

# Harnack Inequalities and Applications for Ornstein-Uhlenbeck Semigroups with Jump\*

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## Abstract

The Harnack inequality established in [11] for generalized Mehler semigroup is improved and generalized. As applications, the log-Harnack inequality, the strong Feller property, the hyper-bounded property, and some heat kernel inequalities are presented for a class of O-U type semigroups with jump. These inequalities and semigroup properties are indeed equivalent, and thus sharp, for the Gaussian case. As an application of the log-Harnack inequality, the HWI inequality is established for the Gaussian case. Perturbations with linear growth are also investigated.

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## 1 Introduction

In this paper we aim to establish Harnack inequalities and applications for a class of Ornstein-Uhlenbeck type SDEs driven by Lévy noises on Hilbert spaces. This problem has been investigated in [11] by using Mehler type formula for the associated semigroups and gradient estimates for dimension-free Harnack inequalities developed in [12]. In this paper, we shall adopt a measure transformation argument to derive a more general and sharper Harnack inequality, and to present finer estimates of heat kernels. This method

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was initiated in [1] by using coupling and Girsanov transformation to establish Harnack inequalities for diffusion semigroups on manifolds with unbounded below curvature, and has been applied in [13, 8, 3] for non-linear SPDEs driven by Gaussian noises and also in [10] for diffusions with singular drifts and multivalued stochastic evolution equations. In this paper we shall modify this argument to SPDEs with jumps.

Let us first recall the Harnack inequality derived in [11]. Let  $\mathbb{H}$  be a real separable Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\|$ . Consider the following Lévy driven stochastic differential equation

$$(1.1) \quad dX_t = AX_t dt + dZ_t, \quad X_0 = x \in \mathbb{H},$$

where  $A$  is the infinitesimal generator of a strongly continuous semigroup  $(T_t)_{t \geq 0}$  on  $\mathbb{H}$ ,  $Z_t := \{Z_t^u, u \in \mathbb{H}\}$  is a cylindrical Lévy process with characteristic triplet  $(a, R, M)$  on some filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ , that is, for every  $u \in \mathbb{H}$  and  $t \geq 0$

$$\begin{aligned} \mathbb{E} \exp(i\langle Z_t, u \rangle) &= \exp \left[ it\langle a, u \rangle - \frac{t}{2} \langle Ru, u \rangle \right. \\ &\quad \left. - \int_{\mathbb{H}} [1 - \exp(i\langle x, u \rangle) + i\langle x, u \rangle 1_{\{|x| \leq 1\}}(x)] M(dx) \right], \end{aligned}$$

where  $a \in \mathbb{H}$ ,  $R$  is a symmetric linear operator on  $\mathbb{H}$  such that

$$R_t := \int_0^t T_s R T_s^* ds$$

is trace class for each  $t > 0$ , and  $M$  is a Lévy measure on  $\mathbb{H}$ . (For simplicity, we shall write  $Z_t^u = \langle Z_t, u \rangle$  for every  $u \in \mathbb{H}$ .) In this case, (1.1) has a unique mild solution

$$X_t = T_t x + \int_0^t T_{t-s} dZ_s, \quad t \geq 0.$$

Let

$$P_t f(x) = \mathbb{E} f(X_t), \quad x \in \mathbb{H}, \quad f \in \mathcal{B}_b(\mathbb{H}),$$

where  $\mathcal{B}_b(\mathbb{H})$  is the space of all bounded measurable functions on  $\mathbb{H}$ . Similarly, let  $\mathcal{B}_b^+(\mathbb{H})$ ,  $\mathcal{C}_b(\mathbb{H})$ ,  $\mathcal{C}^\infty(\mathbb{H})$  be the classes of bounded positive, bounded continuous, and smooth functions on  $\mathbb{H}$  respectively. Let  $G$  be the orthogonal complement of  $\text{Ker} R^{1/2}$ . Then the inverse  $R^{-1/2}$  of  $R^{1/2}$  is well defined from  $R^{1/2}\mathbb{H}$  to  $G$ . The following is the main result derived in [11].

**Theorem 1.1.** ([11]) *Assume that there exists a sequence of eigenvectors of  $A^*$  separating the points of  $\mathbb{H}$ ,  $R$  is of trace class, and  $T_t R \mathbb{H} \subset R^{1/2} \mathbb{H}$  holds for all  $t > 0$ . If*

$$(1.2) \quad \|R^{-1/2} T_t R x\| \leq \sqrt{h(t)} \|R^{1/2} x\|, \quad x \in \mathbb{H}, t \geq 0$$

*holds for some positive function  $h \in C[0, \infty)$ . Then for any  $f \in \mathcal{B}_b^+(\mathbb{H})$ ,*

$$(1.3) \quad (P_t f)^2(x) \leq \exp \left[ \frac{\|R^{-1/2}(x-y)\|^2}{\int_0^t h(s)^{-1} ds} \right] P_t f^2(y), \quad t > 0, x-y \in R^{1/2} \mathbb{H}.$$

If  $M = 0$ , then for any  $\alpha > 1$  and  $f \in \mathcal{B}_b^+(\mathbb{H})$ ,

$$(1.4) \quad (P_t f)^\alpha(x) \leq \exp \left[ \frac{\alpha \|R^{-1/2}(x-y)\|^2}{2(\alpha-1) \int_0^t h(s)^{-1} ds} \right] P_t f^\alpha(y), \quad t > 0, x-y \in R^{1/2}\mathbb{H}.$$

We note that due to the absence of a chain rule, for the case with jump (i.e.  $M \neq 0$ ), the Harnack inequality was proved in [11] only for  $\alpha = 2$  (i.e. (1.3)) by using gradient estimates.

To improve this result, we shall adopt a measure transformation argument and the null controllability of the associated deterministic equation (see Section 2). As a result, we obtain the Harnack inequality by using the image norm  $\|\Gamma_t x\|$  of the operator

$$\Gamma_t := R_t^{-1/2} T_t \text{ with domain } \mathcal{D}(\Gamma_t) := \{x \in \mathbb{H} : T_t x \in R_t^{1/2}\mathbb{H}\}.$$

As explained above,  $R_t^{-1/2}$  is defined from  $R_t^{1/2}\mathbb{H}$  to the orthogonal complement of  $\text{Ker } R_t^{1/2}$ . By letting  $\|\Gamma_t x\| = \infty$  for  $x \notin \mathcal{D}(\Gamma_t)$  and  $\inf \emptyset = \infty$ , we have

$$\|\Gamma_t x\| = \inf \{ \|z\| : z \in \mathbb{H}, R_t^{1/2} z = T_t x \}, \quad x \in \mathbb{H}.$$

Our first result is an improvement of Theorem 1.1: our Harnack inequality generalizes (1.3) without the assumptions of Theorem 1.1. Moreover, our argument also implies the following inequality (1.6), which in particular implies the strong Feller property (even  $\|\Gamma_t \cdot\|$ -Lipschitz strong Feller property) of  $P_t$  if  $\Gamma_t$  is bounded.

