THE CLASS OF THE LOCUS OF INTERMEDIATE JACOBIANS OF CUBIC THREEFOLDS

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ABSTRACT. We study the locus of intermediate Jacobians of cubic threefolds within the moduli space $\mathcal{A}_5$ of complex principally polarized abelian fivefolds, and its generalization to arbitrary genus — the locus of abelian varieties with a singular odd two-torsion point on the theta divisor. Assuming that this locus has expected codimension g (which we show to be true for $g \le 5$, and conjecturally for any g), we compute the class of this locus, and of is closure in the perfect cone toroidal compactification $\mathcal{A}_g^{\mathrm{Perf}}$, in the Chow, homology, and the tautological ring.

Finally, we work out the cases of genus up to 5 in detail, obtaining explicit expressions for the classes of the closures of the loci $\mathcal{A}_1 \times \theta_{\mathrm{null}}$ in $\mathcal{A}_4^{\mathrm{Perf}}$ and of the locus of intermediate Jacobians (together with the same locus of products) — in $\mathcal{A}_5^{\mathrm{Perf}}$.

In the course of our computation we also deal with various intersections of boundary divisors of a level toroidal compactification, which is of independent interest in understanding the cohomology and Chow rings of the moduli spaces.

0. INTRODUCTION

The moduli spaces $\mathcal{M}_g$ of curves of genus g and $\mathcal{A}_g$ of principally polarized abelian varieties (ppav) of dimenson g are at the heart of algebraic geometry. Understanding their geometry includes the question of computing the cohomology ring and the Chow ring of these varieties and their compactifications.

The investigation of the cohomology ring of $\mathcal{M}_g$ and its compactification $\overline{\mathcal{M}_g}$, the moduli space of stable curves, was pioneered by Mumford and later by Faber and is the contents of numerous papers. The cohomology ring of $\mathcal{A}_2$ was first computed by Igusa [Igu62]. It was Mumford [Mum83] who computed the cohomology and Chow ring of $\overline{\mathcal{M}_2}$, or what is the same, of the perfect cone compactification $\mathcal{A}_2^{\mathrm{Perf}}$ (which coincides with both the Igusa and the second Voronoi compactification

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in this case). In this paper he also sets up the general framework in which such computations are done nowadays. The cohomology rings of $\mathcal{A}_3$ and $\mathcal{A}_3^{\mathrm{Sat}}$ were computed by Hain [Ha02], the Chow rings of $\mathcal{A}_3$ and $\mathcal{A}_3^{\mathrm{Perf}}$, with rational coefficients, were computed by van der Geer [vdG99], and finally the second-named author and Tommasi [HT10] computed the cohomology ring of $\mathcal{A}_3^{\mathrm{Perf}}$ and showed that it is equal to the Chow ring.

For higher genus very little is known about the cohomology and Chow rings of $\mathcal{A}_g^{\mathrm{Perf}}$, beyond the fact that the Picard group has rank 2. The cohomology for the $g = 4$ case is the subject of a forthcoming work by Tommasi and the second-named author [HT11], but for higher genus very little is known. One approach to this problem, motivated partly by the success of the similar method for the moduli of curves, is to study the tautological ring generated by the Chern classes λ_i of the Hodge bundle. This ring is known explicitly — the classes λ_i satisfy one fundamental relation

$$(1) \qquad (1 + \lambda_1 + \ldots + \lambda_g)(1 - \lambda_1 + \ldots + (-1)^g \lambda_g) = 1$$

as proven by van der Geer [vdG99] in the Chow group with rational coefficients and by Esnault and Viehweg [EV02] in the Chow ring. One can then compute the projections of various classes to the tautological ring — for example, this was done by Faber [Fab99] for the locus of Jacobians, and by van der Geer [vdG99], respectively Ekedahl and van der Geer [EvdG05] for the locus of products $\mathcal{A}_0 \times \mathcal{A}_{g-1}$.

To the best of our knowledge for $g \geq 4$ no classes of geometrically meaningful cycles (of codimension higher than one — the case of divisors is much easier) have been computed in the Chow ring proper in any toroidal compactification of $\mathcal{A}_g$, in particular the classes of $\mathcal{A}_i \times \mathcal{A}_{g-i}$ are unknown.

In this paper we concentrate on the locus of intermediate Jacobians of cubic threefolds, its closure in $\mathcal{A}_5^{\mathrm{Perf}}$, and its generalization to arbitrary genus. Assuming that such a locus has codimension exactly g (for a detailed discussion of this condition see below), we compute its class in $\mathcal{A}_g^{\mathrm{Perf}}$, and the projection of this class to the tautological ring. In particular for $g = 4$ and $g = 5$ we determine the classes of the loci $\overline{\mathcal{A}_1 \times \theta_{\mathrm{null}}^{(3)}} \subset \mathcal{A}_4^{\mathrm{Perf}}$ and of the closure of the locus of intermediate Jacobians of cubic threefolds, together with $\overline{\mathcal{A}_1 \times \theta_{\mathrm{null}}^{(4)}}$, in $\mathcal{A}_5^{\mathrm{Perf}}$. We emphasize that we compute the classes of these loci precisely, not just their tautological parts. In the course of our computation we also study the combinatorics of intersections of boundary divisors of a level cover of $\mathcal{A}_g^{\mathrm{Perf}}$, describing the classes of various geometric loci contained in the

boundary of $\mathcal{A}_g^{\mathrm{Perf}}$, which is of independent interest for understanding the structure of the Chow and cohomology rings.

Throughout the paper we work over the field of complex numbers.

Acknowledgements

1. Statement of results

The intermediate Jacobians of cubic threefolds were studied in detail by Clemens and Griffiths [CG72] who used them to prove that a non-singular cubic threefold is not rational. The locus IJ of intermediate Jacobians of cubic threefolds within the moduli space $\mathcal{A}_5$ is then a natural 10-dimensional geometrically defined subvariety of a 15-dimensional algebraic variety. Similar to the Schottky problem for Jacobians of curves, it is interesting to try to describe (the closure of) this locus $IJ \subset \mathcal{A}_5$ by geometric or analytic conditions.

Clemens and Griffiths describe the geometry of the intermediate Jacobian in terms of the geometry of the threefold, and in particular show that the theta divisor of such an intermediate Jacobian has a unique triple point. Much later Casalaina-Martin and Friedman [CMF05], [CM08a] used a detailed analysis of theta divisors of Prym varieties, and degenerations, to show that intermediate Jacobians of cubic threefolds are in fact characterized among all indecomposable ppav's (A, Θ) of dimension 5 by the existence of a unique triple point on the theta divisor Θ. This statement can thus be interpreted as a geometric solution to the Schottky problem for intermediate Jacobians of cubic threefolds. Note that such a triple point must be at a 2-torsion point on the ppav, more precisely an odd 2-torsion point. We denote by $A[2]$ the set of 2-torsion points on A, and add a subscript to distinguish the parity.

The above characterization of $IJ \subset \mathcal{A}_5$ can then be generalized to arbitrary genus to define the loci

$$I^{(g)} = \{(A, \Theta) \in \mathcal{A}_g \mid \Theta \text{ is singular at some point m} \in \mathrm{A}[2]_{\mathrm{odd}}\}.$$

The characterization of intermediate Jacobians then amounts to saying that within the locus of indecomposable ppav's on $\mathcal{A}_5$ we have $I^{(5)} = IJ$. It is thus natural to ask to describe the closure $\overline{IJ}$ of the locus IJ within $\mathcal{A}_5$, and the possible corresponding degenerations of the cubic threefold itself. This was accomplished geometrically by Casalaina-Martin and Laza [CML09], while an analytic description was given by the first author and Salvati Manni in [GSM09], using some of the ideas of their earlier work [GSM04]. It turns out that in fact $I^{(5)} \supsetneq \overline{IJ}$ — the extra irreducible component of this locus is $\mathcal{A}_1 \times \theta_{\mathrm{null}}^{(4)}$ (where $\theta_{\mathrm{null}}^{(g)}$ denotes the divisor of ppav for which the theta divisor is singular at an even 2-torsion point).

The locus $I^{(g)}$ has expected codimension g in $\mathcal{A}_g$, and in [GSM09] it is conjectured that it is indeed of pure codimension g (see section 2 for a more detailed discussion). It is thus natural to try to compute the class of $I^{(g)}$ in the Chow ring of $\mathcal{A}_g$. The result in fact follows naturally from interpreting singularities of Θ at odd 2-torsion points as vanishing loci of gradients of theta functions, and thus as zero loci of certain vector-valued Siegel modular forms. The resulting expression is the content of our first main result:

Theorem 1.1. *The virtual class of $I^{(g)}$ in $CH^g(\mathcal{A}_g)$ is given by*

$$[I^{(g)}] = 2^{g-1}(2^g - 1) \sum_{i=0}^{g} \lambda_{g-i} \left(\frac{\lambda_1}{2} \right)^i$$

where $\lambda_i = c_i(\mathbb{E})$ are the Chern classes of the Hodge bundle on $\mathcal{A}_g$.

By virtual class we mean that if the codimension of the locus $I^{(g)}$ is g, as expected, then its class in the Chow group is given by the stated formula. Note that this class lies in the tautological ring of the Chow ring generated by the classes λ_i. The locus $I^{(5)}$ is known to have expected codimension. This follows from the work of Casalaina-Martin [CMF05], [CM08b], for a different proof see [GK11]. Using the relations in $CH^*(\mathcal{A}_g)$ it then follows that we have

Corollary 1.2. *The class of $I^{(5)}$ in $CH^5(\mathcal{A}_5)$ is equal to*

$$93 \cdot (4\lambda_1^2 \lambda_3 + \lambda_1^5/2).$$

Recall that the moduli space of ppav $\mathcal{A}_g$ is not compact. In fact it admits many different toroidal compactifications. It is then natural to investigate the closure $\overline{I^{(g)}}$ in some toroidal compactification, and to compute its class there. The first steps in that direction were done in [GSM09] (see also [CML09]) where the intersection of $\overline{I^{(g)}}$ with the boundary of the partial compactification were investigated.

The second (and much harder) main result of this paper is a computation of the class of $\overline{I^{(g)}}$ in the so-called perfect cone compactification $\mathcal{A}_g^{\mathrm{Perf}}$. Recall also that in genus $g \leq 3$ the perfect cone (also called first Voronoi) compactification coincides with the second Voronoi, and the Igusa (also called central cone) compactification. The perfect cone compactification has the advantage that its boundary divisor is irreducible. We also recall that Shepherd-Barron [SB06] has shown that $\mathcal{A}_g^{\mathrm{Perf}}$ is a canonical model in the sense of the minimal model program. Computing the class $\overline{I^{(g)}}$ requires determining the vanishing behavior of the gradients of the theta functions on various loci of semiabelic varieties, and relies on the main results of our recent preprint [GK11]. We obtain the following

Theorem 1.3. *For $g \leq 5$ (and for any genus in which the statements of* [GK11, theorems 1.2 and 1.3] *hold), we have the following expression for the class of* $\overline{I^{(g)}}$ *in* $CH^g(\mathcal{A}_g^{\mathrm{Perf}})$:

$$(2) \qquad [\overline{I^{(g)}}] = \frac{1}{N} \sum_{m \in (\mathbb{Z}/2\mathbb{Z})_{\mathrm{odd}}^{2g}} \sum_{i=0}^{g} p_* \left(\lambda_{g-i} \left(\frac{\lambda_1}{2} - \frac{1}{4} \sum_{n \in Z_m} \delta_n \right)^i \right)$$

where $p : \mathcal{A}_g^{\mathrm{Perf}}(2) \to \mathcal{A}_g^{\mathrm{Perf}}$ is the level cover, $N = |\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})|$ and Z_m is the set of pairs of non-zero vectors $\pm n \in (\mathbb{Z}/2\mathbb{Z})^{2g}$ such that $m + n$ is even, and we recall that the irreducible components δ_n of the boundary of $\mathcal{A}_g^{\mathrm{Perf}}(2)$ correspond to non-zero elements of $(\mathbb{Z}/2\mathbb{Z})^{2g}$.

At this point some words about intersection theory are in order. We denote by $CH^k(\mathcal{A}_g^{\mathrm{Perf}})$ the Chow group of cycles of codimension k. This is always meant in the sense of the stack, i.e. we are free to work with invariant classes on level covers and then taking the pushforward to $\mathcal{A}_g^{\mathrm{Perf}}$. It should be pointed out that $\mathcal{A}_g^{\mathrm{Perf}}$ is not smooth, also in the stack sense. This comes from the fact that the perfect cone decomposition contains non-basic cones. Hence we cannot assume that $CH^*(\mathcal{A}_g^{\mathrm{Perf}})$ has a ring structure. We shall consider Chern classes as elements in the operational Chow groups of Fulton and MacPherson. As such they act on cycle classes by taking the cup product. Taking the cup product with the fundamental class we can also associate Chow homology classes to Chern classes. By abuse of notation we will not distinguish between a Chern class and its associated Chern homology class. All operations of Chern classes will be performed in the operational Chow ring and the result will then be applied to cycles. In the above theorem the λ_i are the Chern classes of the Hodge bundle and the sum of the boundary components is Cartier. Hence we can

perform the usual operations with these classes and then take the associated Chow homology class. This is what is meant by the right hand side in the formula of the above theorem. Although there is no ring structure on $CH^*(\mathcal{A}_g^{\mathrm{Perf}})$ we will still label classes by their codimension, as this makes the formulae more readable and easier to compare with the existing literature.

We shall also compute the projection of the class computed above to the *tautological ring*, i.e. the ring generated by the Hodge classes: recall that the tautological ring of a toroidal compactification $\mathcal{A}_g^{\mathrm{tor}}$ is the polynomial ring generated by the classes $\lambda_1, \ldots, \lambda_g$ subject to the one fundamental relation (1). This is defined for every toroidal compactification of $\mathcal{A}_g$ and independent of the chosen compactification, as is its pushforward to the Satake compactification (see [EvdG05]).

The tautological ring is also defined for the open part $\mathcal{A}_g$ where the extra relation $\lambda_g = 0$ holds, which was shown by van der Geer [vdG99] in cohomology and by Esnault and Viehweg [EV02] in the Chow ring. We recall that the tautological ring has a perfect pairing, and thus that there is a projection from the Chow ring to the tautological ring. For details we refer the reader to [Fab99], [vdG99],[EvdG05, section 3].

Theorem 1.4. *If* [GK11, theorems 1.2 and 1.3] *hold in genus g (in particular for any $g \leq 5$) the projection of the class $[\overline{I^{(g)}}]$ to the tautological ring is equal to*

$$[I^{(g)}]^{\mathrm{taut}} = \frac{(-1)^{g-1}(g-1)!}{8\zeta(1-2g)}\lambda_g + 2^{g-1}(2^g-1)\sum_{i=0}^{g}\lambda_{g-i}\left(\frac{\lambda_1}{2}\right)^i.$$

We will discuss the above statements, level covers, and their boundary components in detail in the following sections.

We apply the above theorems to compute the classes of $I^{(g)}$ and $\overline{I^{(g)}}$. for all $g \leq 5$. For $g = 2$ these loci are empty, and we get zero as a valid consistency check for our computations. For genus 3 our results agree with the computation of the class $[\overline{\mathrm{Sym}^3(\mathcal{A}_1)}] \in CH^*(\mathcal{A}_3^{\mathrm{Perf}})$ obtained by van der Geer in [vdG98], and corrected in [vdG09]. For genus 4 we compute the class of the locus $\mathcal{A}_1 \times \theta_{null}^{(3)}$ and of its closure in $\mathcal{A}_4^{\mathrm{Perf}}$. In particular, we confirm that the class of the open part of this locus lies in the tautological subring of $CH^*(\mathcal{A}_4)$. Finally, for genus 5 our results are completely new, and give a formula for the class of the locus of intermediate Jacobians of cubic threefolds together with the locus $\mathcal{A}_1 \times \theta_{\mathrm{null}}^{(4)}$, and also of the compactification. We further compute the tautological projections and obtain for the closure of the locus of intermediate Jacobians of cubic threefolds $\overline{IJ}$:

Proposition 1.5. *The projection of the class of the closure of the locus of intermediate Jacobians of cubic threefolds to the tautological ring is given by*

$$[\overline{IJ}]^{\mathrm{taut}} = 140\lambda_1^5 - 376\lambda_1^2\lambda_3 + 848\lambda_5.$$

Except the calculations of the projections of various classes to the tautological ring (which depend on the existence of a perfect pairing and thus require working in the Chow ring with rational coefficients), all our calculations hold in the Chow ring with integer coefficients.

