^\alpha\right)\mathrm{d}r + \int_{-1}^0 e^{-Cr} g_r^{(n)}\mathrm{d}r \\
&\le \int_{-1}^0 \left(e^{-Cr}\|Z_r\|_S^2 + Ce^{-Cr}\|Z_r\|_V^\alpha\right)\mathrm{d}r + C_1,
\end{aligned}$$

where C_1 is a finite constant.

Note that $\alpha \ge 2$ and $\|\cdot\|_S \le C\|\cdot\|_V$, hence by taking $n \to \infty$ and using (2.6) we have

$$\begin{aligned}
\|Z_0\|_S^2 &\le C\int_{-1}^0 e^{-Cr}\left(1 + \|Z_r\|_V^\alpha\right)\mathrm{d}r + C_1 \\
&\le C_2\|Z_{-1}\|_H^2 + C_2,
\end{aligned}$$

where $C_2 > 0$ is a constant. Now the assertion follows from Lemma 2.1. □

Proof of Theorem 1.3: By Lemma 2.3 there exists $r_2(\omega) > 0$ such that for all $\rho > 0$ there exists $\bar{s} \leq -1$ in such a way that $\mathbb{P} - a.s.$

$$\begin{aligned}\|S(0,s;\omega)x\|_S &= \|Z(0,s;\omega)x + N_0(\omega)\|_S \\ &\leq \|Z(0,s;\omega)x\|_S + \|N_0(\omega)\|_S \\ &\leq r_2(\omega) + \|N_0(\omega)\|_S\end{aligned}$$

holds for all $s \leq \bar{s}$ and all $x \in H$ with $\|x\|_H \leq \rho$.

Hence $S(t,s;\omega)x$ is absorbed at time $t = 0$ by the compact random set

$$K(\omega) = \overline{B_S(0, r_2(\omega) + \|N_0(\omega)\|_S)}.$$

By Proposition 1.2 this implies the existence of a random attractor for the RDS φ associated with (1.1). □

2.3 Proof of Theorem 1.4

The proof of first Lemma is mainly based on [9, Theorem 5.1]. The strong monotonicity condition $(H2')$ leads to the following strong contraction property.

Lemma 2.4. *Under the assumptions of Theorem 1.4 with $\beta > 2$, for $s_1 \leq s_2 < t$, $\omega \in \Omega$ and $x, y \in H$ we have*

$$\begin{aligned}\|S(t,s_1;\omega)x - S(t,s_2;\omega)y\|_H^2 &\leq \left\{\|S(s_2,s_1;\omega)x - y\|_H^{2-\beta} + \frac{\lambda}{2}(\beta-2)(t-s_2)\right\}^{-\frac{2}{\beta-2}} \\ &\leq \left\{\frac{\lambda}{2}(\beta-2)(t-s_2)\right\}^{-\frac{2}{\beta-2}}.\end{aligned}$$

In particular, for each $t \in \mathbb{R}$ there exists η_t (independent of x) such that

$$\lim_{s\to-\infty} S(t,s;\omega)x = \eta_t(\omega),$$

where the convergence holds uniformly in x and ω.

Proof. Let $\omega \in \Omega$, $x, y \in H$ and $s_1 \leq s_2 \leq s < t$, then

$$\begin{aligned}&S(t,s_1;\omega)x - S(t,s_2;\omega)y \\ =&S(s,s_1;\omega)x - S(s,s_2;\omega)y + \int_s^t (A(S(r,s_1;\omega)x) - A(S(r,s_2;\omega)y))\,\mathrm{d}r.\end{aligned}$$

Note that $t \mapsto S(t,s_1;\omega)x - S(t,s_2;\omega)y$ is continuous in H. By Itô's formula and $(H2')$

$$\begin{aligned}&\|S(t,s_1;\omega)x - S(t,s_2;\omega)y\|_H^2 \\ =&\|S(s,s_1;\omega)x - S(s,s_2;\omega)y\|_H^2 \\ &+ 2\int_s^t {}_{V^*}\langle A(S(r,s_1;\omega)x) - A(S(r,s_2;\omega)y), S(r,s_1;\omega)x - S(r,s_2;\omega)y\rangle_V \mathrm{d}r \qquad (2.8) \\ \leq&\|S(s,s_1;\omega)x - S(s,s_2;\omega)y\|_H^2 - \lambda\int_s^t \|S(r,s_1;\omega)x - S(r,s_2;\omega)y\|_H^\beta \mathrm{d}r.\end{aligned}$$

The idea of the rest of the proof is to compare $\|S(t,s_1;\omega)x - S(t,s_2;\omega)y\|_H^2$ with the solution to the ordinary differential equation

$$h'(t) = -\lambda h(t)^{\frac{\beta}{2}},\ t \geq s_2;\ h(s_2) = \|S(s_2,s_1;\omega)x - y\|_H^2. \tag{2.9}$$

However, since $\|S(t,s_1;\omega)x - S(t,s_2;\omega)y\|_H^2$ is not necessarily differentiable in t we cannot apply classical comparison results.

Let

$$h_\epsilon(t) = \left\{ (\|S(s_2,s_1;\omega)x - y\|_H + \epsilon)^{2-\beta} + \frac{\lambda}{2}(\beta-2)(t-s_2) \right\}^{-\frac{2}{\beta-2}}.$$

It is easy to show that h_ϵ is a solution of (2.9) with $h_\epsilon(s_2) = (\|S(s_2,s_1;\omega)x - y\|_H + \epsilon)^2$. Now we prove that

$$\|S(t,s_1;\omega)x - S(t,s_2;\omega)y\|_H^2 \leq h_\epsilon(t), \quad t \geq s_2. \tag{2.10}$$

Let

$$\begin{aligned} \Phi_\epsilon(t) &= h_\epsilon(t) - \|S(t,s_1;\omega)x - S(t,s_2;\omega)y\|_H^2, \\ \tau_\epsilon &= \inf\left\{ t \geq s_2 |\ \Phi_\epsilon(t) \leq 0 \right\}. \end{aligned}$$

Because $\Phi_\epsilon(s_2) > 0$ and by the continuity of Φ_ϵ we know that $\tau_\epsilon > s_2$. Furthermore, note that by definition we have

$$\begin{aligned} &h_\epsilon(t) \geq \|S(t,s_1;\omega)x - S(t,s_2;\omega)y\|_H^2,\ t \in [s_2,\tau_\epsilon]; \\ &h_\epsilon(t) \leq (\|S(s_2,s_1;\omega)x - y\|_H + \epsilon)^2 =: c_\epsilon,\ t \geq s_2. \end{aligned}$$

If $\tau_\epsilon < \infty$, then $\Phi_\epsilon(\tau_\epsilon) \leq 0$ by the continuity of Φ_ϵ. Therefore, by the mean value theorem and (2.8) for all $s_2 \leq s \leq t \leq \tau_\epsilon$ we have,

$$\begin{aligned} \Phi_\epsilon(t) &= h_\epsilon(t) - \|S(t,s_1;\omega)x - S(t,s_2;\omega)y\|_H^2 \\ &\geq \Phi_\epsilon(s) - \lambda \int_s^t \left(h_\epsilon(r)^{\frac{\beta}{2}} - \left(\|S(r,s_1;\omega)x - S(r,s_2;\omega)y\|_H^2 \right)^{\frac{\beta}{2}} \right) \mathrm{d}r \\ &\geq \Phi_\epsilon(s) - \frac{\lambda\beta c_\epsilon^{\frac{\beta-2}{2}}}{2} \int_s^t \Phi_\epsilon(r)\mathrm{d}r. \end{aligned}$$

Using Gronwall's Lemma we obtain

$$\Phi_\epsilon(\tau_\epsilon) \geq \Phi_\epsilon(s_2) \exp\left[-\frac{\lambda\beta}{2} c_\epsilon^{\frac{\beta-2}{2}} (\tau_\epsilon - s_2) \right] > 0.$$

This contradiction implies that $\tau_\epsilon = \infty$, i.e. (2.10) holds.