**Theorem 1.2.** For any  $\alpha > 1$  and  $f \in \mathcal{B}_b^+(\mathbb{H})$ ,

$$(1.5) \quad (P_t f(x))^\alpha \leq \exp \left[ \frac{\alpha \|\Gamma_t(x-y)\|^2}{2(\alpha-1)} \right] P_t f^\alpha(y), \quad x, y \in \mathbb{H}, t > 0.$$

Consequently, (1.2) implies (1.4) for any  $f \in \mathcal{B}_b^+(\mathbb{H})$ . Moreover, for any  $f \in \mathcal{B}_b(\mathbb{H})$  and  $x, y \in \mathbb{H}$ ,

$$(1.6) \quad \begin{aligned} & |P_t f(x) - P_t f(y)|^2 \\ & \leq (e^{\|\Gamma_t(x-y)\|^2} - 1) \min \{ P_t f^2(x) - (P_t f(x))^2, P_t f^2(y) - (P_t f(y))^2 \}. \end{aligned}$$

When  $T_t \mathbb{H} \subset R_t^{1/2}\mathbb{H}$ ,  $\Gamma_t$  is a bounded operator by the closed graph theorem. In this case Theorem 1.2 implies the following result.

**Theorem 1.3.** Let  $t > 0$ . The following statements are gradually weaker, i.e. statement (i) implies statement (i+1) for  $1 \leq i \leq 4$ :

- (1)  $T_t \mathbb{H} \subset R_t^{1/2}\mathbb{H}$ ;
- (2)  $\|\Gamma_t\| < \infty$  and for any  $\alpha > 1$  and  $f \in \mathcal{B}_b^+(\mathbb{H})$ ,

$$(1.7) \quad (P_t f(x))^\alpha \leq \exp \left[ \frac{\alpha (\|\Gamma_t\| \cdot \|x-y\|)^2}{2(\alpha-1)} \right] P_t f^\alpha(y), \quad x, y \in \mathbb{H};$$

(3)  $\|\Gamma_t\| < \infty$  and there exists  $\alpha > 1$  such that (1.7) holds for all  $f \in \mathcal{B}_b^+(\mathbb{H})$ ;

(4)  $\|\Gamma_t\| < \infty$  and for any  $f \in \mathcal{B}_b^+(\mathbb{H})$  with  $f \geq 1$ ,

$$(1.8) \quad P_t \log f(x) \leq \log P_t f(y) + \frac{\|\Gamma_t\|^2}{2} \|x - y\|^2, \quad x, y \in \mathbb{H};$$

(5)  $P_t$  is strong Feller.

If, in particular,  $M = 0$ , then all the above statements are equivalent.

According to [6, Theorem 3.1],  $(P_t)$  has a unique invariant probability measure  $\mu$  provided

(A)  $\lim_{t \rightarrow \infty} \|T_t x\| = 0$  for  $x \in \mathbb{H}$ ;  $\sup_{t > 0} \text{Tr } R_t < \infty$ ;  $\int_0^\infty ds \int_{\mathbb{H}} (1 \wedge \|T_s x\|^2) M(dx) < \infty$ ;

$$\lim_{t \rightarrow \infty} \left\{ \int_0^t T_s a \, ds + \int_0^t ds \int_{\mathbb{H}} T_s x \left( \frac{1}{1 + \|T_s x\|^2} - \frac{1}{1 + \|x\|^2} \right) M(dx) \right\}$$

exists in  $\mathbb{H}$ ; and  $R$  is of trace class.

In this case, if  $P_t$  is strong Feller then it has a density  $p_t(x, y)$  w.r.t.  $\mu$  on  $\text{supp } \mu$ , the support of  $\mu$ . As observed in the recent paper [14], the Harnack inequality (1.7) and the log-Harnack inequality (1.8) are equivalent to the following inequalities for  $p_t(x, y)$  respectively:

$$(1.9) \quad \int_{\mathbb{H}} p_t(x, z) \left( \frac{p_t(x, z)}{p_t(y, z)} \right)^{\frac{1}{\alpha-1}} \mu(dz) \leq \exp \left[ \frac{\alpha (\|\Gamma_T\| \cdot \|x - y\|)^2}{2(\alpha - 1)^2} \right], \quad x, y \in \text{supp } \mu;$$

$$(1.10) \quad \int_{\mathbb{H}} p_t(x, y) \log \frac{p_t(x, z)}{p_t(y, z)} \mu(dz) \leq \frac{\|\Gamma_t\|^2}{2} \|x - y\|^2, \quad x, y \in \text{supp } \mu.$$

Moreover, if  $M = 0$ , by e.g. [5, Theorem 10.3.5], Theorem 1.3 (1) implies that  $P_t L^p(\mathbb{H}; \mu) \subset \mathcal{C}^\infty(\mathbb{H})$  for  $p > 1$  and  $t > 0$ . So, we have the following consequence of Theorem 1.3.

**Corollary 1.4.** *Let  $M = 0$  and assume that  $(P_t)$  has a invariant probability measure  $\mu$  with full support. Then for any  $t > 0$ , (1)–(5) in Theorem 1.3 and the following statements are equivalent:*

- (6) For any  $\alpha > 1$ , (1.9) holds;
- (7) For some  $\alpha > 1$ , (1.9) holds;
- (8) The entropy inequality (1.10) holds;
- (9) For any  $p > 1$ ,  $P_t L^p(\mathbb{H}; \mu) \subset \mathcal{C}^\infty(\mathbb{H})$ .

The following result is a standard consequence of the Harnack inequality (1.7), where (i) follows from [3, Proposition 4.1], (ii) follows from Lemma [11, Lemma 2.2], and the proof of (iii) is similar to the those of [11, Theorem 1.5 and Proposition 1.6] (see also [10] for details).