2. Gradients of theta functions

We denote by $\mathcal{H}_g$ the Siegel upper half space of genus g — the set of symmetric complex $g \times g$ matrices with positive-definite imaginary part. Recall that the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ acts on $\mathcal{H}_g$ by $\gamma \circ \tau = (A\tau + B)(C\tau + D)^{-1}$.

We recall that the level subgroups of $\Gamma_g := \mathrm{Sp}(2g, \mathbb{Z})$ are defined as follows:

$$\Gamma_g(n) := \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \,\middle|\, \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod n \right\}$$

$$\Gamma_g(n, 2n) := \left\{ \gamma \in \Gamma_g(n) \mid \mathrm{diag}(A^t B) \equiv \mathrm{diag}(C^t D) \equiv 0 \mod 2n \right\}.$$

The moduli space of ppav is then $\mathcal{A}_g = \mathcal{H}_g / \Gamma_g$, while the level moduli spaces $\mathcal{A}_g(n) := \mathcal{H}_g / \Gamma_g(n)$ and $\mathcal{A}_g(n, 2n) := \mathcal{H}_g / \Gamma_g(n, 2n)$ are finite covers of $\mathcal{A}_g$.

We denote by $\theta(\tau, z)$ the Riemann theta function of $\tau \in \mathcal{H}_g$ and $z \in \mathbb{C}^g$

$$\theta(\tau, z) := \sum_{n \in \mathbb{Z}^g} \mathbf{e}(n^t \tau n / 2 + n^t z)$$

where for future use we denote $\mathbf{e}(x) := \exp(2\pi i x)$ the exponential function.

For an abelian variety A, we denote by $A[2]$ the set of two-torsion points on it; as a group, $A[2] \cong (\mathbb{Z}/2\mathbb{Z})^{2g}$. Analytically the points in $A[2]$ can be labeled $m = (\tau\varepsilon + \delta)/2$, where $\tau \in \mathcal{H}_g$ projects to $A \in \mathcal{A}_g$, and $\varepsilon, \delta \in (\mathbb{Z}/2\mathbb{Z})^g$. For future use we denote $\sigma(m) := \varepsilon \cdot \delta \in \mathbb{Z}/2\mathbb{Z}$ and call it the parity of m. Accordingly we call m even or odd depending on whether $\sigma(m)$ is 0 or 1, respectively. This is equivalent to the point m not lying (resp. lying) on the theta divisor for a generic τ (i.e. for a two-torsion point m the function $\theta(\tau, m)$ is identically zero if and only if m is odd).

For a point $m = (\tau\varepsilon + \delta)/2$ we denote the theta function with (half-integer) characteristic

$$\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau, z) := \theta_m(\tau, z) :=$$

$$= \sum_{n \in \mathbb{Z}^g} \mathbf{e}((n + \varepsilon/2)^t \tau(n + \varepsilon/2)/2 + (n + \varepsilon/2)^t(z + \delta/2)).$$

As a function of z, the theta function with characteristic is even or odd depending on the parity of the characteristic. In particular, all theta constants (the values of theta functions with characteristics at $z = 0$) with odd characteristics vanish identically.

Let $\pi : \mathcal{X}_g \to \mathcal{A}_g$ be the universal family, which exists over the stack, and let $\mathbb{E} := \pi_* \Omega_{\mathcal{X}_g/\mathcal{A}_g}$ be the Hodge bundle. The gradient

$$(3) \qquad\qquad F_m := \mathrm{grad}_z \theta_m(\tau, z)|_{z=0}$$

with respect to z of the theta function vanishes identically in τ for even m, and is generically non-zero for m odd. This gradient is a vector-valued modular form for $\Gamma_g(4, 8)$ for the representation $\det^{\otimes 1/2} \otimes \mathrm{std} : \Gamma_g \to GL(g, \mathbb{C})$. In other words, we have

$$(4) \qquad\qquad F_m \in H^0(\mathcal{A}_g(4, 8), \det \mathbb{E}^{\otimes 1/2} \otimes \mathbb{E})$$

(see [GSM04] for more details).

We recall from [Igu72, p. 50], that up to a simple exponential factor, the theta function with characteristic m is equal to the Riemann theta function shifted by the point m:

$$\theta(\tau, z + (\tau\varepsilon + \delta)/2) = \mathbf{e}(-\varepsilon^t \tau\varepsilon/8 - \varepsilon^t \delta/4 - \varepsilon^t z/2)\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau, z).$$

We can thus compute for an odd point $m \in A[2]$

$$(5) \qquad f_m(\tau) := \mathrm{grad}_z \theta(\tau, z)|_{z=m} = \mathrm{grad}_z \theta(\tau, z + (\tau\varepsilon + \delta)/2)_{z=0}$$

$$= \mathbf{e}(-\varepsilon^t \tau\varepsilon/8 - \varepsilon^t \delta/4 - \varepsilon^t z/2)\mathrm{grad}_z \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau, z)|_{z=0}$$

$$= \mathbf{e}(-\varepsilon^t \tau\varepsilon/8 - \varepsilon^t \delta/4)F_m(\tau)$$

since $\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau, 0) = 0$ for the odd two-torsion point. Thus f_m and F_m differ by a nowhere vanishing holomorphic function on $\mathcal{H}_g$, and thus their zero loci are the same. Moreover, the line bundle on $\mathcal{A}_g(4, 8)$ defined by the exponential factor is trivial since it has a nowhere vanishing section. (In what follows, it will be crucial that this exponential

factor, while non-vanishing on $\mathcal{A}_g$, in fact vanishes on some irreducible components of the boundary of $\mathcal{A}_g^{\mathrm{Perf}}(2)$.) Thus we also have

$$f_m \in H^0(\mathcal{A}_g(4,8), \det \mathbb{E}^{\otimes 1/2} \otimes \mathbb{E}).$$

The group $\Gamma_g(2)/\Gamma_g(4,8)$ acts on the gradients by certain signs, and thus the zero locus

(6) $$G_m := G_{\epsilon,\delta} = \{\tau | F_m(\tau) = 0\} = \{\tau | f_m(\tau) = 0\}$$

is a well-defined subvariety of $\mathcal{A}_g(2)$. We refer to [GSM04, GSM09] for a more detailed discussion of the properties of the gradients of the theta function and further questions on loci of ppav with points of high multiplicity on the theta divisor.

Finally we denote by

$$I^{(g)} := p(G_m) \subset \mathcal{A}_g$$

the locus of ppav for which *some* F_m vanishes. Geometrically, as explained in the introduction, this is the locus of ppav having a singularity at an odd two-torsion point. We will omit the index (g) when no confusion is possible. Note that it follows from the fact that Γ_g permutes the F_m that the projection of G_m to $\mathcal{A}_g$ does not depend on m.

The multiplicity of the theta function for ppav of low dimension has been studied extensively. Recall that a ppav is called decomposable if it is a product of two lower-dimensional ppav. For genus up to 4 it is known that no indecomposable ppav has a point of multiplicity 3 on the theta divisor [CM08b], and thus by studying the multiplicity of points on the reducible theta divisors for decomposable ppav, we see that as schemes

$$I^{(3)} = \mathrm{Sym}^3(\mathcal{A}_1)$$

and

$$I^{(4)} = \mathcal{A}_1 \times \theta_{\mathrm{null}}^{(3)}$$

where $\theta_{\mathrm{null}}^{(3)}$ denotes the theta-null divisor in $\mathcal{A}_3$: the locus of those ppav for which there exists a point $m \in A[2]_{\mathrm{even}}$ lying on the theta divisor (or, equivalently, for which some theta constant vanishes).

It was recently shown in [CMF05, CM08a] that within the locus of indecomposable abelian 5-folds the locus $I^{(5)}$ coincides with the locus IJ of intermediate Jacobians of cubic threefolds, while in [CML09] degenerations of intermediate Jacobians were studied. Combining this with results of [GSM09] one gets scheme-theoretically

$$I^{(5)} = \overline{IJ} \cup \mathcal{A}_1 \times \theta_{\mathrm{null}}^{(4)};$$

where $\overline{IJ}$ denotes the closure of IJ in $\mathcal{A}_5$. In [CML09] the boundary of IJ was described, and in particular it was shown that within $\mathcal{A}_5$ one has

$$\overline{IJ} \setminus IJ = \overline{\mathcal{J}_5^h} \cup \mathcal{A}_1 \times (\overline{\mathcal{J}_4} \cap \theta_{\mathrm{null}}^{(4)}),$$

where $\overline{\mathcal{J}_g} \subset \mathcal{A}_g$ denotes the closure of the locus of Jacobians of curves, and $\overline{\mathcal{J}_g^h} \subset \mathcal{A}_g$ denotes the closure of the locus of hyperelliptic Jacobians (see also [GSM09] for a discussion). In particular one sees that $\mathcal{A}_1 \times (\overline{\mathcal{J}_4} \cap \theta_{\mathrm{null}}^{(4)})$ is the intersection of the two irreducible components of the locus $I^{(5)}$.

Remark 2.1. The explicit descriptions of the loci $I^{(g)}$ given above for $g \leq 5$ are a priori only set-theoretic. To make these descriptions scheme-theoretic, one needs to ascertain that along each of the above-mentioned components the gradient of the theta function F_m vanishes with multiplicity 1, and not higher. In general this would be a rather hard computation, as one would need to establish that at a generic point of each component of $I^{(g)}$ precisely one F_m vanishes, and that this F_m vanishes with multiplicity one.

Note, however, that the closure in a suitable toroidal compactification of all of the components described above intersects the boundary of the partial compactification (see the following sections for a more precise discussion and definitions). This intersection can be computed (see [GSM09, Prop. 12] and [GK11]). It turns out that this intersection is related to the locus of singularities of the universal theta divisor (which is known to be reduced, see [Deb92]), and the locus $I^{(g-1)}$. It thus follows inductively that for $g \leq 5$ the intersection of $I^{(g)}$ with the boundary of the partial compactification is reduced, and thus that the locus itself is reduced (note that [GSM09, Prop. 12]) holds scheme-theoretically. Notice that for $g > 5$ we do not know this to be the case, and thus theorem 1.3 should be a priori interpreted as giving the class of the scheme $I^{(g)}$ which may have non-reduced components.

In fact, in higher genus the locus $I^{(g)}$ is not well understood. Indeed, even the following question is open:

Conjecture 2.2. [GSM09, Conjecture 1] *The locus $I^{(g)}$ has pure codimension g in $\mathcal{A}_g$ for any g.*

Notice that since locally $I^{(g)}$ is given by the vanishing of g partial derivatives of the theta function at an odd two-torsion point, we know that codimension of each its irreducible component is at most g. What the conjecture says is thus that the vanishing of the partial derivatives imposes g independent conditions. In this paper we will concentrate on

the case of $g \leq 5$ (when this conjecture is known to hold (see [CM08b] and [GK11, Thm 1.2]). The above analytic description of $I^{(g)}$ allows us to prove our first main result.

Proof of theorem 1.1. This is an immediate consequence of the fact that $I^{(g)}$ is the zero set of the sections F_m, resp. f_m of the rank g vector bundle $\mathbb{E} \otimes \det \mathbb{E}^{\otimes 1/2}$. Provided $I^{(g)}$ vanishes in codimension g it follows that $[I^{(g)}] = c_g(\mathbb{E} \otimes \det \mathbb{E}^{\otimes 1/2})$. This Chern class can be computed using $c_i(\mathbb{E}) = \lambda_i$, and then the claim follows, since there are $2^{g-1}(2^g - 1)$ odd theta characteristics. $\square$

Recall that the moduli spaces $\mathcal{A}_g$ are not compact. By $\mathcal{A}_g^{\mathrm{Perf}}$ we denote the perfect cone toroidal compactification of $\mathcal{A}_g$ and by $\mathcal{A}_g^{\mathrm{Vor}}$ we denote the second Voronoi toroidal compactification. The rest of this paper is devoted to studying the closure of the locus $I^{(g)}$ in $\mathcal{A}_g^{\mathrm{Perf}}$. The boundary degenerations are much harder, and the final result is the computation of the class of the closure in theorem 1.3.

3. EXTENSION OF THETA GRADIENTS ON THE BOUNDARY

The boundary of the perfect cone compactification $\mathcal{A}_g^{\mathrm{Perf}}$ is an irreducible divisor $D \subset \mathcal{A}_g^{\mathrm{Perf}}$. We denote by $p : \mathcal{A}_g(\ell) \to \mathcal{A}_g$ and $\bar{p} : \mathcal{A}_g^{\mathrm{Perf}}(\ell) \to \mathcal{A}_g^{\mathrm{Perf}}$ the level ℓ covers of the moduli spaces, and by D_i the irreducible divisorial components of the boundary of $\mathcal{A}_g^{\mathrm{Perf}}(\ell)$. We denote δ_i the class of the boundary divisor in $CH^1(\mathcal{A}_g^{\mathrm{Perf}}(\ell))$. Note that the cover $\bar{p} : \mathcal{A}_g^{\mathrm{Perf}}(\ell) \to \mathcal{A}_g^{\mathrm{Perf}}$ branches to order ℓ along each D_i, and thus we have $\bar{p}_*(\delta_i) = \delta/\ell$.

We have seen that the gradient of the theta function can be interpreted as a section of the vector bundle $\mathbb{E} \otimes \det \mathbb{E}^{\otimes 1/2}$. The aim of this section is to prove that we can extend this to the perfect cone toroidal compactification $\mathcal{A}_g^{\mathrm{Perf}}$ of $\mathcal{A}_g$.

We first remark that the Hodge bundle extends as a vector bundle over any toroidal compactification $\mathcal{A}_g^{\mathrm{tor}}$ of $\mathcal{A}_g$ (see [Mum77]), as well as over any toroidal compactification of any level cover. Indeed, to define the Hodge bundle we note that any toroidal compactification admits a universal family of (non-compact) semiabelian varieties $\mathcal{G}_g^{\mathrm{tor}}$. This has the "zero" section (which is really $1 \in \mathbb{C}^*$ on each torus) $s : \mathcal{A}_g^{\mathrm{tor}} \to \mathcal{G}_g^{\mathrm{tor}}$, and the Hodge bundle is defined by $\mathbb{E} := s^*(\Omega^1_{\mathcal{G}_g^{\mathrm{tor}}/\mathcal{A}_g^{\mathrm{tor}}})$. In particular, the fiber of $\mathbb{E}$ over $[A] \in \mathcal{A}_g$ is given by $\mathbb{E}_{[A]} = \Omega^1_{A,0}$.

In this section we shall work over $\mathcal{A}_g(4, 8)$ and its perfect cone compactification $\mathcal{A}_g^{\mathrm{Perf}}(4, 8)$. We will, by abuse of notation, denote the

Hodge bundle on this level cover, as well as its extension to the compactification, by $\mathbb{E}$.

Proposition 3.1. *The gradients of theta functions with characteristics at zero, F_m, extend on $\mathcal{A}_g^{\mathrm{Perf}}(4, 8)$ to sections of the extension of the Hodge bundle twisted by a square root of its determinant, i.e. we have some extensions*

$$\overline{F_m} \in H^0\left(\mathcal{A}_g^{\mathrm{Perf}}(4, 8), \det \mathbb{E}^{\otimes 1/2} \otimes \mathbb{E}\right).$$

Proof. Before entering into the necessary computations, we will make some comments. It is enough to prove extension to the generic point on each boundary component, as extension over codimension 2 sets is then automatic by Hartogs' extension theorem on normal analytic spaces. Here we shall make use of the fact that the perfect cone decomposition $\mathcal{A}_g^{\mathrm{Perf}}$ has only one boundary component. This is no longer true for $\mathcal{A}_g^{\mathrm{Perf}}(4, 8)$, but the group $\Gamma_g/\Gamma_g(4, 8)$ acts transitively on the set of boundary components D_i of $\mathcal{A}_g^{\mathrm{Perf}}(4, 8)$. Hence it will be sufficient to consider one of them — we shall work with the so-called standard boundary component. Since the Voronoi and the perfect cone compactification coincide in genus $g \leq 3$ and since $\mathcal{A}_g^{\mathrm{Vor}}$ is a blow-up of $\mathcal{A}_g^{\mathrm{Perf}}$ in genus $g = 4, 5$ ([ER88, RB78]) we also obtain extension to the Voronoi compactification for genus $g \leq 5$.