Since (2.10) holds for any $\epsilon > 0$ we can conclude that

$$\begin{aligned} \|S(t,s_1;\omega)x - S(t,s_2;\omega)y\|_H^2 &\leq \left\{ \|S(s_2,s_1;\omega)x - y\|_H^{2-\beta} + \frac{\lambda}{2}(\beta-2)(t-s_2) \right\}^{-\frac{2}{\beta-2}} \\ &\leq \|S(s_2,s_1;\omega)x - y\|_H^2 \wedge \left\{ \frac{\lambda}{2}(\beta-2)(t-s_2) \right\}^{-\frac{2}{\beta-2}} \\ &\leq \left\{ \frac{\lambda}{2}(\beta-2)(t-s_2) \right\}^{-\frac{2}{\beta-2}} \end{aligned}$$

holds for any $t > s_2$. □

Lemma 2.5. *Suppose the assumptions of Theorem 1.4 with $\beta = 2$ and $(S4)$ hold. Then for each $\eta \in (0, \lambda)$ there is an $\mathbb{R}_+$-valued random variable K_η such that*

$$\|S(t, s_1; \omega)x - S(t, s_2; \omega)y\|_H^2 \leq 2\left(\|x\|_H^2 e^{\frac{\eta}{2}s_1} + K_\eta(\omega) + \|y\|_H^2\right) e^{(\lambda-\eta)s_2} e^{-\lambda t}$$

for all $s_1 \leq s_2 < t$, $\omega \in \Omega$ and $x, y \in H$. In particular, for each $t \in \mathbb{R}$ there exists η_t (independent of x) such that

$$\lim_{s \to -\infty} S(t, s; \omega)x = \eta_t(\omega),$$

where the convergence holds locally uniformly in x.

Proof. As in Lemma 2.4 for $\omega \in \Omega$, $x, y \in H$ and $s_1 \leq s_2 \leq s < t$ we obtain

$$\begin{aligned}
&\|S(t, s_1; \omega)x - S(t, s_2; \omega)y\|_H^2 \\
\leq &\|S(s, s_1; \omega)x - S(s, s_2; \omega)y\|_H^2 - \lambda \int_s^t \|S(r, s_1; \omega)x - S(r, s_2; \omega)y\|_H^2 \mathrm{d}r.
\end{aligned}$$

Thus, by Gronwall's Lemma

$$\begin{aligned}
\|S(t, s_1; \omega)x - S(t, s_2; \omega)y\|_H^2 &\leq \|S(s_2, s_1; \omega)x - y\|_H^2 e^{-\lambda(t-s_2)} \\
&\leq 2\left(\|S(s_2, s_1; \omega)x\|_H^2 + \|y\|_H^2\right) e^{-\lambda(t-s_2)}.
\end{aligned}$$

By [40, Lemma 4.3.8] $(H3)$, $(H4)$ and $(H2')$ imply that for each $\eta \in (0, \lambda)$ there exists a $C_\eta > 0$ such that for all $v \in V$

$$2\ {}_{V^*}\langle A(v), v\rangle_V \leq -\eta\|v\|_H^2 + C_\eta. \tag{2.11}$$

Let $\eta \in (0, \lambda)$ and $\tilde{\eta} = \frac{\eta+\lambda}{2} \in (\eta, \lambda)$. We use (2.11) with $\tilde{\eta}$, $(H3)$, $(H4)$ and Young's inequality to obtain

$$\begin{aligned}
&2\ {}_{V^*}\langle A(v + N_r), v\rangle_V \\
=&2\ {}_{V^*}\langle A(v + N_r), v + N_r - N_r\rangle_V \\
\leq &2\varepsilon_1\ {}_{V^*}\langle A(v + N_r), v + N_r\rangle_V + 2(1 - \varepsilon_1)\ {}_{V^*}\langle A(v + N_r), v + N_r\rangle_V \\
&+ 2\|A(v + N_r)\|_{V^*}\|N_r\|_V \\
\leq &\varepsilon_1 K\|v + N_r\|_H^2 - \delta\varepsilon_1\|v + N_r\|_V^\alpha + \varepsilon_1 C - \tilde{\eta}(1 - \varepsilon_1)\|v + N_r\|_H^2 + (1 - \varepsilon_1)C_{\tilde{\eta}} \\
&+ \varepsilon_2\|A(v + N_r)\|_{V^*}^{\frac{\alpha}{\alpha-1}} + C_{\varepsilon_2}\|N_r\|_V^\alpha \\
\leq &(\varepsilon_1 K - \tilde{\eta}(1 - \varepsilon_1))\|v + N_r\|_H^2 + (\varepsilon_2 C - \delta\varepsilon_1)\|v + N_r\|_V^\alpha + \varepsilon_1 C + (1 - \varepsilon_1)C_{\tilde{\eta}} \\
&+ \varepsilon_2 C + C_{\varepsilon_2}\|N_r\|_V^\alpha,
\end{aligned}$$

where $\varepsilon_1 \in [0, 1]$, $\varepsilon_2 > 0$ and $C, C_{\varepsilon_1}, C_{\varepsilon_2} > 0$ are some constants.

Now by taking $0 < \varepsilon_1 \le \frac{\tilde{\eta}-\eta}{\tilde{\eta}+K} \wedge 1$ and $\varepsilon_2 = \frac{\delta\varepsilon_1}{C}$ we have

$$\begin{aligned} 2\ {}_{V^*}\langle A(v+N_r), v\rangle_V &\le -\eta\|v+N_r\|_H^2 + \varepsilon_1 C + (1-\varepsilon_1)C_{\tilde{\eta}} + \varepsilon_2 C + C_{\varepsilon_2}\|N_r\|_V^\alpha \\ &\le -\frac{\eta}{2}\|v\|_H^2 + \eta\|N_r\|_H^2 + \varepsilon_1 C + (1-\varepsilon_1)C_{\tilde{\eta}} + \varepsilon_2 C + C_{\varepsilon_2}\|N_r\|_V^\alpha \\ &\le -\frac{\eta}{2}\|v\|_H^2 + C_\eta(r), \end{aligned}$$

where $C_\eta(r) = \eta\|N_r\|_H^2 + C_{\varepsilon_2}\|N_r\|_V^\alpha + \varepsilon_1 C + (1-\varepsilon_1)C_{\tilde{\eta}} + \varepsilon_2 C$.