**Corollary 1.5.** *Assume that  $(P_t)$  has a invariant measure and that  $\Gamma_t$  is bounded for a fixed  $t > 0$ , and let  $p_t(x, y)$  be the density of  $P_t$  w.r.t.  $\mu$ . Then:*

(i)  $P_t L^p(\mathbb{H}; \mu) \subset \mathcal{C}(\mathbb{H})$  for any  $p > 1$ .

(ii) For any  $\alpha > 1$ ,

$$\|p_t(x, \cdot)\|_{L^{\alpha/(\alpha-1)}(\mathbb{H}; \mu)} \leq \left[ \int_{\mathbb{H}} \exp\left(-\frac{\alpha \|\Gamma_t\|^2}{2(\alpha-1)} \|x-y\|^2\right) \mu(dy) \right]^{-1/\alpha}, \quad x \in \text{supp}\mu.$$

(iii) If there exist some constants  $\varepsilon > 0$  and  $\alpha > 1$  such that

$$C(t, \alpha, \varepsilon) := \int_{\mathbb{H}} \left[ \int_{\mathbb{H}} \exp\left(-\frac{\alpha \|\Gamma_t\|^2}{2(\alpha-1)} \|x-y\|^2\right) \mu(dy) \right]^{-(1+\varepsilon)} \mu(dx) < \infty,$$

then  $P_t$  is hyper-bounded with

$$\|P_t\|_{\alpha \rightarrow \alpha(1+\varepsilon)} \leq C(t, \alpha, \varepsilon)^{\frac{1}{\alpha(1+\varepsilon)}}.$$

If  $C(t, \alpha, 0) < \infty$  then  $P_t$  is uniformly integrable in  $L^\alpha(\mathbb{H}; \mu)$  and hence  $P_s$  is compact on  $L^\alpha(\mathbb{H}, \mu)$  for every  $s > t$ .

We shall prove Theorems 1.2 and 1.3 in the next section, and present in Section 3 applications of the log-Harnack inequality to cost-entropy inequalities of the semigroup and the HWI inequality in the Gaussian case. Finally, in Section 4 we investigate the Harnack inequality and strong Feller property for a class of semi-linear stochastic equations by using a perturbation argument.

## 2 Proofs of Theorems 1.2 and 1.3

As explained in the last section, Corollary 1.4 is a direct consequence of Theorem 1.3. Since (2) implying (3) is trivial, (3) implying (4) and (4) implying (5) have been proved in [14] for Markov semigroups on abstract Polish spaces, and (5) implying (1) follows from [4, Theorem 9.19], it suffices to prove Theorem 1.2.

Consider the following linear control system on  $\mathbb{H}$

$$(2.1) \quad dx_t = Ax_t dt + R^{1/2}u_t dt, \quad x_0 = x \in \mathbb{H}.$$

According [15, Part IV, Theorem 2.3] (ref. also the appendix of [4] or [5]),

$$(2.2) \quad \|\Gamma_t x\|^2 = \inf \left\{ \int_0^t \|u_s\|^2 ds : u \in L^2([0, t] \rightarrow \mathbb{H}; ds), x_0 = x, x_t = 0 \right\}.$$

This implies the following upper bounds of  $\|\Gamma_t x\|$ .

**Proposition 2.1.** *Let  $t > 0$ . Then for any strictly positive  $\xi \in C([0, t])$ ,*

$$(2.3) \quad \|\Gamma_t x\|^2 \leq \frac{\int_0^t \|R^{-1/2} T_s x\|^2 \xi_s^2 ds}{\left(\int_0^t \xi_s ds\right)^2}, \quad x \in \mathbb{H},$$

where  $\|R^{-1/2} x\| = \infty$  if  $x \notin R^{1/2}\mathbb{H}$ . Consequently, (1.2) implies

$$(2.4) \quad \|\Gamma_t x\|^2 \leq \frac{\|R^{-1/2} x\|^2}{\int_0^t h(s)^{-1} ds}, \quad x \in \mathbb{H}.$$

*Proof.* We only need to consider the case that  $T_s x \in R^{1/2}\mathbb{H}$  for a.e.  $s \in [0, t]$  and  $\{\xi_s R^{-1/2} T_s x\}_{s \in [0, t]} \in L^2([0, t] \rightarrow \mathbb{H}; ds)$ . In this case, for

$$u_s := -\frac{\xi_s}{\int_0^t \xi_r dr} R^{-1/2} T_s x, \quad s \in [0, t],$$

one has a null control of the system (2.1); that is,  $u \in L^2([0, t] \rightarrow \mathbb{H}; ds)$  and

$$x_t := T_t x + \int_0^t T_{t-s} R^{1/2} u_s ds = 0.$$

Then (2.3) follows from (2.2) by taking  $\xi(s) = h(s)^{-1}$ ,  $s \in [0, t]$ .  $\square$

To prove the desired Harnack inequality, we adopt the following Girsanov theorem for Lévy processes. Let  $\|\cdot\|_0$  be the norm on  $\mathbb{H}_0 := R^{1/2}(\mathbb{H})$  with inner product  $\langle x, y \rangle_0 := \langle R^{-1/2} x, R^{-1/2} y \rangle$  for  $x, y \in \mathbb{H}_0$ .

**Proposition 2.2.** *Let  $t > 0$ . Suppose that  $(Z_s)_{0 \leq s \leq t}$  is an  $\mathbb{H}$ -valued Lévy process on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{0 \leq s \leq t}, \mathbb{P})$  with characteristic triplet  $(a, R, M)$ . Denote by  $Z'$  the Gaussian part of  $Z$ . For any  $\mathbb{H}_0$ -valued predictable process  $\psi_s$ , independent of  $Z_s - Z'_s$  such that*

$$s \mapsto \rho_s := \exp \left( \int_0^s \langle \psi_r, dZ'_r \rangle_0 - \frac{1}{2} \int_0^s \|\psi_r\|_0^2 dr \right)$$

is a  $\mathcal{F}_s$ -martingale, the process

$$[0, t] \ni s \mapsto \tilde{Z}_s := Z_s - \int_0^s \psi_r dr$$

is also a Lévy process with characteristic triplet  $(a, R, M)$  under the probability measure  $d\tilde{\mathbb{P}} := \rho_t d\mathbb{P}$ .