Recall that the boundary components of $\mathcal{A}_g^{\mathrm{Perf}}(4, 8)$ correspond to lines in $\mathbb{Q}^{2g}$ modulo the action of the group $\Gamma_g(4, 8)$. We shall work with the standard cusp corresponding to the line l_0 generated by the vector $(0, \ldots, 0; 1, 0, \ldots, 0)$. Let $P(l_0)$ be the corresponding parabolic subgroup and let $P'(l_0)$ be the center of the unipotent radical of $P(l_0)$. Moreover let $U(l_0) = P'(l_0)\backslash\mathcal{H}_g$ be the partial quotient with respect to $P'(l_0)$ and let $V(l_0)$ be the partial compactification of $U(l_0)$ given by adding the cusp corresponding to l_0 (see below for details). The partial compactification of $\mathcal{A}_g^{\mathrm{Perf}}(4, 8)$ in a neighborhood of the standard cusp is then obtained by taking the quotient of $V(l_0)$ by the group $P(l_0)/P'(l_0)$. Clearly it makes sense to speak about the Hodge bundle $\mathbb{E}_{\mathcal{H}_g}$ over the Siegel space $\mathcal{H}_g$ (where we have a universal family) as well as about the Hodge bundle $\mathbb{E}_{U(l_0)}$ resp. its extension $\mathbb{E}_{V(l_0)}$ over $U(l_0)$ and $V(l_0)$ respectively. The Hodge bundle $\mathbb{E}_{\mathcal{H}_g}$ is trivial as follows immediately from the construction of the universal family over $\mathcal{H}_g$. More precisely the universal family over $\mathcal{H}_g$ is given as the quotient of $\mathcal{H}_g \times \mathbb{C}^g$ by the group $\mathbb{Z}^{2g}$ where $(M, N) \in \mathbb{Z}^{2g}$ acts on $\mathcal{H}_g \times \mathbb{C}^g$ by $(\tau, z) \mapsto (\tau, z + \tau M + N)$. The differentials $dz_1, \ldots, dz_g$ then define a trivialization of $\mathbb{E}_{\mathcal{H}_g}$.

Lemma 3.2. *The following holds*

  (i) *The trivialization of $\mathbb{E}_{\mathcal{H}_g}$ over $\mathcal{H}_g$ descends to a trivialization of $\mathbb{E}_{U(l_0)}$ over $U(l_0)$.*
  (ii) $j_*(\mathbb{E}_{U(l_0)}) = \mathbb{E}_{V(l_0)}$ *where $j : U(l_0) \hookrightarrow V(l_0)$ is the inclusion.*

Proof of the lemma. We recall that the universal abelian variety over $\mathcal{A}_g(4, 8)$ is defined by taking the quotient of $\mathcal{H}_g \times \mathbb{C}^g$ with respect to the semi-direct product $\mathbb{Z}^{2g} \rtimes \Gamma_g(4, 8)$, which acts as follows:

$$\left((M, N), \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) : \mathcal{H}_g \times \mathbb{C}^g \to \mathcal{H}_g \times \mathbb{C}^g$$

$$(\tau, z) \mapsto ((A\tau + B)(C\tau + D)^{-1}, (z + \tau M + N)(C\tau + D)^{-1}).$$

The center of the unipotent radical of the parabolic subgroup associated to a line in $\mathbb{Q}^{2g}$ is a rank 1 lattice. For the standard cusp it consists of the matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of the form

$$\begin{pmatrix} 1 & 0 & s & 0 \\ 0 & 1_{g-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{g-1} \end{pmatrix} \quad \text{where } s \in 8\mathbb{Z}.$$

In particular $A = D = 1_g$ and $C = 0$. From this we can immediately deduce the first assertion of the lemma.

   In order to prove the second assertion we have to understand how the universal semi-abelian variety can be extended over the generic point of the boundary divisor associated to the standard cusp. The quotient of $\mathcal{H}_g$ by P' is given by

$$\mathcal{H}_g \to \mathcal{H}_{g-1} \times \mathbb{C}^{g-1} \times \mathbb{C}^*$$

$$\tau = \begin{pmatrix} \omega & b^t \\ b & \tau' \end{pmatrix} \mapsto (\tau', b, \mathbf{e}(\omega/8)).$$

We denote the image of this quotient map by U. This is an open subset (in the analytic topology) of $\mathcal{H}_{g-1} \times \mathbb{C}^{g-1} \times \mathbb{C}$. Let V be the interior of the closure of U in $\mathcal{H}_{g-1} \times \mathbb{C}^{g-1} \times \mathbb{C}$. The difference between V and U is the set $\mathcal{H}_{g-1} \times \mathbb{C}^{g-1} \times \{0\}$. Adding this set is adding the divisor associated to the standard cusp. We consider $q_8 := \mathbf{e}(\omega/8)$ as the coordinate on $\mathbb{C}^*$. Thus adding the boundary divisor corresponds to adding the set $\{q_8 = 0\}$. We now have to understand the universal semi-abelian variety over U and its extension to V. Let $N_1 \cong \mathbb{Z}^{2g}$ be the lattice given by $\{M\tau + N \mid M, N \in \mathbb{Z}^g\}$ and let $N_2 = \{M; M \in \mathbb{Z}^g\} \cong \mathbb{Z}^g$.

   In order to construct the universal semi-abelian variety we consider the action of $N_1 \rtimes P'$ on $\mathcal{H}_g \times \mathbb{C}^g$. For this we first consider the action

of the subgroup $N_2 \rtimes P'$. Clearly, the quotient of $\mathcal{H}_g \times \mathbb{C}^g$ by this subgroup is the trivial $(\mathbb{C}^*)^g$-bundle on U. We denote the coordinates on $(\mathbb{C}^*)^g$ by $x_i = \mathbf{e}(z_i); i = 1, \dots, g$. The action of $N_3 = N_1/N_2 \cong \mathbb{Z}^g$ on $U \times (\mathbb{C}^*)^g$ is trivial on the base and by multiplication with powers of $(t_{k,1}, t_{k,2}, \dots, t_{k,g})$ where $t_{k,j} = \mathbf{e}(\tau_{kj})$ on the torus $(\mathbb{C}^*)^g$. In order to construct the semi-universal abelian variety we extend the trivial $(\mathbb{C}^*)^g$-bundle on U trivially to V and also extend the action of N_3 to $V \times (\mathbb{C}^*)^g$. Note that the action of N_3 on $V \times (\mathbb{C}^*)^g$ is no longer free on the first coordinate when $k = 1$. In order to overcome this difficulty one considers a toroidal embedding $V \times (\mathbb{C}^*)^g \hookrightarrow X_\Sigma$. This construction is analogous to the construction of Shioda modular surfaces (see [HKW93, Part I, 2B and 3D]) where the case $g = 2$ is treated in detail. There is a projection $X_\Sigma \to V$ whose restriction to U is just projection of the trivial torus bundle $U \times (\mathbb{C}^*)^g \to U$. Over the boundary, i.e. over points with $q_8 = 0$ one has a chain of countably many copies of $(\mathbb{C}^*)^g$. The action of N_3 on $U \times (\mathbb{C}^*)^g \to U$ extends to an action on X_Σ and the quotient is a semi-abelian group scheme over V. The semi-abelian group scheme over the partial compactification of $\mathcal{A}_g(4,8)$ in the direction of the standard cusp is then obtained by taking a further quotient with respect to P/P'. Note that this group acts freely due to the presence of the level structure.

To prove the second assertion of the lemma we have to understand the relation of $\mathbb{E}_V$ and $\mathbb{E}_U$. We claim that we can take $dx_1, \dots, dx_g$ as a basis for the fibers of $\mathbb{E}_V$. This indeed proves assertion (ii). To see the claim it is enough to consider the case of the universal elliptic curve with fiber coordinate x_1, as the other coordinates are not affected by the construction. Thus we consider the torus $(\mathbb{C}^*)^2$ with coordinates $(w_1, t_{1,1}) = (x, q)$. For this we define a torus embedding $(\mathbb{C}^*)^2 \hookrightarrow X'_\Sigma$. As we are only interested in the situation over the section given by the origin it suffices to look at one chart X'_{σ_0} of X'_Σ. We are thus in the situation of [HKW93, p. 29] and $X'_{\sigma_0} \cong \mathbb{C}^2$ with embedding $(x, q) \mapsto (x, x^{-1}q) = (u, v)$. The projection onto the base is given by $(u, v) \mapsto uv = q$. The section given by the origin is the set $\{(1, q); q \in \mathbb{C}\}$. In order to describe the Hodge bundle we have to consider the relative tangent bundle $\Omega^1_{\mathbb{C}^2/\mathbb{C}}$ restricted to the zero-section. This is generated by du, dv modulo the pullback $d(uv) = udv + vdu$. Restricting the latter to the zero-section gives $dv + vdu$ and hence the relative cotangent bundle is generated by $du = dx$ which we can take as trivializing section. $\square$

This argument shows that, in particular, the Hodge bundle over V is trivial. Hence all we have to do to prove our claim is to show that

the functions F_m are invariant with respect to P' (which is obvious) and that they extend without poles to V. For this we have to compute the Fourier-Jacobi expansion. This was done in detail in [GSM09], and we now summarize the results for completeness. Indeed, denote $z = (z_1, z')$ where $z_1 \in \mathbb{C}$ and $z' \in \mathbb{C}^{g-1}$ and write $\tau = \begin{pmatrix} \omega & b^t \\ b & \tau' \end{pmatrix}$. The Fourier-Jacobi expansion at the standard cusp amounts to writing the Taylor series in q_8 (the coordinates transverse to the boundary of $\mathcal{A}_g^{\mathrm{Perf}}(8)$) as $\tau_{11} \to i\infty$ For the characteristics of the gradient being $\varepsilon = \varepsilon_1 \varepsilon'$ and $\delta = \delta_1 \delta'$ we then get for the case $\varepsilon_1 = 0$

$$(7) \qquad \partial_{z_1}\theta \begin{bmatrix} 0 & \varepsilon' \\ \delta_1 & \delta' \end{bmatrix} = q_8^4 \cdot 4\pi i (-1)^{\delta_1} \theta \begin{bmatrix} \varepsilon' \\ \delta' \end{bmatrix} (\tau, b) + O(q_8^{16})$$

$$(8) \qquad \partial_z \theta \begin{bmatrix} 0 & \varepsilon' \\ \delta_1 & \delta' \end{bmatrix} = 2\partial_z \theta \begin{bmatrix} \varepsilon' \\ \delta' \end{bmatrix} (\tau, z)|_{z=0} + O(q_8^4).$$

This is to say that in the case of $\varepsilon_1 = 0$ (this can be said invariantly: see Proposition 3.3 below) the gradient does not vanish identically. On the other hand, for $\varepsilon_1 = 1$ we get

$$(9) \qquad \partial_{z_1}\theta \begin{bmatrix} 1 & \varepsilon' \\ \delta_1 & \delta' \end{bmatrix} = q_8 \cdot 4\pi i \mathbf{e}(\delta_1/4) \theta \begin{bmatrix} \varepsilon' \\ \delta' \end{bmatrix} (\tau, b/2) + O(q_8^9)$$

$$(10) \qquad \partial_z \theta \begin{bmatrix} 1 & \varepsilon' \\ \delta_1 & \delta' \end{bmatrix} = q_8 \cdot 2\mathbf{e}(\delta_1/4) \partial_z \theta \begin{bmatrix} \varepsilon' \\ \delta' \end{bmatrix} (\tau, b/2) + O(q_8^9)$$

which shows that in this case the generic vanishing order of $\overline{F_m}$ in q_8 is precisely equal to one. In any case the Fourier-Jacobi expansion is holomorphic and in view of Lemma 3.2 this proves Proposition 3.1. □

The above computations actually give us more information. To explain this, we prefer to work on the full level-8 cover $\mathcal{A}_g^{\mathrm{Perf}}(8)$ rather than $\mathcal{A}_g^{\mathrm{Perf}}(4, 8)$. Recall from [Nam80, Ch. 4] (or eg. [Erd07, Sec. 3]) that the boundary components of $\mathcal{A}_g^{\mathrm{Perf}}(2)$ are in bijective correspondence with the primitive vectors in $((\mathbb{Z}/8\mathbb{Z})^{2g} \setminus \{0\})/ \pm 1$, and Γ_g acts transitively on the set of boundary components of $\mathcal{A}_g^{\mathrm{Perf}}(8)$. Under this correspondence the standard boundary component corresponds to the vector $(0, 0, \ldots, 0; 1, 0, \ldots, 0)$. Now let D_n be a boundary component of $\mathcal{A}_g^{\mathrm{Perf}}(8)$ labeled by a primitive vector $\pm n \in (\mathbb{Z}/8\mathbb{Z})^{2g}$. We denote the reduction of n modulo 2 (which simply sends odd entries to 1 and even entries to 0) by n_2. We identify an odd 2-torsion point $m = (\tau\varepsilon + \delta)/2$ with its characteristic $(\varepsilon, \delta) \in (\mathbb{Z}/2\mathbb{Z})^{2g}$. The above calculations show that F_m does not vanish on the standard component if and only if $\varepsilon_1 = 0$. Note that in this case $\sigma(m + n_2) = \varepsilon\delta + \varepsilon_1 = 1$ since m is odd,

i.e. $\varepsilon\delta = 1 \mod 2$ and $\varepsilon_1 = 0$. Using the action of the finite group $\Gamma_g/\Gamma_g(8)$ on $\mathcal{A}_g^{\mathrm{Perf}}(8)$ we thus obtain

Proposition 3.3. *The gradient $\overline{F_m}$ does not vanish generically on the boundary component D_n of $\mathcal{A}_g^{\mathrm{Perf}}(8)$ if and only if $\sigma(m + n_2) = 1$. Otherwise it vanishes on D_n with generic vanishing order equal to one.*

We would like to interpret the above proposition by saying that

(11)
$$\tilde{F}_m \in H^0\left(\mathcal{A}_g^{\mathrm{Perf}}(8), \det(\mathbb{E})^{\otimes 1/2} \otimes \mathbb{E} \otimes \mathcal{O}\left(-\sum_{\{\pm n | m + n_2 \text{ even}\}} D_n\right)\right).$$

To be able to say this we must show that $\sum_{\{\pm n | m + n_2 \text{ even}\}} D_n$ is a Cartier divisor and not just a Weil divisor. This is not immediate as $\mathcal{A}_g^{\mathrm{Perf}}(8)$ is not a smooth space for $g \geq 4$. Analytically this means that we want to divide the gradient by the product of the defining equations for all boundary components of $\mathcal{A}_g^{\mathrm{Perf}}(8)$ where it vanishes identically with multiplicity one.

The gradients of theta functions with characteristics were studied in [GSM04], and their boundary degenerations were also considered in [GSM09]. It seems, however, that the gradients f_m of the theta function at odd two-torsion points were not as extensively studied, while for us they turn out to be more important.