Hence for all $t_2 \ge t_1 \ge s$,

$$\begin{aligned} \|Z(t_2,s;\omega)x\|_H^2 &= \|Z(t_1,s;\omega)x\|_H^2 + 2\int_{t_1}^{t_2} {}_{V^*}\langle A(Z(r,s;\omega)x + N_r(\omega)), Z(r,s;\omega)x\rangle_V \mathrm{d}r \\ &\le \|Z(t_1,s;\omega)x\|_H^2 - \frac{\eta}{2}\int_{t_1}^{t_2} \|Z(r,s;\omega)x\|_H^2 \mathrm{d}r + \int_{t_1}^{t_2} C_\eta(r)\mathrm{d}r. \end{aligned}$$

By Gronwall's Lemma

$$\begin{aligned} \|S(s_2,s_1;\omega)x\|_H^2 &\le 2\left(\|Z(s_2,s_1;\omega)x\|_H^2 + \|N_{s_2}(\omega)\|_H^2\right) \\ &\le 2(\|x\|_H^2 e^{-\frac{\eta}{2}(s_2-s_1)} + \int_{s_1}^{s_2} e^{-\eta(s_2-r)} C_\eta(r)\mathrm{d}r + \|N_{s_2}(\omega)\|_H^2). \end{aligned}$$

For $s_1 \le s_2 \le 0$ we conclude that

$$\begin{aligned} &\|S(t,s_1;\omega)x - S(t,s_2;\omega)y\|_H^2 \\ &\le 2\left(\|S(s_2,s_1;\omega)x\|_H^2 + \|y\|_H^2\right) e^{-\lambda(t-s_2)} \\ &\le 4\left(\|x\|_H^2 e^{-\frac{\eta}{2}(s_2-s_1)} + \int_{s_1}^{s_2} e^{-\frac{\eta}{2}(s_2-r)} C_\eta(r)\mathrm{d}r + \|N_{s_2}(\omega)\|_H^2 + \frac{1}{2}\|y\|_H^2\right) e^{-\lambda(t-s_2)} \\ &\le 4\left(\|x\|_H^2 e^{\frac{\eta}{2}s_1} e^{(\lambda-\frac{\eta}{2})s_2} + e^{(\lambda-\frac{\eta}{2})s_2}\int_{s_1}^{s_2} e^{\frac{\eta}{2}r} C_\eta(r)\mathrm{d}r + e^{\lambda s_2}\|N_{s_2}(\omega)\|_H^2 + \frac{e^{\lambda s_2}}{2}\|y\|_H^2\right) e^{-\lambda t} \\ &\le 4\left(e^{(\lambda-\frac{\eta}{2})s_2}\|x\|_H^2 + e^{(\lambda-\frac{\eta}{2})s_2} K_\eta + e^{\lambda s_2}\|N_{s_2}(\omega)\|_H^2 + \frac{e^{\lambda s_2}}{2}\|y\|_H^2\right) e^{-\lambda t} \\ &\to 0 \quad \text{as} \quad s_1, s_2 \to -\infty, \end{aligned}$$

where $K_\eta = \int_{-\infty}^0 e^{\frac{\eta}{2}r} C_\eta(r)\mathrm{d}r$ is finite by $(S4)$, i.e. by the subexponential growth of $\|N_t\|_V$.

Therefore, for all $t \in \mathbb{R}$ and $\omega \in \Omega$ there exists a limit $\eta_t(\omega)$ (independent of x) such that

$$\lim_{s\to-\infty} S(t,s;\omega)x = \eta_t(\omega)$$

holds locally uniformly in x. □

Now we can finish the proof of Theorem 1.4.

Proof of Theorem 1.4 By Lemma 2.4 and 2.5 we may define

$$\mathcal{A}(\omega) = \{\eta_0(\omega)\}.$$

We shall show that this defines a global random attractor for the RDS associated with (1.1).

Since $\eta_0(\omega)$ is measurable, $\mathcal{A}(\omega)$ is a random compact set. Hence we only need to check the invariance and attraction properties for $\mathcal{A}(\omega)$.

The Continuity of $x \mapsto S(t,0;\omega)x$ and the flow property imply that

$$\begin{aligned}\varphi(t,\omega)\mathcal{A}(\omega) &= \left\{ S(t,0;\omega) \lim_{s\to-\infty} S(0,s;\omega)x \right\} = \left\{ \lim_{s\to-\infty} S(t,s;\omega)x \right\} \\ &= \left\{ \lim_{s\to-\infty} S(0,s-t,\theta_t\omega)x \right\} = \{\eta_0(\theta_t\omega)\} = \mathcal{A}(\theta_t\omega),\ t>0, x\in H.\end{aligned}$$

Since the convergence in Lemma 2.4 is uniform (locally uniform resp. in Lemma 2.5) with respect to $x \in H$, for any bounded set $B \subseteq H$ we have

$$\begin{aligned}d(\varphi(t,\theta_{-t}\omega)B, \mathcal{A}(\omega)) &= \sup_{x\in B} \|S(t,0,\theta_{-t}\omega)x - \eta_0(\omega)\|_H \\ &= \sup_{x\in B} \|S(0,-t;\omega)x - \eta_0(\omega)\|_H \to 0 (t\to\infty),\end{aligned}$$

i.e. $\mathcal{A}(\omega)$ attracts all deterministic bounded sets.

Therefore, $\mathcal{A}$ is a global random attractor for the RDS associated with (1.1).

We now deduce the unique existence of an invariant random measure $\mu_\cdot \in \mathcal{P}_\Omega(H)$. For the notion of an invariant random measure we refer to [17, Definition 4.1]. By [17, Corollary 4.4] the existence of a random attractor implies the existence of an invariant random measure. Moreover, by [15, Theorem 2.12] every invariant measure for φ is supported by $\mathcal{A} = \{\eta_0\}$, i.e. $\mu_\omega(\{\eta_0(\omega)\}) = 1$ for $\mathbb{P}$-a.a. ω.

The bounds on the speed of attraction follow immediately from the respective bounds in Lemma 2.4 and 2.5. □

3 Applications to concrete SPDE

In this section we present several examples of admissible random perturbations N_t and also show that $(H1)-(H5)$ and $(H2')$ can be verified for many concrete SPDE. Hence Theorem 1.3 and 1.4 can be applied to show the existence of a random attractor for those examples.

We will first show that all cádlág processes with stationary increments satisfy $(S1)$-$(S3)$ and thus Theorem 1.1 and Theorem 1.4 are applicable. Of course, this contains all Lévy processes as well as fractional Brownian Motion.

Lemma 3.1. *Let $(N_t)_{t\in\mathbb{R}}$ be a V-valued process with stationary increments and a.s. cádlág paths. Then there is a metric dynamical system $(\Omega,\mathcal{F},\mathbb{P},\theta_t)$ and a version $\tilde{N}_t$ (cf. [42, Definition 1.6]) on $(\Omega,\mathcal{F},\mathbb{P},\theta_t)$ such that $\tilde{N}_t$ satisfies $(S1)$-$(S3)$.*

Proof. We choose $\Omega = D(\mathbb{R};V)$ to be the set of all cádlág functions endowed with the Skorohod topology (cf. [2], pp. 545), $\mathcal{F} = \mathcal{B}(\Omega)$, $\theta_t(\omega) = \omega(t+\cdot) - \omega(t)$ and $\mathbb{P} = \mathcal{L}(N)$ to be the law of N_t (or more precisely its restriction on Ω). Note that $\mathcal{F}$ is the trace in Ω of the product σ-algebra $\mathcal{B}(V)^{\mathbb{R}}$ and $(t,\omega) \mapsto \theta_t(\omega)$ is measurable. Since N_t has stationary increments we know that $\theta_t\mathbb{P} = \mathbb{P}$. Hence $(\Omega,\mathcal{F},\mathbb{P},\theta_t)$ defines a metric dynamical system and the coordinate process $\tilde{N}_t$ on Ω is a version of N_t satisfying $(S1),(S2)$ and $(S3)$. □

We will prove the asymptotic bound $(S4)$ for two classes of processes. The first class consists of all processes with independent increments (e.g. Lévy processes) where the proof is based on the strong law of large numbers, and the second class consists of all processes with Hölder continuous paths (e.g. fractional Brownian Motion), for which we use Kolmogorov's continuity theorem and the dichotomy of linear growth for stationary processes.