*Proof.* We write

$$\mathbb{E} \exp(i\langle Z_s, z \rangle) = \exp[-s\vartheta_1(z) - z\vartheta_2(z)], \quad z \in \mathbb{H},$$

where

$$\vartheta_1(z) := \frac{1}{2} \langle Rz, z \rangle$$

and

$$\vartheta_2(z) := -i \langle z, a \rangle + \int_{\mathbb{H}} [1 - \exp(i \langle z, x \rangle) + i \langle z, x \rangle 1_{\{|x| \leq 1\}}(x)] M(dx).$$

Correspondingly, the process  $Z_s$  is decomposed by  $Z_s = Z'_s + Z''_s$ , where  $Z'_s$  is the Gaussian part of  $Z_s$  with symbol  $\vartheta_1$ , and  $Z''_s$  is the jump process with symbol  $\vartheta_2$ .

By the Girsanov theorem for Wiener processes on Hilbert space (see [4, Theorem 10.14]),

$$\tilde{Z}'_s = Z'_s - \int_0^s \psi_r dr, \quad 0 \leq s \leq t$$

is an  $R$ -Wiener process under the probability measure  $\tilde{\mathbb{P}}$ . Consequently, for all  $0 \leq s \leq t$  and all  $z \in \mathbb{H}$ , by the martingale property of  $\rho_s$  we have

$$\mathbb{E}_{\tilde{\mathbb{P}}} [\exp(i \langle z, \tilde{Z}'_s \rangle)] = \mathbb{E} [\rho_s \exp(i \langle z, \tilde{Z}'_s \rangle)] = \mathbb{E} \exp(i \langle z, Z'_s \rangle) = \exp[-s \vartheta_1(z)],$$

where  $\mathbb{E}_{\tilde{\mathbb{P}}}$  is the expectation taken for  $\tilde{\mathbb{P}}$ . Combining this with the independence of  $Z'$  and  $Z''$ , we obtain

$$\mathbb{E}_{\tilde{\mathbb{P}}} \exp(i \langle z, \tilde{Z}_s \rangle) = \left( \mathbb{E} \rho_s \exp(i \langle z, \tilde{Z}'_s \rangle) \right) \mathbb{E} \exp(i \langle z, Z''_s \rangle) = \exp[-s \vartheta_1(z) - t \vartheta_2(z)].$$

Thus, under  $\tilde{\mathbb{P}}$  the characteristic symbol of  $\tilde{Z}_s$  is also  $\vartheta_1 + \vartheta_2$ . This completes the proof.  $\square$

By Proposition 2.2, we are able to establish the Harnack inequality by using the null controllability of the deterministic equation (2.1).

**Proposition 2.3.** *Let  $t > 0$  and  $x, y \in \mathbb{H}$ . Suppose that there exists  $u \in L^2([0, t] \rightarrow \mathbb{H}; ds)$  such that  $x_t = 0$ , where  $x_s$  solves (2.1) with  $x_0 = y - x$ . Then for any  $\alpha > 1$ ,*

$$(2.5) \quad (P_t f)^\alpha(x) \leq \exp\left(\frac{\alpha}{2(\alpha-1)} \int_0^t \|u_s\|^2 ds\right) P_t f^\alpha(y), \quad f \in \mathcal{B}_b^+(\mathbb{H}).$$

Moreover, for any  $f \in \mathcal{B}_b(\mathbb{H})$  and  $x, y \in \mathbb{H}$ ,

$$(2.6) \quad |P_t f(x) - P_t f(y)|^2 \leq (e^{\int_0^t \|u_s\|^2 ds} - 1) \{P_t f^2(y) - (P_t f(y))^2\}.$$

*Proof.* Let  $(Z'_s)_{0 \leq s \leq t}$  be the Gaussian part of the Lévy process  $Z_s$ , which is an  $R$ -Wiener process on  $\mathbb{H}$ . Let  $\psi_s = R^{1/2} u_s \in \mathbb{H}_0$  for  $s \in [0, t]$ . Then by Proposition 2.2,

$$\tilde{Z}_s := Z_s - \int_0^s \psi_r dr, \quad 0 \leq s \leq t$$

is a Lévy process with characteristic triplet  $(a, R, M)$  under the probability measure  $\tilde{\mathbb{P}}$ .

Let

$$\begin{aligned} Y_t^y &= S_t y + \int_0^t S_{t-s} dZ_s, \\ X_t^x &= S_t x + \int_0^t S_{t-s} d\tilde{Z}_s. \end{aligned}$$

Then, by the definition of  $P_t$  and since  $Z_s$  and  $\tilde{Z}_s$  are cylindrical Lévy processes with characteristic triplet  $(a, R, M)$  under  $\mathbb{P}$  and  $\tilde{\mathbb{P}}$  respectively, we have

$$(2.7) \quad P_t f(y) = \mathbb{E}f(Y_t^y), \quad P_t f(x) = \mathbb{E}_{\tilde{\mathbb{P}}}f(X_t^x) = \mathbb{E}[\rho_t f(X_t^x)], \quad f \in \mathcal{B}_b(\mathbb{H}).$$

Moreover, it is easy to see that

$$X_s^x = Y_s^y - x_s, \quad s \in [0, t].$$

So,  $X_t^x = Y_t^y$  due to  $x_t = 0$ . Combining this with (2.7), for any  $f \in \mathcal{B}_b^+(\mathbb{H})$  we have

$$\begin{aligned} \mathbb{P}_t f(x) &= \mathbb{E}[\rho_t f(X_t^x)] = \mathbb{E}[\rho_t f(Y_t^y)] \\ &\leq (\mathbb{E}\rho_t^{\alpha/(\alpha-1)})^{(\alpha-1)/\alpha} (\mathbb{E}f^\alpha(Y_t^y))^{1/\alpha} = (\mathbb{E}\rho_t^{\alpha/(\alpha-1)})^{\alpha-1/\alpha} (P_t f^\alpha(y))^{1/\alpha}. \end{aligned}$$

This implies (2.5) by noting that

$$\mathbb{E}\rho_t^{\alpha/(\alpha-1)} = \exp \left[ \frac{\alpha}{2(\alpha-1)^2} \int_0^t \|\psi_s\|_0^2 ds \right].$$