Proposition 3.4. *The divisor $\sum_{\{\pm n | m + n_2 \text{ even}\}} D_n$ is a Cartier divisor and we can also extend the sections f_m to sections*

$$\tilde{f}_m \in H^0\left(\mathcal{A}_g^{\mathrm{Perf}}(8), \det(\mathbb{E})^{\otimes(1/2)} \otimes \mathbb{E} \otimes \mathcal{O}\left(-\sum_{\{\pm n | m + n_2 \text{ even}\}} D_n\right)\right)$$

not vanishing identically on any component D_i of the boundary of $\mathcal{A}_g^{\mathrm{Perf}}(8)$. Moreover $\tilde{f}_m$ and $\tilde{F}_m$ are equal up to a non-zero constant and thus we have

$$\overline{G_m^{(g)}}(8) := \{\tau \in \mathcal{A}_g^{\mathrm{Perf}}(8) \mid \tilde{F}_m = 0\} = \{\tau \in \mathcal{A}_g^{\mathrm{Perf}}(8) \mid \tilde{f}_m = 0\}.$$

Proof. Indeed, by formula (5) we have on $\mathcal{A}_g(8)$ the relation

$$f_m(\tau) = \mathbf{e}(-\varepsilon^t \tau \varepsilon/8 - \varepsilon^t \delta/4) F_m(\tau),$$

and thus to determine the extension of f_m to $\mathcal{A}_g^{\mathrm{Perf}}(8)$ it is enough to determine the properties of the exponential factor, and to combine them with the results obtained above about $\overline{F_m}$. Note also that the functions $\mathbf{e}(-\varepsilon^t \tau \varepsilon/8 - \varepsilon^t \delta/4)$ descend to any partial quotient which arises in the

construction of the toroidal compactification and thus $\sum_{\{\pm n \mid m+n_2 \text{ even}\}} D_n$ is a Cartier divisor on $\mathcal{A}_g^{\text{Perf}}(8)$. To determine the extension of f_m it is again enough to work with the standard component. Noticing that $\mathbf{e}(-\varepsilon^t \delta/4)$ is a constant independent of τ and thus irrelevant for our computations, we compute in the above notation

$$\mathbf{e}\left(-\frac{1}{8}\begin{pmatrix}\varepsilon_1 & \varepsilon'^t\end{pmatrix}\begin{pmatrix}\omega & b^t \\ b & \tau'\end{pmatrix}\begin{pmatrix}\varepsilon_1 \\ \varepsilon'\end{pmatrix}\right)$$

$$= \mathbf{e}\left(-\frac{1}{8}\left(\varepsilon_1^2\omega + 2\varepsilon_1 b^t\varepsilon' + \varepsilon'^t\tau'\varepsilon'\right)\right) = q_8^{-\varepsilon_1^2}\mathbf{e}(-\varepsilon_1 b^t\varepsilon'/4 - \varepsilon'^t\tau'\varepsilon'/8).$$

It thus follows that for $\varepsilon_1 = 0$ this exponential factor is independent of q_8, while for $\varepsilon_1 = 1$ it has a pole q_8^{-1}. Noticing that in these cases $\overline{F_m} = O(1)$ and $\overline{F_m} = O(q_8)$, respectively, it thus follows that the extension $\tilde{f}_m$ does not vanish generically on any boundary component of $\mathcal{A}_g^{\text{Perf}}(8)$, and thus finally that $\tilde{f}_m$ and $\tilde{F}_m$ coincide up to a non-zero constant. $\qquad\square$

Since the group $\Gamma_g/\Gamma_g(8)$ permutes the sets $\overline{G_m^{(g)}(8)} \subset \mathcal{A}_g^{\text{Perf}}(8)$, this defines a locus $\overline{G} = \overline{G^{(g)}}$ such that $\cup_m \overline{G_m^{(g)}(8)} = p^*(\overline{G^{(g)}})$.

Geometrically, the above computation amounts to saying that we have determined the vanishing orders of $\overline{F_m}$ on the open part of the boundary of $\mathcal{A}_g^{\text{Perf}}(8)$, and in particular noticed that the zero loci of $\tilde{f}_m$ do not have any irreducible components contained in this locus. It is then natural to wonder whether if we consider the full compactification $\mathcal{A}_g^{\text{Perf}}(8)$, the zero locus of any $\tilde{f}_m$ may have any irreducible component contained in the boundary (from the above, it would then have to be disjoint from the partial compactification). We conjecture that this in fact impossible:

Conjecture 3.5. *For any genus we have the equality* $\overline{G^{(g)}} = \overline{I^{(g)}}$.

While we were unable to prove this conjecture in full generality, one of the main results of [GK11] is theorem 1.3 there, which is a proof of the above conjecture for $g \leq 5$. This result is obtained by explicitly describing the geometry of various types of principally polarized semi-abelic varieties in detail, and checking that there are no components of $\overline{G^{(g)}}$ contained in the loci of semi-abelic varieties of any of the types described. We thus obtain our second main result:

Proof of theorem 1.3. Recall that what this theorem claims is an expression for the class of the locus $\overline{I^{(g)}}$ in $CH^g(\mathcal{A}_g^{\text{Perf}})$. Note, however, that the locus $\overline{G_m^{(g)}(8)} \subset \mathcal{A}_g^{\text{Perf}}(8)$ is by definition the zero locus of $\tilde{f}_m$,

which is a section of a certain vector bundle given by proposition 3.4. Thus the class of $\overline{G_m^{(g)}}(8)$, if it has expected codimension g, is given by the top Chern class of this vector bundle. Thus we need to first understand the geometry of the cover, and then to compute the top Chern class.

Indeed, we are working with the full level ℓ Galois cover $p : \mathcal{A}_g^{\mathrm{Perf}}(\ell) \to \mathcal{A}_g^{\mathrm{Perf}}$, with the Galois group $\mathrm{Sp}(2g, \mathbb{Z}/\ell\mathbb{Z})$. As long as one considers projective varieties this action factors through $\mathrm{PSp}(2g, \mathbb{Z}/\ell\mathbb{Z})$. However, as we will be doing all our calculations in the stack set-up it is important to work with $\mathrm{Sp}(2g, \mathbb{Z}/\ell\mathbb{Z})$ as $\pm\mathbf{1}$ acts non-trivially on $\mathcal{A}_g^{\mathrm{Perf}}$ (as a stack). For any $\mathrm{Sp}(2g, \mathbb{Z}/\ell\mathbb{Z})$-invariant cycle Z with class $[Z]$ on $\mathcal{A}_g^{\mathrm{Perf}}(n)$ we define

$$(12) \qquad \overline{p}_*([Z]) = \frac{1}{|\mathrm{Sp}(2g, \mathbb{Z}/\ell\mathbb{Z})|}[p_*(Z)].$$

For classes Z which are the pullback of a stacky class W we have $\overline{p}_*(Z) = W$ and in particular we have $\overline{p}_*(\lambda_i) = \lambda_i$ where the λ_i are the classes of the Hodge bundle(s).

Now, to prove the theorem we first perform the computation on $\mathcal{A}_g^{\mathrm{Perf}}(8)$, and then argue invariance to reduce this to $\mathcal{A}_g^{\mathrm{Perf}}(2)$. Denote by $\alpha_1, \cdots, \alpha_g$ the Chern roots of the Hodge vector bundle $\mathbb{E}$ on $\mathcal{A}_g^{\mathrm{Perf}}(8)$, so that the Chern polynomial is $c(\mathbb{E}) = \prod(1+\alpha_i)$, which means that for the symmetric polynomials we have $s_i(\alpha_1, \ldots, \alpha_g) = \lambda_i$. From formula (11) we know that $\tilde{f}_m$ is a section of $\mathbb{E} \otimes L_m$ for a certain line bundle L_m. The cycle $\overline{G^{(g)}}(8)$ (which is the pre-image of $\overline{G^{(g)}}$) is the union of the vanishing loci of the sections F_m. Now, if $\tilde{f}_m$ vanish in codimension g, so that $\overline{G}$ is pure of codimension g — and this is our assumption (or the content of [GK11, Thm. 1.2 and 1.3] for $g \leq 5$), we recall that the vanishing locus is reduced, as noted in remark 2.1. It now follows, see [Ful98, Chapter 14], that

$$[\overline{G^{(g)}}(8)] = \sum_{m \in (\mathbb{Z}/2\mathbb{Z})_{\mathrm{odd}}^{2g}} c_g(\mathbb{E} \otimes L_m).$$

Denoting

$$\ell_m := c_1(L_m) = \frac{\lambda_1}{2} - \sum_{\pm n \in (\mathbb{Z}/8\mathbb{Z})_{\mathrm{primitive}}^{2g}, n_2+m \text{ even}} \delta_n = \frac{\lambda_1}{2} - \sum_{n \in Z_m} \delta_n$$

we compute on $\mathcal{A}_g^{\mathrm{Perf}}(8)$ that

$$c_g(\mathbb{E} \otimes L_m) = \left[\prod_{i=1}^{g}(1 + \alpha_i + \ell_m) \right]_{\deg=g} = \sum_{i=0}^{g} s_{g-i}(\alpha_1, \ldots, \alpha_g)\ell_m^i$$

$$= \sum_{i=0}^{g} \lambda_{g-i}\ell_m^i = \sum_{i=0}^{g} \lambda_{g-i} \left(\frac{\lambda_1}{2} - \sum_{n \in (\mathbb{Z}/8\mathbb{Z})_{\mathrm{primitive}}^{2g}, n_2+m \ \mathrm{even}} \delta_n \right)^i$$

and the formula on $\mathcal{A}_g^{\mathrm{Perf}}(8)$ follows.

Finally, to obtain the stated formula, as a pushforward from $\mathcal{A}_g^{\mathrm{Perf}}(2)$, we note that the expression above is certainly invariant under the Galois action $\Gamma_g(2)/\Gamma_g(8)$, as only the reduction n_2 modulo two matters. Remembering that $\mathcal{A}_g^{\mathrm{Perf}}(8) \to \mathcal{A}_g^{\mathrm{Perf}}(2)$ branches along the boundary to order 4, we get the result as stated. Note also that as we remarked before, the vanishing locus of each F_m is well-defined as a subvariety of $\mathcal{A}_g(2)$, even though F_m itself is only a section of a well-defined vector bundle over $\mathcal{A}_g(4, 8)$. $\qquad\square$

4. THE TAUTOLOGICAL RING

We will now use various properties of the tautological ring of $\mathcal{A}_g^{\mathrm{Perf}}$ (this section uses the perfect pairing, and we thus have to work with rational coefficients) to compute the projection of the class $\left[\overline{I^{(g)}}\right]$ given by theorem 1.3 to the tautological ring, thus proving theorem 1.4.

We recall that the perfect cone (and in fact any toroidal) compactification $\mathcal{A}_g^{\mathrm{Perf}}$ admits a stratification induced by the natural map $P : \mathcal{A}_g^{\mathrm{Perf}} \to \mathcal{A}_g^{\mathrm{Sat}}$ to the Satake compactification. Writing the Satake compactification as

$$\mathcal{A}_g^{\mathrm{Sat}} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \mathcal{A}_{g-2} \ldots \sqcup \mathcal{A}_0,$$

following van der Geer [vdG98], we denote $\beta_k^0 := P^{-1}(\mathcal{A}_{g-k})$ the open strata, and denote by $\beta_k := \beta_k^0 \sqcup \ldots \sqcup \beta_g^0 = P^{-1}(\mathcal{A}_{g-k}^{\mathrm{Sat}})$ the closed strata — and by abuse of notation, their classes in $CH^k(\mathcal{A}_g^{\mathrm{Perf}})$. We also recall that δ_i denote the classes of the irreducible components of the boundary of $\mathcal{A}_g^{\mathrm{Perf}}(2)$, and we follow van der Geer in denoting by σ_k the degree k elementary symmetric polynomial in the classes δ_i, which we would like to consider as an element of $CH^k(\mathcal{A}_g^{\mathrm{Perf}}(2))$. This, however, is problematic, as in general the boundary components δ_i are not Cartier divisors (this can be seen for $g = 4$ by looking at the second perfect cone which is a 10-dimensional cone with 12 extremal rays, see [HS04] for a discussion of this), and thus this is also the case

for $g > 4$. However, the sum of all boundary components, as well as certain partial sums, are Cartier as we saw in section 3. In what follow, whenever we work with expressions such as the σ_i we shall mean by this the restriction of this cycle to the smooth part $\mathcal{A}_g^{\mathrm{perf},0}$ of $\mathcal{A}_g^{\mathrm{Perf}}$ where it is well defined. We observe that the singular locus of $\mathcal{A}_g^{\mathrm{Perf}}$ is contained in β_4. We also notice that σ_k is invariant under the action of Γ_g, and we can thus also denote σ_k the corresponding class in $CH^k(\mathcal{A}_g^{\mathrm{perf},0})$.

We shall frequently make implicit use of the following very useful fact, see [Ful98, Proposition 1.1.8]. If Z is a closed subvariety of $\mathcal{A}_g^{\mathrm{Perf}}$ then there is an exact sequence

$$(13) \qquad CH_k(Z) \to CH_k(\mathcal{A}_g^{\mathrm{Perf}}) \to CH_k(\mathcal{A}_g^{\mathrm{Perf}} - Z) \to 0.$$

We start with two useful lemmas concerning the projection of classes supported on the boundary to the tautological ring.

Lemma 4.1. *Let $[X]$ be a class of codimension at most $g - 1$ in the Chow of a toroidal compactification $\mathcal{A}_g^{\mathrm{tor}}$, supported over the boundary β_1. Then $(P(\lambda)[X])^{\mathrm{taut}} = 0$ for every non-constant polynomial $P(\lambda)$ in the tautological classes λ_i.*

Proof. We first notice that it is enough to prove that $[X]^{\mathrm{taut}} = 0$: if $[X]$ pairs to zero with every polynomial of complementary degree in the λ_i, then the same is true for $P(\lambda)[X]$.

We write $[X] = Q(\lambda) + [Y]$ where $Q(\lambda)$ is a polynomial in the $\lambda_i; i \leq g - 1$ by the assumption on the codimension of $[X]$, and $[Y]$ pairs to zero with all tautological classes of complementary degree (that is, polynomials in the λ_i of complementary degree). Restricting to $\mathcal{A}_g$ we obtain $[X]|_{\mathcal{A}_g} = 0$ and $[Y]|_{\mathcal{A}_g}$ still projects to 0 in the tautological ring of $\mathcal{A}_g$. But this is a contradiction, since $Q(\lambda)$ defines a non-zero class in the tautological ring of $\mathcal{A}_g$, the only new relation in the tautological ring of $\mathcal{A}_g$ versus that in the tautological ring of $\mathcal{A}_g^{\mathrm{tor}}$ being $\lambda_g = 0$ in degree g (see [vdG99]). $\qquad\square$

Lemma 4.2. *Let $[X]$ be a Chow cohomology class of codimension at most g on a toroidal compactification $\mathcal{A}_g^{\mathrm{tor}}$ supported over β_2. Then $[X]^{\mathrm{taut}} = 0$.*

Proof. The idea of the proof is similar to the proof of the previous lemma — basically these lemmas together are based on the fact that in a sense λ_g is the only degree g tautological class supported on β_1. Rigorously, suppose that we have $[X] = Q(\lambda) + [Y]$ in $CH^*(\mathcal{A}_g^{\mathrm{Perf}})$ with $[Y]^{\mathrm{taut}} = 0$. Unless we have $Q(\lambda) = c\lambda_g$, for some $c \in \mathbb{C}$, the image of $Q(\lambda)$ in $CH^*(\mathcal{A}_g)$ (obtained from $CH^*(\mathcal{A}_g^{\mathrm{Perf}})$ by imposing one extra

condition $\lambda_g = 0$) is non-zero, which contradicts the fact that the class is supported over β_2. But if we have $Q(\lambda) = c\lambda_g$, then we must have

$$0 = \lambda_1^{\frac{g(g-1)}{2}}[X] = c\lambda_1^{\frac{g(g-1)}{2}}\lambda_g + 0 \neq 0$$

where we have used the fact that $\lambda_1^{\frac{g(g-1)}{2}}$ is zero on β_2 for dimension reason, and the last intersection number is non-zero by Hirzebruch-Mumford proportionality. (This can also be seen explicitly without a reference to proportionality, since by [vdG99, Theorem 4.7] the class λ_g is proportional to the class $[B_g]$ of the closure of the zero section of $\mathcal{X}_{g-1} \to \mathcal{A}_{g-1}$, modulo classes supported on β_2, which have zero pairing with the top power of λ_1 for dimension reasons. By [SB06] the normalization of this closure of the locus of trivial extensions is isomorphic to $\mathcal{A}_{g-1}^{\mathrm{Perf}}$, where the top self-intersection number of λ_1 is non-zero). $\qquad\square$

We now compute the projection to the tautological ring of the one degree g monomial in σ and λ not covered by the above lemmas.

Proposition 4.3. *The projection of σ_1^g to the tautological ring is*

$$(14) \qquad [\sigma_1^g]^{\mathrm{taut}} = (-2)^{g-1}(g-1)![B_g]^{\mathrm{taut}} = \frac{-2^{g-1}(g-1)!}{\zeta(1-2g)}\lambda_g,$$

where B_g is the class of the closure of the zero section of $\mathcal{X}_{g-1} \to \mathcal{A}_{g-1}$ in the boundary of the partial compactification, as above.