Lemma 3.2. *Let V be a separable Banach space and N_t be a V-valued Lévy process with Lévy characteristics (m, R, ν) (e.g. cf. [39, Corollary 4.59]). Assume that $\int_V (\|x\|_V \vee \|x\|_V^2)\, d\nu(x) < \infty$, then we have $\mathbb{P}$-a.s.*

$$\frac{N_t}{|t|} \to \pm \mathbb{E}N_1 \ (t \to \pm\infty).$$

Proof. Since $\bar{N}_t := N_{-t}$ is also a Lévy process satisfying the assumptions and $\mathbb{E}\bar{N}_1 = -\mathbb{E}N_1$, it is sufficient to prove the assertion for $t \to +\infty$. By the Lévy-Itô decomposition for Banach space valued Lévy processes (cf. [1, Theorem 4.1]) we have

$$N_t = mt + W_t + \int_{B_1(0)} x\tilde{N}(t, dx) + \int_{B_1^c(0)} xN(t, dx),$$

where $m \in V$, W_t is a V-valued Wiener process and

$$\int_{B_1(0)} x\tilde{N}(t, dx) := \lim_{n\to\infty} \int_{\{\frac{1}{n+1} \le \|x\|_V < 1\}} x\tilde{N}(t, dx) = \lim_{n\to\infty} \sum_{k=1}^{n} \int_{\{\frac{1}{k+1} \le \|x\|_V < \frac{1}{k}\}} x\tilde{N}(t, dx) \tag{3.1}$$

is a $\mathbb{P}$-$a.s.$ limit of compensated compound Poisson processes.

By an analogous calculation to [39, pp. 49] we have

$$\mathbb{E}\| \int_{\{\epsilon \le \|x\|_V < 1\}} x\tilde{N}(t, dx)\|_V^2 \le 2t \int_{\{\epsilon \le \|x\|_V < 1\}} \|x\|_V^2 d\nu(x) + 4\left(t \int_{\{\epsilon \le \|x\|_V < 1\}} \|x\|_V d\nu(x)\right)^2,$$

and

$$\mathbb{E}\| \int_{B_1^c(0)} xN(t, dx)\|_V^2 \le t \int_{B_1^c(0)} \|x\|_V^2 d\nu(x) + \left(t \int_{B_1^c(0)} \|x\|_V d\nu(x)\right)^2.$$

Thus,

$$\begin{aligned} \sup_n \mathbb{E}\| \sum_{k=1}^{n} \int_{\{\frac{1}{k+1} \le \|x\|_V < \frac{1}{k}\}} x\tilde{N}(t, dx)\|^2 &= \sup_n \mathbb{E}\| \int_{\{\frac{1}{n+1} \le \|x\|_V < 1\}} x\tilde{N}(t, dx)\|^2 \\ \le 2t \int_{B_1(0)} \|x\|_V^2 d\nu(x) + 4\left(t \int_{B_1(0)} \|x\|_V d\nu(x)\right)^2 &< \infty. \end{aligned} \tag{3.2}$$

By (3.1) $\int_{B_1(0)} x\tilde{N}(t, dx)$ is the limit of a $\mathbb{P}$-$a.s.$ converging series of independent random variables and by [32, Theorem 3.4.2] the bound (3.2) implies that the convergence in (3.1) also holds in $L^2(\Omega; V)$. Hence $N_t \in L^2(\Omega; V)$ and

$$\mathbb{E}N_t = t\left(m + \int_{B_1^c(0)} x d\nu(x)\right) = t\mathbb{E}N_1.$$

Let now N_t be centered (i.e. $\mathbb{E}N_t = 0$) and note

$$N_n = N_n - N_{n-1} + N_{n-1} - N_{n-2} + ... + N_1,$$

then by the law of large numbers for Banach space valued random vectors (cf. [25, Theorem III.1.1]) we have $\mathbb{P}$-$a.s.$

$$\frac{N_n}{n} \to \mathbb{E}[N_1] = 0 \ (n \to \infty).$$

It remains to derive the bound for $N_t - N_{[t]}$. Let

$$S_n := \sup_{s\in[0,1]} ||N_{n+s} - N_n||_V.$$

Since N_t is centered and has first moment, it is a martingale. Thus $\|N_t\|_V$ is a non-negative cádlág submartingale and Doob's maximal inequality implies that

$$\mathbb{E}S_0 = \mathbb{E} \sup_{s\in[0,1]} ||N_s||_V \leq 2(\mathbb{E}\|N_1\|_V^2)^{\frac{1}{2}} < \infty.$$

Since S_n are i.i.d., by the strong law of large numbers we have for $N \to \infty$

$$\frac{\sum_{n=1}^N S_n}{N} \to \mathbb{E}[S_1], \ \mathbb{P}\text{-a.s..}$$

In particular, we have $\frac{S_N}{N} \to 0$, $\mathbb{P}$-$a.s.$. Consequently,

$$\begin{aligned}\frac{\|N_t\|_V}{t} &\leq \frac{[t]}{t}\left(\frac{\|N_t - N_{[t]}\|_V}{[t]} + \frac{\|N_{[t]}\|_V}{[t]}\right) \\ &\leq \frac{[t]}{t}\left(\frac{S_{[t]}}{[t]} + \frac{\|N_{[t]}\|_V}{[t]}\right) \to 0 \ (t \to \infty), \ \mathbb{P}\text{-a.s. .}\end{aligned}$$

For N_t not necessarily centered we have

$$\frac{N_t}{t} = \frac{N_t - \mathbb{E}N_t}{t} + \frac{\mathbb{E}N_t}{t} = \frac{N_t - \mathbb{E}N_t}{t} + \mathbb{E}N_1 \to \mathbb{E}N_1, \ (t \to \infty), \ \mathbb{P}\text{-a.s. .}$$

□

We now prove an asymptotic bound for processes satisfying the assumptions of Kolmogorov's continuity theorem. The proof is similar to [37, Lemmas 2.4 and 2.6] where the case of fractional Brownian Motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ is considered. However, note that we do not require $\gamma = 2$ in (3.3), hence here we can include fractional Brownian Motion with any Hurst parameter $H \in (0, 1)$ (see Lemma 3.5).