Similarly, since  $\mathbb{E}\rho_t = 1$ , we have

$$\begin{aligned} |P_t f(x) - P_t f(y)|^2 &= |\mathbb{E}f(\rho_t X_t^x) - \mathbb{E}f(Y_t^y)|^2 \\ &= |\mathbb{E}\{(\rho_t - 1)(f(Y_t^y) - P_t f(y))\}|^2 \leq \{P_t f^2(y) - (P_t f(y))^2\} \mathbb{E}|\rho_t - 1|^2. \end{aligned}$$

This implies (2.6) by noting that

$$\mathbb{E}(\rho_t - 1)^2 = \mathbb{E}\rho_t^2 - 1 = e^{\int_0^t \|u_s\|^2 ds} - 1.$$

□

*Proof of Theorem 1.2.* Combining (2.2) with (2.5), we obtain (1.5). If (1.2) holds, then (1.4) follows from (2.5) according to Proposition 2.1. Finally, (1.6) follows from (2.2) and (2.6), where the latter holds also by exchanging the positions of  $x$  and  $y$ . □



### 3 Application to the HWI inequality

The HWI inequality, established in [9] and reproved in [2] for symmetric diffusions on finite dimensional Riemannian manifolds, links the entropy, information and the transportation-cost. In this section, we shall prove it for the present non-symmetric infinite-dimensional model.

Throughout this section we assume that

(A')  $P_t$  has an invariant probability measure  $\mu$ .

This assumption follows from assumption (A) as explained in Section 1. We first observe that the log-Harnack inequality (1.8) implies an entropy-cost inequality for  $P_t^*$ , the joint operator of  $P_t$  on  $L^2(\mathbb{H}; \mu)$ .

**Proposition 3.1.** *Assume (A'). Let  $P_t^*$  be the adjoint operator of  $P_t$  on  $L^2(\mathbb{H}; \mu)$ . If  $\|\Gamma_t\| < \infty$ , then*

$$\mu((P_t^* f) \log P_t^* f) \leq \frac{\|\Gamma_t\|^2}{2} W_2(f\mu, \mu)^2, \quad f \geq 0, \mu(f) = 1$$

holds, where  $W_2^2$  is the Warsserstein distance induced by the cost-function  $(x, y) \mapsto \|x - y\|^2$ ; that is,

$$W_2(f\mu, \mu)^2 = \inf_{\pi \in \mathcal{C}(f\mu, \mu)} \int_{\mathbb{H} \times \mathbb{H}} \|x - y\|^2 \pi(dx, dy)$$

for  $\mathcal{C}(f\mu, \mu)$  the set of all couplings of  $f\mu$  and  $\mu$ . Consequently, (1.2) implies

$$\mu((P_t^* f) \log P_t^* f) \leq \frac{W_2(f\mu, \mu)^2}{2 \int_0^t h(s)^{-1} ds}, \quad t > 0.$$

*Proof.* Due to Proposition 2.1, it suffices to prove the first assertion. We shall adopt an argument in [2] by using the log-Harnack inequality (1.8). Let  $f \geq 0$  such that  $\mu(f) = 1$ . By an approximation argument, we may assume that  $f$  is bounded. So, by Theorem 1.3 we have

$$P_t(\log P_t^* f)(x) \leq \log(P_t P_t^* f)(y) + \frac{\|\Gamma_t\|^2}{2} \|x - y\|^2, \quad x, y \in \mathbb{H}.$$

Integrating both sides w.r.t.  $\pi \in \mathcal{C}(f\mu, \mu)$ , and minimizing in  $\pi$ , we arrive at

$$\mu((P_t^* f) \log P_t^* f) \leq \mu(\log(P_t P_t^* f)) + \frac{\|\Gamma_t\|^2}{2} W_2(f\mu, \mu)^2.$$

This completes the proof by noting that, since  $\mu$  is invariant for  $P_t$  and  $P_t^*$ ,

$$\mu(\log(P_t P_t^* f)) \leq \log \mu(P_t P_t^* f) = \log 1 = 0.$$

□

According to the above result, to derive the entropy-cost inequality for  $P_t$ , we shall need the log-Harnack inequality for the adjoint semigroup  $P_t^*$ . To ensure that  $P_t^*$  is again

an O-U type semigroup, we shall simply consider the Gaussian case (i.e.  $M = 0$ ), and assume **(A)**. In this case,  $\mu$  is a Gaussian measure with co-variance

$$R_\infty := \int_0^\infty T_s R T_s^* ds.$$

To see that  $P_t^*$  as a generalized Mehler semigroup (in the sense of [6]), we assume further

**(B)**  $M = 0, R_\infty \mathbb{H} \subset \mathcal{D}(A^*)$ , and the operator  $\tilde{A} = R_\infty A^* R_\infty^{-1}$  with domain

$$\mathcal{D}(\tilde{A}) := \{x \in R_\infty \mathbb{H} : R_\infty^{-1} x \in \mathcal{D}(A^*)\}$$

generates a  $C_0$ -semigroup  $\tilde{T}_t$  on  $\mathbb{H}$  such that

$$\tilde{R}_t = \int_0^t \tilde{T}_s R \tilde{T}_s^* ds$$

is of trace class for  $t > 0$ .

In this case,  $P_t^*$  can be calculated explicitly as (see [5, Proposition 10.1.9])

$$P_t^* f(x) = \int_{\mathbb{H}} f(\tilde{T}_t x + y) N_{\tilde{R}_t}(dy), \quad f \in \mathcal{B}_b(\mathbb{H}),$$

where  $N_{\tilde{R}_t}$  is the centered Gaussian measure with co-variance  $\tilde{R}_t$ . Thus,  $P_t^*$  is the transition semigroup of the solution to

$$d\tilde{X}_t = \tilde{A}\tilde{X}_t dt + R^{1/2} dW_t$$

for  $W_t$  the cylindrical Brownian motion on  $\mathbb{H}$ . So, the following is a direct consequence of Theorems 1.2 and 1.3 and Proposition 3.1.