Proof. We first observe that it suffices to prove the first equality, since it follows from [vdG99, Theorem 4.7] and the above lemma 4.2 that $[B_g]^{\mathrm{taut}} = (-1)^g\lambda_g/\zeta(1-2g)$. Recall $\sigma_1|_{\beta_1^0} = -2T$, where $\beta_1^0 = \beta_1 \setminus \beta_2$ is the open part (isomorphic to the universal Kummer family), and $T \subset \mathcal{X}_{g-1}$ is the universal theta divisor trivialized along the zero section (see [EGH10] for a more precise description). Thus we compute $\sigma_1^g = \overline{(-2T)^{g-1}} + X$ for some class X supported within β_2. Here $\overline{(-2T)^{g-1}}$ is an extension of the class $(-2T)^{g-1}$ to $\mathcal{A}_g^{\mathrm{Perf}}$ (which is unique up to a cycle supported on β_2). Since the class $[\sigma_1^g] - (-2)^{g-1}(g-1)![B_g]$ is supported on β_1 we can argue as in the proof of lemma 4.2 that $[\sigma_1^g]^{\mathrm{taut}} - (-2)^{g-1}(g-1)![B_g]^{\mathrm{taut}} = c\lambda_g$. Now note that on each fiber of the boundary of the partial compactification $\mathcal{X}_{g-1} \to \mathcal{A}_{g-1}$ the divisor T restricts to the principal polarization, with top self-intersection number $(g-1)!$. Thus the degree of T^{g-1} on each fibre is $(g-1)!$. Intersecting with $\lambda_1^{\frac{g(g-1)}{2}}$ then shows that $c = 0$ as claimed. $\qquad\square$

We are now ready to compute the projection $[\overline{I^{(g)}}]^{\mathrm{taut}}$.

Proof of theorem 1.4. Let us expand the expression in formula (2) in theorem 1.3, and take one of the terms: this will then be of the form $p_*(Q(\lambda)P(\delta_n))$, where $Q(\lambda)$ is a monomial in the classes λ_i (i.e. lies in the tautological ring), P is a monomial in classes δ_n of various irreducible components of the boundary of $\mathcal{A}_g^{\mathrm{Perf}}(2)$, and the total codimension is equal to g.

If P contains a product of at least two different boundary components, $\delta_{n_1}\delta_{n_2}$, it is supported within β_2, and thus by lemma 4.2 in this case we have $[P]^{\mathrm{taut}} = 0$. This implies that all pairing of P with polynomials in λ classes are zero, and thus this is also the case for QP, so we have $[p_*(Q(\lambda)P(\delta_n))]^{\mathrm{taut}} = 0$. Thus to have a non-zero projection to the tautological ring we must have $P = \delta_n^i$. Moreover, if $0 < i < g$, then lemma 4.1 applies, and we also get $[p_*(Q(\lambda)P(\delta_n))]^{\mathrm{taut}} = 0$. Thus the only monomials in the expansion of (2) that can have a non-zero projection to the tautological ring are either those that do not contain any δ_n (and thus these are the ones giving the expression on the open part in theorem 1.1), or the ones of the form δ_n^g. Thus the extra term compared to theorem 1.1 that we need to compute is

$$\frac{1}{N} \sum_{m \in (\mathbb{Z}/2\mathbb{Z})_{\mathrm{odd}}^{2g}} (-4)^{-g} p_* \left(\sum_{n \in Z_m} \delta_n^g \right).$$

We know by proposition 3.3 that $n \in Z_m$ if and only if $\sigma(n+m) = 0$. Thus for n fixed there are 2^{2g-2} such odd m. The projection $p_*\delta_n$ for any n is equal to $\sigma_1/2$ (recall the branching order 2 of the cover $\mathcal{A}_g^{\mathrm{Perf}}(2) \to \mathcal{A}_g^{\mathrm{Perf}}$ along the boundary). Thus the above pushforward is equal to

$$\frac{1}{N}(-1)^g 2^{-2g} 2^{2g-2} \sum_{n \in (\mathbb{Z}/2\mathbb{Z})^{2g} \setminus \{0\}} p_*(\delta_n^g) = (-1)^g 2^{-2-g} \sigma_1^g.$$

We now use formula (14) to obtain

$$(-1)^g 2^{-2-g} [\sigma_1^g]^{\mathrm{taut}} = \frac{(-1)^{g-1}(g-1)!}{8\zeta(1-2g)} \lambda_g + [X]^{\mathrm{taut}},$$

where, however, the class $[X]$ is of codimension g, supported over β_2, and thus has zero tautological projection by lemma 4.2. $\square$

5. COMBINATORICS OF BOUNDARY DIVISORS

The rest of this paper will be devoted to using theorem 1.3 to compute the classes of the loci $\overline{I^{(g)}}$ explicitly for $g \leq 5$ (and not only its tautological part given by theorem 1.4).

In this section we will deal with the combinatorics of the terms appearing in the expression in theorem 1.3, enabling us to further perform the explicit class computations.

Proposition 5.1. *Formula (2) can be written in a different way, grouping by powers of δ, as follows:*

$$(15) \quad [\overline{I^{(g)}}] = \sum_{m \text{ odd}} \overline{p}_* \left(\sum_{j=0}^{g} \left(-\sum_{n \in Z_m} \delta_n \right)^j \sum_{i=j}^{g} \binom{i}{j} \left(\frac{\lambda_1}{2} \right)^{i-j} \lambda_{g-i} \right).$$

This expression is easier to use, as similar terms involving δ_n's are grouped together. To use the proposition we thus need to study the combinatorics of the intersections of the components D_i of the boundary $\partial \mathcal{A}_g^{\mathrm{Perf}}(2)$, and the vanishing of a given $\overline{F_m}$ on a set of components.

The combinatorial description of the boundary components of $\mathcal{A}_g^{\mathrm{Perf}}(2)$ and their intersection behavior can be deduced from the correspondence between cusps and isotropic subspaces of $\mathbb{Q}^{2g}$. For a detailed discussion of the combinatorics of the boundary we refer the reader to [Erd07]. Recall that the group $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z}) = \Gamma_g/\Gamma_g(2)$ has an affine transitive action on the set of odd theta characteristics m. The action of $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ on the set of boundary components of $\mathcal{A}_g^{\mathrm{Perf}}(2)$, i.e. the linear action on $(\mathbb{Z}/2\mathbb{Z})^{2g} \setminus \{0\}$ is also transitive. Recall that the symplectic product of $n_1, n_2 \in (\mathbb{Z}/2\mathbb{Z})^{2g}$ with $n_i = [\alpha_i, \beta_i]$ is defined as $n_1 \cdot n_2 = \alpha_1 \cdot \beta_2 + \alpha_2 \cdot \beta_1 \in \mathbb{Z}/2\mathbb{Z}$. Then boundary components D_{n_1} and D_{n_2} of $\mathcal{A}_g^{\mathrm{Perf}}(2)$ intersect if and only if $n_1 \cdot n_2 = 0$, i.e. n_1, n_2 span an isotropic plane (see. [Erd07, Proposition 3.3.15]). Moreover recall that the intersection $D_{n_1} \cap \ldots \cap D_{n_k} \subset \partial \mathcal{A}_g^{\mathrm{Perf}}(2)$ lies in the stratum β_j and intersects β_j^0, where j is the dimension of the linear span of the vectors $n_1, \ldots, n_k$ in $(\mathbb{Z}/2\mathbb{Z})^{2g}$ (i.e. an open part of this intersection lies in the preimage of $\mathcal{A}_{g-j} \subset \partial \mathcal{A}_g^{\mathrm{Sat}}$).

Proposition 5.2. *For any set of vectors $n_1, \ldots, n_k \in (\mathbb{Z}/2\mathbb{Z})^{2g} \setminus \{0\}$ such that the intersection $D_{n_1} \cap \cdots \cap D_{n_k}$ is non-empty the number of gradients F_m vanishing on each D_{n_i}, i.e. the number of odd $m \in (\mathbb{Z}/2\mathbb{Z})^{2g}$ such that $m + n_i$ is even for all i, is*

  (1) *zero if there exists a linear relation $n_{i_1} + \ldots + n_{i_j} = 0$ with odd number of terms $j = 2l + 1$*
  (2) *equal to 2^{2g-k-1} if $n_1, \ldots, n_k$ are a basis for an isotropic subspace*

Proof. For part (1), relabel the n's so that the linear dependence is $n_1 + \ldots + n_{2l+1} = 0$. For the intersection $D_{n_1} \cap \ldots \cap D_{n_{2l+1}}$ to be non-empty, we must have $n_i \cdot n_j = 0$ for all $1 \le i < j \le 2l+1$. On the other

hand, for $\overline{F_m}$ to vanish on D_{n_i}, we must have $n_i \in Z_m$, i.e. $\sigma(m+n_i) = 0$. Recall that by definition we have $\sigma(m+n) = \sigma(m) + \sigma(n) + m \cdot n$, and thus

$$\sigma(m_1 + \ldots + m_k) = \sum_{i=1}^{k} \sigma(m_i) + \sum_{1 \leq i < j \leq k} m_i \cdot m_j.$$

Suppose now that some $\overline{F_m}$ vanished on all of these D_{n_i}. We can then use the linear dependence to compute

$$1 = \sigma(m) = \sigma\left(m + \sum_{i=1}^{2l+1} n_i\right) =$$

$$\sigma(m) + \sum_{i=1}^{2l+1} \sigma(n_i) + m \cdot \sum_{i=1}^{2l+1} n_i + \sum_{1 \leq i < j \leq 2l+1} n_i \cdot n_j =$$

$$= (2l+1)\sigma(m) + \sum_{i=1}^{2l+1} \sigma(n_i) + m \cdot \left(\sum_{i=1}^{2l+1} n_i\right) = \sum_{i=1}^{2l+1} \sigma(m + n_i) = 0,$$

which is a contradiction.

For part (2), note that the action of $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ is transitive on the set of bases of k-dimensional isotropic subspaces (this is well known, eg. [Erd07, Remark 3.3.9]), and thus any such basis k-tuple is conjugate to the standard one, i.e. in which $n_i = [0, e_i]$ with e_i being the unit vector in $(\mathbb{Z}/2\mathbb{Z})^g$ in the i'th direction. Thus it is enough to count the number of m such that $\sigma(m + n_i) = 0$ for the standard isotropic tuple. Indeed, such m are all of the form

$$\begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} = \begin{bmatrix} 1 \ldots 1 * \ldots * \\ * \ldots * * \ldots * \end{bmatrix}$$

where $*$ denote arbitrary entries and there are k 1's in the top row. Recalling that m must be odd means we can choose all entries except the first bottom one arbitrarily, and then there would be a unique choice of the first bottom one to make m odd. Thus the total number of such m is equal to 2^{2g-k-1} as claimed. $\square$

6. Known cases: genus 2 and 3

We shall <u>now</u> demonstrate how our formula works by computing the classes $[\overline{G^{(g)}}]$ in genus 2 and 3. Note that in general it is an extremely interesting question to define a meaningful "enlarged" tautological ring of the Chow ring for some toroidal compactification of $\mathcal{A}_g$ (see [vdG99],[EvdG05], and we hope our computations here may also shed further light on this.

Note that for $g \leq 3$ the stack $\mathcal{A}_g^{\mathrm{perf}}$ is smooth and thus we can work in the (classical) Chow ring of this space.

6.1. **Genus 2.** Recall that the locus $I^{(2)}$ is empty, and also $\overline{G^{(2)}} = \emptyset$. It is convenient to introduce some ordering on the set $(\mathbb{Z}/2\mathbb{Z})^{2g}$, so that the following formulas are easier to write using $n_1 < n_2$, etc. notation. We now use Theorem 1.3 to compute

$$[\overline{G^{(2)}}] = \sum_{m \text{ odd}} \bar{p}_* \left(\lambda_2 + \lambda_1 \left(\frac{\lambda_1}{2} - \frac{1}{4} \sum_{n \in Z_m} \delta_n \right) + \left(\frac{\lambda_1}{2} - \frac{1}{4} \sum_{n \in Z_m} \delta_n \right)^2 \right)$$

$$= \bar{p}_* \left(\sum_{m \text{ odd}} \left(\frac{5}{4}\lambda_1^2 - \frac{2}{4}\lambda_1 \sum_{n \in Z_m} \delta_n + \frac{1}{16} \sum_{n \in Z_m} \delta_n^2 + \frac{2}{16} \sum_{n_1 < n_2 \in Z_m; n_1 \cdot n_2 = 0} \delta_{n_1} \delta_{n_2} \right) \right)$$

where we used the relation $\lambda_2 = \lambda_1^2/2$. In genus 2 there are 6 odd characteristics. By proposition 5.2 four of $\overline{F_m}$ vanish on any boundary component D_i. Any intersection $D_i \cap D_j$ of two boundary components, if non-empty, is codimension two, and there exist then two $\overline{F_m}$ vanishing on both D_i and D_j. We thus compute

$$[\overline{G^{(2)}}] = \bar{p}_* \left(6 \cdot \frac{5}{4}\lambda_1^2 - 4 \cdot \frac{1}{2}\lambda_1 \sum_n \delta_n + 4 \cdot \frac{1}{16} \cdot \sum_n \delta_n^2 + 2 \cdot \frac{1}{8} \sum_{n_1 < n_2} \delta_{n_1} \delta_{n_2} \right)$$

$$= \frac{15}{2}\lambda_1^2 - \lambda_1 \sigma_1 + \frac{\sigma_1^2 - 2\sigma_2}{16} + \frac{\sigma_2}{16}$$

where we again used the fact that $\mathcal{A}_g^{\mathrm{Perf}}(2) \to \mathcal{A}_g^{\mathrm{Perf}}$ has branching order 2 along the boundary, so that the pushforwards are

(16) $$\bar{p}_*(\sum \delta_n) = \frac{\sigma_1}{2}, \qquad \bar{p}_*(\sum_{n_1 < n_2} \delta_{n_1} \delta_{n_2}) = \frac{\sigma_2}{4}$$

as well as

(17) $$\bar{p}_* \left(\sum_n \delta_n^2 \right) = \bar{p}_* \left((\sum_n \delta_n)^2 - 2 \sum_{n_1 < n_2} \delta_{n_1} \delta_{n_2} \right) = \frac{\sigma_1^2}{4} - \frac{\sigma_2}{2}.$$

Here σ_2 is as in [vdG99], namely the stacky cycle defined by the union of the intersections of two boundary components on a level cover. Finally from the computation of $CH^*(\mathcal{A}_2^{\mathrm{Perf}})$ in [vdG98] we have

$$\sigma_2 = 6\lambda_1 \sigma_1, \qquad \sigma_1^2 = 22\sigma_1 \lambda_1 - 120\lambda_1^2$$

and substituting this into the expression above gives $[\overline{G^{(2)}}] = 0$ as expected.

In general these computations allow us to compute the terms with $i < 3$ in (15), obtaining for any genus

Lemma 6.1. *For any genus we have the formula*

$$\left[\overline{G^{(g)}}\right] = 2^{g-1}(2^g - 1) \sum_{i=0}^{g} \lambda_{g-i} \left(\frac{\lambda_1}{2}\right)^i - 2^{2g-5}\sigma_1 \sum_{i=1}^{g} i\lambda_{g-i} \left(\frac{\lambda_1}{2}\right)^{i-1}$$

$$+ 2^{2g-8}(\sigma_1^2 - \sigma_2) \sum_{i=2}^{g} \frac{i(i-1)}{2}\lambda_{g-i} \left(\frac{\lambda_1}{2}\right)^{i-2} + P$$

where the last term denotes some polynomial in δ_n having degree at least 3.

6.2. **The genus 3 case: the locus** $\mathrm{Sym}^3(\mathcal{A}_1)$. In genus 3 it is in fact known [CM08b] that $I^{(3)} = \mathrm{Sym}^3(\mathcal{A}_1)$ as a set (in the literature on the subject it is customary to write $\mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1$ for the locus of such reducible ppav, while in fact it is really the symmetric product, and we try to make this distinction). The multiplicity is in fact equal to one as discussed in remark 2.1.