Lemma 3.3. *Let $(N_t)_{t\in\mathbb{R}}$ be a process on a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ with values in a Banach space V such that $(S1)$ holds. Assume that there exist constants $\gamma > 1$, $\alpha > 0$ and $C \in \mathbb{R}$ such that*

$$\mathbb{E}\|N_t - N_s\|_V^\gamma \leq C|t - s|^{1+\alpha}, \ \forall t, s \in \mathbb{R}. \tag{3.3}$$

Then there exists a θ_t-invariant set $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$ and for any $\epsilon > 0$, $\omega \in \Omega_0$, $0 < \beta < \frac{\alpha}{\gamma}$ and any interval $[s_0, t_0] \subseteq \mathbb{R}$ there exist constants $C_1 = C_1(\epsilon, \omega, \beta), C_2 = C_2(\omega, \beta, s_0, t_0) > 0$ such that

$$\begin{aligned} &\|N_t(\omega)\|_V \leq \epsilon|t|^2 + C_1, \ \forall t \in \mathbb{R} \text{ and} \\ &\|N_\cdot(\omega)\|_{C^\beta([s_0,t_0];V)} \leq C_2. \end{aligned}$$

In particular, N_t satisfies $(S4)$.

Proof. Since $\bar{N}_t := N_{-t}$ also satisfies the assumptions, it is enough to prove the assertion for $t > 0$. Firstly, we have

$$\begin{aligned} \|N_t(\omega)\|_V &= \|N_t(\omega) - N_{[t]}(\omega) + N_{[t]}(\omega) - N_{[t]-1}(\omega) + ... + N_1(\omega) - N_0(\omega) + N_0(\omega)\|_V \\ &= \|N_{t-[t]}(\theta_{[t]}\omega) - N_0(\theta_{[t]}\omega) + N_1(\theta_{[t]-1}\omega) - N_0(\theta_{[t]-1}\omega) + ... \\ &\quad + N_1(\omega) - N_0(\omega) + N_0(\omega)\|_V \\ &\leq \|N_\cdot(\theta_{[t]}\omega)\|_{C^\beta([0,1];V)} + \|N_\cdot(\theta_{[t]-1}\omega)\|_{C^\beta([0,1];V)} + ... + \|N_\cdot(\omega)\|_{C^\beta([0,1];V)} \\ &\quad + \|N_0(\omega)\|_V, \ t \in \mathbb{R}_+. \end{aligned} \tag{3.4}$$

Hence we need to derive a bound for $\|N_\cdot(\theta_r\omega)\|_{C^\beta([0,1];V)}$ as a function of r.

Using Kolmogorov's continuity theorem (cf. [29, Theorem 1.4.1]) and (3.3) we obtain

$$\|N_\cdot(\omega)\|_{C^\beta([s,t];V)} \leq K(\omega, \beta, s, t) \in L^\gamma(\Omega, \mathcal{F}^{\mathbb{P}}, \mathbb{P}) \subseteq L^1(\Omega, \mathcal{F}^{\mathbb{P}}, \mathbb{P}), \ \forall \ s < t,$$

where $\mathcal{F}^{\mathbb{P}}$ is the completion of $\mathcal{F}$ with respect to $\mathbb{P}$. Note that

$$\begin{aligned} \|N_\cdot(\theta_r\omega)\|_{C^\beta([s,t];V)} &= \sup_{u \neq v, u, v \in [s,t]} \frac{\|N_u(\theta_r\omega) - N_v(\theta_r\omega)\|_V}{|u - v|^\beta} \\ &= \sup_{u \neq v, u, v \in [s,t]} \frac{\|N_{u+r}(\omega) - N_{v+r}(\omega)\|_V}{|u - v|^\beta} \\ &= \|N_\cdot(\omega)\|_{C^\beta([s+r,t+r];V)}, \end{aligned} \tag{3.5}$$

for all $s < t$ and $r \in \mathbb{R}$. Hence

$$\sup_{r \in [0,1]} \|N_\cdot(\theta_r\omega)\|_{C^\beta([s,t];V)} \leq \|N_\cdot(\omega)\|_{C^\beta([s,t+1];V)} \in L^1(\Omega), \ \forall \ s < t.$$

The dichotomy of linear growth for stationary processes (cf. [2, Proposition 4.1.3 (ii)]) states that any measurable map $f : \Omega \to \mathbb{R}$ on a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ with $\sup_{t \in [0,1]} f^+(\theta_t \cdot) \in L^1(\Omega)$ grows sublinearly, i.e.

$$\limsup_{t \to \pm\infty} \frac{1}{|t|} f(\theta_t\omega) = 0,$$

on an invariant set of full $\mathbb{P}$ measure. We conclude that there is a θ_t-invariant set $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that

$$\lim_{|t| \to \infty} \frac{1}{|t|} \|N_\cdot(\theta_t\omega)\|_{C^\beta([0,1];V)} = 0, \ \forall \ \omega \in \Omega_0.$$

Hence for every $\epsilon > 0$, $\omega \in \Omega_0$ there exists a constant $T := T(\epsilon, \omega) \in \mathbb{N}$ such that

$$\|N_\cdot(\theta_t\omega)\|_{C^\beta([0,1];V)} \leq \epsilon|t|, \ |t| \geq T.$$

By (3.4) this implies

$$\begin{aligned}\|N_t(\omega)\|_V &\leq \sum_{k=T}^{[t]} \|N_\cdot(\theta_k\omega)\|_{C^\beta([0,1];V)} + \sum_{k=0}^{T-1} \|N_\cdot(\theta_k\omega)\|_{C^\beta([0,1];V)} + \|N_0(\omega)\|_V \\ &\leq \epsilon[t]^2 + T\|N_\cdot(\omega)\|_{C^\beta([0,T];V)} + \|N_0(\omega)\|_V \\ &\leq \epsilon[t]^2 + TK(\omega, \beta, 0, T) + \|N_0(\omega)\|_V, \ t \in \mathbb{R}_+.\end{aligned}$$

□

Corollary 3.4. *Let $(N_t)_{t\in\mathbb{R}}$ be a V-valued process with stationary increments and a.s. cádlág paths. Assume that (3.3) or the assumptions of Lemma 3.2 hold, then there is a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ and a version $\tilde{N}_t$ on $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ such that $\tilde{N}_t$ satisfies $(S1)$-$(S4)$.*

Now we show that (3.3) holds for fractional Brownian Motion (fBM) with any Hurst parameter. We first recall the definition of Banach space-valued fBM.

Definition 3.1 (Fractional Brownian Motion)**.** Let $H \in (0, 1)$ and $R : V^* \to V$ be a bounded linear and non-negative symmetric operator. A V-valued $\mathbb{P}$-a.s. continuous centered Gaussian process B_t^H starting at 0 is called an R-fBM with Hurst parameter H if the covariance is given by

$$\mathbb{E}[\ {}_{V^*}\langle x, B_t^H\rangle_V \ {}_{V^*}\langle y, B_s^H\rangle_V] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})\ {}_{V^*}\langle x, Ry\rangle_V$$

for all $x, y \in V^*$, $t, s \in \mathbb{R}_+$.

It is easy to see that B_t^H has stationary increments. Thus, according to Lemma 3.1 we will always consider the canonical realization of fBM in this paper.

Let

$$V \subseteq H \subseteq V^*$$

be a Gelfand triple, B_t^H be an R-fBM in H and $\lambda_k \geq 0$, $e_k \in H$ such that $Re_k = \lambda_k e_k$. Then B_t^H has the representation

$$B_t^H = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k^H(t)e_k, \tag{3.6}$$

where β_k^H are independent real-valued fBM and the convergence holds $\mathbb{P}$-a.s. as well as in each $L^p(\Omega; H)$.