**Proposition 3.2.** *Assume **(A)**, **(B)**. Let  $\tilde{\Gamma}_t = \tilde{R}_t^{-1/2} \tilde{T}_t$ . Then*

$$(P_t^* f)^\alpha(x) \leq P_t^* f^\alpha(y) \exp\left(\frac{\alpha \|\tilde{\Gamma}_t(x - y)\|^2}{2(\alpha - 1)}\right), \quad f \in \mathcal{B}_b^+(\mathbb{H}), x, y \in \mathbb{H}.$$

If  $\tilde{\Gamma}_t$  is bounded, then (1.8) holds for  $P_t^*, \tilde{\Gamma}_t$  in place of  $P_t$  and  $\Gamma_t$  respectively. In particular,

$$(3.1) \quad \mu((P_t f) \log P_t f) \leq \frac{\|\tilde{\Gamma}_t\|^2}{2} W_2(\mu, f\mu)^2, \quad f \geq 0, \mu(f) = 1.$$

Let  $W_0$  be the space of functions  $f$  of the form

$$f(x) = F(\langle \xi_1, x \rangle, \dots, \langle \xi_m, x \rangle), \quad x \in \mathbb{H}$$

for some  $m \geq 1$  and  $F \in \mathcal{S}(\mathbb{R}^m, \mathbb{C})$  (i.e. the Schwartz space of complex-valued functions, “rapidly decreasing” at infinity as well as their derivatives). Let  $W$  be the real-valued elements of  $W_0$ . According to [7],  $W$  is dense in  $L^p(\mu)$  for any  $p \geq 1$  and is a core of  $D(L)$ , the  $L^2(\mu)$ -domain of the generator  $L$  of  $P_t$ . We are now able to present the following result on the HWI inequality.

**Theorem 3.3.** *Assume (A) and (B). Assume further that  $A^*$  has a sequence of eigenvectors separating the points in  $\mathbb{H}$ . If (1.2) holds then*

$$\mu(f^2 \log f^2) \leq 2\mu(\langle \mathbb{R}Df, Df \rangle) \int_0^t h(s)ds + \frac{\|\tilde{\Gamma}_t\|^2}{2} W_2(\mu, f^2\mu)^2, \quad t > 0, f \in W, \mu(f^2) = 1,$$

where  $Df$  is the Fréchet derivative of  $f$ .

*Proof.* Let  $f \in W$  such that  $\mu(f^2) = 1$ . By [11, Theorem 1.3(2)], we have

$$P_t(f^2 \log f^2) \leq (P_t f^2) \log P_t f^2 + 2 \left( \int_0^t h(s)ds \right) P_t \langle \mathbb{R}Df, Df \rangle.$$

Integrating w.r.t.  $\mu$  we obtain

$$\mu(f^2 \log f^2) \leq 2\mu(\langle \mathbb{R}Df, Df \rangle) \int_0^t h(s)ds + \mu((P_t f^2) \log P_t f^2).$$

The proof is then completed by combining this with Proposition 3.2.  $\square$

If in particular  $P_t$  is symmetric (it is the case iff  $AR^{1/2} = R^{1/2}A^*$ ), then (1.2) implies

$$\mu(f^2 \log f^2) \leq 2\mu(\langle \mathbb{R}Df, Df \rangle) \int_0^t h(s)ds + \frac{1}{2 \int_0^t h(s)^{-1}ds} W_2(\mu, f^2\mu)^2$$

for all  $f \in W, \mu(f^2) = 1, t > 0$ .

## 4 Semi-linear stochastic equations

Consider the equation

$$(4.1) \quad dX_t^x = AX_t^x dt + F(X_t^x)dt + R^{1/2}dW_t, \quad X_0^x = x \in \mathbb{H},$$

where  $F$  is a measurable map on  $\mathbb{H}$  such that  $F(\mathbb{H}) \subset R^{1/2}\mathbb{H}$ , and  $W_t$  is the cylindrical Brownian motion on  $\mathbb{H}$ . We shall establish the Harnack inequality for the associated semigroup by regarding (4.1) as a perturbation to (1.1) with  $Z_t = R^{1/2}W_t$ , i.e.  $b = 0, M = 0$ . Since we do not assume that  $F$  is dissipative, the study is not included in [3]. In general, this equation only admits a weak solution. In this paper we shall consider the weak solution for (4.1) constructed from (1.1) with  $Z_t = R^{1/2}W_t$  by Girsanov transformations.

Let  $\tilde{X}_t^x$  be the mild solution to

$$d\tilde{X}_t^x = A\tilde{X}_t^x dt + R^{1/2}dW_t, \quad X_0^x = x.$$

We have  $\tilde{X}_t^x = W_A(t) + T_t x$ , where

$$W_A(t) = \int_0^t T_{t-s} R^{1/2} dW_s.$$

Since  $\int_0^t T_s R T_s^* ds$  is of trace class,  $W_A \in L^2([0, t]; \mathbb{H})$  for any  $t > 0$ . Let

$$\begin{aligned}\psi_x(t) &= R^{-1/2} F(W_A(t) + T_t x), \\ \tilde{W}_t^x &= W_t - \int_0^t \psi_x(s) ds, \\ \rho_t^x &= \exp \left( \int_0^t \langle \psi_x(s), dW_s \rangle - \frac{1}{2} \int_0^t \|\psi_x(s)\|^2 ds \right).\end{aligned}$$

Assume that

$$(4.2) \quad \|R^{-1/2} F(x)\|^2 \leq k_1 + k_2 \|x\|^2, \quad x \in \mathbb{H}$$

holds for some  $k_1, k_2 \geq 0$ . Then by [4, Theorem 10.20] and its proof,  $\mathbb{Q}_x := \rho_t^x \mathbb{P}$  is a probability measure and  $\tilde{X}_t^x$  is a weak solution to (4.1) under  $\mathbb{Q}_x$  with respect to the cylindrical Brownian motion  $\tilde{W}_t^x$ . Denote the corresponding ‘‘semigroup’’ by

$$(4.3) \quad P_t^F f(x) = \mathbb{E}_{\mathbb{Q}_x} f(\tilde{X}_t^x), \quad f \in \mathcal{B}_b(\mathbb{H}).$$

We note that due to the lack of uniqueness, in general  $P_t^F$  may not provide a semigroup (but cf. also [7]). Let  $P_t$  be the semigroup of  $\tilde{X}_t^i$  under  $\mathbb{P}$ . By Theorem 1.2 we have

$$(4.4) \quad (P_t f)^\alpha(x) \leq P_t f^\alpha(y) \exp \left( \frac{\alpha \|\Gamma_t(x - y)\|^2}{2(\alpha - 1)} \right), \quad f \in \mathcal{B}_b^+(\mathbb{H}),$$

where  $\Gamma_t := R_t^{-1/2} T_t$ . Moreover, by [4, (10.42)], for any  $p > 0$  there exists  $t_p > 0$  such that

$$C_{p, k_2}(t) := \mathbb{E} \exp \left( 2p(2p + 1) k_2 \int_0^t \|W_A(s)\|^2 ds \right) < \infty, \quad t \in [0, t_p].$$