We know from [GK11, Th. 1.3] that $\overline{G^{(3)}} = \overline{I^{(3)}}$. We shall now compute the class of $\overline{G^{(3)}}$ and show that it coincides indeed with the class $[\overline{\mathrm{Sym}^3(\mathcal{A}_1)}] \in CH^*(\mathcal{A}_3^{\mathrm{Perf}})$ as computed in [vdG98], and corrected in [vdG09], thus demonstrating the way our machinery works.

Indeed, if we use Lemma 6.1, the only new ingredient needed is an expression for the terms with at least three δ's. We perform this computation in arbitrary genus for further use. To this end we first compute

$$\left(\sum_{n \in Z_m} \delta_n\right)^3 = \sum_{a \in Z_m} \delta_a^3 + 3 \sum_{a \neq b \in Z_m} \delta_a^2\delta_b + 6 \sum_{a < b < c \neq a \in Z_m} \delta_a\delta_b\delta_c.$$

By Proposition 5.2 we have 2^{2g-2} sections $\overline{F_m}$ generically vanishing on any given boundary component D_a, 2^{2g-3} vanishing on any non-empty intersection $D_a \cap D_b$, and 2^{2g-4} vanishing on any non-empty intersection $D_a \cap D_b \cap D_c$ that is *global*, i.e. over $\beta_3 \subset \mathcal{A}_g^{\mathrm{Sat}}$ (by part 1 of Proposition 5.2 there do not exist any $\overline{F_m}$ generically vanishing on a *local* triple intersection of boundary divisors, over $\beta_2^0 \subset \mathcal{A}_g^{\mathrm{Sat}}$, which would correspond to the case $a + b + c = 0$). By definition the locus $\beta_3 \in \mathcal{A}_g^{\mathrm{Perf}}(2)$ is the union of all global triple intersections, while σ_3 is the class of the union of *all* triple intersections, whether global or local. Altogether this yields

$$\sum_{m \text{ odd}} \overline{p}_* \left(\sum_{n \in Z_m} \delta_n^3\right) = 2^{2g-2}\overline{p}_*(\sum_a \delta_b^3) + 3 \cdot 2^{2g-3}\overline{p}_* \left(\sum_{a \neq b} \delta_a^2\delta_b\right) + 2^{2g-4} \cdot 6\frac{\beta_3}{2^3}$$

where on the right-hand-side the summation is over all boundary components labeled by $a, b, c \in (\mathbb{Z}/2)^{2g} \setminus 0$. The factor 6 in front of the last summand comes from the fact that every set $\{a, b, c\}$ appears 6 times in the summation, whereas the factor 8^3 comes from the branching of order 8 along each boundary component.

We now need to deal with the combinatorics of the intersections of the boundary components. To this end, observe the identity

$$\overline{p}_* \left((\sum_a \delta_a)(\sum_{a<b} \delta_a \delta_b) \right) = \frac{\sigma_1 \sigma_2}{2^3} = \overline{p}_* \left(\sum_{a \neq b} \delta_a^2 \delta_b + 3 \sum_{a<b<c} \delta_a \delta_b \delta_c \right)$$

We can thus deduce

$$\overline{p}_* \left(\sum_{a \neq b} \delta_a^2 \delta_b \right) = \frac{\sigma_1 \sigma_2 - 3\sigma_3}{8},$$

and similarly using

$$\frac{\sigma_1^3}{8} = \overline{p}_* \left((\sum_a \delta_a)^3 \right) = \overline{p}_* \left(\sum_a \delta_a^3 + 3 \sum_{a \neq b} \delta_a^2 \delta_b + 6 \sum_{a<b<c} \delta_a \delta_b \delta_c \right)$$

we deduce

$$\overline{p}_* \left(\sum_a \delta_a^3 \right) = \frac{\sigma_1^3 - 3(\sigma_1\sigma_2 - 3\sigma_3) - 6\sigma_3}{8} = \frac{\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3}{8}.$$

This finally yields, after grouping similar terms

Lemma 6.2. *The term of the polynomial P in Lemma 6.1 involving products of three boundary divisors equals*

$$- \sum_{m \text{ odd}} \overline{p}_* \left(\frac{1}{4} \sum_{n \in Z_m} \delta_n \right)^3 \sum_{i=3}^{g} \binom{i}{3} \left(\frac{\lambda_1}{2} \right)^{i-3} \lambda_{g-i}$$

$$= -2^{2g-12} \left(2\sigma_1^3 - 3\sigma_1\sigma_2 - 3\sigma_3 + 3\beta_3 \right) \sum_{i=3}^{g} \binom{i}{3} \left(\frac{\lambda_1}{2} \right)^{i-3} \lambda_{g-i}.$$

Combining this expression with the result of Lemma 6.1 we finally get for $g = 3$

$$[\overline{G^{(3)}}] = 28 \left(\lambda_3 + \lambda_2 \frac{\lambda_1}{2} + \lambda_1 \frac{\lambda_1^2}{4} + \frac{\lambda_1^3}{8} \right) - 2\sigma_1 \left(\lambda_2 + 2\lambda_1 \frac{\lambda_1}{2} + 3\frac{\lambda_1^2}{4} \right)$$

$$+ \frac{1}{4}(\sigma_1^2 - \sigma_2) \left(\lambda_1 + 3\frac{\lambda_1}{2} \right) - \frac{1}{32}(\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3) - \frac{3}{64}(\sigma_1\sigma_2 - 3\sigma_3) - \frac{3}{64}\beta_3.$$

The Chow ring $CH^*(\mathcal{A}_3^{\mathrm{Perf}})$ was computed in [vdG98], and corrected in [vdG09], and in particular it was shown (as corrected in the erratum) that

$$[\overline{\mathrm{Sym}^3(\mathcal{A}_1)}] = -35\lambda_3 + \frac{35}{2}\lambda_1^3 - \frac{25}{4}\lambda_1^2\sigma_1 + \frac{5}{8}\lambda_1\sigma_2 + \frac{5}{8}\lambda_1\sigma_1^2 - \frac{1}{12}\sigma_1\sigma_2.$$

Simplifying the expression for $[\overline{G^{(3)}}]$ above and using the relations determined in [vdG98], shows at the end of a lengthy, but straightforward, computation that we in fact have

$$[\overline{G^{(3)}}] = [\overline{\mathrm{Sym}^3(\mathcal{A}_1)}]$$

as expected.

7. Genus 4 case: the class of the locus $\mathcal{A}_1 \times \theta_{\mathrm{null}}^{(3)}$ is tautological

We will now treat the first new case, that of $g = 4$. As remarked above, for $g = 4$ the locus $\overline{I^{(4)}}$ is simply the closure of the locus of products $\mathcal{A}_1 \times \theta_{\mathrm{null}}^{(3)}$. Unlike the genus 3 case, $CH^*(\mathcal{A}_4)$ and $CH^*(\mathcal{A}_4^{\mathrm{Perf}})$ are not known (this is currently under investigation, see [HT11]), and in particular the classes of the locus $\mathcal{A}_1 \times \theta_{\mathrm{null}}^{(3)}$ and its closure are not known. However, the projections to the tautological ring of these classes can be computed, providing a good consistency check for theorem 1.4, which for $g = 4$ gives the formula

$$[I^{(4)}] = 45\lambda_1^4.$$

In this section we first compute $[\mathcal{A}_1 \times \theta_{\mathrm{null}}^{(3)}]^{\mathrm{taut}}$ directly geometrically, thus confirming our results, and then proceed to use theorem 1.3 to obtain a complete expression for this class — and not only for its tautological part.

We start by considering the locus of products $\mathcal{A}_1 \times \mathcal{A}_3$.

Lemma 7.1. *The projection of the class $[\overline{\mathcal{A}_1 \times \mathcal{A}_3}] \in CH_{\mathbb{Q}}^*(\mathcal{A}_4^{\mathrm{Perf}})$ to the tautological ring is given by*

$$[\overline{\mathcal{A}_1 \times \mathcal{A}_3}]^{\mathrm{taut}} = 20\lambda_3$$

Proof. We note that by [vdG99] there are only two degree 3 classes in the tautological ring, λ_1^3 and λ_3, and thus we know that we must have an expression of the form

$$[\overline{\mathcal{A}_1 \times \mathcal{A}_3}]^{\mathrm{taut}} = A\lambda_1^3 + B\lambda_3,$$

and it remains to determine the coefficients A and B. This can be done by pairing with the complementary dimension tautological classes. Indeed, by using the Hirzebruch-Mumford proportionality principle

[Mum77], [vdG99], these can be computed (we got the number for the top self-intersection number of λ_1 from [vdG99], and used the relation $(1 + \lambda_1 + \ldots + \lambda_g)(1 - \lambda_1 + \ldots (-1)^g\lambda_g) = 1$ from there to obtain $\lambda_3^2 = \lambda_1^3\lambda_3 - \lambda_1^6/8$ and $\lambda_1^5\lambda_3 = 7\lambda_1^8/48$)

$$\langle\lambda_1^3 \cdot \lambda_1^7\rangle_{\mathcal{A}_4^{\mathrm{Perf}}} = \frac{1}{1814400}; \quad \langle\lambda_3 \cdot \lambda_1^7\rangle_{\mathcal{A}_4^{\mathrm{Perf}}} = \frac{7}{48} \cdot \frac{1}{1814400}$$

$$\langle\lambda_1^3 \cdot \lambda_1^4\lambda_3\rangle_{\mathcal{A}_4^{\mathrm{Perf}}} = \frac{7}{48} \cdot \frac{1}{1814400}; \quad \langle\lambda_3 \cdot \lambda_1^4\lambda_3\rangle_{\mathcal{A}_4^{\mathrm{Perf}}} = \frac{1}{48} \cdot \frac{1}{1814400}.$$

We now need to compute the restrictions of the tautological classes to $\overline{\mathcal{A}_1 \times \mathcal{A}_3}$ and compute the corresponding intersection numbers there. Note that the perfect cone compactification is multiplicative [SB06], i.e. $\overline{\mathcal{A}_1 \times \mathcal{A}_3} = \mathcal{A}_1^{\mathrm{Perf}} \times \mathcal{A}_3^{\mathrm{Perf}}$. Moreover, the Hodge bundle on the locus of products is the sum of the pullbacks of the Hodge bundles on the two factors, so that we in particular have

$$(18) \qquad \lambda_1|_{\mathcal{A}_1^{\mathrm{Perf}} \times \mathcal{A}_3^{\mathrm{Perf}}} = 1 \times \lambda_1 + \lambda_1 \times 1$$

and

$$(19) \qquad \lambda_3|_{\mathcal{A}_1^{\mathrm{Perf}} \times \mathcal{A}_3^{\mathrm{Perf}}} = 1 \times \lambda_3 + \lambda_1 \times \lambda_2.$$

These expressions allow us to compute

$$\lambda_1^7[\overline{\mathcal{A}_1 \times \mathcal{A}_3}] = 7\langle\lambda_1\rangle_{\mathcal{A}_1^{\mathrm{Perf}}} \cdot \langle\lambda_1^6\rangle_{\mathcal{A}_3^{\mathrm{Perf}}} = 7 \cdot \frac{1}{24} \cdot \frac{1}{181440}$$

and

$$\lambda_1^4\lambda_3[\overline{\mathcal{A}_1 \times \mathcal{A}_3}] = \langle\lambda_1\rangle_{\mathcal{A}_1^{\mathrm{Perf}}} \cdot \langle\lambda_1^4\lambda_2\rangle_{\mathcal{A}_3^{\mathrm{Perf}}} + 4\langle\lambda_1\rangle_{\mathcal{A}_1^{\mathrm{Perf}}} \cdot \langle\lambda_1^3\lambda_3\rangle_{\mathcal{A}_3^{\mathrm{Perf}}}$$

$$= \frac{1}{24} \cdot \frac{1}{2} \cdot \frac{1}{181440} + 4\frac{1}{24} \cdot \frac{1}{8} \cdot \frac{1}{181440} = \frac{1}{24} \cdot \frac{1}{181440}$$

where for the last line we used $\lambda_2 = \lambda_1^2/2$ and computed the $\lambda_1^3\lambda_3$ intersection number from $2\lambda_1\lambda_3 = \lambda_2^2 = \lambda_1^4/4$. Combining the above computations, we then get the following two equations for A and B (after factoring out the common factor of $1/1814400$):

$$A + \frac{7}{48}B = 10 \cdot \frac{7}{24}; \qquad \frac{7}{48}A + \frac{1}{48}B = 10 \cdot \frac{1}{24}$$

and solving these gives the result of the proposition. $\qquad\square$

Corollary 7.2. *The projection of the class* $[\overline{\mathcal{A}_1 \times \theta_{\mathrm{null}}^{(3)}}]$ *to the tautological ring (with rational coefficients) is given by*

$$[\overline{\mathcal{A}_1 \times \theta_{\mathrm{null}}^{(3)}}]^{\mathrm{taut}} = 45\lambda_1^4.$$

Remark 7.3. To obtain the corollary we are basically going to intersect with a divisor, and deduce the tautological part of the product. It is tempting to say that this can be done in the tautological ring, but notice that a priori Chow is not a ring, and the projection from the Chow to the tautological ring cannot be a ring homomorphism.

Indeed, for example on $\overline{\mathcal{A}}_2$ we have $\lambda_1^2 \sigma_1 = 0$, and thus the projection of σ_1 to the tautological ring is equal to zero, while $\lambda_1 \sigma_1^2 \neq 0$, and thus the projection of σ_1^2 to the tautological ring is non-zero (in fact we have $\sigma_1^2 = 22\sigma_1\lambda_1 - 120\lambda_1^2$, see [vdG98]).

Proof. From our previous proposition we know that

$$[\overline{\mathcal{A}_1 \times \mathcal{A}_3}] = 20\lambda_3 + [X]$$

where the class $[X]$ has intersection 0 with all monomials in the λ_i of degree 7. The class of the theta-null divisor $[\overline{\theta_{\mathrm{null}}^{(3)}}]$ in $CH^*(\mathcal{A}_3^{\mathrm{Perf}})$ equals $18\lambda_1 - 2\sigma_1^{(3)}$ where $\sigma_1^{(3)}$ is the class of the boundary. We thus obtain

$$18\lambda_1[\overline{\mathcal{A}_1 \times \mathcal{A}_3}] = [18\lambda_1|_{\mathcal{A}_1^{\mathrm{Perf}}} \times \mathcal{A}_3^{\mathrm{Perf}}] + [\overline{\mathcal{A}_1 \times \theta_{\mathrm{null}}^{(3)}}]$$

$$+[\mathcal{A}_1^{\mathrm{Perf}} \times 2\sigma_1^{(3)}] + 18\lambda_1[X].$$

We now deal with the summands term by term. For the first term, we compute

$$[18\lambda_1|_{\mathcal{A}_1^{\mathrm{Perf}}} \times \mathcal{A}_3^{\mathrm{Perf}}] = \frac{3}{2}[\mathcal{A}_0 \times \mathcal{A}_3^{\mathrm{Perf}}]$$

(keeping in mind the stackiness, so that λ_1 has degree $1/24$ on $\mathcal{A}_1$ and $\mathcal{A}_0$ has degree $1/2$). The tautological projection of the class $[\mathcal{A}_0 \times \mathcal{A}_3^{\mathrm{Perf}}]$ was computed in [vdG99, Prop. 4.3], resp. [EvdG05, Theorem 3.4], and we thus get

$$\frac{3}{2}[\mathcal{A}_0 \times \mathcal{A}_3^{\mathrm{Perf}}]^{\mathrm{taut}} = \frac{3\lambda_4}{2\zeta(-7)} = 360\lambda_1\lambda_3 - 45\lambda_1^4.$$

Let us now compute the tautological part of $[\mathcal{A}_1^{\mathrm{Perf}} \times 2\sigma_1^{(3)}]$ (note that lemmas 4.1 and 4.2 do not work here). Indeed, this is a codimension 4 locus, and since we have $\lambda_1^6 = 8\lambda_1^3\lambda_3 - 8\lambda_3^2$, to compute the tautological part we need to know the intersections with λ_3^2 and with $\lambda_1^3\lambda_3$. Using formulae (18) and (19) and recalling from [vdG99, Lemma 3.11] that in $CH^*(\mathcal{A}_3^{\mathrm{Perf}})$ we have $\lambda_3\sigma_1 = 0$, we thus get

$$\lambda_3[\mathcal{A}_1^{\mathrm{Perf}} \times 2\sigma_1^{(3)}] = [\lambda_1|_{\mathcal{A}_1^{\mathrm{Perf}}} \times \lambda_1^2\sigma_1^{(3)}],$$

from which the intersection numbers are

$$\lambda_3^2[\mathcal{A}_1^{\mathrm{Perf}} \times 2\sigma_1^{(3)}] = 0$$

since λ_1^2 is zero in dimension one on $\mathcal{A}_1^{\mathrm{Perf}}$ and

$$\lambda_1^3 \lambda_3 [\mathcal{A}_1^{\mathrm{Perf}} \times 2\sigma_1^{(3)}] = 0$$

since $\lambda_3^5 \sigma_1 = 0$ on $\mathcal{A}_3^{\mathrm{Perf}}$. Thus $[\mathcal{A}_1^{\mathrm{Perf}} \times 2\sigma_1^{(3)}]^{\mathrm{taut}} = 0$.