Lemma 3.5. *Let B_t^H be a fBM in H with representation (3.6) and assume that $K = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\|e_k\|_V < \infty$. Then B_t^H satisfies (3.3), more precisely, for each $m \in \mathbb{N}$ there is a constant $C > 0$ such that*

$$\mathbb{E}\ \|B_t^H - B_s^H\|_V^{2m} \leq CK|t-s|^{2Hm}, \ s, t \in \mathbb{R}.$$

Proof. By the comparability of Gaussian moments (cf. [30, Corollary 3.2]) we have

$$\big[\mathbb{E}\|B_t^H - B_s^H\|_V^{2m}\big]^{\frac{1}{2m}} = \big[\mathbb{E}\|B_{t-s}^H\|_V^{2m}\big]^{\frac{1}{2m}} \le C\mathbb{E}\|B_{t-s}^H\|_V,$$

where $C > 0$ is a constant depending only on m.

By our assumption we know that the convergence in (3.6) also holds in $L^1(\Omega; V)$. Hence we have

$$\begin{aligned}\big[\mathbb{E}\|B_t^H - B_s^H\|_V^{2m}\big]^{\frac{1}{2m}} &\le C\mathbb{E}\|B_{t-s}^H\|_V \\ &= C\lim_{N\to\infty}\mathbb{E}\|\sum_{k=1}^N \sqrt{\lambda_k}\beta_k^H(t-s)e_k\|_V \\ &\le C\lim_{N\to\infty}\sum_{k=1}^N \sqrt{\lambda_k}\|e_k\|_V\mathbb{E}|\beta_k^H(t-s)| \\ &\le CK|t-s|^H,\ s,t\in\mathbb{R}.\end{aligned}$$

In particular, choosing m such that $2Hm > 1$ we get (3.3). □

We now proceed to examples of SPDE satisfying $(H1)-(H5)$ and $(H2')$. Note that most of those assumptions are well known and have been used extensively in recent years for investigating SPDE within the variational framework, e.g. see $(H1)-(H4)$ in [23, 24, 33, 34, 35, 40] and $(H2')$ in [18, 33, 35]. It has also been proved that (1.5) in $(H5)$ holds for many SPDE in [34]. Hence, we only need to verify (1.6) in $(H5)$.

The following elementary lemma is crucial for verifying $(H2')$ (cf. [33, 35]). For the proof see e.g. [35].

Lemma 3.6. *Let $(E, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\|\cdot\|$ denote its norm. Then for any $r \ge 0$ we have*

$$\langle\|a\|^r a - \|b\|^r b, a-b\rangle \ge 2^{-r}\|a-b\|^{r+2},\ a,b\in E. \tag{3.7}$$

Example 3.7. *Let Λ be an open bounded domain in $\mathbb{R}^d$ and $L^p := L^p(\Lambda)$ for some fixed $p \ge 2$. Consider the following triple*

$$V := L^p \subseteq H := L^2 \subseteq (L^p)^* \equiv L^{\frac{p}{p-1}}$$

and the stochastic equation

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + \mathrm{d}N_t,\ t\in\mathbb{R}, \tag{3.8}$$

where N_t is an L^p-valued process with stationary increments and a.s. cádlág paths, $f:\mathbb{R}^d\to\mathbb{R}^d$ is continuous and satisfies the following conditions:

$$\begin{aligned}&\langle f(x)-f(y), x-y\rangle \le -\lambda|x-y|^\beta; \\ &\langle f(x), x\rangle \le -\delta|x|^p + K|x|^2 + C; \\ &|f(x)| \le C(|x|^{p-1}+1),\ x,y\in\mathbb{R}^d,\end{aligned} \tag{3.9}$$

where $C, \lambda > 0, \delta > 0, \beta > 2$ are some constants and $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^d$. Then the RDS generated by (3.8) *has a unique random fixed point and the other assertions in Theorem 1.4 also hold. If $\beta = 2$ in (3.9), then the conclusions still hold, provided N_t also satisfies* $(S4)$.

Proof. Using a similar argument as in [40, Example 4.1.5], one can show that $(H1)$,$(H2')$, $(H3)$ and $(H4)$ hold for (3.8) with $\alpha = p$. Hence Theorem 1.4 applies. □

Remark 3.1. (i) A typical example for f is as follows (cf. [34, 35, 40]):

$$f(x) = -|x|^{p-2}x + \eta x, \ \eta \leq 0.$$

(ii) The first inequality in (3.9) implies that

$$\langle f(x), x \rangle \leq -\frac{\lambda}{2}|x|^{\beta} + C.$$

Therefore, if $\beta \geq p$, then the second inequality (so called coercivity condition) in (3.9) automatically holds.

(iii) If N_t is a finite-dimensional fBM, the existence of a random fixed point for (3.8) has also been studied in [21]. Compared with the result in [21], we only require a coercivity condition (the second inequality in (3.9)) on f instead of assuming f to be continuously differentiable as in [21]. Another improvement is that we can allow equation (3.8) to be driven by infinitely many fractional Brownian motions or by Lévy noise.

Example 3.8. *(Stochastic reaction-diffusion equation)*
Let Λ be an open bounded domain in $\mathbb{R}^d$. We consider the following triple

$$V := W_0^{1,2}(\Lambda) \subseteq L^2(\Lambda) \subseteq (W_0^{1,2}(\Lambda))^*$$

and the stochastic reaction-diffusion equation

$$\mathrm{d}X_t = (\Delta X_t - |X_t|^{p-2}X_t + \eta X_t)\mathrm{d}t + \mathrm{d}N_t, \tag{3.10}$$

where $1 \leq p \leq 2$ and η are some constants, N_t is a V-valued process with stationary increments and a.s. cádlág paths.

(1) If $\eta \leq 0$ and $(S4)$ holds, then all assertions in Theorem 1.4 hold for (3.10) *with $\beta = 2$.*

(2) If $\eta > 0$, $N_\cdot(\omega) \in L^2([-1, 0]; W^{3,2}(\Lambda))$ for $\mathbb{P}$-a.e. ω and satisfies $(S4)$, then the stochastic flow associated with (3.10) *has a compact random attractor.*

Proof. (1) By Lemma 3.1 we know that $(S1)$-$(S3)$ hold. It is also well known that $(H1)$-$(H4)$ hold for (3.10) (cf. [33, 35, 40]). If $\eta \leq 0$, then it is easy to show that $(H2')$ holds with $\beta = 2$. Therefore, all assertions in Theorem 1.4 hold for (3.10).