In particular, if  $k_2 = 0$  then  $C_{p, k_2}(t) = 1$ ,  $t \geq 0$ . More precisely, let

$$\theta = \text{Tr} \int_0^1 T_s R T_s^* ds.$$

We have

$$C_0 := \sup_{s \in [0, 1]} \mathbb{E} e^{\|W_A(s)\|^2 / 4\theta} < \infty.$$

Thus, for any  $\lambda > 0$ ,

$$(4.5) \quad \begin{aligned}\mathbb{E} e^{\lambda \int_0^t \|W_A(s)\|^2 ds} &= \mathbb{E} e^{\frac{1}{t} \int_0^t \lambda t \|W_A(s)\|^2 ds} \leq \frac{1}{t} \int_0^t \mathbb{E} e^{\lambda t \|W_A(s)\|^2} ds \\ &\leq \frac{1}{t} \int_0^t (\mathbb{E} e^{\|W_A(s)\|^2 / 4\theta})^{4\theta \lambda t} ds \leq C_0^{4\theta \lambda t}, \quad t \in [0, 1 \wedge (4\theta \lambda)^{-1}].\end{aligned}$$

Combining this with (4.3) we obtain the following result.

**Theorem 4.1.** *If (4.2) holds, then for any  $t > 0$ ,  $\alpha > 1$ ,  $x, y \in \mathbb{H}$ ,  $p, q > 1$  with  $\alpha/(pq) > 1$ , and  $f \in \mathcal{B}_b^+(\mathbb{H})$*

$$\begin{aligned} (P_t^F f)^\alpha(x) &\leq \left(C_{\frac{p}{p-1}, k_2}(t)\right)^{\alpha p/(2(p-1))} \left(C_{\frac{1}{q-1}, k_2}(t)\right)^{\alpha q/(2(q-1))} P_t^F f^\alpha(y) \exp\left(\frac{\alpha q \|\Gamma_t(x-y)\|^2}{2(\alpha-q)}\right) \\ &\quad + \alpha \left[\frac{p+1}{p-1} + \frac{q+1}{q(q-1)}\right] \int_0^t [k_1 + k_2(\|T_s x\|^2 + \|T_s y\|^2)] ds. \end{aligned}$$

Consequently, if  $\|\Gamma_t\| < \infty$  for  $t > 0$ , then  $P_t^F$  is strong Feller provided it is a semigroup.

*Proof.* For simplicity, we denote  $p' = \frac{p}{p-1}$ ,  $q' = \frac{q}{q-1}$ ,  $\theta = \alpha/(pq)$ .

$$\begin{aligned} P_t^F f(x) &= \mathbb{E}_{\mathbb{Q}_x} f(\tilde{X}_t^x) = \mathbb{E}_{\rho_t^x} f(\tilde{X}_t^x) \leq (\mathbb{E} f^p(\tilde{X}_t^x))^{1/p} (\mathbb{E}(\rho_t^x)^{p'})^{1/p'} \\ &= (P_t f^p(x))^{1/p} (\mathbb{E}(\rho_t^x)^{p'})^{1/p'} \\ &\leq \left[ P_t f^{\theta p}(y) \exp\left(\frac{\theta \|\Gamma_t(x-y)\|^2}{2(\theta-1)}\right) \right]^{1/(\theta p)} (\mathbb{E}(\rho_t^x)^{p'})^{1/p'}. \end{aligned}$$

On the other hand, for any  $g \in \mathcal{C}_b^+(\mathbb{H})$ ,

$$P_t g(y) \leq \mathbb{E}_{\mathbb{P}} g(\tilde{X}_t^y) = \mathbb{E}_{\mathbb{Q}_y} g(\tilde{X}_t^y) (\rho_t^y)^{-1} \leq (P_t^F g^q(y))^{1/q} (\mathbb{E}(\rho_t^y)^{1-q'})^{1/q'}.$$

So by taking  $g = f^{\theta p}$

$$(P_t^F f)^\alpha(x) \leq P_t^F f^\alpha(y) \exp\left(\frac{\alpha \|\Gamma_t(x-y)\|^2}{2p(\theta-1)}\right) (\mathbb{E}(\rho_t^x)^{p'})^{\alpha/p'} (\mathbb{E}(\rho_t^y)^{1-q'})^{\alpha/q'}.$$

This implies the desired Harnack inequality according to the following Lemma 4.2.

Now, assume that  $\Gamma_t$  is bounded for  $t > 0$  and assume that  $P_t^F$  is a semigroup. Let  $f \in \mathcal{B}_b^+(\mathbb{H})$ . By the first assertion and (4.5), for any  $\alpha > 1$  there exist constants  $t_\alpha, c_\alpha > 0$  and a positive function  $H_\alpha$  on  $(0, t_\alpha)$  such that

$$(4.6) \quad P_t^F f(x) \leq (P_t^F f^\alpha(y))^{1/\alpha} e^{c_\alpha t + \|x-y\|^2 H_\alpha(t)}, \quad t \in (0, t_\alpha).$$

Then, for any  $t > 0$ ,

$$\begin{aligned} \limsup_{x \rightarrow y} P_t^F f(x) &\leq \limsup_{\alpha \rightarrow 1} \limsup_{s \rightarrow 0} \limsup_{x \rightarrow y} \left\{ P_s^F (P_{t-s}^F f)^\alpha(y) \right\}^{1/\alpha} e^{c_\alpha s + \|x-y\|^2 H_\alpha(s)} \\ &\leq \limsup_{\alpha \rightarrow 1} \limsup_{s \rightarrow 0} \limsup_{x \rightarrow y} \left\{ P_t^F f^\alpha(y) \right\}^{1/\alpha} e^{c_\alpha s + \|x-y\|^2 H_\alpha(s)} = P_t^F f(y). \end{aligned}$$

On the other hand, (4.6) also implies

$$\begin{aligned} P_t^F f(x) &\geq \left\{ P_s^F (P_{t-s}^F f)^{1/\alpha}(y) \right\}^\alpha e^{-\alpha c_\alpha s - \alpha H_\alpha(s) \|x-y\|^2} \\ &\geq \left\{ P_t^F f^{1/\alpha}(y) \right\}^\alpha e^{-\alpha c_\alpha s - \alpha H_\alpha(s) \|x-y\|^2}, \quad s \in (0, t_\alpha). \end{aligned}$$