Since $[X]$ lies in the orthogonal complement of the tautological ring the same holds for $\lambda_1[X]$ and the assertion follows. $\qquad\square$

The above computation matches the result of our theorem 1.4 for $g = 4$. We now proceed to compute the class completely.

Proposition 7.4. *The class of the locus is*

$$[\overline{I^{(4)}}] = \overline{[\mathcal{A}_1 \times \theta_{\mathrm{null}}^{(3)}]} = 180\lambda_1\lambda_3 + \frac{45}{2}\lambda_1^4 - 8\sigma_1\lambda_3 - 14\sigma_1\lambda_1^3 + \frac{7}{2}\lambda_1^2\sigma_1^2$$

$$-\frac{7}{2}\lambda_1^2\sigma_2 - \frac{3}{8}\lambda_1\sigma_1^3 + \frac{9}{16}\lambda_1\sigma_1\sigma_2 + \frac{9}{16}\lambda_1\sigma_3 - \frac{9}{16}\lambda_1\beta_3 + \frac{3}{64}Y + \frac{1}{64}\sigma_4$$

$$-\frac{1}{16}\sigma_1\sigma_3 + \frac{3}{64}\sigma_1\beta_3 + \frac{1}{64}\sigma_2^2 - \frac{1}{32}\sigma_1^2\sigma_2 + \frac{1}{64}\sigma_1^4 \quad \in CH^*(\mathcal{A}_4^{Perf})$$

where the class Y is defined as

$$Y := \sum_{n_1 < n_2 < n_3 < n_4; n_1+n_2+n_3+n_4=0} \delta_{n_1}\delta_{n_2}\delta_{n_3}\delta_{n_4}.$$

Remark 7.5. One can understand the locus with class Y geometrically, similarly to the way the discriminant locus $\Delta = \sigma_3 - \beta_3$ can be described as the class of the stratum of semiabelic varieties whose normalization has two components, each of which is a $\mathbb{P}^2$ bundle over some ppav $B \in \mathcal{A}_{g-2}$. Indeed, by inspecting the table of perfect cones for all possible small codimension strata in $\mathcal{A}_g^{\mathrm{Perf}}$ we see that Y is in fact equal to the locus of semiabelic varieties of torus rank 3 (since the dimension of the linear span of n_i is three), for which the normalization is a union of two $\mathbb{P}^3$ bundles and a $F(2,2)$ bundle. To prove this, we just observe that the corresponding cone is generated by $x_1^2, x_2^2, x_3^2, (x_1 + x_2 + x_3)^2$, giving the relation among the n_i as in the definition of Y. We shall discuss this in more detail in remark 8.2.

Remark 7.6. It is a very appealing question to try to define and describe a suitably "extended" tautological ring of a toroidal compactification of $\mathcal{A}_g$ (in particular, of the perfect cone compactification). In addition to the tautological classes λ_i, presumably such an extended ring would include the classes $\lambda_i, \sigma_i, \beta_i$. In view of the above computation, it would also be natural to include Y and the classes of all strata in the boundary corresponding to various types of semi-abelic varieties. Note that the proof below shows that Y does not lie in the

ring generated by σ_i and β_i, but it could lie in the ring generated by these together with λ_i — we were unable to determine this.

Proof of proposition 7.4. Note that one a priori technical difficulty is that $\mathcal{A}_4^{\mathrm{Perf}}$ is singular. However, it is only singular in codimension 10 (at the one point corresponding to the only non basic cone in the perfect cone compactification of genus 4, namely the second perfect cone, see [HS04]). The computations we do are for classes of codimension at most 4, and thus the singularity does not matter for these computations.

Indeed, applying the formulas from lemmas 6.1 and 6.2 and using the relations among the λ classes from [vdG99] we get

$$[G^{(4)}] = 180\lambda_1\lambda_3 + \frac{45}{2}\lambda_1^4 - 8\sigma_1\lambda_1 - 14\sigma_1\lambda_1^3 + \frac{7}{2}\lambda_1^2\sigma_1^2 - \frac{7}{2}\lambda_1^2\sigma_2$$

$$-\frac{3}{16}\lambda_1(2\sigma_1^3 - 3\sigma_1\sigma_2 - 3\sigma_3 + 3\beta_3)$$

$$+\frac{1}{4^4}\sum_{m \text{ odd}} \bar{p}_*\Big(24\sum_{n_1<n_2<n_3<n_4\in Z_m}\delta_{n_1}\delta_{n_2}\delta_{n_3}\delta_{n_4} + 12\sum_{n_1;n_2<n_3\in Z_m}\delta_{n_1}^2\delta_{n_2}\delta_{n_3}$$

$$+6\sum_{n_1<n_2\in Z_m}\delta_{n_1}^2\delta_{n_2}^2 + 4\sum_{n_1;n_2\in Z_m}\delta_{n_1}^3\delta_{n_2} + \sum_{n\in Z_m}\delta_n^4\Big).$$

where all the sums are taken over all $n_i \in Z_m$ that are pairwise distinct, and the coefficient of some monomial $\delta_{n_1}^{a_1}\cdot\ldots\cdot\delta_{n_k}^{a_k}$ is equal to the binomial coefficient $\binom{\sum a_i}{a_1,\ldots,a_k}$

We now need to go through the possible combinatorics of the intersections. Note that any intersection $D_a \cap D_b$ for $a \neq b$, if non-empty, "lives over β_2" (which is to say that it intersects β_2^0 and does not intersect β_1^0), while for triple intersections we know that $D_a \cap D_b \cap D_{a+b}$ lives over β_2, and all other triple intersections live over β_3. Similarly the quadruple intersections that live over β_3 are $D_a \cap D_b \cap D_c \cap D_{a+b}$ and $D_a \cap D_b \cap D_c \cap D_{a+b+c}$, while all the other ones live over β_4, and in fact the other intersections together give β_4.

Using Proposition 5.2, and in particular that we cannot have $n_1 + n_2 + n_3 = 0$ for elements of Z_m, we thus compute for arbitrary g (notice that the sums on the right are now over all n_i, not just those in Z_m; we suppress the fact that n_i are ordered, and use n for the index that is not ordered relative to n_i):

$$\sum_m \sum_{n_1<n_2<n_3<n_4\in Z_m}\delta_{n_1}\delta_{n_2}\delta_{n_3}\delta_{n_4} = 2^{2g-4}\sum_{n_1+n_2+n_3+n_4=0}\delta_{n_1}\delta_{n_2}\delta_{n_3}\delta_{n_4} + 2^{2g-5}\beta_4;$$

$$\sum_m \sum_{n_1;n_2<n_3\in Z_m}\delta_{n_1}^2\delta_{n_2}\delta_{n_3} = 2^{2g-4}\sum_{n_1+n_2+n_3\neq 0;n_2<n_3}\delta_{n_1}^2\delta_{n_2}\delta_{n_3};$$

$$\sum_m \sum_{n_1 < n_2 \in Z_m} \delta_{n_1}^2 \delta_{n_2}^2 = 2^{2g-3} \sum \delta_{n_1}^2 \delta_{n_2}^2,$$

$$\sum_m \sum_{n_1, n_2 \in Z_m} \delta_{n_1}^3 \delta_{n_2} = 2^{2g-3} \sum \delta_{n_1}^3 \delta_{n_2}.$$

We now try to express the right-hand-sides as linear combinations of the degree four classes in the algebra generated by $\sigma_1 = \beta_1, \sigma_2 = \beta_2, \sigma_3, \beta_3, \sigma_4$, the corresponding expressions for which are

$$\sigma_4 - \beta_4 = \sum_{n_1+n_2+n_3+n_4=0} \delta_{n_1}\delta_{n_2}\delta_{n_3}\delta_{n_4} + \sum_{n_1+n_2+n_3=0;n} \delta_{n_1}\delta_{n_2}\delta_{n_3}\delta_n;$$

$$\sigma_1\beta_3 = 4\beta_4 + 4 \sum_{n_1+n_2+n_3+n_4=0} \delta_{n_1}\delta_{n_2}\delta_{n_3}\delta_{n_4} + 3 \sum_{n_1+n_2+n_3=0;n} \delta_{n_1}\delta_{n_2}\delta_{n_3}\delta_n$$

$$+ \sum_{n_1+n_2+n_3 \neq 0; n_2 < n_3} \delta_{n_1}^2 \delta_{n_2}\delta_{n_3},$$

$$\sigma_1(\sigma_3 - \beta_3) = \sum_{n_1+n_2+n_3=0;n} \delta_{n_1}\delta_{n_2}\delta_{n_3}\delta_n + \sum \delta_{n_1+n_2}^2 \delta_{n_1}\delta_{n_2}$$

for the terms involving some combinatorics. Notice that on the right-hand-side we have 4 different unknown terms, and thus we would not be able to express all of them in terms of these classes, so that the resulting expression will have to involve Y.

For the rest of the terms there is no combinatorics of the intersections involved, and we compute

$$\sigma_2^2 = 6 \sum_{\text{all}} \delta_{n_1}\delta_{n_2}\delta_{n_3}\delta_{n_4} + 2 \sum_{\text{all}} \delta_{n_1}^2 \delta_{n_2}\delta_{n_3} + \sum_{\text{all}} \delta_{n_1}^2 \delta_{n_2}^2;$$

$$\sigma_1^2\sigma_2 = 12 \sum_{\text{all}} \delta_{n_1}\delta_{n_2}\delta_{n_3}\delta_{n_4} + 5 \sum_{\text{all}} \delta_{n_1}^2 \delta_{n_2}\delta_{n_3} + 2 \sum_{\text{all}} \delta_{n_1}^2 \delta_{n_2}^2 + \sum_{\text{all}} \delta_{n_1}^3 \delta_{n_2};$$

$$\sigma_1^4 = 24 \sum_{\text{all}} \delta_{n_1}\delta_{n_2}\delta_{n_3}\delta_{n_4} + 12 \sum_{\text{all}} \delta_{n_1}^2 \delta_{n_2}\delta_{n_3} + 6 \sum_{\text{all}} \delta_{n_1}^2 \delta_{n_2}^2;$$

$$+ 4 \sum_{\text{all}} \delta_{n_1}^3 \delta_{n_2} + \sum_{\text{all}} \delta_n^4$$

for the remaining terms (in the last three expressions we sum over all possible dimensions of linear spans of $\{n_i\}$ on the right).

Combining all of these allows us to first express all symmetric polynomials in D_n in terms of the standard symmetric polynomials, and then we express all the terms involving various combinatorics of the intersections in terms of these classes and Y, obtaining the expression as claimed. □

For further use we record the result of the above computation for arbitrary g.

Lemma 7.7. *For arbitrary g the order 4 term in proposition 6.1 is equal to*

$$\frac{2^{2g-2}}{8^4}\left(6Y + 3\beta_4 + 3(\sigma_1\beta_3 - \beta_4 - Y - 3\sigma_4) + 3(\sigma_2^2 - 2\sigma_1\sigma_3 + 2\sigma_4)\right.$$

$$\left.+2(\sigma_1^2\sigma_2 - 2\sigma_2^2 - \sigma_1\sigma_3 + 4\sigma_4) + \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 4\sigma_1\sigma_3 - 4\sigma_4\right)$$

$$= 2^{2g-14}\left(\sigma_4 - 4\sigma_3\sigma_1 + 3Y + 3\sigma_1\beta_3 + \sigma_2^2 - 2\sigma_2\sigma_1^2 + \sigma_1^4\right)$$

times the corresponding polynomial in λ classes.

8. Genus 5: the locus of intermediate Jacobians

In this section we finally compute the class in the Chow ring of the locus $I^{(5)}$, that is of the locus of intermediate Jacobians of cubic threefolds together with the locus of products $\mathcal{A}_1 \times \theta_{\text{null}}^{(4)}$. We then compute the projection to the tautological ring of each of these two loci. The computation is similar to the one in the previous section, but much more involved, with more new classes appearing. Again we shall first work on the smooth part of $\mathcal{A}_5^{\text{perf}}$. In this case we observe that the codimension of the singular locus is at least 6, which is sufficient for our purpose: away from β_5 we know it to be 10, whereas on β_5 there is only one stratum of codimension 5 and this corresponds to the principal cone, which is basic.

Proposition 8.1. *We have in $CH^*(\mathcal{A}_5^{\text{Perf}})$ the formula*

$$[\overline{IJ}] + [\overline{\mathcal{A}_1 \times \theta_{\text{null}}^{(4)}}] = 496\lambda_5 + 372\lambda_3\lambda_1^2 + \frac{93}{2}\lambda_1^5 - 64\lambda_3\lambda_1\sigma_1 - 34\lambda_1^4\sigma_1$$

$$+(4\lambda_3 + 14\lambda_1^3)(\sigma_1^2 - \sigma_2) + \frac{5}{4}\lambda_1^2\left(3\sigma_3 + 3\sigma_2\sigma_1 - 2\sigma_1^3\right)$$

$$+\frac{7}{32}\lambda_1(\sigma_4 - 4\sigma_3\sigma_1 + 3Y + 3\beta_3\sigma_1 + \sigma_2^2 - 2\sigma_2\sigma_1^2 + \sigma_1^4)$$

$$-\frac{1}{256}\left(-95\sigma_5 - 30\beta_5 - 45A_2 - 30A_3 - 15A_4 + 15C_1 + 10D_1\right.$$

$$+45\sigma_4\sigma_1 + 15\beta_4\sigma_1 + 30Y\sigma_1 + 5\sigma_3\sigma_2 - 15\sigma_3\sigma_1^2 + 5\sigma_2^2\sigma_1 - 5\sigma_2\sigma_1^3 + 2\sigma_1^5)$$

where the classes A_2, A_3, A_4, C_1, D_1 are defined below, and we refer to remark 8.2 for a geometric interpretation of the classes A_2, A_3, A_4.