(2) According to Theorem 1.3 one only needs to verify $(H5)$. Let $S = W_0^{1,2}(\Lambda)$ and Δ be the Laplace operator on $L^2(\Lambda)$ with Dirichlet boundary conditions. We define

$$T_n = -\Delta(1 - \frac{\Delta}{n})^{-1}.$$

Let $\{P_t\}_{t\geq 0}$ and $\mathcal{E}$ denote the semigroup and Dirichlet form corresponding to Δ. It is easy to show that T_n are continuous operators on $W_0^{1,2}(\Lambda)$ by noting that

$$T_n = n\left(I - \left(I - \frac{\Delta}{n}\right)^{-1}\right).$$

Then we have

$$\begin{aligned}
{}_{V^*}\langle \Delta u, T_n u\rangle_V &= {}_{V^*}\langle \Delta u, -\Delta(1-\frac{\Delta}{n})^{-1}u\rangle_V \\
&= {}_{V^*}\langle \Delta u, nu - n(1-\frac{\Delta}{n})^{-1}u\rangle_V \\
&= -n\int_0^\infty e^{-t}\langle \nabla u, \nabla u - \nabla P_{\frac{t}{n}}u\rangle_{L^2(\Lambda)}\mathrm{d}t \\
&\leq -n\int_0^\infty e^{-t}(\mathcal{E}(u,u) - \mathcal{E}(u, P_{\frac{t}{n}}u))\mathrm{d}t \\
&\leq 0,
\end{aligned}$$

where the last step follows from the contraction property of the Dirichlet form $\mathcal{E}$.

By using a similar argument one can show that

$${}_{V^*}\langle -|u|^{p-2}u + \eta u, T_n u\rangle_V \leq \eta\|u\|_n^2,\ u \in W_0^{1,2}(\Lambda).$$

Hence (1.5) holds. Using the fact that P_t is bounded on $W_0^{1,2}(\Lambda)$ and $N_\cdot(\omega) \in L^2([-1,0]; W^{3,2}(\Lambda))$ for $\mathbb{P}$-*a.e.* ω we have

$$\begin{aligned}
\int_{-1}^0 \|T_n N_t\|_V^2 dt &= \int_{-1}^0 \|-\Delta(I-\frac{\Delta}{n})^{-1}N_t\|_V^2 dt \\
&= \int_{-1}^0 \|(I-\frac{\Delta}{n})^{-1}(\Delta N_t)\|_V^2 dt \\
&\leq C\int_{-1}^0 \|\Delta N_t\|_V^2 dt < \infty,
\end{aligned} \tag{3.11}$$

where the third step follows from the following formula

$$(I-\frac{\Delta}{n})^{-1}v = \int_0^\infty e^{-t}P_{\frac{t}{n}}v\mathrm{d}t,\ v \in V.$$

Hence (1.6) holds. Then the existence of the random attractor for (3.10) follows from Theorem 1.3. □

Remark 3.2. In Example 3.8 we had to restrict to reaction terms of at most linear growth. This restriction is due to the fact that the variational approach to SPDE as presented in [27, 40] does not apply to nonlinearities of arbitrary high order. However, we only used the results from [27, 40] to construct the associated RDS. Therefore, as soon as we can obtain the corresponding RDS by some other method, our arguments can be used without change

to prove the existence of the random attractor. More precisely, let Λ be an open bounded domain in $\mathbb{R}^d$. We consider the following triple

$$V := W_0^{1,2}(\Lambda) \cap L^p(\Lambda) \subseteq H := L^2(\Lambda) \subseteq (W_0^{1,2}(\Lambda) \cap L^p(\Lambda))^*$$

and the stochastic reaction-diffusion equation

$$\mathrm{d}X_t = (\Delta X_t - |X_t|^{p-2}X_t + \eta X_t)\mathrm{d}t + \mathrm{d}N_t, \tag{3.12}$$

where $2 < p$, $\eta \in \mathbb{R}$ are some constants and N_t is a V-valued process with stationary increments and a.s. cádlág paths. Note that (3.12) does not satisfy $(H3)$-$(H4)$ with the same parameter α. Nevertheless, the associated RDS can be defined by an analogous transformation into a random PDE. The existence and uniqueness of solutions for the transformed equation (1.2) follows by a standard proof via Galerkin approximations (cf. [45, pp. 91]). The proof of condition $(H5)$ carries over without change. If $\eta \leq 0$ then Theorem 1.4 can be applied with $\beta = p$. If $\eta > 0$, $N_\cdot(\omega) \in L^2([-1,0]; W_0^{3,2}(\Lambda)) \cap L^p([-1,0]; W_0^{2,p}(\Lambda))$ for $\mathbb{P}$-*a.e.* ω and satisfies $(S4)$, then the same arguments as for Theorem 1.3 yield the existence of the random attractor.

In [17, Section 5] the existence of a random attractor for stochastic reaction diffusion equations perturbed by finite dimensional Brownian noise is obtained under the assumption that the noise takes values in $H^2(\Lambda) \cap H_0^1(\Lambda) \cap W^{2,\frac{p}{p-1}}(\Lambda)$. In comparison, we can allow infinite dimensional noise and include fractional Brownian motion as well as Lévy type noise, but we need to require slightly more regular noise taking values in $H^3(\Lambda) \cap H_0^1(\Lambda) \cap W^{2,p}(\Lambda)$.

Remark 3.3. Simple examples of noises satisfying the assumptions are given by finite dimensional noise. Let $N \in \mathbb{N}$ and

$$N_t = \sum_{n=1}^{N} \varphi_n \beta_n^H(t) \left(\text{or} \quad N_t = \sum_{n=1}^{N} \varphi_n L_n(t) \right), \ t \in \mathbb{R}, \tag{3.13}$$

where $\varphi_n \in W_0^{3,2}(\Lambda) \cap W_0^{2,p}(\Lambda)$ and β_n^H are independent two-sided fractional Brownian motions with Hurst parameter $H \in (0,1)$ (or L_n are independent two-sided Lévy processes). It is easy to show that the noise (3.13) satisfies all assumptions required in the above example. Noise of this form can also be used for those examples below as well by choosing appropriate spaces for φ_n.

Example 3.9. *(stochastic porous media equation)*
Let Λ be an open bounded domain in $\mathbb{R}^d$. For $r > 1$ we consider the following triple

$$V := L^{r+1}(\Lambda) \subseteq H := W_0^{-1,2}(\Lambda) \subseteq V^*$$

and the stochastic porous media equation

$$\mathrm{d}X_t = \left(\Delta(|X_t|^{r-1}X_t) + \eta X_t\right)\mathrm{d}t + \mathrm{d}N_t, \tag{3.14}$$

where η is a constant, N_t is a V-valued process with stationary increments and a.s. cádlág paths.

(1) If $N.(\omega) \in L^{r+1}([-1,0];W^{2,r+1}(\Lambda))$ for $\mathbb{P}$-a.e. ω and satisfies $(S4)$, then the stochastic flow associated with (3.14) has a compact random attractor.

(2) If $\eta \leq 0$, then all assertions in Theorem 1.4 hold for (3.14).

Proof. (1) According to [40, Example 4.1.11; Remark 4.1.15] we know that $(H1)$-$(H4)$ hold for (3.14). By Lemma 3.1 and the assumptions we know $(S1)$-$(S4)$ also hold. Hence we only need to verify $(H5)$ in Theorem 1.3.

Let $S = L^2(\Lambda)$ and Δ be the Laplace operator on $L^2(\Lambda)$ with Dirichlet boundary conditions. We define

$$T_n = -\Delta(I - \frac{\Delta}{n})^{-1} = n\left(I - (I - \frac{\Delta}{n})^{-1}\right).$$

It is well known that the heat semigroup $\{P_t\}$ (generated by Δ) is contractive on $L^p(\Lambda)$ for any $p > 1$. Then by the same argument as in (3.11) we know that (1.6) holds.