So, by first letting  $x \rightarrow y$  then  $s \rightarrow 0$  and finally  $\alpha \rightarrow 1$  we arrive at

$$\liminf_{x \rightarrow y} P_t^F f(x) \geq P_t^F f(y).$$

Therefore,  $P_t^F f$  is continuous on  $\mathbb{H}$ . □

**Lemma 4.2.** *Assume (4.2). For any  $p > 1$ ,  $\delta > 0$  and  $x \in \mathbb{H}$ , then*

$$\begin{aligned}\mathbb{E}(\rho_t^x)^p &\leq (C_{p,k_2}(t))^{1/2} \exp\left(\frac{p(2p-1)}{2} \int_0^t (k_1 + 2k_2 \|T_s x\|^2) ds\right) \\ \mathbb{E}(\rho_t^x)^{-\delta} &\leq (C_{\delta,k_2}(t))^{1/2} \exp\left(\frac{\delta(2\delta+1)}{2} \int_0^t (k_1 + 2k_2 \|x\|^2) ds\right).\end{aligned}$$

*Proof.* According to the proof of [4, Theorem 10.20], for any  $\lambda \in \mathbb{R}$ , the process

$$t \mapsto \exp\left[\lambda \int_0^t \langle \psi_x(s), dW_s \rangle - \frac{\lambda^2}{2} \int_0^t \|\psi_x(s)\|^2 ds\right]$$

is a martingale. So,

$$\begin{aligned}&\mathbb{E}(\rho_t^x)^p \\ &= \mathbb{E} \exp\left(p \int_0^t \langle \psi_x(s), dW_s \rangle - p^2 \int_0^t \|\psi_x(s)\|^2 ds\right) \exp\left(\frac{p(2p-1)}{2} \int_0^t \|\psi_x(s)\|^2 ds\right) \\ &\leq \left[\mathbb{E} \exp\left(2p \int_0^t \langle \psi_x(s), dW_s \rangle - 2p^2 \int_0^t \|\psi_x(s)\|^2 ds\right)\right]^{1/2} \\ &\quad \cdot \left[\mathbb{E} \exp\left(p(2p-1) \int_0^t \|\psi_x(s)\|^2 ds\right)\right]^{1/2} \\ &= \left[\mathbb{E} \exp\left(p(2p-1) \int_0^t \|\psi_x(s)\|^2 ds\right)\right]^{1/2}.\end{aligned}$$

This implies the first inequality since (4.2) and the boundedness of  $T_s$  imply

$$\|\psi_x(s)\|^2 \leq k_1 + 2k_2 \|W_A(s)\|^2 + 2k_2 \|x\|^2.$$

Similarly, the second inequality follows by noting that

$$\begin{aligned}&\mathbb{E}(\rho_t^x)^{-\delta} \\ &= \mathbb{E} \exp\left(-\delta \int_0^t \langle \psi_x(s), dW_s \rangle - \delta^2 \int_0^t \|\psi_x(s)\|^2 ds\right) \exp\left(\frac{\delta(2\delta+1)}{2} \int_0^t \|\psi_x(s)\|^2 ds\right) \\ &\leq \left[\mathbb{E} \exp\left(-2\delta \int_0^t \langle \psi_x(s), dW_s \rangle - 2\delta^2 \int_0^t \|\psi_x(s)\|^2 ds\right)\right]^{1/2} \\ &\quad \cdot \left[\mathbb{E} \exp\left(\delta(2\delta+1) \int_0^t \|\psi_x(s)\|^2 ds\right)\right]^{1/2} \\ &= \left[\mathbb{E} \exp\left(\delta(2\delta+1) \int_0^t \|\psi_x(s)\|^2 ds\right)\right]^{1/2}.\end{aligned}$$

□

## References

- [1] M. Arnaudon, A. Thalmaier, F.-Y. Wang, *Harnack inequality and heat kernel estimates on manifolds with curvature unbounded below*, Bull. Sci. Math. 130(2006), 223–233.
- [2] S.G. Bobkov, I. Gentil, M. Ledoux, *Hypercontractivity of Hamilton-Jacobi equations*, J. Math. Pures Appl. (9) 80(2001), 669–696.
- [3] G. Da Prato, M. Röckner, F.-Y. Wang, *Singular stochastic equations on Hilbert spaces: Harnack inequalities for their transition semigroups*, J. Funct. Anal. 257(2009), 992–1017.
- [4] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, Vol. 44 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1992.
- [5] G. Da Prato and J. Zabczyk, *Second order partial differential equations in Hilbert spaces*, London Mathematical Society Lecture Note Series, vol. 293, Cambridge University Press, Cambridge, 2002.
- [6] M. Fuhrman and M. Röckner, *Generalized Mehler semigroups: the non-Gaussian case*, Potential Anal. **12** (2000), no. 1, 1–47.
- [7] P. Lescot and M. Röckner, *Perturbations of generalized Mehler semigroups and applications to stochastic heat equations with Lévy noise and singular drift*, Potential Anal. 20(2004), 317–344.
- [8] W. Liu and F.-Y. Wang, *Harnack inequality and strong Feller property for stochastic fast-diffusion equations*, J. Math. Anal. Appl. 342(2008), 651–662.
- [9] F. Otto and C. Villani, *Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality*, J. Funct. Anal. 173(2000), 361–400.
- [10] S.-X. Ouyang, *Harnack inequalities and applications for stochastic equations*, Ph.D. thesis, Bielefeld University, available on <http://bieson.ub.uni-bielefeld.de/volltexte/2009/1463/pdf/ouyang.pdf> (2009).
- [11] M. Röckner, F.-Y. Wang, *Harnack and functional inequalities for generalized Mehler semigroups*, J. Funct. Anal. 203(2003), 237–261.
- [12] F.-Y. Wang, *Logarithmic Sobolev inequalities on noncompact Riemannian manifolds*, Probab. Theory Related Fields 109(1997), 417–424.
- [13] F.-Y. Wang, *Harnack inequality and applications for stochastic generalized porous media equations*, Ann. Probab. 35(2007), 1333–1350.
- [14] F.-Y. Wang, *Heat kernel inequalities for convexity of manifold and curvature condition*, 2009.

- [15] J. Zabczyk, *Mathematical Control Theory*, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2008, an introduction, Reprint of the 1995 edition.