Proof. We shall again use formula (15). Similar to the above computations, the new term we need to deal with here is $\sum_m (\sum_{n \in Z_m} \delta_n)^5$. Thus we will need to handle all products of five δ_n, with indices distinct or equal, and with all possible combinatorics of the intersections. The computations will be still more involved, and we proceed systematically, first listing all the terms that could appear. Recall that by convention the indices $n_1, \ldots, n_k$, and separately the indices $m_1, \ldots, m_\ell$ are ordered, and that all the linear relations among indices are stated under the sums. We then have the following types of products of 5 boundary components:

$$A_1 := \sum \delta_{n_1} \delta_{n_2} \delta_{n_3} \delta_{n_4} \delta_{n_5}; \quad A_2 := \sum_{n_1+n_2+n_3+n_4+n_5=0} \delta_{n_1} \delta_{n_2} \delta_{n_3} \delta_{n_4} \delta_{n_5};$$

$$A_3 := \sum_{n_1+n_2+n_3+n_4=0} \delta_{n_1} \delta_{n_2} \delta_{n_3} \delta_{n_4} \delta_n; \quad A_4 := \sum_{n_1+n_2+n_3=0} \delta_{n_1} \delta_{n_2} \delta_{n_3} \delta_{m_1} \delta_{m_2};$$

$$A_5 := \sum_{n+n_1+n_2=n+m_1+m_2=0} \delta_n \delta_{n_1} \delta_{n_2} \delta_{m_1} \delta_{m_2};$$

$$B_1 := \sum \delta_n^2 \delta_{n_1} \delta_{n_2} \delta_{n_3}; \quad B_2 := \sum_{n+n_1+n_2+n_3=0} \delta_n^2 \delta_{n_1} \delta_{n_2} \delta_{n_3};$$

$$B_3 := \sum_{n+n_1+n_2=0} \delta_n^2 \delta_{n_1} \delta_{n_2} \delta_m; \quad B_4 := \sum_{n_1+n_2+n_3=0} \delta_n^2 \delta_{n_1} \delta_{n_2} \delta_{n_3};$$

$$C_1 := \sum \delta_{n_1}^2 \delta_{n_2}^2 \delta_n; \quad C_2 := \sum_{n_1+n_2+n=0} \delta_{n_1}^2 \delta_{n_2}^2 \delta_n;$$

$$D_1 := \sum \delta_n^3 \delta_{n_1} \delta_{n_2}; \quad D_2 := \sum_{n+n_1+n_2=0} \delta_n^3 \delta_{n_1} \delta_{n_2};$$

$$E := \sum \delta_n^3 \delta_k^2; \quad F := \sum \delta_n^4 \delta_k; \quad G := \sum \delta_n^5$$

Remark 8.2. Some of these loci — the ones where each δ_n appears to power one — have geometric interpretations. Indeed, recall that the strata of a toroidal compactification correspond to orbits of cones in the corresponding fan. The perfect cone compactification $\mathcal{A}_g^{\mathrm{Perf}}$ is given, as the name indicates, by the perfect cone decomposition of $\mathrm{Sym}_{\geq 0}^2(\mathbb{R}^g)$ and $\mathcal{A}_g^{\mathrm{Vor}}$ is given by the second Voronoi decomposition. For genus $g \leq 3$ these two toroidal compactifications coincide, whereas for $g = 4, 5$, but not in general, the second Voronoi decomposition is a refinement of the perfect cone decomposition. In other words, $\mathcal{A}_g^{\mathrm{Vor}}$ is a blow-up of $\mathcal{A}_g^{\mathrm{Perf}}$ for $g = 4, 5$ and by inspection of the decompositions one can see that the center of the blow-up $\mathcal{A}_g^{\mathrm{Vor}} \to \mathcal{A}_g^{\mathrm{Perf}}$ has codimension > 5. We also recall that, due to the moduli interpretation of $\mathcal{A}_g^{\mathrm{Vor}}$ (see [Ale02]),

the strata of this toroidal compactification have an interpretation in terms of polarized semi-abelic varieties. In [GK11, section 3] we have enumerated the relevant cones of small codimension and described the corresponding semi-abelic varieties. We can use this description to give a geometric interpretation for all of the above loci where each δ_n appears to power one.

First, we note that $A_1 = \beta_5$. Then observe that A_2 is the class of the locus of semi-abelic varieties of torus rank 4 the normalization of which is a union of two $\mathbb{P}^4$ bundles and one X bundle (the toric polytope corresponding to X is the 4-dimensional cube with two simplices removed) — the corresponding cone is generated by $x_1^2, x_2^2, x_3^2, x_4^2, (x_1 + x_2 + x_3 + x_4)^2$ and has the correct linear dependency. The class A_3 is the class of the stratum where the corresponding cone is generated by $x_1^2, x_2^2, x_3^2, x_4^2, (x_1 + x_2 + x_3)^2$, and the normalization of the semiabelic variety is the union of two $\mathbb{P}^1 \times \mathbb{P}^3$ bundles and one $\mathbb{P}^1 \times F(2,2)$ bundle. Similarly, A_4 corresponds to the stratum where the corresponding cone is $x_1^2, x_2^2, x_3^2, x_4^2, (x_1 + x_2)^2$. The normalization of such a semiabelic variety is two copies of a $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ bundle. Finally, A_5 corresponds to the cone $x_1^2, x_2^2, x_3^2, (x_1 + x_2)^2, (x_1 + x_3)^2$, and the normalization of the corresponding semiabelic varieties consists of two copies of a $\mathbb{P}^3$ bundle and two copies of an F_2 (singular cone over $\mathbb{P}^1 \times \mathbb{P}^1$) bundle.

Denoting by A the sum of all A_i, etc., we now express all degree 5 elements of the ring generated by σ_i and β_i in terms of the classes above. The expressions we get for polynomials in σ_i are standard expressions in the algebra of symmetric polynomials, while in expressions involving β_i the combinatorics of the indices plays a role — some summands are missing, and the coefficients differ depending on the combinatorics. Indeed we have

$$\sigma_5 = A; \quad \sigma_1\sigma_4 = 5A + B; \quad \sigma_2\sigma_3 = 10A + 3B + C;$$
$$\sigma_1^2\sigma_3 = 20A + 7B + 2C + D; \quad \sigma_1\sigma_2^2 = 30A + 12B + 5C + 2D + E;$$
$$\sigma_1^3\sigma_2 = 60A + 27B + 12C + 7D + 3E + F;$$
$$\sigma_1^5 = 120A + 60B + 30C + 20D + 10E + 5F + G$$

for the symmetric polynomials, and also

$$\beta_5 = A_1; \quad \sigma_1\beta_4 = 5A_1 + 5A_2 + 4A_3 + 3A_4 + B_1;$$
$$\sigma_2\beta_3 = 10A_1 + 10A_2 + 10A_3 + 9A_4 + 8A_5 + 3B_1 + 3B_2 + 2B_3 + 3B_4 + C_1;$$
$$\sigma_1^2\beta_3 = 20A_1 + 20A_2 + 20A_3 + 18A_4 + 16A_5 + 7B_1 + 7B_2 + 5B_3 + 6B_4 + 2C_1 + D_1.$$

We want to use these identities to express all these unknown classes in terms of the smallest possible number of unknowns. We first note that from symmetric polynomial expressions we get expressions for

A, B, C, D, E, F, G in the ring generated by σ_i (and thus writing down $A = A_1 + A_2 + A_3 + A_4 + A_5$ gives a non-trivial identity among the codimension 5 boundary strata — which we then use to eliminate A_5 from the formulas). From the expression for $\sigma_1 \beta_4$ we can express B_1 in terms of the geometrically defined classes A_i. We can further note that $Y \sigma_1 = A_3 + A_5 + B_2$, which provides a geometric description of B_2.

Using proposition 5.2, we can now deal with the fifth order term appearing in the class computation in Theorem 1.3. Indeed, we note that no terms where there is a linear relation of odd length occur, and thus compute

$$\frac{1}{4^5} p_* \sum_{m \text{ odd}} \left(\sum_{n \in Z_m} \delta_n \right)^5 = \frac{1}{8^5} \left(120(2^{2g-6}A_1 + 2^{2g-5}A_3) \right.$$

$$+60(2^{2g-5}B_1 + 2^{2g-4}B_2) + 30(2^{2g-4}C_1) + 20(2^{2g-4}D_1)$$

$$\left. +10 \cdot 2^{2g-3}E + 5 \cdot 2^{2g-3}F + 2^{2g-2}G \right)$$

$$= 2^{2g-18} \left(15A_1 + 30A_3 + 15B_1 + 30B_2 + 15C_1 + 10D_1 \right.$$

$$\left. +10E + 5F + 2G \right).$$

To obtain an expression for this class in terms of the classes previously introduced, we note that B_1 and B_2 can be expressed in terms of the A_i and $Y \sigma_1$, that we have eliminated A_5, but that there is no way to avoid using C_1 and D_1. Doing this computation in Maple yields the result of the proposition. Of course using the formulas for $\sigma_2 \beta_3$ and $\sigma_1^2 \beta_3$, we can express C_1 and D_1 in terms of B_3, B_4, and the other classes, but it is not clear why this would be a better expression. $\qquad \square$

The proposition above computes the class of the locus $[I^{(5)}]$. To be able to compute the class of the locus of intermediate Jacobians, we need to compute the class of the other component. This is accomplished similarly to the computations in the previous section.

Proposition 8.3. *The projection of the class $[\overline{\mathcal{A}_1 \times \mathcal{A}_4}] \in CH^*_{\mathbb{Q}}(\mathcal{A}_5^{\text{Perf}})$ to the tautological ring is given by*

$$[\overline{\mathcal{A}_1 \times \mathcal{A}_4}]^{\text{taut}} = -\frac{11}{8}\lambda_1^4 + 11\lambda_1\lambda_3.$$

In particular, if the locus of products $[\mathcal{A}_1 \times \mathcal{A}_4] \in CH^*(\mathcal{A}_5)$ is tautological, then $[\mathcal{A}_1 \times \mathcal{A}_4] = -\frac{11}{8}\lambda_1^4 + 11\lambda_1\lambda_3$.

Proof. The proof proceeds similarly to the proof of lemma 7.1 for genus 4. From van der Geer's defining relation [vdG99, (1)] of the tautological ring we get the following explicit identities in $CH^*(\mathcal{A}_5^{\mathrm{Perf}})$:

$$\lambda_3^2 = \lambda_1^3\lambda_3 - \frac{\lambda_1^6}{8} - 2\lambda_1\lambda_5; \qquad \lambda_1^5\lambda_3 = \frac{7}{48}\lambda_1^8 + \frac{8\lambda_5}{3}(\lambda_1^3 + \lambda_3); \qquad \lambda_5^2 = 0.$$

Multiplying the first of these by λ_1^5 and the second by λ_3 and equating the results further gives

$$\lambda_1^3\lambda_3\lambda_5 = \frac{1}{5}\lambda_1^6\lambda_5 - \frac{1}{7040}\lambda_1^{11};$$

multiplying this one by λ_1^2 and the second by λ_5 and equating the results finally yields

$$\lambda_1^8\lambda_5 = \frac{3}{1144}\lambda_1^{13}.$$

Using these relations, we can compute explicitly all the top intersection numbers in the ring generated by the λ_i classes (of course, these also follow from the Hirzebruch-Mumford proportionality, but we could not find an easily available reference for intersection numbers on the symplectic Grassmannian).

Indeed, from the Hirzebruch-Mumford proportionality (see [vdG99] for the formula) we get

$$\langle \lambda_1^{15} \rangle_{\mathcal{A}_5^{\mathrm{Perf}}} = \frac{13}{16329600}.$$

Using the above relations we compute step by step

$$\langle \lambda_1^{12}\lambda_3 \rangle_{\mathcal{A}_5^{\mathrm{Perf}}} = \langle \frac{8}{3}\lambda_1^7\lambda_3\lambda_5 + \frac{8}{3}\lambda_1^{10}\lambda_5 + \frac{7}{48}\lambda_1^{15} \rangle_{\mathcal{A}_5^{\mathrm{Perf}}}$$

$$= \langle \frac{16}{5}\lambda_1^{10}\lambda_5 + \frac{8}{55}\lambda_1^{15} \rangle_{\mathcal{A}_5^{\mathrm{Perf}}}$$

$$= \frac{2}{13}\langle \lambda_1^{15} \rangle_{\mathcal{A}_5^{\mathrm{Perf}}} = \frac{2}{16329600}$$

and then

$$\langle \lambda_1^9\lambda_3^2 \rangle_{\mathcal{A}_5^{\mathrm{Perf}}} = \langle \lambda_1^{12}\lambda_3 - 2\lambda_1^{10}\lambda_5 - \frac{1}{8}\lambda_1^{15} \rangle_{\mathcal{A}_5^{\mathrm{Perf}}}$$

$$= \left(\frac{2}{13} - \frac{3}{572} - \frac{1}{8} \right)\langle \lambda_1^{15} \rangle_{\mathcal{A}_5^{\mathrm{Perf}}} = \frac{1}{53222400}.$$

From the fact that the Hodge bundle on $\mathcal{A}_5$ restricts to the sum of Hodge bundles on the factors of a decomposable ppav, we get, as in lemma 7.1,

$$\lambda_1^{11}[\overline{\mathcal{A}_1 \times \mathcal{A}_4}] = 11\langle \lambda_1 \rangle_{\mathcal{A}_1^{\mathrm{Perf}}} \cdot \langle \lambda_1^{10} \rangle_{\mathcal{A}_4^{\mathrm{Perf}}} = \frac{11}{24} \cdot \frac{1}{1814400}$$

and

$$\lambda_1^8 \lambda_3 [\overline{\mathcal{A}_1 \times \mathcal{A}_4}] = \langle \lambda_1 \rangle_{\mathcal{A}_1^{\mathrm{Perf}}} \cdot \langle \lambda_1^8 \lambda_2 \rangle_{\mathcal{A}_4^{\mathrm{Perf}}} + 8 \langle \lambda_1 \rangle_{\mathcal{A}_1^{\mathrm{Perf}}} \cdot \langle \lambda_1^7 \lambda_3 \rangle_{\mathcal{A}_4^{\mathrm{Perf}}}$$

$$= \frac{1}{24} \cdot \frac{1}{2} \cdot \frac{1}{1814400} + 8 \cdot \frac{1}{24} \cdot \frac{7}{48} \cdot \frac{1}{1814400} = \frac{5}{72} \cdot \frac{1}{1814400}$$

where we have used the computation of $\langle \lambda_1^7 \lambda_3 \rangle_{\mathcal{A}_4^{\mathrm{Perf}}}$ from lemma 7.1.

Thus if we have $[\overline{\mathcal{A}_1 \times \mathcal{A}_4}] = A\lambda_1^4 + B\lambda_1\lambda_3$ in the tautological ring, we can compute the coefficients from the above relations, and the result is as stated. $\qquad\square$

Similarly to the $g = 4$ case, we can then compute the projection of the class of the theta-null divisor to the tautological ring.

Corollary 8.4. *The projection of the class $[\overline{\mathcal{A}_1 \times \theta_{\mathrm{null}}^{(4)}}]$ to the tautological ring with rational coefficients is given by*

$$[\overline{\mathcal{A}_1 \times \theta_{\mathrm{null}}^{(4)}}]^{\mathrm{taut}} = 187(-\lambda_1^5/2 + 4\lambda_1^2\lambda_3 - 4\lambda_5).$$

Proof. Indeed, similarly to the proof of corollary 7.2, let $[X] \in CH^4(\mathcal{A}_5)$ be the part of the class of $[\overline{\mathcal{A}_1 \times \mathcal{A}_4}]$ orthogonal to the tautological ring. Since the class of $[\overline{\theta_{\mathrm{null}}^{(4)}}]$ is equal to $68\lambda_1 - 8\sigma_1^{(4)}$, we compute

$$68\lambda_1[\overline{\mathcal{A}_1 \times \mathcal{A}_4}] = [68\lambda_1|_{\mathcal{A}_1^{\mathrm{Perf}} \times \mathcal{A}_3^{\mathrm{Perf}}}] + [\overline{\mathcal{A}_1 \times \theta_{\mathrm{null}}^{(4)}}] + 68\lambda_1[X] + 8[\mathcal{A}_1^{\mathrm{Perf}} \times \sigma_1^{(4)}].$$

The class $\lambda_1[X]$ is orthogonal to the tautological ring, and we need to argue that so is the last class. Indeed, it is codimension 5 in $\mathcal{A}_5^{\mathrm{Perf}}$, but by the restriction property of the Hodge bundle it is enough to argue that $\sigma_1^{(4)}$ is orthogonal to the tautological ring of $\mathcal{A}_4^{\mathrm{Perf}}$, which follows from lemma 4.1.

Thus using the above proposition together with the formula $[\mathcal{A}_0 \times \mathcal{A}_4^{\mathrm{Perf}}]^{\mathrm{taut}} = -\lambda_5/\zeta(-9)$ from [vdG99, prop. 4.3] (and recalling again that $\deg \mathcal{A}_0 = 1/2$, as of a stack, and thus $\lambda_1|_{\mathcal{A}_0} = 1/12[\mathcal{A}_0]$), we obtain

$$-2 \cdot \frac{68\lambda_5}{24\zeta(-9)} + [\overline{\mathcal{A}_1 \times \theta_{\mathrm{null}}^{(4)}}]^{\mathrm{taut}} = 187(-\lambda_1^5/2 + 4\lambda_1^2\lambda_3).$$

Recalling $\zeta(-9) = -1/132$ proves the claim. $\qquad\square$

Combining this with theorem 1.4 finally yields proposition 1.5, giving a formula for $[\overline{IJ}]^{\mathrm{taut}}$.

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