In order to show that T_n are continuous operators on $L^{r+1}(\Lambda)$ we use the formula

$$(I - \frac{\Delta}{n})^{-1}u = \int_0^\infty e^{-t}P_{\frac{t}{n}}u\mathrm{d}t.$$

By Hölder's inequality and the contractivity of $\{P_t\}$ on $L^{r+1}(\Lambda)$ we have

$$\begin{aligned}
& {}_{V^*}\langle \Delta(|u|^{r-1}u) + \eta u, -\Delta(1 - \frac{\Delta}{n})^{-1}u\rangle_V \\
= \;& \langle |u|^{r-1}u, nu - n(1 - \frac{\Delta}{n})^{-1}u\rangle_{L^2} + \eta\|u\|_n^2 \\
= \;& -n\int_0^\infty e^{-t}\left(\int_\Lambda |u|^{r+1}\mathrm{d}x - \int_\Lambda |u|^{r-1}u \cdot P_{\frac{t}{n}}u\mathrm{d}x\right)\mathrm{d}t + \eta\|u\|_n^2 \\
\leq \;& \eta\|u\|_n^2, \ \forall u \in L^{r+1}(\Lambda).
\end{aligned}$$

Hence (1.5) holds and the assertion follows from Theorem 1.3.

(2) If $\eta \leq 0$, then by Lemma 3.6 it is easy to show that $(H2')$ holds with $\beta = r + 1$ (cf. [33, 35]). Hence all assertions in Theorem 1.4 hold for (3.14). □

Remark 3.4. In [9] the existence of a random attractor for generalized porous media equations perturbed by finite dimensional Brownian noise has been proven under the assumption that the noise takes values in $W_0^{1,r+1}$. In the case of the standard porous medium equation our results thus extend [9] to infinite dimensional noise and fractional Brownian motion as well as Lévy type noise, if the noise is more regular, i.e. takes values in $W^{2,r+1}$.

Example 3.10. *(Stochastic p-Laplace equation)*
Let Λ be an open bounded domain in $\mathbb{R}^d$ with convex and smooth boundary. We consider the following triple

$$V := W^{1,p}(\Lambda) \subseteq H := L^2(\Lambda) \subseteq (W^{1,p}(\Lambda))^*$$

and the stochastic p-Laplace equation

$$\mathrm{d}X_t = \left[\mathbf{div}(|\nabla X_t|^{p-2}\nabla X_t) - \eta_1|X_t|^{\tilde{p}-2}X_t + \eta_2 X_t\right]\mathrm{d}t + \mathrm{d}N_t, \tag{3.15}$$

where $2 < p < \infty, 1 \leq \tilde{p} \leq p$, $\eta_1 \geq 0$, $\eta_2 \in \mathbb{R}$ are some constants and N_t is a V-valued process with stationary increments and a.s. cádlág paths.

(1) If $N.(\omega) \in L^p([-1,0];W^{3,p}(\Lambda))$ for $\mathbb{P}$-a.e. ω and satisfies $(S4)$, then the stochastic flow associated with (3.15) *has a compact random attractor.*

(2) If $\eta_2 \leq 0$, then all assertions in Theorem 1.4 hold for (3.15).

Proof. (1) According to [40, Example 4.1.9] and the assumptions, we only need to verify $(H5)$ in Theorem 1.3.

Let $S = W^{1,2}(\Lambda) = \mathcal{D}(\sqrt{-\Delta})$, where Δ is the Laplace operator on $L^2(\Lambda)$ with Neumann boundary conditions. It is well known that the corresponding semigroup $\{P_t\}$ is the Neumann heat semigroup (i.e. the corresponding Markov process is Brownian Motion with reflecting boundary conditions). Moreover, we know that P_t maps $L^p(\Lambda)$ into $W^{1,p}(\Lambda)$ continuously (see [13, Section 2] for more general results). Then for all $t \geq 0$, $P_t : W^{1,p}(\Lambda) \to W^{1,p}(\Lambda)$ is continuous.

Now we define

$$T_n = -\Delta(I - \frac{\Delta}{n})^{-1} = n\left(I - (I - \frac{\Delta}{n})^{-1}\right).$$

It is easy to show that T_n are also continuous operators on $W^{1,p}(\Lambda)$ since

$$(I - \frac{\Delta}{n})^{-1}u = \int_0^\infty e^{-t} P_{\frac{t}{n}} u \mathrm{d}t.$$

Moreover, since the boundary of the domain is convex and smooth, we have the following gradient estimate (cf. [48, Theorem 2.5.1])

$$|\nabla P_t u| \leq P_t |\nabla u|, \quad u \in W^{1,p}(\Lambda). \tag{3.16}$$

Since $\{P_t\}$ is a contractive semigroup on $L^p(\Lambda)$, it is easy to see that $\{P_t\}$ is also a contractive semigroup on $W^{1,p}(\Lambda)$ by (3.16). Therefore,

$$\begin{aligned}
& {}_{V^*}\langle \mathbf{div}(|\nabla u|^{p-2}\nabla u), T_n u\rangle_V \\
= \ & {}_{V^*}\langle \mathbf{div}(|\nabla u|^{p-2}\nabla u), nu - n(1 - \frac{\Delta}{n})^{-1}u\rangle_V \\
= \ & n\int_0^\infty e^{-t} {}_{V^*}\langle \mathbf{div}(|\nabla u|^{p-2}\nabla u), u - P_{\frac{t}{n}}u\rangle_V \mathrm{d}t \\
= \ & -n\int_0^\infty e^{-t}\left(\int_\Lambda |\nabla u|^p \mathrm{d}x - \int_\Lambda |\nabla u|^{p-2}\nabla u \cdot \nabla P_{\frac{t}{n}} u \mathrm{d}x\right)\mathrm{d}t \\
\leq \ & 0, \ u \in W^{1,p}(\Lambda),
\end{aligned}$$

where in the last step we used Hölder's inequality and the contractivity of $\{P_t\}$ on $W^{1,p}(\Lambda)$

to conclude

$$\begin{aligned}
&\int_\Lambda |\nabla u|^{p-2}\nabla u\cdot\nabla P_s u\mathrm{d}x\\
\leq\ &\left(\int_\Lambda |\nabla u|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\cdot\left(\int_\Lambda |\nabla P_s u|^p\mathrm{d}x\right)^{\frac{1}{p}}\\
\leq\ &\left(\int_\Lambda |\nabla u|^p\right)^{\frac{p-1}{p}}\cdot\left(\int_\Lambda |P_s|\nabla u\|^p\mathrm{d}x\right)^{\frac{1}{p}}\\
\leq\ &\int_\Lambda |\nabla u|^p\mathrm{d}x.
\end{aligned}$$

Using the same argument we obtain

$$_{V^*}\langle -\eta_1|u|^{\tilde{p}-2}u-\eta_2 u, T_n u\rangle_V \leq \eta_2\|u\|_n^2,\ u\in W^{1,p}(\Lambda).$$

Hence (1.5) holds.

Note that (1.6) also holds due to the same argument as in (3.11). Therefore, the assertion follows from Theorem 1.3.

(2) If $\eta_2\leq 0$, then by Lemma 3.6 it is easy to show that $(H2')$ holds with $\beta=p$ (cf. [33, 35]). Hence all assertions in Theorem 1.4 hold for (3.15). □

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